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Equation of Motion for Compact Binaries with Strong Internal Gravity

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§1 Introduction

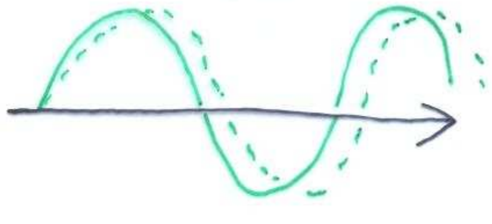
- Grav. Waves Astrophysics

Large Interferometric Detectors
LIGO, VIRGO, GEO600, TAMA300

- The most promising source is a Compact Binary.

  ⋯ Inspiralling / Merging neutron stars

⇒ Accurate theoretical templates of waveforms are needed



for · improving S/N

· determining astrophysical parameters.

$(M_{NS}, \cdots ; H_0, \cdots)$

- "Accurate" Eq. of Motion for Binary

$\Rightarrow$ Some Approximations

$\because$) G.R. • Non-linear

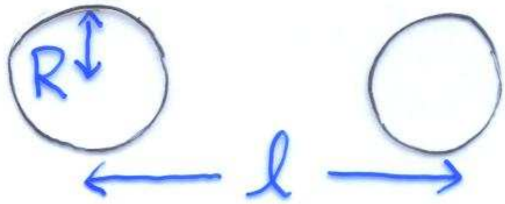
• Radiation Reaction

- Post-Newtonian Approximation

"$\frac{1}{c} \longrightarrow 0$"

① Slow Motion $\left|\frac{\vec{v}}{c}\right| \ll 1$

② Weak Field



$\left|\frac{GM}{c^2 \ell}\right| \ll 1$

$\left|\frac{GM}{c^2 R}\right| \ll 1$ $\cdots$ Weak Internal Gravity

(X $\cdots$ N.S. ~ 0.1)

Q. How to treat the Strong Internal Gravity in derivation of Eq. of Motion ?

$\Rightarrow$ This talk

- Modelling Component Stars

① δ- fn.

$\Box \phi(x) = \delta(x)$ well-defined at linear order

However, Einstein Eq. is non-linear.

$\Rightarrow$ Divergences appear.

$\Rightarrow$ Regularization is needed to mediate them.

cf. Damour & Deruelle (1981) Will & Wiseman (1996)
Blanchet, Faye & Ponsot (1998) Jaranowski & Schäfer (1998)
. . . . . .

However, its relation to the real system is not clear.

② Fluid Distribution

- Spherical Balls — Grischuck & Kopeikin (1983), Kopeikin (1985)

  ✓ Tidal Distortion at higher orders

  ✓ Only for Weak Internal Gravity

$\Rightarrow$ Fluid but not Sphere

$\sim$ 1980's $\leqq 2.5$ PN $\left(\frac{1}{c^5}\right)$

1990's $\geqq 3$ PN $\left(\frac{1}{c^6}\right)$

◎ δ-fn.

- ~~Harmonic Gauge~~ ADM Gauge

  Jaranowski & Schäfer (1998) (1999)

  Ambiguities in Regularisation schemes at 3PN

  Blanchet & Faye (2000) (2001) Harmonic Gauge

  Generalized Hadamard Regularization

  (1 undetermined coefficient)

- ADM Gauge + Dimensional Regularization

  Damour, Jaranowski & Schäfer (2001)

◎ Fluid Model (Dust)

Pati & Will (2000) (2002)

Direct Integration

◎ Point Particle Limit Approach

Futamase (1985)

Itoh, Futamase & Asada (1999) (2000)

Itoh, Futamase (2003)

Itoh (2003)

