Spacetime Dynamics of Rotating Compact Stars

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A) Quasi-normal modes (w-modes) of rotating compact stars

- Mode calculations in the frequency domain
- Time-evolutions of test-scalar fields
- The ergoregion instability
- Excitation of marginally stable/unstable modes

B) Late-time tails (in progress)

- Rotating stars vs rotating BHs
- Tails of slowly rotating stars

Some background: uniform density stars

- As a stellar model, we choose the celebrated Schwarzschild interior solution which describes a *uniform-density, spherically-symmetric star, made of perfect fluid.* In addition we take into account the *leading order rotational corrections* (the star retains its spherical shape at that order, metric stationary and axisymmetric).

- The resulting metric takes the form:

$$ds^2 = -e^{\nu} dt^2 - 2\omega r^2 \sin^2\theta dt d\phi + e^{\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

  where ν,λ and ω are functions of r only. If we denote by R the stellar radius then for $r > R$ we have,

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2M}{r}, \quad M = 4\pi \int_0^R \rho r^2 dr = \frac{4\pi}{3}\rho R^3$$

  where M is the star's total mass and ρ is the (constant) density. In the stellar interior,

$$\begin{aligned} e^{-\lambda} &= 1 - \frac{2m(r)}{r} = 1 - \frac{2Mr^2}{R^3} \\ e^{\nu} &= \left[\frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}\right]^2 \end{aligned}$$

  with $m(r)$ denoting the mass contained within a sphere of radius r.

- Stellar configuration stable for $R/M > 2.25$. Recall that a typical neutron star has $R/M \sim 5$ while for a white dwarf $R/M \sim 10^4$. Ultracompact stars are still speculative objects.

- The star's angular velocity can be expressed as,

$$\Omega = \epsilon\sqrt{\frac{M}{R^3}}$$

where the parameter ϵ can take any value up to ~ 0.7; (which justifies the adoption of a slow-rotation framework $\omega \sim \mathcal{O}(\Omega)$). In the vacuum outside the star we have,

$$\omega = \frac{2J}{r^3}$$

where J is the total angular momentum of the star. In the interior, the function $\omega(r)$ satisfies a simple ODE.

- The scalar field Φ satisfies the wave equation $\nabla^a \nabla_a \Phi = 0$. *Note that scalar field perturbations do not couple to any fluid perturbations.* Assuming a decomposition,

$$\Phi = \frac{1}{r}\psi(r,t) Y_{\ell m}(\theta, \phi)$$

For the spacetime under discussion we get the following "master" equation:

$$\frac{\partial^2 \psi}{\partial t^2} + 2im\omega \frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial r_*^2} + V(r)\psi = 0$$

where we have introduced the "tortoise" radial coordinate

$$dr_*/dr = e^{(\lambda-\nu)/2}$$

Assuming a harmonic time-dependence $\sim e^{-i\sigma t}$ it gives,

$$\frac{\partial^2 \psi}{\partial r_*^2} + \{\sigma^2 - 2m\omega\sigma - V(r)\}\psi = 0$$

At infinity the solution behaves as,

$$\psi \sim A^{\mathrm{in}} e^{-i\sigma r_*} + A^{\mathrm{out}} e^{+i\sigma r_*}$$

- The QNM condition is $A^{\mathrm{in}}/A^{\mathrm{out}} = 0$ for some frequency $\sigma = \sigma_n$ (together with regularity of ψ at the origin).

- We can make several "predictions" regarding w-modes just by looking how the effective potential $Q = V(r) - \sigma^2 + 2m\omega\sigma$, changes with rotation:

