Dirty
Black Hole Evolutions
with Creative Engineering

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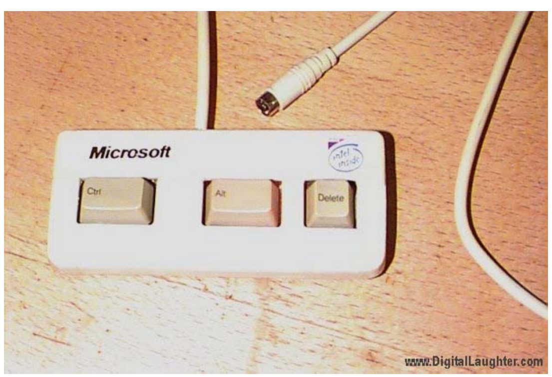


Examples:

- Conformally flat initial data
- Thin-sandwich data
- Hydro-without-hydro
- Bonazzola’s talk

This Talk:

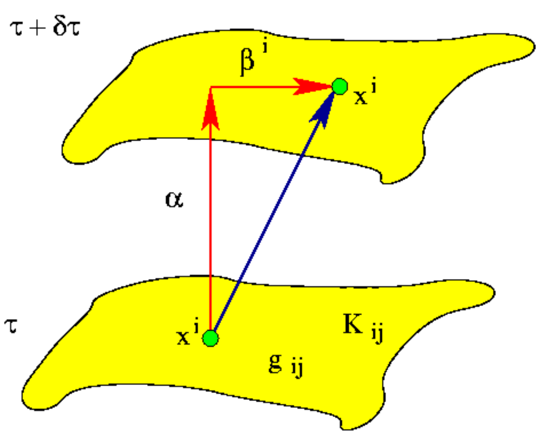
- 3+1 Conformal Formulations
- Black Hole Excision
- Wobbling Black Hole
- Partial BBH Orbit
- Constraint Violating Modes

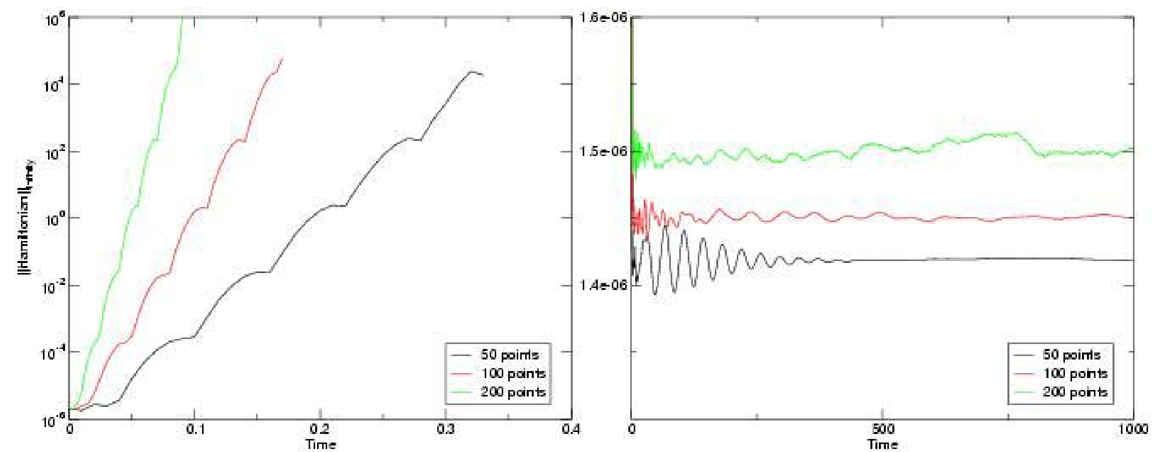
Classical ADM

$$R_{\mu\nu} = 0 \qquad \begin{cases} \partial_o g_{ij} = -2\,\alpha\, K_{ij} \\ \partial_o K_{ij} = -\nabla_i \nabla_j \alpha + \alpha (R_{ij} - 2\, K_i{}^k\, K_{kj} + K_{ij}\, K) \end{cases}$$

$$G_{\mu\nu} = 0 \qquad \begin{cases} R + K^2 - K_{ij}\, K^{ij} = 0 \\ \nabla_j K^{ij} - \nabla^i K = 0 \end{cases}$$

$$\partial_o \equiv \partial_t - \mathcal{L}_\beta$$





Apples with Apples I

Windows vs Linux

$$\partial_t \phi = \eta$$
$$\partial_t \eta = \Delta\phi + \nabla_i \phi \nabla^i \phi - \eta^2$$

- Re-write equations in a manifestly hyperbolic form.
- Well-posedness will tell you that deviations are bounded by exp(ct)
- Theorems and rigorous mathematics are available.
- Formal excision and outer boundaries.
- One can talk about initial boundary value problem.

- Re-write equations with **some** hyperbolic flavor.
- Identify and eliminate troublesome non-linear terms.
- Sorry, no theorems.
- Excision and outer boundaries become an art (or cooking).

BSSN Formulation

$$
\begin{aligned}
g_{ij} &= g^{1/3}\,\hat{g}_{ij} = e^{4\,\Phi}\hat{g}_{ij} \\
g &= e^{12\,\Phi} \\
K_{ij} &= e^{4\,\Phi}\hat{A}_{ij} + \frac{1}{3} g_{ij}\,K \\
\hat{\Gamma}^{i} &= \hat{g}^{jk}\hat{\Gamma}^{i}{}_{jk} = -\partial_j \hat{g}^{ij}
\end{aligned}
$$

$$
\begin{aligned}
\partial_o \Phi &= -\frac{1}{6}\hat{\alpha}\,\hat{K}\,e^{-6\,(n+k)\,\Phi} \\
\partial_o \hat{g}_{ij} &= -2\,\hat{\alpha}\,\hat{A}_{ij}\,e^{-6\,(n+a)\,\Phi} \\
\partial_o \hat{K} &= -\nabla_i \nabla^i \alpha + \alpha\left(\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{3}\,\hat{K}^2\right) \\
\partial_o \hat{A}_{ij} &= e^{-4\Phi}[-\nabla_i \nabla_j \alpha + \alpha R_{ij}]^{TF} + \alpha\,(K\,\hat{A}_{ij} - 2\,\hat{A}_{ik}\hat{A}^{k}{}_{j}) \\
\partial_o \hat{\Gamma} &= -2\,\hat{A}^{ij}\nabla_j \alpha + 12\,\alpha \hat{A}^{ij}\nabla_j \Phi \ldots
\end{aligned}
$$

Laguna & Shoemaker (2002)

$$
\begin{aligned}
\hat{g}_{ij} &= g^{-1/3}\, g_{ij} \\
\hat{\Gamma}^i &= \hat{g}^{jk}\hat{\Gamma}^i{}_{jk} \\
\hat{A}^i{}_j &= g^{a/2} A^i{}_j \\
\hat{K} &= g^{k/2} K \\
\hat{\beta}^i &= \beta^i \\
\hat{\alpha} &= g^{n/2}\alpha \\
\Phi &= \frac{1}{12}\ln g
\end{aligned}
$$

Why?