§2 Formalism

2-1 Scaling

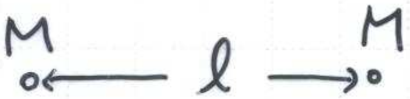
- Newtonian dynamical time   Futamase & Schutz 1983

$$\tau \equiv \epsilon t$$

$$\frac{dx^i}{dt} = \epsilon \frac{dx^i}{d\tau} = O(\epsilon)$$

Newtonian limit : $\epsilon \to 0$

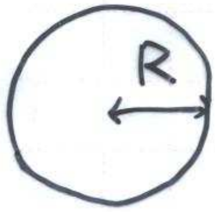
- Binary



$$\underbrace{\left|\frac{d^2x^i}{dt^2}\right|}_{= O(\epsilon^2)} \sim \frac{M}{l^2}$$

We fix l $\Rightarrow$ $M \propto \epsilon^2$

- Strong Internal Gravity

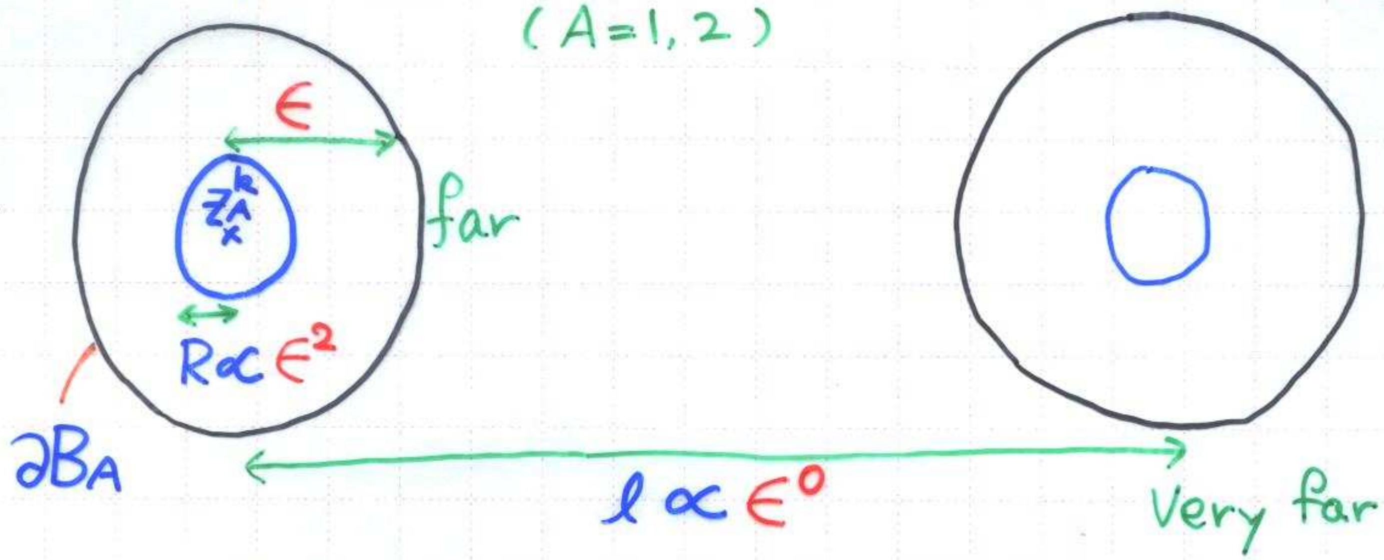


$$\frac{M}{R} = O(1) \Rightarrow R \propto \epsilon^2$$

$$\therefore \quad \rho \propto \frac{M}{R^3} \propto \epsilon^{-4}$$

cf. Standard PNA $\quad \frac{GM}{Rc^2} \ll 1$

- Body Zone : $B_A \equiv \{x^k \mid |x^k - z_A^k| < \epsilon R_A\ ,\ R_A = \mathrm{const.}\}$
  $(A = 1, 2)$



Point Particle Limit : $\epsilon \to 0$

- Body Zone coordinate : $\alpha_A^{\underline{i}} \equiv \frac{1}{\epsilon^2}(x^i - z_A^i)$

- Slowly Rotating Stars : Pressure $\leftrightarrow$ Grav.