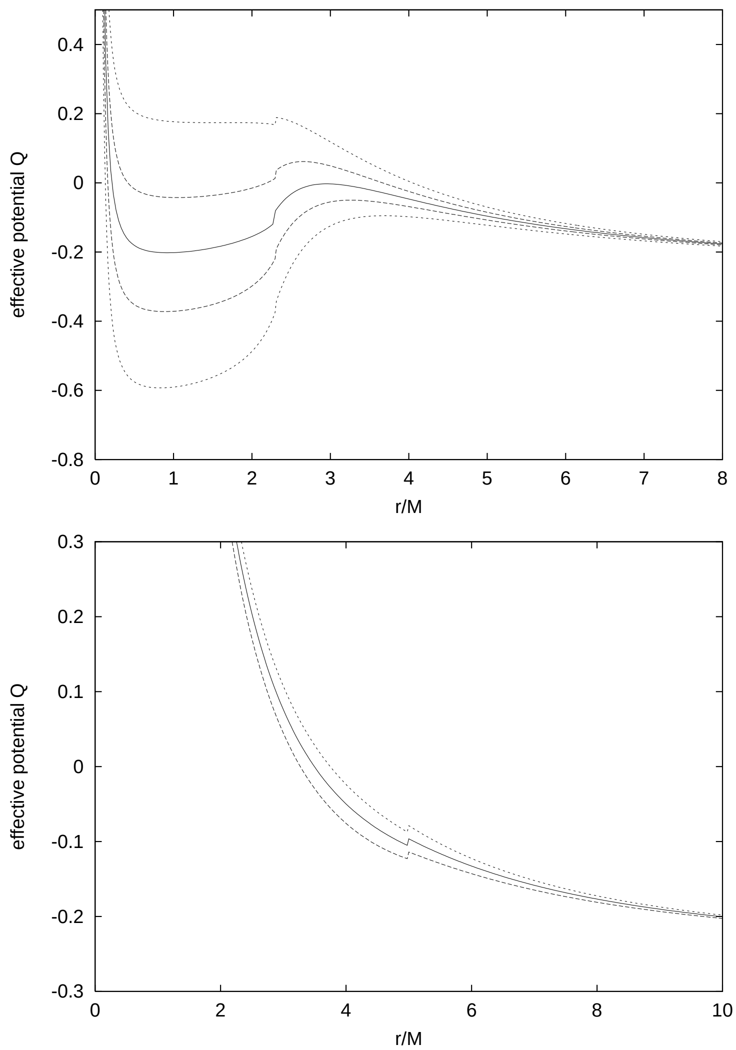


Figure 1: **Top**: $R/M = 2.3$, $\epsilon = 0, 0.3, 0.7$. **Bottom**: $R/M = 5$, $\epsilon = 0, 0.5$. For both cases $\sigma M = 0.5$ and $m = \pm 2$.

- Rotation has a stronger influence for ultracompact stars than (say) canonical neutron stars.
- The potential "well" becomes deeper (swallower) with increasing ϵ, for $m < 0 (m > 0)$. Hence, we expect the number of trapped w-modes to change accordingly.
- A given w-mode will become shorter or longer lived, with increasing rotation, for $m > 0$ (prograde) and $m < 0$ (retrograde) respectively.

w-modes: frequency-domain

- Families of w-modes : **curvature** modes, **trapped** modes (present only in ultracompact stars) and **interface** modes (originated by the density discontinuity at $r = R$; we shall not discuss them here).

- We use two different techniques for computing modes: the **Prüfer method** (which is a complex-coordinate phase-function method) for modes with $Re\sigma \sim Im\sigma$ and a **resonance method** devised by Andersson & Comer for modes with $Re\sigma \gg Im\sigma$.

- When rotation is switched on:

  (i) "Zeeman" splitting of $2l + 1$ degeneracy of the azimuthal number m.

  (ii) Modes with $m > 0$ ($m < 0$) have their real parts increased (decreased) while their imaginary parts (absolute values) increase (decrease) with increasing rotation. This is a CFS-like behaviour.

  Instead, modes of rotating BHs with $m > 0$ ($m < 0$) have their real parts increased (decreased) and their imaginary parts decreased.

  (iii) The number of trapped modes increases (decreases) for a given $m < 0$ ($m > 0$) with increasing rotation and fixed compactness R/M.