Natural choice? n = -1, a = 1, k = 1/3

$$\partial_o \Phi = -\frac{1}{6} \hat{\alpha}\, \hat{K}\, e^{-6\,(n+k)\,\Phi}$$

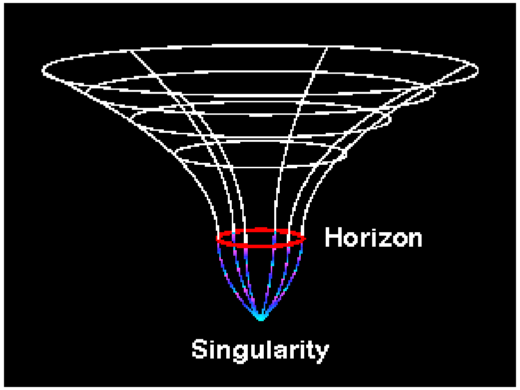
$$\partial_o \hat{g}_{ij} = -2\, \hat{\alpha}\, \hat{A}_{ij}\, e^{-6\,(n+a)\,\Phi}$$

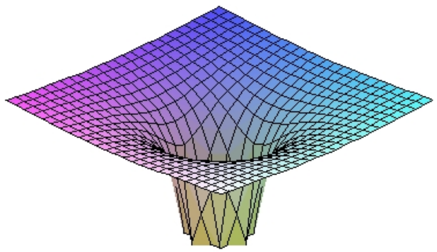
$$\begin{aligned}\partial_o \hat{K} &= -\nabla_i \nabla^i \alpha\, e^{6\,k\,\Phi} + \hat{\alpha}\, \hat{A}^i{}_j \hat{A}^j{}_i\, e^{-6\,(n+2\,a-k)\,\Phi} \\ &+ \left(\frac{1}{3} - k\right) \hat{\alpha}\, \hat{K}^2\, e^{-6\,(n+k)\,\Phi}\end{aligned}$$

$$\begin{aligned}\partial_o \hat{A}^i{}_j &= -\left(\nabla^i \nabla_j \alpha - \frac{1}{3} \delta^i{}_j\, \nabla^k \nabla_k \alpha\right) e^{6\,a\,\Phi} \\ &+ \alpha \left[R^i{}_j - \frac{1}{3} \delta^i{}_j \left(\hat{A}^k{}_l \hat{A}^l{}_k - \frac{2}{3} \hat{K}^2\right)\right] e^{6\,a\,\Phi} \\ &+ (1-a)\, \hat{\alpha}\, \hat{K} \hat{A}^i{}_j\, e^{-6\,(n+k)\,\Phi}\end{aligned}$$

$$\begin{aligned}\partial_o \hat{\Gamma}^i &= \left(8\,k\, \hat{\alpha}\, \hat{K}\, \hat{g}^{ij}\, \partial_j \Phi - \frac{4}{3} \hat{\alpha}\, \hat{g}^{ij}\, \partial_j \hat{K}\right) e^{-6\,(n+a)\,\Phi} \\ &+ \left[2\, \hat{\alpha}\, \hat{A}^{jk} \hat{\Gamma}^i{}_{jk} - 2\, \hat{A}^{ij} \partial_j \hat{\alpha} + 12\,(1+n)\, \hat{\alpha}\, \hat{A}^{ij}\, \partial_j \Phi\right] e^{-6\,(n+a)\,\Phi} \\ &- \left(\frac{2}{3} + \chi\right) \partial_l \beta^l\, (\hat{\Gamma}^i - \hat{g}^{jk} \hat{\Gamma}^i_{jk})\end{aligned}$$

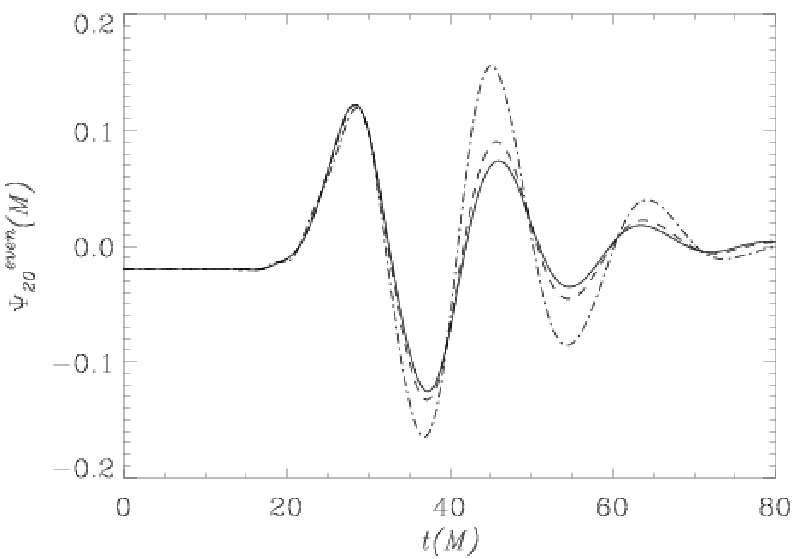
Black Hole Singularity





Puncture

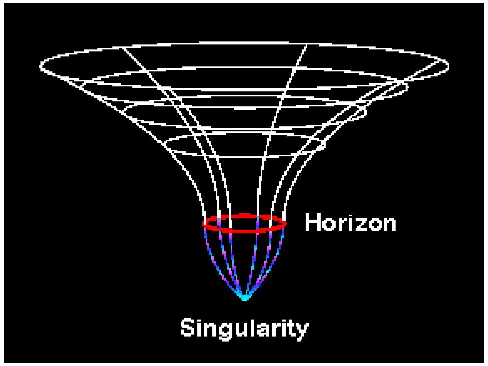
$$
\begin{aligned}
ds^2 &= -\alpha^2\, dt^2 + \phi^4\,(dr^2 + r^2\, d\Omega^2) \\
\alpha &= \frac{1 - M/2\,r}{1 + M/2\,r} \\
\phi &= 1 + M/2\,r
\end{aligned}
$$

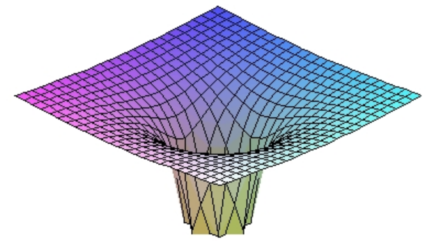


BBH Puncture Initial Data
Brandt and Brugmann (1997)

BBH Puncture Evolutions
Alcubierre et al (2000,2002)

Black Hole Singularity



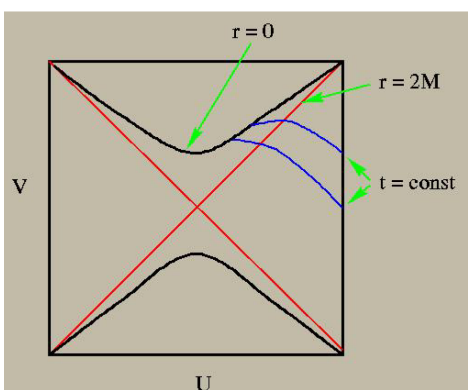


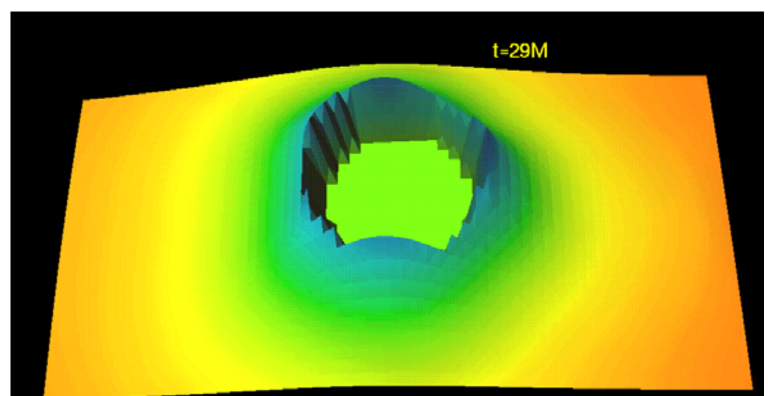
Excision

Unruh (1984)

Ingoing-Eddington-Finkelstein

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right)\,dt^2 + \frac{4\,M}{r}\,dt\,dr + \left(1 + \frac{2\,M}{r}\right)\,dr^2 + r^2\,d\Omega^2$$





Student: Are boundary conditions needed at the excision boundary?
Death: No, if the field variables are purely outgoing.

S: How can I be sure that this is indeed the case?
D: Work with manifestly hyperbolic formulations and apply rigorous numerical analysis.

S: Are the BSSN-based codes in AEI, UIUC and Penn State of this kind?
D: I am afraid not my son!

S: But their codes seem to work! Is there anything certain in this field ?
D: Only death. Bye.

S: Wait a moment!
D: You all say that. But I grant no reprieves.

Outgoing Boundary Conditions?