$$\frac{1}{R}\left(\frac{P}{\rho}\right) \sim \frac{M}{R^2} \Rightarrow P \propto \epsilon^{-4}$$

$$\therefore\ T^{\tau\tau} = \epsilon^2 T^{tt} \sim \epsilon^2 \rho \propto \epsilon^{-2}$$

$$T^{\tau\underline{i}} = \epsilon^{-1} T^{ti} \sim \epsilon^{-1} \rho v^i \propto \epsilon^{-4}$$

$$T^{\underline{i}\underline{j}} = \epsilon^{-4} T^{ij} \sim \epsilon^{-4} P \delta^{ij} \propto \epsilon^{-8}$$

2-2 Einstein Eq.

- $\bar{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}\, g^{\mu\nu}$

- harmonic condition : $\bar{h}^{\mu\nu}{}_{,\nu} = 0$

- Einstein Eq.

$$\Box \bar{h}^{\mu\nu} = -16\pi \Lambda^{\mu\nu}$$

... solved iteratively

$$\begin{cases} \Box \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu \\ \Lambda^{\mu\nu} \equiv \Theta^{\mu\nu} + \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta} \\ \Theta^{\mu\nu} \equiv (-g)(T^{\mu\nu} + t^{\mu\nu}_{LL}) \\ \chi^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi}(\bar{h}^{\alpha\nu}\bar{h}^{\beta\mu} - \bar{h}^{\alpha\beta}\bar{h}^{\mu\nu}) \end{cases}$$

- Conservation Law

$$\Lambda^{\mu\nu}{}_{,\nu} = 0$$

2-3 Eq. of Motion

- Four Momentum of Body A

$$P^{\mu}_{A}(t) \equiv \int_{B_A} d^3x\, \Lambda^{t\mu}$$

$$M_A \equiv P^{\mu=0}_{A}$$

- Dipole Moment

$$D^{i}_{A}(t) \equiv \int_{B_A} d^3x\, \Lambda^{tt}\left[x^i - z^i_A(t)\right]$$

- $v^i_A \equiv \frac{d}{dt} z^i_A$

- From $\frac{d}{dt} D^i_A(t)$, we obtain the velocity-momentum relation

$$\underline{\underline{P^i_A}} = M_A \underline{\underline{v^i_A}} + Q^i_A + \frac{d}{dt} D^i_A \qquad \text{——— ①}$$

where

$$Q^i_A \equiv \oint_{\partial B_A} dS_k\, \Lambda^{tk}\left[x^i - z^i_A(t)\right] - v^k_A \oint_{\partial B_A} dS_k\, \Lambda^{tt}\left[x^i - z^i_A(t)\right]$$

- We obtain, using $\Lambda^{\mu\nu}{}_{,\nu} = 0$,

$$\frac{d}{dt} P^{\mu}_{A} = -\oint_{\partial B_A} dS_k\, \Lambda^{k\mu} + v^k_A \oint_{\partial B_A} dS_k\, \Lambda^{t\mu} \qquad \text{——— ②}$$

- Substituting ① to ②, we obtain

$$
\boxed{
\begin{aligned}
M_A \frac{d}{dt} v_A^i = &-\oint_{\partial B_A} dS_k \Lambda^{ki} + v_A^k \oint_{\partial B_A} dS_k \Lambda^{ti} \\
&+ v_A^i \left[\oint_{\partial B_A} dS_k \Lambda^{kt} - \oint_{\partial B_A} dS_k \Lambda^{tt} \right] \\
&- \frac{d}{dt} Q_A^i - \frac{d^2}{dt^2} D_A^i
\end{aligned}
}
$$

This becomes the Eq. of Motion,

when we specify D_A^i as $D_A^i = 0$ for instance.

① This does not depend on choices of B_A,

although each term does depend.