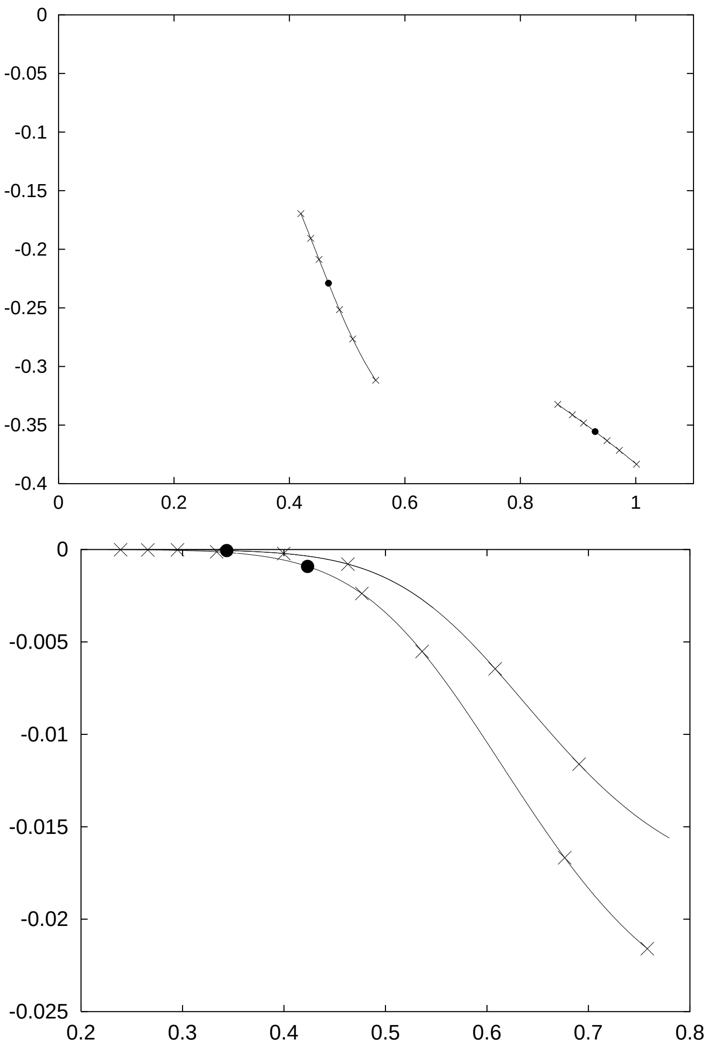
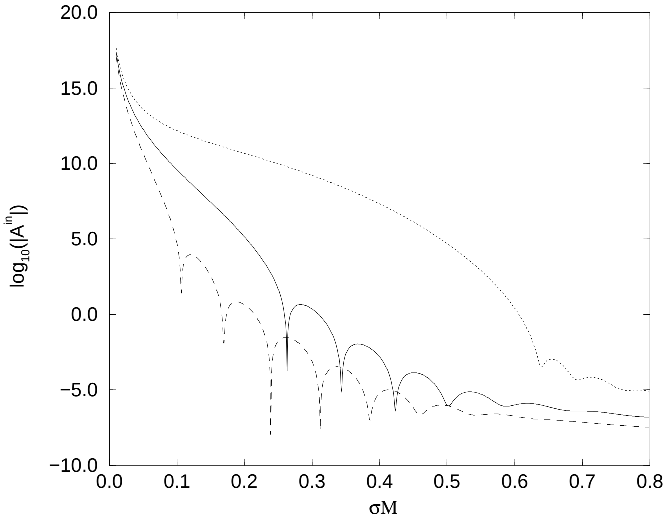


Figure 2: **Top**: the first two w-modes for a star of $R/M = 5$ under the influence of rotation. Crosses represent the mode location for $\epsilon = 0.3, 0.6, 1$ and for $m = \pm 2$. The black dot is the nonrotating mode. **Bottom**: Two trapped w-modes for a star of $R/M = 2.3$ with varying rotation and for $m = \pm 2$. Crosses mark $\epsilon = 0.1, 0.2, 0.4, 0.5$ and dots mark the nonrotating modes.