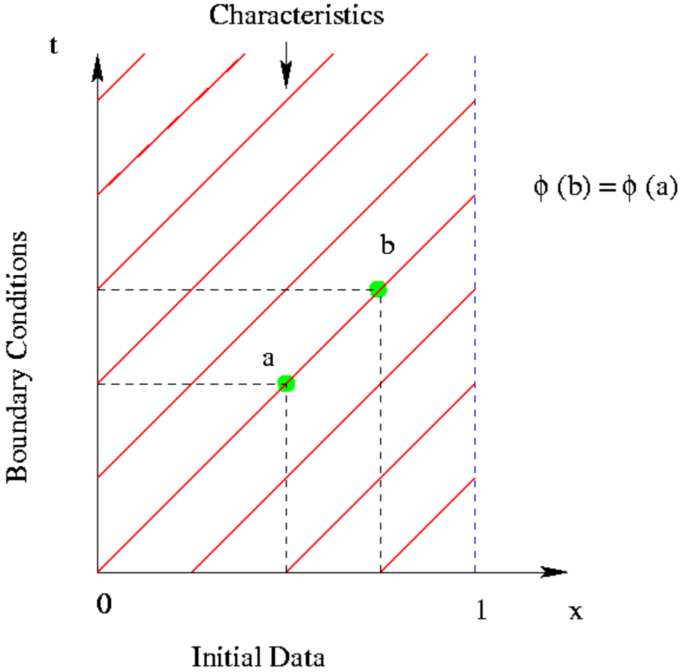
Consider,

$$\partial_t \phi + \partial_x \phi = 0$$

Approximate,

$$\partial_x \phi_i \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\,\Delta x}$$

What to do at x = 1?



One-sided differencing

$$\partial_x \phi_i \approx \frac{3\,\phi_i - 4\,\phi_{i-1} + \phi_{i-2}}{2\,\Delta x}$$

Centered differencing & Extrapolation

$$\partial_x \phi_i \approx \frac{\phi^*_{i+1} - \phi_{i-1}}{2\,\Delta x}$$

$$\phi^*_i \approx 3\,\phi_i - 3\,\phi_{i-1} + \phi_{i-2}$$

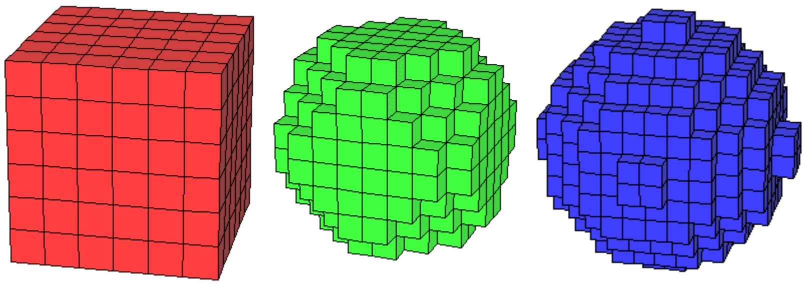
Finite-Differences at the Excision Boundary

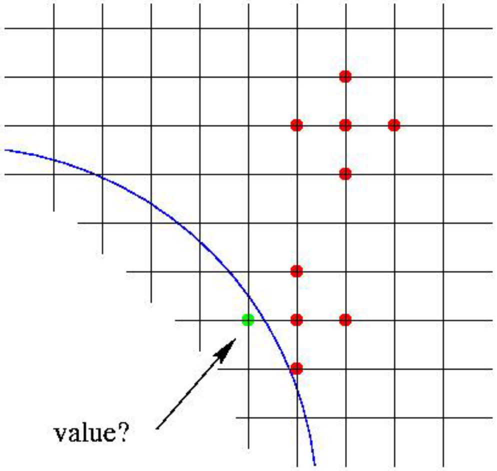
OPTIONS:

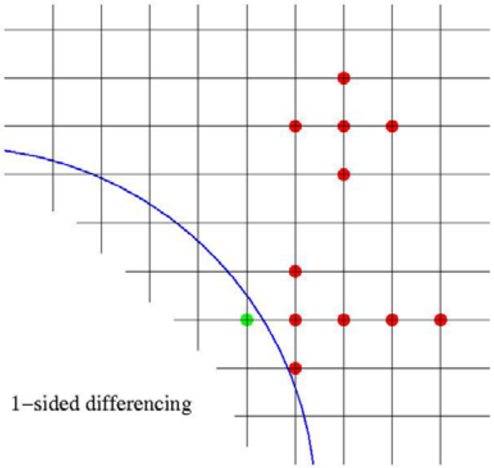
- Modify finite-difference stencils at the excision boundary.
- Preserve centered-difference stencils and use extrapolation.

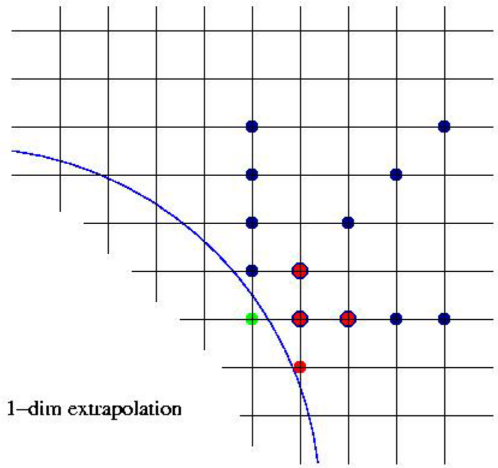
ISSUES:

- Accuracy of extrapolation
- Choice of grid-points
- Stability



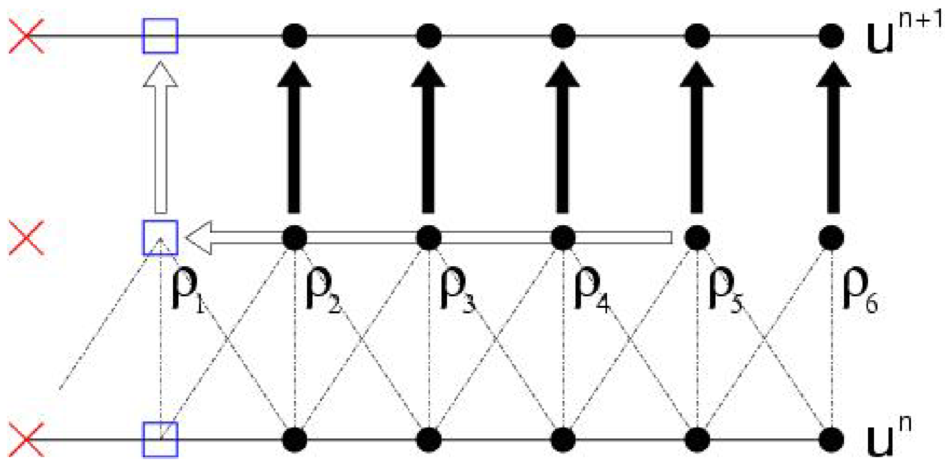




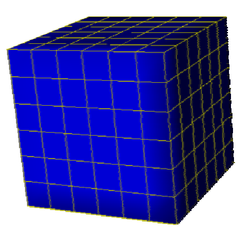


Simple Excision I

$$\partial_t u = \rho \quad \Rightarrow \quad u_i^{n+1} = u_i^n + \Delta t \, \rho_i^{n+1/2}$$