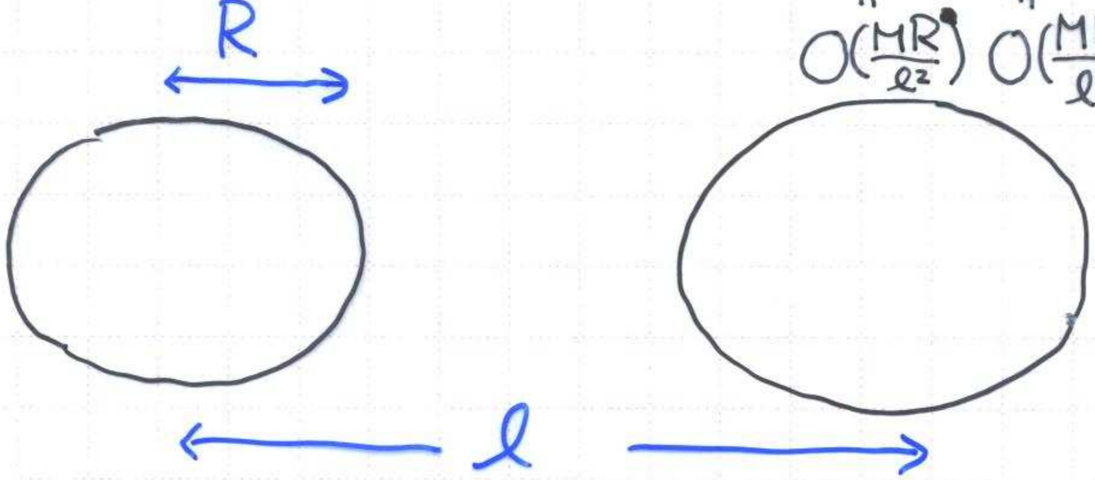
② All integrations are taken only on ∂B_A,

outside the strong field.

cf. Einstein, Infeld & Hoffman (1938)

3. Multipole Expansion : $\frac{M}{\ell} + \frac{D}{\ell^2} + \frac{Q}{\ell^3} + \cdots$

$$\frac{D}{\ell^2} = O\left(\frac{MR}{\ell^2}\right), \qquad \frac{Q}{\ell^3} = O\left(\frac{MR^2}{\ell^3}\right)$$



Expansion in $\frac{R}{\ell}$

$$\frac{R}{\ell} \sim \underbrace{\frac{R}{M}}_{O(1)} \times \underbrace{\frac{M}{\ell}}_{O(\epsilon^2)} \sim O(\epsilon^2) = \text{PN}$$

§3. EOM at the Newtonian order

3-1 Newtonian order

- $D_A^i(\tau) = \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3\alpha_A \, \Lambda_A^{\tau\tau} \, \alpha_A^i$

- $D_A^i = 0 \Rightarrow$ z_A^i is COM.

- Metric :

$$\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_A \frac{m_A}{r_A} + O(\epsilon^5)$$

$$\bar{h}^{\tau i} = 4\epsilon^4 \left[\sum_A \frac{J_A^i}{r_A} + \sum_A \frac{m_A v_A^i}{r_A} \right] + O(\epsilon^5)$$

$$\bar{h}^{ij} = 4\epsilon^2 \sum_A \frac{Z_A^{ij}}{r_A}$$

$$+ \epsilon^4 \Big[8 \sum_A \frac{v_A^{(i} J_A^{j)}}{r_A} + 4 \sum_A \frac{m_A v_A^i v_A^j}{r_A} + \sum_A \frac{m_A^2}{r_A^4} r_A^i r_A^j$$

$$- 8 \frac{m_1 m_2}{r_{12} S} n_{12}^i n_{12}^j - 8 \left(\delta_k^i \delta_\ell^j - \frac{1}{2} \delta^{ij} \delta_{k\ell} \right) \frac{m_1 m_2}{S^2} (n_{12} - n_1)^{(k} (n_{12} + n_2)^{\ell)}$$