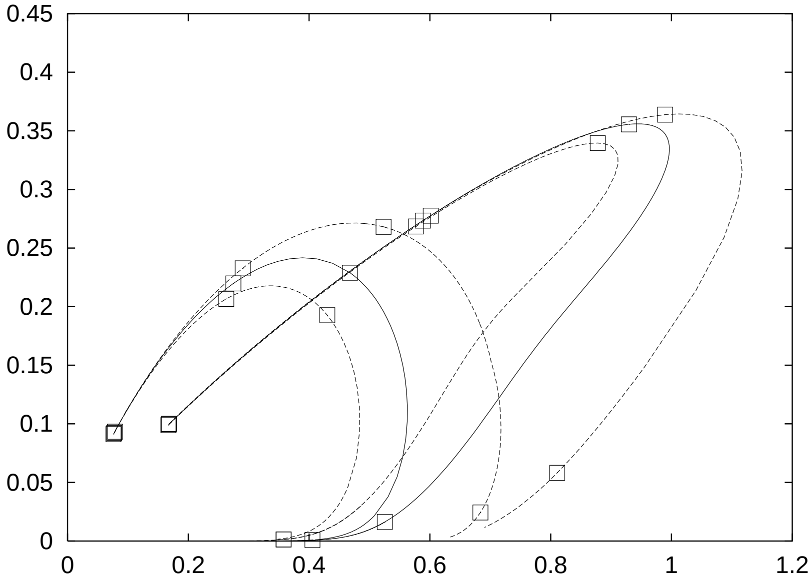


Figure 3: **Top**: Long-lived w-modes manifested as resonances of the $|A^{\rm in}(\sigma)|$ function, for $R/M = 2.3$ and $\epsilon = 0$ (solid curve) and $\epsilon = 0.5$, $m = +2$ (dotted curve) and $m = -2$ (dashed curve). **Bottom**: w-modes on the complex frequency plane (in particular $-Im\sigma$ vs $Re\sigma$) for given rotation $\epsilon = 0.5$ (dashed curves for $m = \pm 2$) and varying compactness R/M. Solid curves refers to the nonrotating case. Boxes mark the values $R/M = 40, 10, 5, 2.4$.