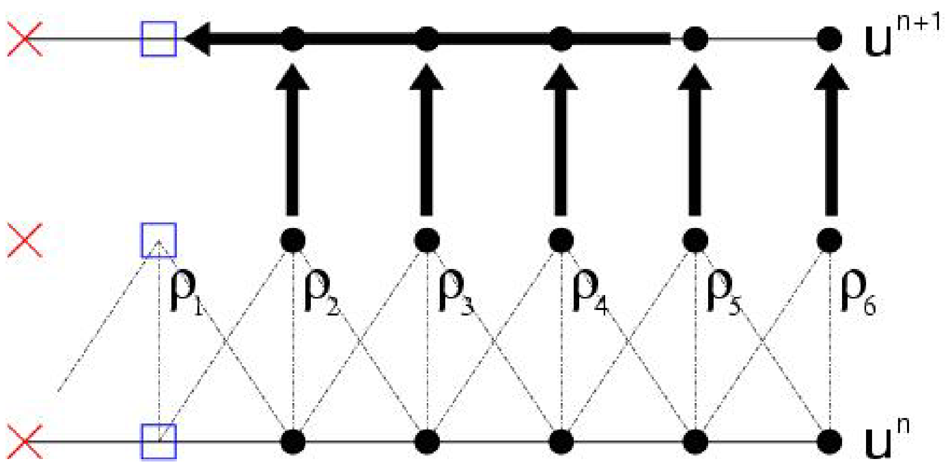


Alcubierre & Brugmann



$$\rho_1^{n+1/2} = \rho_2^{n+1/2} \quad \Rightarrow \quad 0 = \partial_x \rho = \partial_t \partial_x u$$

Simple Excision II

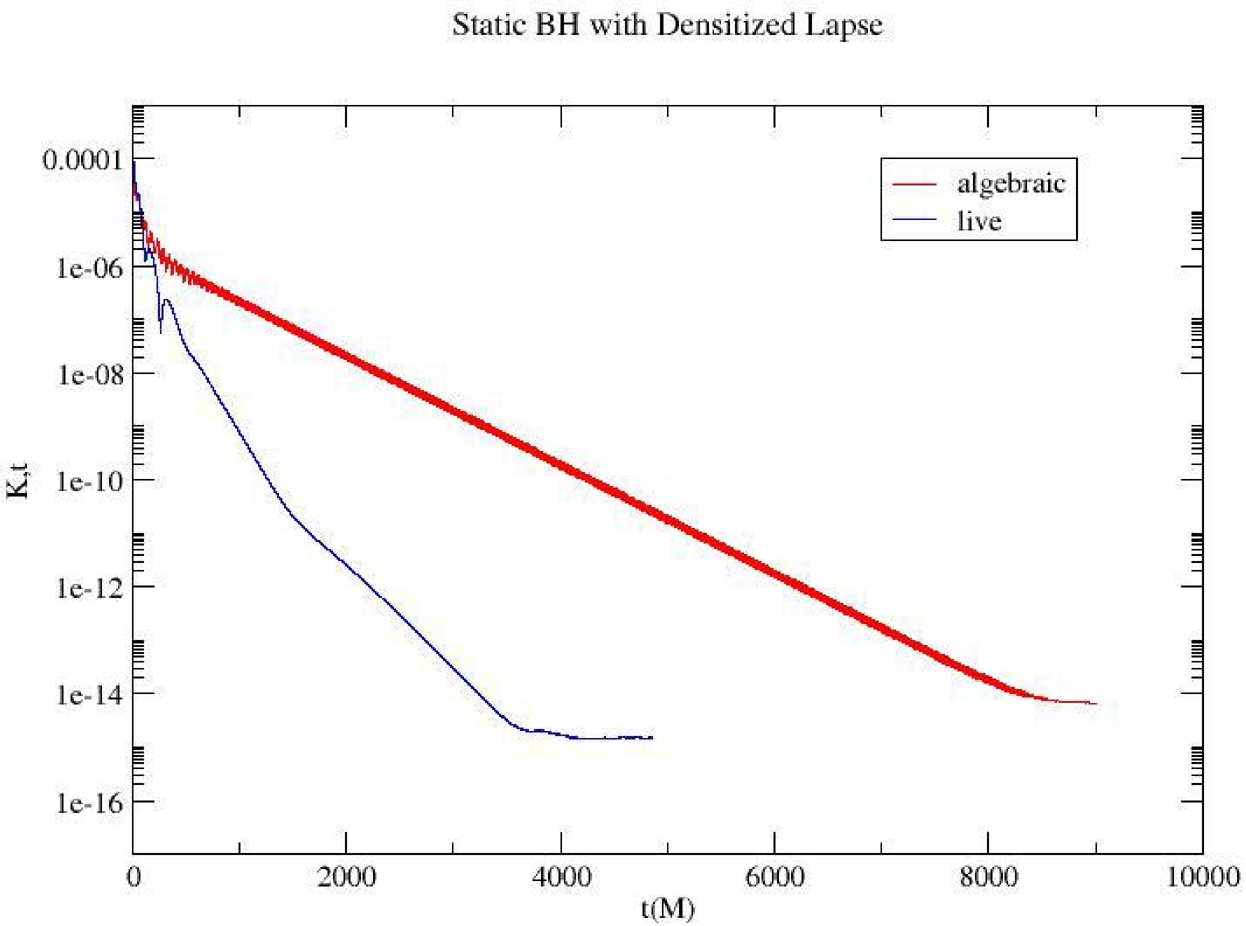


Solution extrapolation

For novel and rigorous excision: O. Reula talk tomorrow.

Single Static Black Hole

with B. Kelly, K. Smith, D. Shoemaker, U. Sperhake

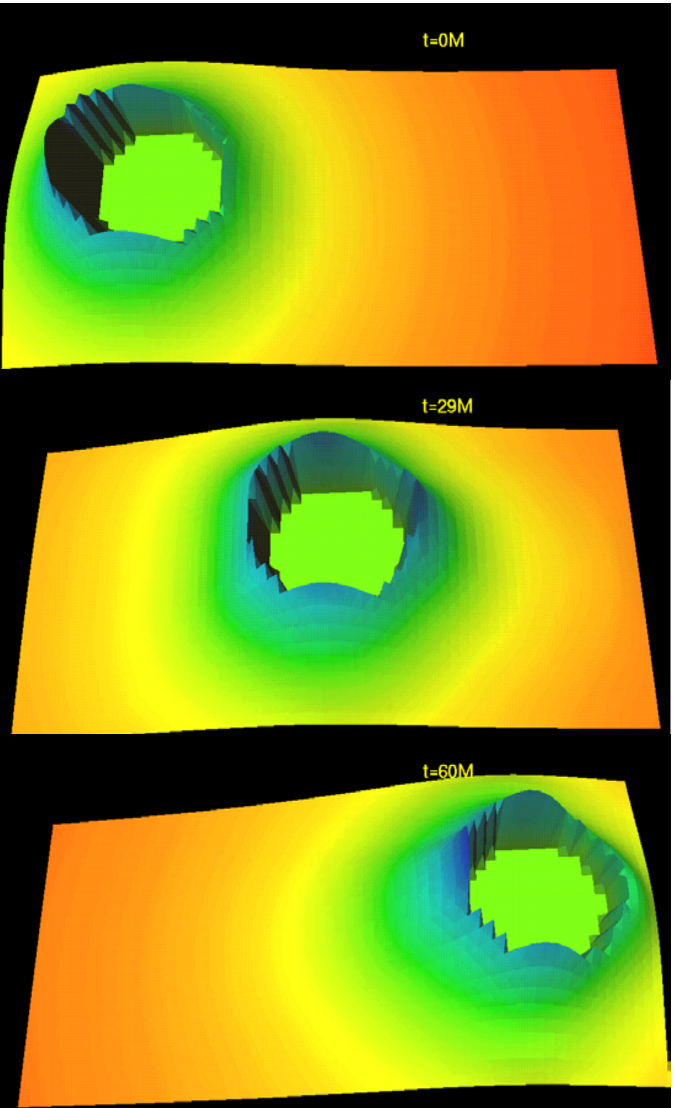


$$\hat{\alpha} = g^{-1/2}\,\alpha$$

A Wobbling Black Hole

with B. Kelly, D. Shoemaker, K. Smith, U. Sperhake

Boosted BH
BBH Alliance



Wobbling BH:

$$\begin{aligned} t &= \bar{t} \\ x^i &= \bar{x}^i + \xi^i(\bar{t}) \\ \beta^i &= \bar{\beta}^i + \dot{\xi}^i \end{aligned}$$