$$+ O(\epsilon^5) \, ,$$

where

$r_A^i \equiv x^i - z_A^i$, $r_A \equiv |\vec{r}_A|$, $r_{12}^i \equiv z_1^i - z_2^i$, $S \equiv r_1 + r_2 + r_{12}$,

$$m_A \equiv \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3\alpha_A \, \Lambda_A^{\tau\tau}$$

$$J_A^i \equiv \lim_{\epsilon \to 0} \epsilon^4 \int_{B_A} d^3\alpha_A \, \Lambda_A^{\tau i}$$

$$Z_A^{ij} \equiv \lim_{\epsilon \to 0} \epsilon^8 \int_{B_A} d^3\alpha_A \, \Lambda_A^{ij}$$

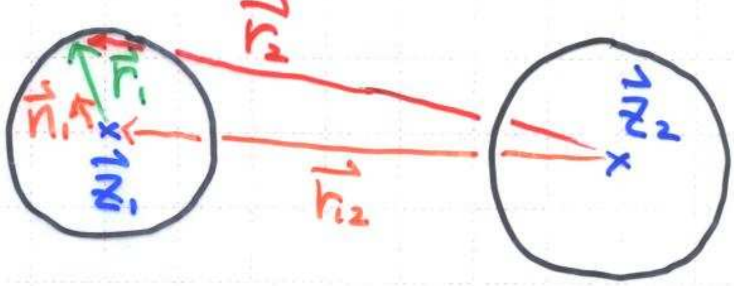
- In order to evaluate the surface integrals, we take

$$\vec{r}_1 \equiv \epsilon R_1 \vec{n}_1$$

$$\vec{r}_2 \equiv \vec{r}_{12} + \epsilon R_1 \vec{n}_1$$

$$\vec{r}_{12} \equiv \vec{z}_1 - \vec{z}_2$$

$$\vec{n}_{12} \equiv \vec{r}_{12} / |\vec{r}_{12}| .$$



Then,

$$\oint_{\partial B_1} dS_k \,{}_{(4)}\left[(-g)\, t^{ik}_{LL} \right]$$

$$= \frac{1}{64\pi} \left(\delta^i_\ell \delta^k_m - \frac{1}{2} \delta^{ik} \delta_{\ell m} \right) \sum_A \sum_B m_A m_B \oint_{\partial B_1} dS_k \frac{r_A^\ell r_B^m}{r_A^3 r_B^3}$$

$$= \frac{m_1 m_2}{r_{12}^2} n_{12}^i$$

$$\therefore \quad m_1 \frac{dv_1^i}{d\tau} = - \frac{m_1 m_2}{r_{12}^2} n_{12}^i$$

$$\left(m_A \neq \int_{B_A} d^3x \, \rho \right)$$

This shows that the Newtonian form of EOM is valid for the strongly self-gravitating compact binaries, when a mass is defined in a proper manner.

§4 1PN

- $$\frac{d}{d\tau} P_1^\tau = -\epsilon^2 \frac{\bar{m}_1 \bar{m}_2}{r_{12}^3}\left[4(\vec{r}_{12}\cdot\vec{v}_1) - 3(\vec{r}_{12}\cdot\vec{v}_2)\right]$$

$\longleftarrow$ Newtonian EoM

$$= \epsilon^2 \bar{m}_1 \frac{d}{d\tau}\left[\frac{1}{2}v_1^2 + \frac{3\bar{m}_2}{r_{12}}\right]$$

$$\therefore\ P_1^\tau = m_1\left[1 + \epsilon^2\left(\frac{1}{2}v_1^2 + \frac{3\bar{m}_2}{r_{12}}\right)\right]$$

(The integration constant is renormalized into m_1.)