w-modes: time-domain

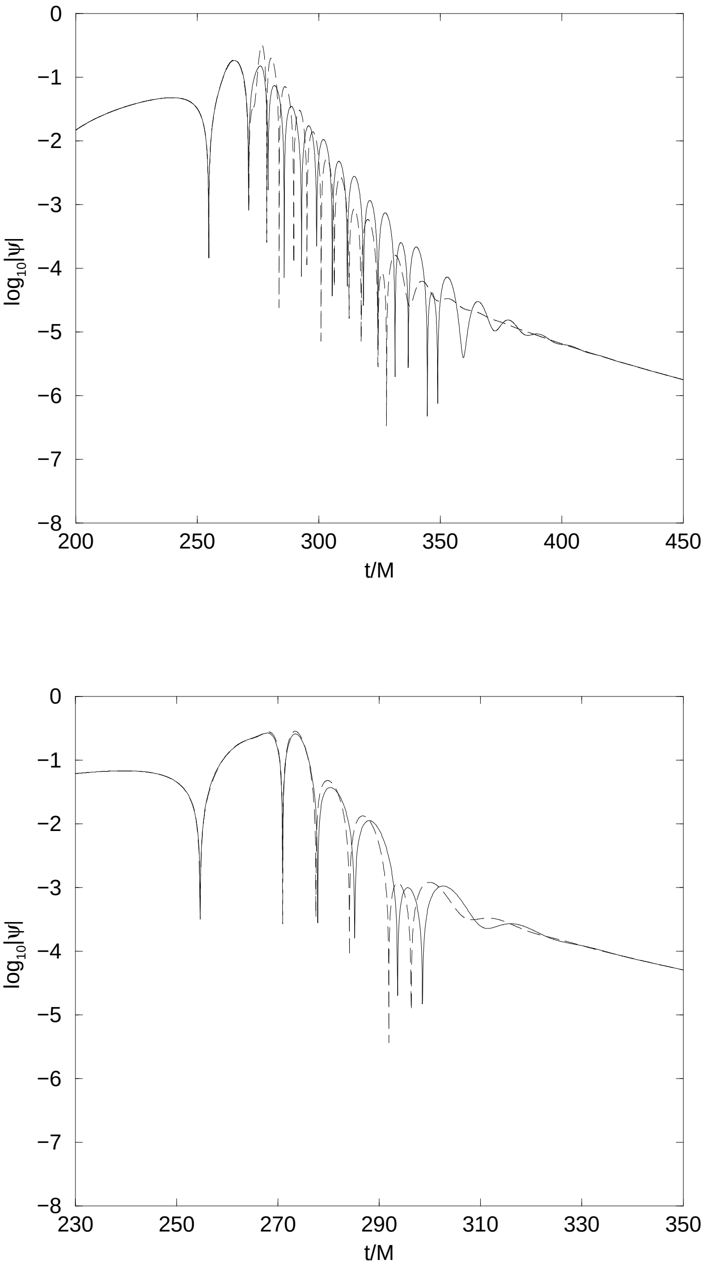


Figure 4: **Top**: $R/M = 3.5$, $\epsilon = 0, 0.5$ (dashed, solid curves respectively). **Bottom**: $R/M = 5$, $\epsilon = 0, 0.5$. Initial pulse at $100M$, observation point at $150M$.