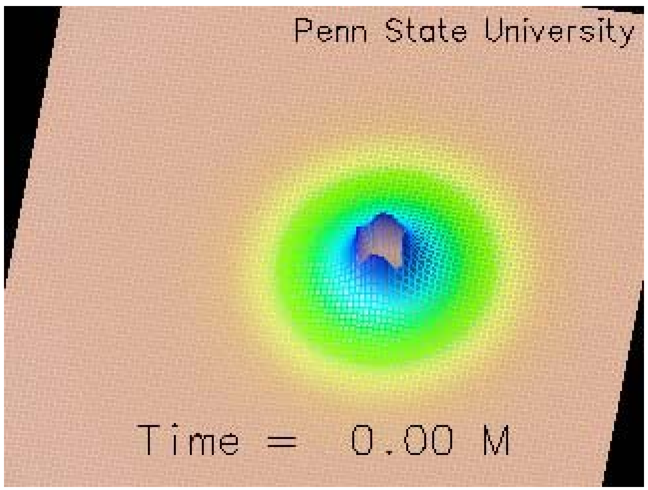
Caution:

- Simple-excision v.1 will not work
- Points appear and disappear
- Better outer boundary is needed

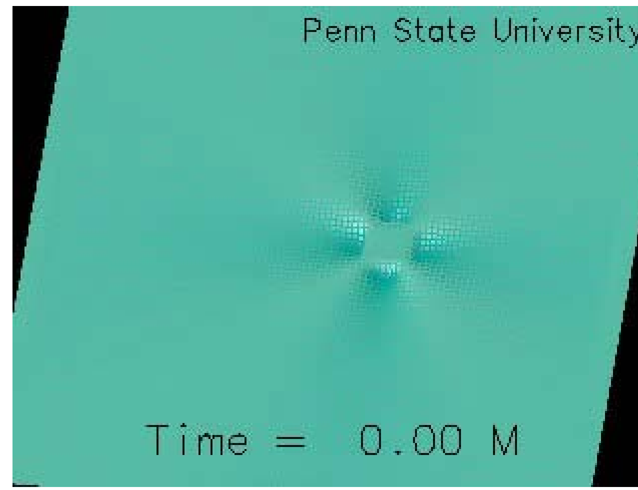
A Wobbling Black Hole

with B. Kelly, D. Shoemaker, K. Smith, U. Sperhake

Circling BH



Trace Extrinsic Curvature

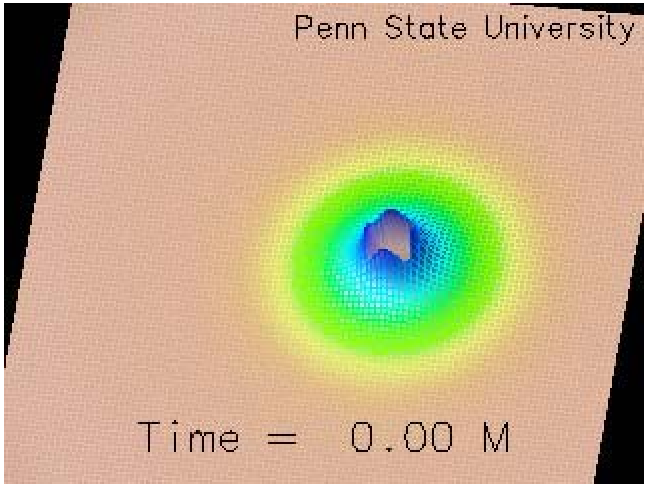


Normalized Hamiltonian Constraint

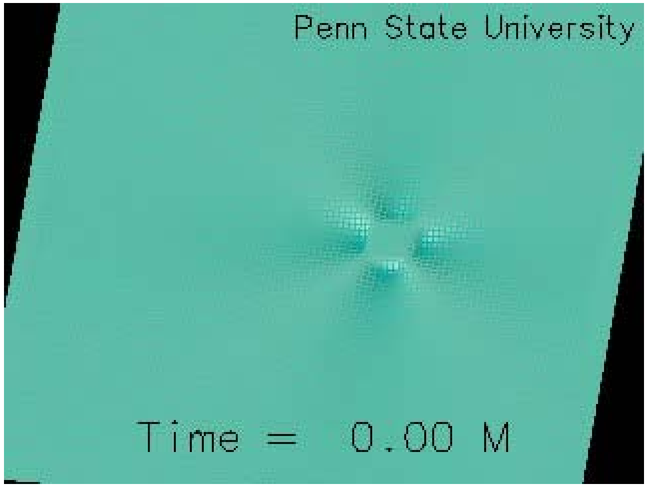
A Wobbling Black Hole

with B. Kelly, D. Shoemaker, K. Smith, U. Sperhake

Spiraling BH

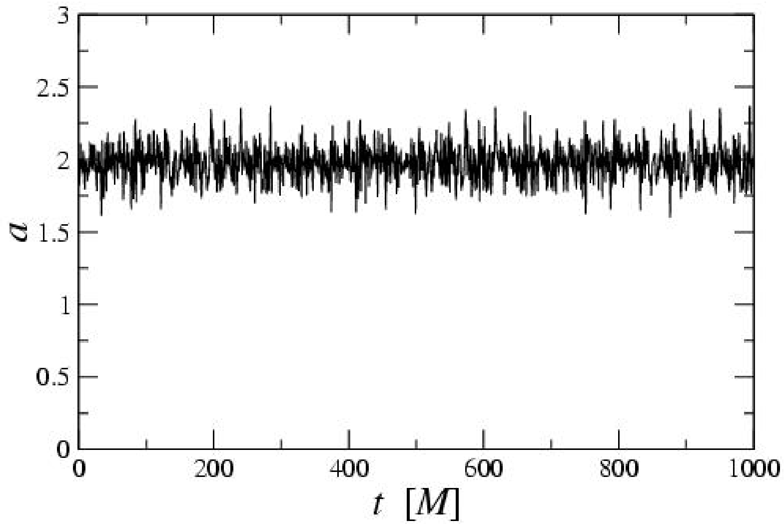


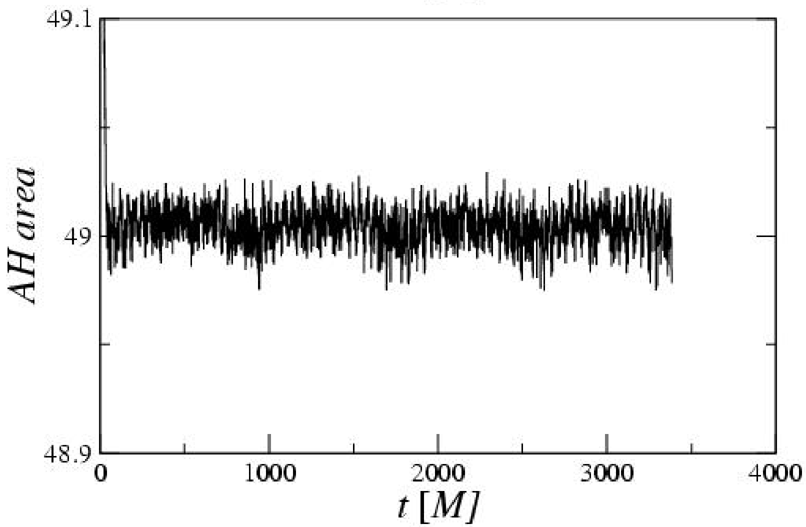
Trace Extrinsic Curvature



Normalized Hamiltonian Constraint

Convergence & AH Areas



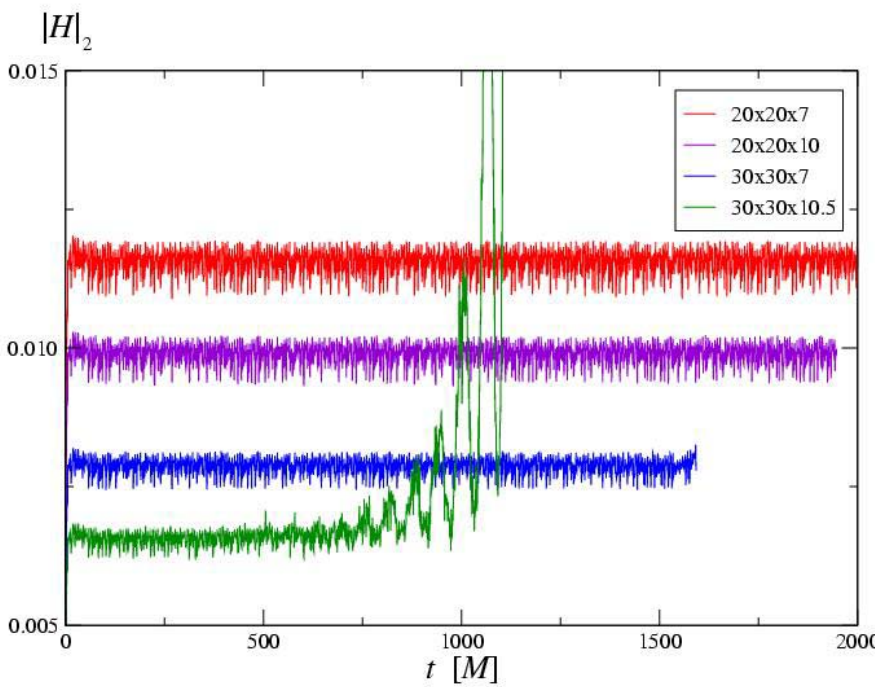


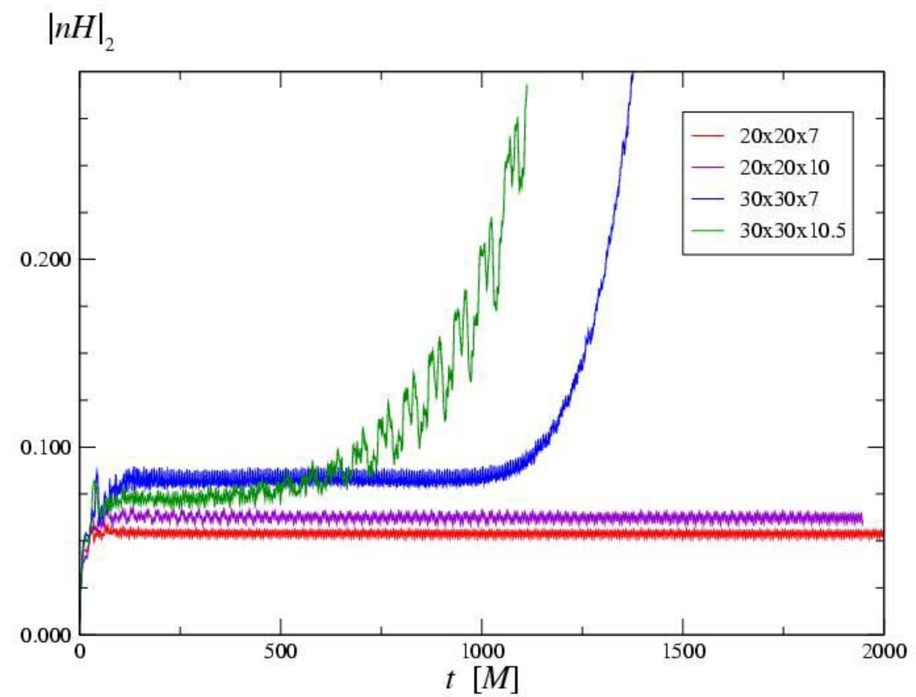