- Boost : Body Zone $\underset{A}{}$ $\longleftrightarrow$ Grobal Coordinate $\underset{N}{}$

$$T_N^{\tau\tau} = (\Gamma_A)^2 T_A^{\tau\tau} + 2\epsilon^4 v_A^i T_A^{i\tau} + O(\epsilon^6)$$

$$T_N^{\tau i} = \epsilon^2 T_A^{\tau i} + T_A^{\tau\tau} v_A^i$$

$$T_N^{ij} = T_A^{\tau\tau} v_A^i v_A^j + 2\epsilon^2 v_A^{(i} T_A^{j)\tau} + \epsilon^4 T_A^{ij}$$

• Metric

$$ {}_{(6)}\bar{h}^{\tau\tau} = -2\sum_A \frac{\bar{m}_A}{r_A}\left[(\vec{n}_A\cdot\vec{v}_A)^2 - v_A^2\right] + 2\frac{\bar{m}_1\bar{m}_2}{r_{12}^2}\vec{n}_{12}\cdot(\vec{n}_1-\vec{n}_2) $$

$$ +7\sum_A \frac{\bar{m}_A^2}{r_A^2} + 14\frac{\bar{m}_1\bar{m}_2}{r_1 r_2} - 14\frac{\bar{m}_1\bar{m}_2}{r_{12}}\sum_A\frac{1}{r_A} $$

$$ {}_{(4)}\bar{h}^{\tau i} = 4\sum_A \frac{\bar{P}_A^{\tau} v_A^i}{r_A} $$

$$ {}_{(4)}\bar{h}^{ij} = 4\sum_A \frac{\bar{P}_A^{\tau} v_A^i v_A^j}{r_A} + \sum_A \frac{\bar{m}_A^2}{r_A^4} r_A^i r_A^j - \frac{8\bar{m}_1\bar{m}_2}{r_{12}S} n_{12}^i n_{12}^j $$

$$ -8\left[\delta_k^i\delta_l^j - \frac{1}{2}\delta^{ij}\delta_{kl}\right]\frac{\bar{m}_1\bar{m}_2}{S^2}(n_{12}-n_1)^{(k}(n_{12}+n_2)^{l)} $$

• EOM

$$ \bar{m}_1\frac{dv_1^i}{d\tau} = (N) + \epsilon^2\frac{\bar{m}_1\bar{m}_2}{r_{12}^2}\Big[n_{12}^i\Big\{-v_1^2-2v_2^2+\frac{3}{2}(\vec{v}_2\cdot\vec{n}_{12})^2+4(\vec{v}_1\cdot\vec{v}_2) $$

$$ +\frac{5\bar{m}_1}{r_{12}}+\frac{4\bar{m}_2}{r_{12}}\Big\} $$

$$ +V^i\left\{4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{v}_2\cdot\vec{n}_{12})\right\}\Big], $$

where $\vec{V} \equiv \vec{v}_1 - \vec{v}_2$.

This agrees with the "standard" 1PN EOM.

This means that it is applicable for binaries of strong self-gravitating stars.

§5 2PN (Spin-Orbit, Quadrupole)

- Metric (Spin)

$$\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_A \Big[\frac{1}{r_A} \int_{B_A} d^3\alpha_A \, \epsilon^2 \Lambda_N^{\tau\tau}$$

$$+ \epsilon^2 \frac{r_A^i}{r_A^3} \int_{B_A} d^3\alpha_A \, \epsilon^2 \alpha_A^i \Lambda^{\tau\tau} + O(\epsilon^4) \Big]$$

+ "irrelevant terms"

$$= 4\epsilon^4 \sum_A \Big[\frac{\bar{P}_A^\tau}{r_A} + \epsilon^2 \frac{r_A^i}{r_A^3} \left(d_A^i + \epsilon^2 \underline{M_A^{ik}} v_A^k \right) \Big]$$

+ " " ,

where

$$d_A^i \equiv \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3\alpha_A \, \Gamma_A^2 \, \alpha_A^i \Lambda_A^{\tau\tau}$$ (Dipole Moment in Body Zone coord.)

$$\underline{M_A^{ij}} \equiv \lim_{\epsilon \to 0} 2\epsilon^4 \int_{B_A} d^3\alpha_A \, \alpha_A^{[i} \Lambda_A^{j]\tau}$$ (Spin tensor)

$$\bar{h}^{\tau i} = 2\epsilon^6 \sum_A \frac{r_A^k}{r_A^3} \underline{M_A^{ki}}$$

$$\bar{h}^{ij} = 4\epsilon^6 \sum_A \frac{r_A^k}{r_A^3} \underline{M_A^{k(i}} v_A^{j)}$$

- We choose $dA^i = 0$ $(\not{D}A^i \neq 0)$.