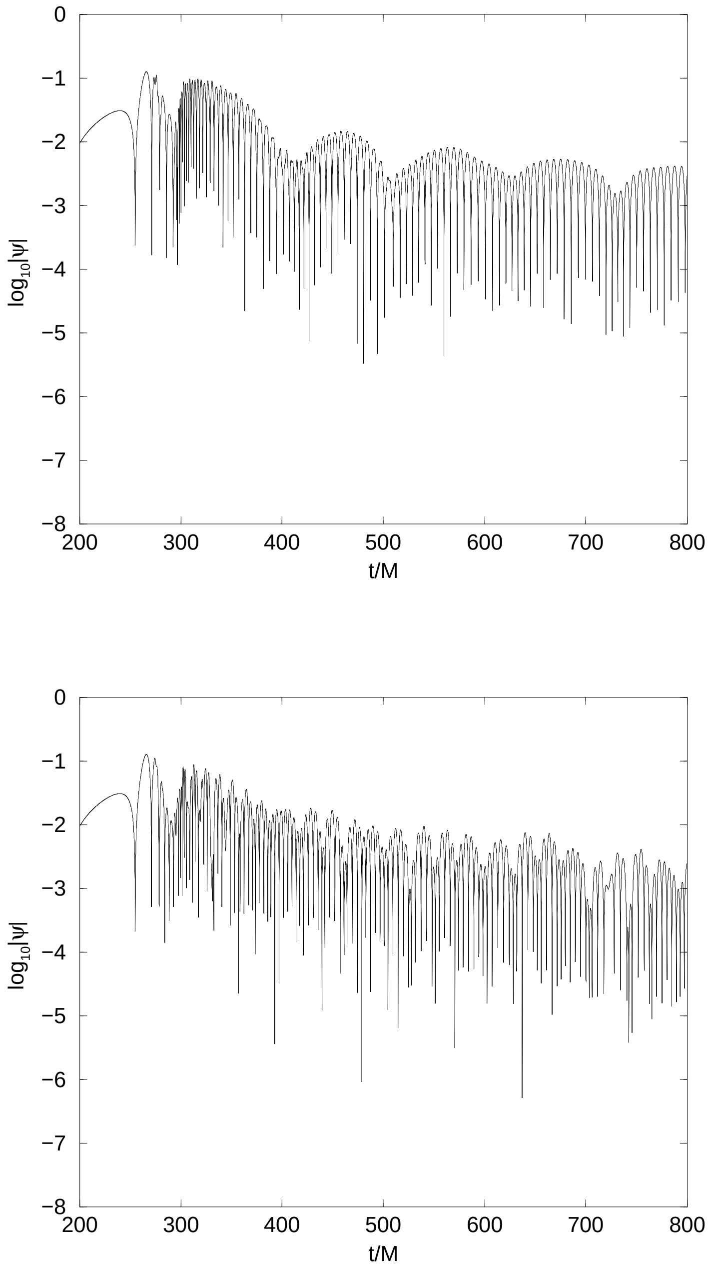


Figure 5: **Top**: $R/M = 2.26$, $\epsilon = 0$. **Bottom**: $R/M = 2.26$, $\epsilon = 0.5$. Initial pulse at $100M$, observation point at $150M$.

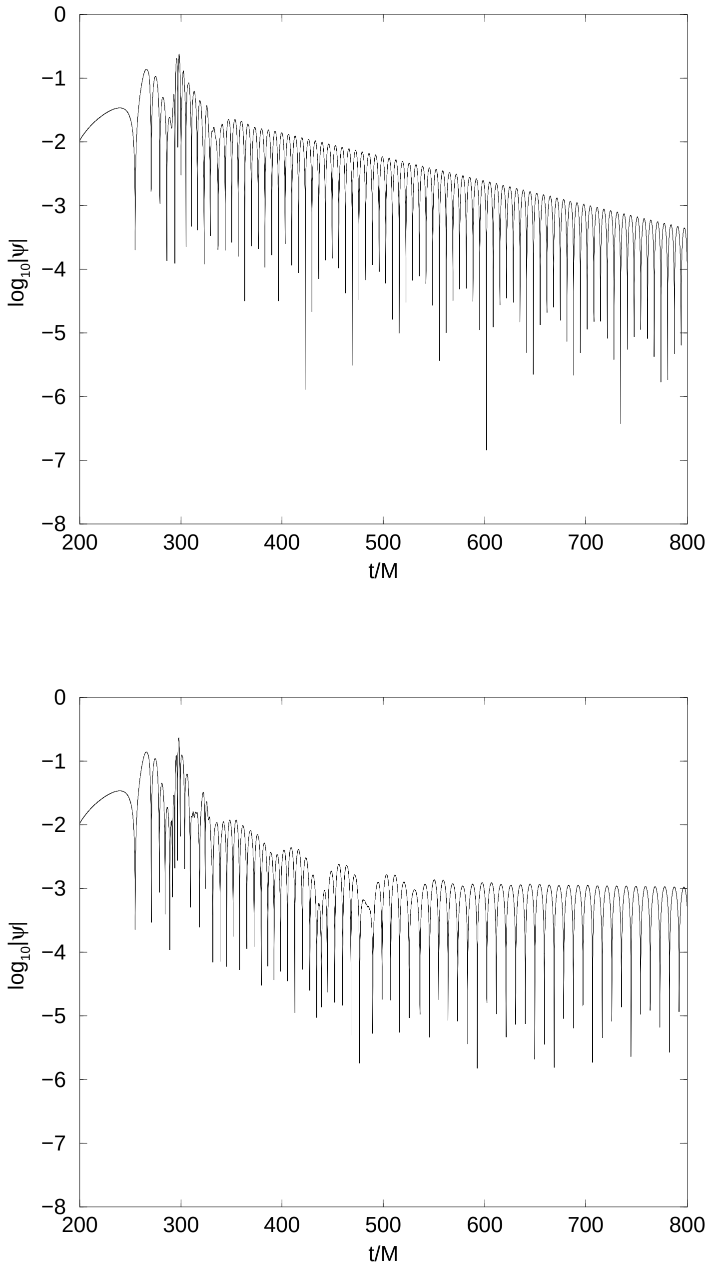


Figure 6: **Top**: $R/M = 2.5$, $\epsilon = 0$. **Bottom**: $R/M = 2.5$, $\epsilon = 0.5$. Initial pulse at $100M$, observation point at $150M$.