$$K_h = K_{ana} + C\, h^a$$

$$a = \frac{1}{\ln 2} \ln \left| \frac{K_h - K_{ana}}{K_{h/2} - K_{ana}} \right|$$

$$A_{AH} = 4\,\pi\,(2\,M)^2 = 50.265$$

Boundary Effects





Approx. Co-rotating BBH Initial Data

- Superposed boosted IEF holes

$$
\begin{aligned}
g_{ij} &= \eta_{ij} + 2\,H_1\,l^1_i\,l^1_j + 2\,H_2\,l^2_i\,l^2_j \\
\beta^i &= \beta^i_1 + \beta^i_2 \\
\alpha &= \alpha_1\,\alpha_2
\end{aligned}
$$

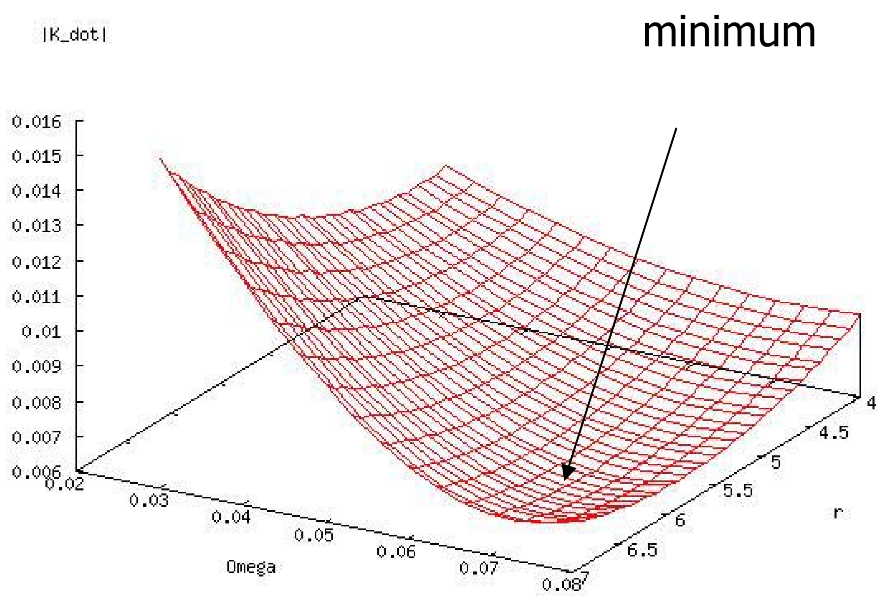
- Transform to a rotating frame (Ω)
- Extrinsic curvature

$$K_{ij} = \frac{-1}{2\,\alpha}\left(\nabla_i\beta_j + \nabla_j\beta_i\right)$$

- Three-parameter family (v, d, Ω)
- Find minimum of

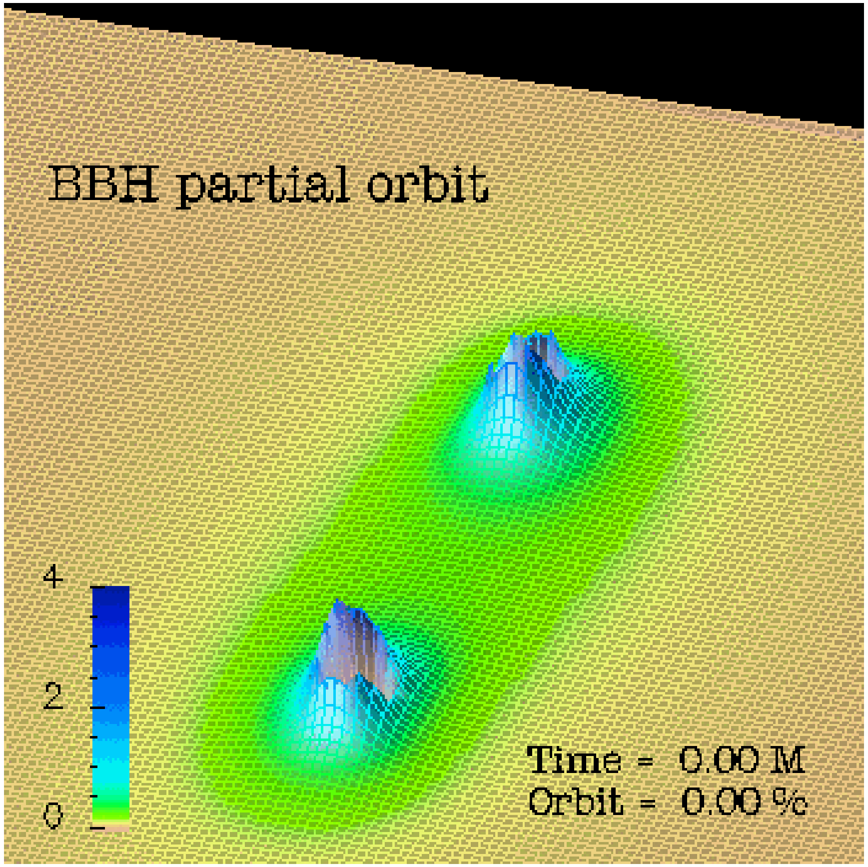
$$S = \int a\,|\partial_t K_{ij}|^2 + b\,|\rho|^2 + b\,|J^i|^2 dV$$