Then,

$$P_1^i = P_1^\tau v_1^i + Q_1^i - \epsilon^4 \frac{m_2 M_1^{ij} n_{12}^j}{r_{12}^2}$$

$$Q_1^i = \epsilon^4 \frac{2\bar{m}_2}{3 r_{12}^2} M_1^{ik} n_{12}^k$$

∴ We obtain

$$\left(\bar{m}_1 \frac{dv_1^i}{d\tau}\right)_{SO} = -\epsilon^4 \frac{V^k}{r_{12}^3}\Big[(2\bar{m}_1 M_2^{i\ell} + \bar{m}_2 M_1^{i\ell})\Delta^{\ell k}$$
$$+ 2(\bar{m}_1 M_2^{\ell k} + \bar{m}_2 M_1^{\ell k})\Delta^{\ell i}\Big],$$

where $\Delta^{ij} \equiv \delta^{ij} - 3 n_{12}^i n_{12}^j$.

The same form as Thorne and Hartle (1985)'s result.

- $\not{D}A^i = 0$ case

$\rightarrow$ $\left(\bar{m}_1 \frac{dv_1^i}{d\tau}\right)_{SO}$ is expressed not only by $\vec{V}$

but also by $\vec{v}_1$ (or $\vec{v}_2$).

(… conditions for fixing the COM)

· Metric (Quadrupole)

$$\bar{h}^{\tau\tau} = 6\,\epsilon^{8} \sum_A \frac{r_A^k r_A^l}{r_A^5}\,\underline{\hat{I}_A^{kl}}$$

$$\hat{I}_A^{ij} \equiv \lim_{\epsilon\to 0} \epsilon^{2} \int_{B_A} d^3\alpha_A\, \Lambda_A^{\tau\tau} \left[\alpha_A^i \alpha_A^j - \frac{1}{3}\delta^{ij} |\alpha_A|^2 \right]$$

$$\Downarrow$$

$$\left(\bar{m}_1 \frac{dv_1^i}{d\tau}\right)_Q = \epsilon^4 \frac{3}{2 r_{12}^4} \left(\bar{m}_1 \underline{\hat{I}_2^{kl}} + \bar{m}_2 \underline{\hat{I}_1^{kl}}\right)\left(2\delta^{il} n_{12}^k - 5 n_{12}^i n_{12}^k n_{12}^l\right).$$

This agrees with the case in the Newtonian gravity.

(but at 2PN)

§6 3PN

Itoh, Futamase (2003) Itoh (2003)

- Sphere in grobal Frame ("moving")

$\neq$ Sphere in generalized Fermi Frame ("at rest")

Ashby & Bertotti (1986)

Leading Contribution ··· Lorentz Contraction

$$\delta I_A^{\langle ij \rangle} = -\epsilon^2 \frac{4}{5} m_A v_A^{\langle i} v_A^{j \rangle}$$

Others > 3PN

$\Rightarrow$ Eq. of Motion at 3PN,

which agrees with · $\omega_{static} = 0$

··· $E = E(\Omega)$

Damour, Jaranowski & Schäfer (2001)

· $\lambda = -\frac{1987}{3080}$

de Andrade, Blanchet & Faye (2001)

§7 Summary

Point Particle Limit

+

Surface Integral Method

give us

Accurate Eq. of Motion

for Compact Binaries

with Strong Internal Gravity

[Same Coefficients as Weak Gravity]