Ergoregion instability

- Rotation leads to “dragging of inertial frames”.
- Sufficiently compact stars (just like Kerr BHs) can support an ergoregion at their vicinity (rotating below the Kepler limit), i.e. a region where the time-related Killing vector ξ_t shifts from timelike to spacelike (or in physical terms: no timelike world-line can remain at rest relative to an observer at infinity). The ergoregion's boundary is given by,

  $$g_{ab}\, \xi_t^a \xi_t^b = g_{tt} = 0$$

  Ergoregion forms when g_{tt} is amended with $\mathcal{O}(\Omega^2)$ corrections and $R/M < 2.6$ for uniform density stars (Schutz & Comins).
- Since ξ_t is spacelike inside the ergoregion, particles and fields can be attributed *negative* energy. This leads to the so-called “ergoregion instability” : negative energy can accumulate inside the ergoregion while positive energy can be radiated to infinity while overall energy is conserved.
- This phenomenon is a close relative to the more familiar “superradiant amplification” known from BH physics. What makes them differ is the different “left” boundary condition for the wavefunction: purely ingoing behaviour at the BH horizon as opposed to total reflection at the star's centre. This is clearly demonstrated in the work of Vilenkin who considered a Kerr BH with a reflecting “wall” placed just outside the horizon.
- The ergoregion instability was first discovered by Friedman and by Schutz & Comins who showed that the associated instability timescale *for scalar field perturbations* is much larger than any dynamical timescale. This result was further established by the work of Yoshida & Eriguchi.

- Indeed, out time-evolutions for a scalar field perturbation (including the appropriate second-order corrections to g_{tt}), show no observable instability over an interval of $\sim 10^3 M$. Equally unsuccesfull were our attempts to find any unstable modes in the frequency domain (presumably due to the extreme proximity of these modes to the real axis).

- Do we expect similar timescales for (the realistic case of) gravitational perturbations ? The answer should be negative, based on our knowledge on superradiance (where the amplification is $\sim 10^3 - 10^4$ stronger for gravitational perturbations as compared to scalar perturbations). Hence, we should expect the ergoregion instability timescale to shrink by a similar factor. Hence, we conjecture for the growth timescales,

$$\frac{\tau_{sc}}{\tau_{grav}} \sim 10^3 - 10^4$$

  Extrapolating from the scalar field results of Yoshida & Eriguchi, we expect the gravitational instability to be significant after a time interval $\sim 10^3 M$ (few minutes in human units) which is reasonably short.

- The recent work by Kokkotas, Ruoff & Andersson (who consider axial gravitational perturbations) indeed verifies our intuition. Moreover, they explicitly show that the ergoregion instability is manifested as *unstable w-modes* (in particular the trapped modes living at the bottom of the potential well).

Excitation of marginally stable/unstable modes

- We consider modes characterised by $Im(\sigma) \ll Re(\sigma)$. This category includes both (very) *slowly* damped and growing modes. One could naively expect that long-lived modes stand a better chance to be detected during the course of a gravitational wave observation. *Is this indeed the case ?*

- To investigate this issue in some detail we study the scalar field initial value problem in the frequency domain following a method familiar from BH perturbation theory. The contribution of a given mode σ_n at the field radiated at infinity can be written as,

$$h_{mode} \sim \frac{A^{\mathrm{out}}}{\alpha_n} e^{-i\sigma_n(t-r^*)}, \quad \alpha_n = (dA^{\mathrm{in}}/d\sigma)_{\sigma_n}$$

  The factor in front of the exponential in the last equation is the mode-excitation amplitude and is independent of the initial data.

- It can be further shown that at the vicinity of σ_n we have,

$$A^{\mathrm{out}}/A^{\mathrm{in}} \approx e^{2i\delta} \frac{(\sigma - \sigma_n^*)}{(\sigma - \sigma_n)}, \quad A^{in} \approx a_n(\sigma - \sigma_n)$$

  from which it follows,

$$\frac{A^{\mathrm{out}}}{a_n} \approx e^{2i\delta}(\sigma - \sigma_n^*) \approx 2ie^{2i\delta} Im\ \sigma_n$$

- In a realistic GW observation, the effective amplitude of a periodic (or quasi-periodic as in the present case) signal grows as $\sqrt{N}$, where N is the number of the observed cycles. Hence, a better measure of the mode's detectability would be,

$$h_{mode} \sim \sqrt{\frac{Re\ \sigma_n}{Im\ \sigma_n}} \frac{A^{\mathrm{out}}}{a_n}(\sigma_n) \sim \sqrt{Re\ \sigma_n Im\ \sigma_n} \quad \to 0$$

  which suggests that for the modes under discussion we should not expect a significant level of excitation as compared (say) to modes with $Im(\sigma) \sim Re(\sigma)$.