- Notice: Data does not satisfy constraints



BBH “Orbits” with D. Shoemaker

- Evolution equations:
  Modified BSSN
  [Laguna-Shoemaker CQG 19, 3679 (2002)]
- Initial data:
  Modified Superposed IEF
  [Matzner-Huq-Shoemaker PRD 59, 024015]
- Gauge conditions:
  Live 1+log lapse and Gamma-driver shift
  [Alcubierre, et al gr-qc/0206072]
- Outer boundary:
  f = fo + u(r-vt)/r

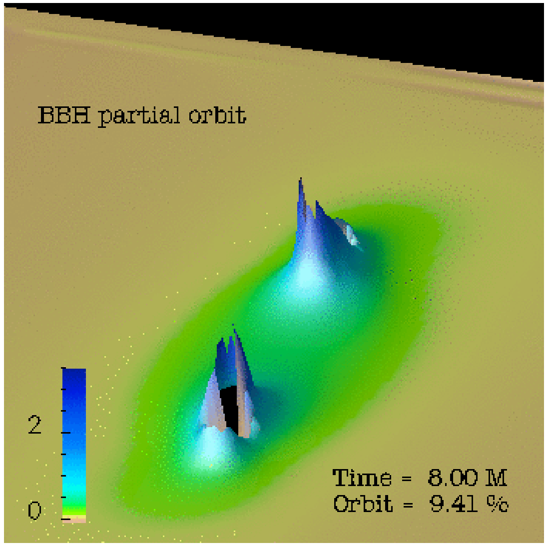


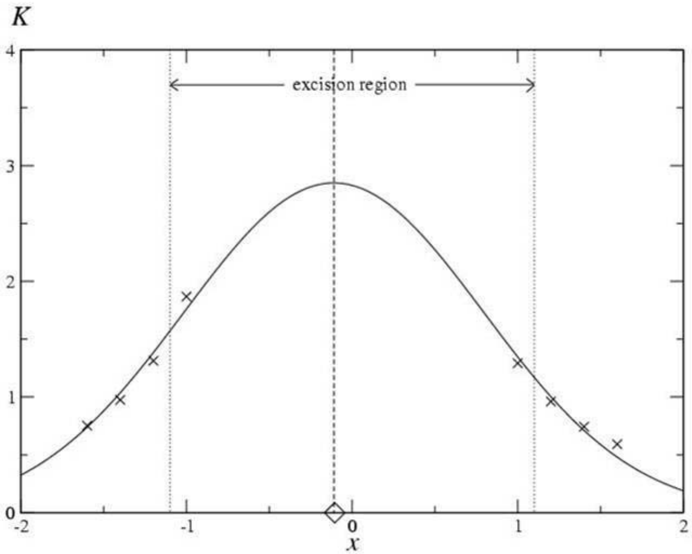
Trace of Extrinsic Curvature
Separation = 12 M
Boost Velocity = 0.6

Tracking Singularities

In principle an apparent horizon tracker is needed, but they are too expensive!

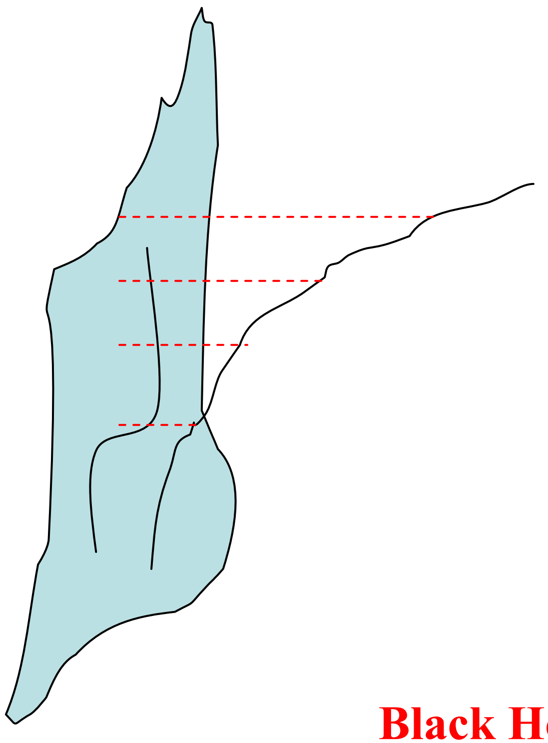
Alternative:
Identify the 1/r fall-off near the singularity and follow its changes





Constraint Violating Modes

(g_{ij}, K_{ij})



Are there solutions to the Einstein evolution equations that violate the constraints and do not exhibit exponential growths?

Black Hole in a Box

Metric Distortion:

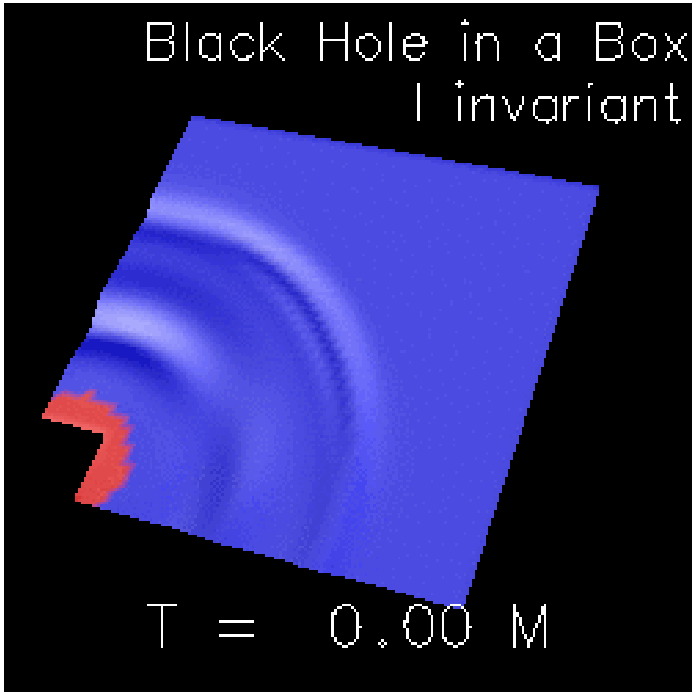
$$K = K_{iEF}\,[1 + a\,R(r)\,Y_{20}]$$
$$\phi = \phi_{iEF}\,[1 + b\,R(r)\,Y_{20}$$
$$R = \exp\left[-\frac{(r-r_o)^2}{\lambda^2}\right]$$

Mass Blending:

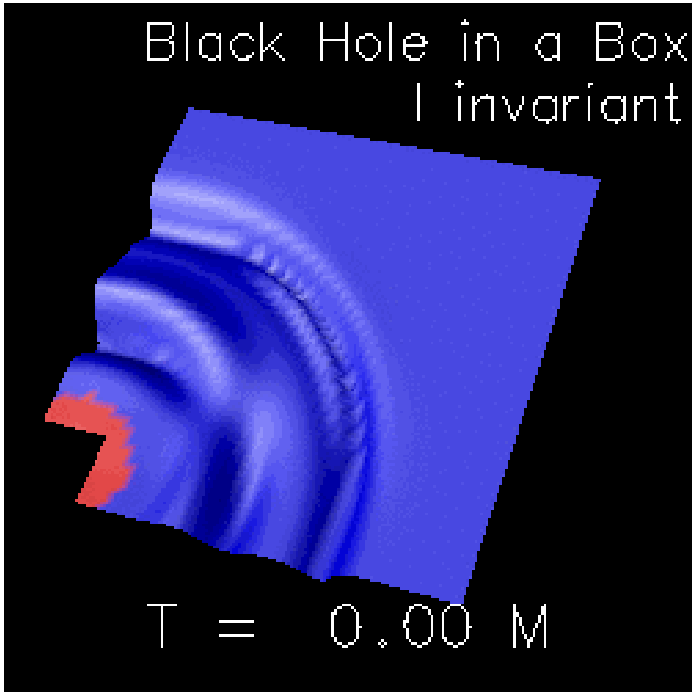
$$M = [1 - c(r)]\,M_1 + c(r)\,M_2$$
$$c = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{(r-r_o)}{\sigma}\right]$$

Black Hole in a Box

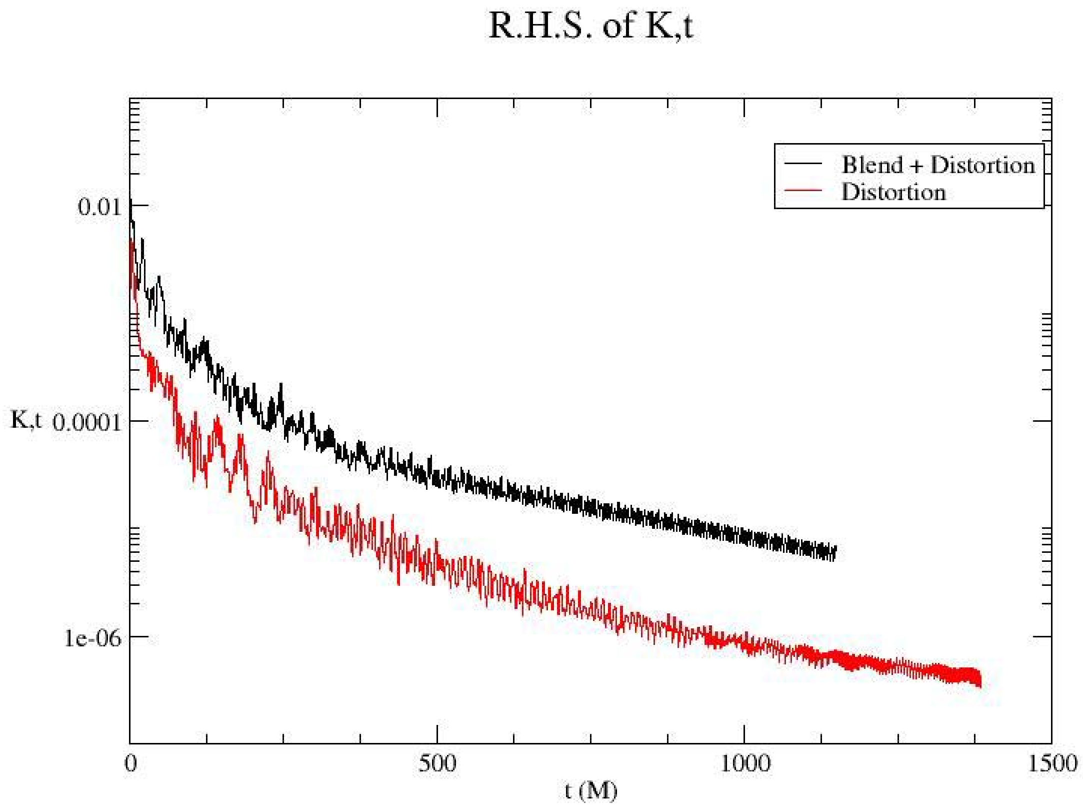
With D. Shoemaker

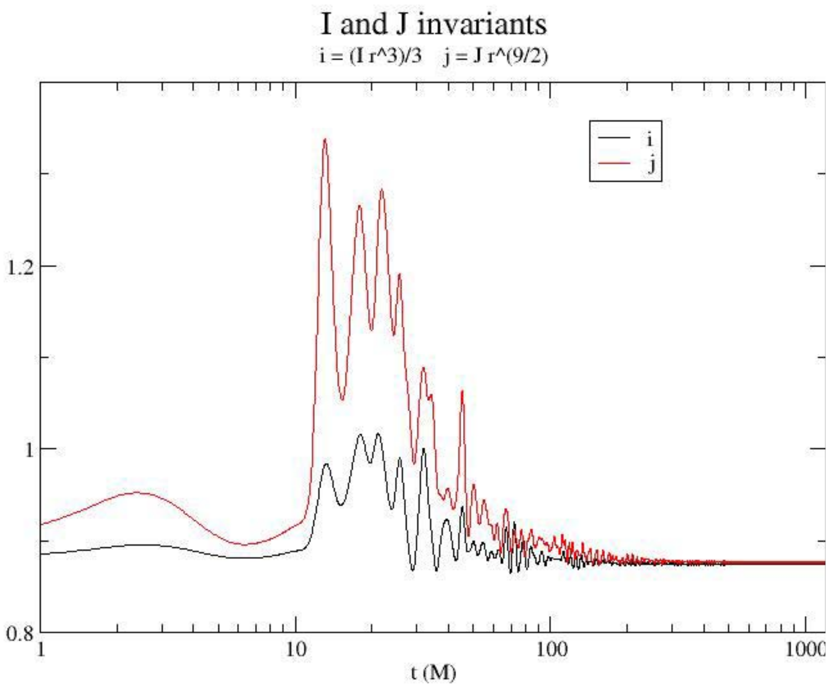


Distortion

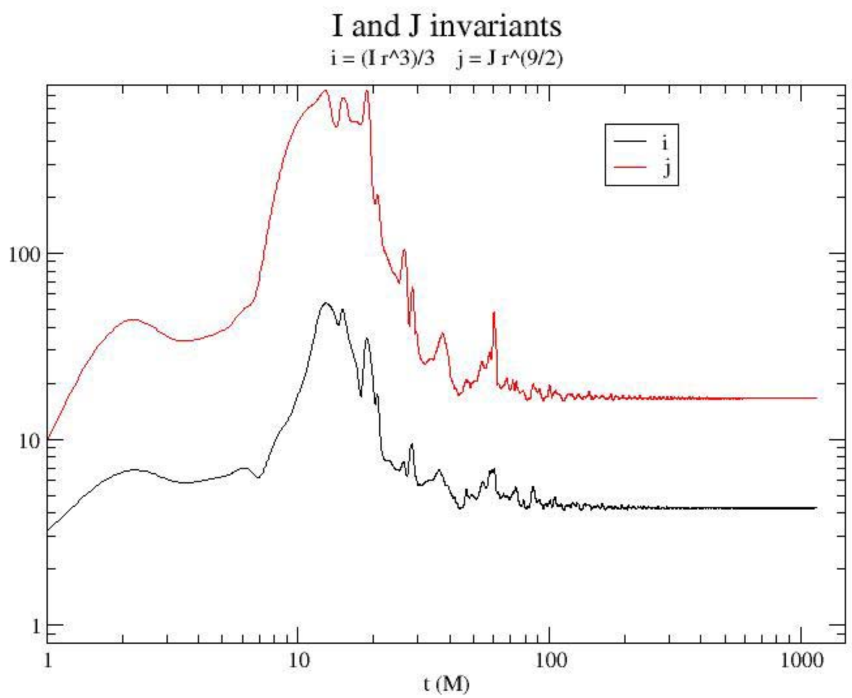


Distortion + Blend


R.H.S. of K,t
Blend + Distortion
Distortion
0.01
K,t
0.0001
1e-06
0
500
1000
1500
t (M)



Distortion



Distortion + Blend

Recall speciality index: $S = 27\, J^2 / I^3$

Conclusions

- Black hole excision seems to be in a good track.
- Stable constraint-violating evolutions exist. How generic?
- More work on gauges, outer boundary, tracking of singularities, AMR, constrained evolutions, etc.
- More creative engineering and formal mathematics are needed.
- We should not be afraid of using approximations
- Not too far into the future, we will have real simulations of …