Late-Time Tails: Rotating Stars vs Kerr BHs

- Untill recently, the standard (?) picture was that both BHs and rotating stars produce the same late-time, power-law fall-off as observed at some (fixed) large distance (for time-asymmetric initial data),

$$\psi(r,t) \sim Y_{l_{min}}^{m} t^{-(2l_{min}+3)}, \quad t \to +\infty$$

  where l_{min} is the lowest existing angular mode. The above statement is certainly true for the non-rotating case (Price's theorem) and is based on two facts: (i) tails represent radiation scattered by the body's asymptotic gravitational field (which is common for spherical symmetric bodies, i.e. Schwarzschild spacetime) and (ii) the absence of angular mode-coupling, i.e. a pure Y_l^m mode will be preserved.

- So how does rotation change the game ?

  (i) Loss of spherical symmetry means that coupling of angular modes can take place, respecting, at the same time, the axisymmetry and equatorial reflection symmetry of the background field.

  (ii) The Kerr metric is not the unique exterior solution, *except at the asymptotic zone (or in a slow-rotation approximation) where only the body's leading moments (mass and spin) appear.*

- The existing analytical (and approximate) results (Hod, Poisson), based on the above two points, suggest that the tail power-law is universal for rotating bodies. However, the power exponent is *not what we would intuitively expect* !

$$\psi(r,t) \sim \begin{cases} Y_l^m \quad t^{-(2l^*+3)} & l^* = m, m+1 \\ Y_m^m \quad t^{-(l^*+m+1)} & l^* - m \geq 2 \text{ (even)} \\ Y_{m+1}^m \quad t^{-(l^*+m+2)} & l^* - m \geq 2 \text{ (odd)} \end{cases}$$

where l^* is the *initial data* angular mode.

- Follow-up high-precision Kerr time-evolutions (Burko-Khanna) disagree with analytical prediction (for BHs) while supporting intuition.

- The reason behind these conflicting results (Poisson, Burko & Khanna) is that the excitation of some angular modes (typically including the dominant one, i.e. the lowest l) is a non-linear process with respect to the body's gravitational potential (i.e. the body's quadrupole (and higher) order multipole moments are involved).

- Hence, bodies with different multipolar structure should give different late-time power-laws, simply because their gravitational field mixes angular modes in a different way.

- Easy way to find out what is the tail power-law for an arbitrary, slow-rotating relativistic star:

  We can use the $\mathcal{O}(\Omega^2)$ *exterior* Hartle metric which includes quadrupole-order moments:

$$\begin{aligned} ds^2 &= -e^{\nu}(1+2h)dt^2 + e^{\lambda}[1+2m/(r-2M)]dr^2 \\ &\quad + r^2(1+2k)[d\theta^2 + \sin^2\theta(d\phi - \omega dt)^2] + \mathcal{O}(\Omega^3) \end{aligned}$$

  where the functions $h(r,\theta), m(r,\theta), k(r,\theta)$ are known if we prescribe values for $\{M, J, Q\}$, the mass, spin and quadrupole moment. Note that the second-order Kerr metric is recovered if we make the special choice $J = Ma$ and $Q = J^2/M$ (leading to $Q = J^2/M$; BHs have only two "hairs").

- The nature of our problem does not require the knowledge of the internal metric (which can only be found numerically). Instead we choose to impose a convenient boundary condition at the star's vicinity (for example, total reflection) and parametrise our spacetime in terms of its multipole moments.