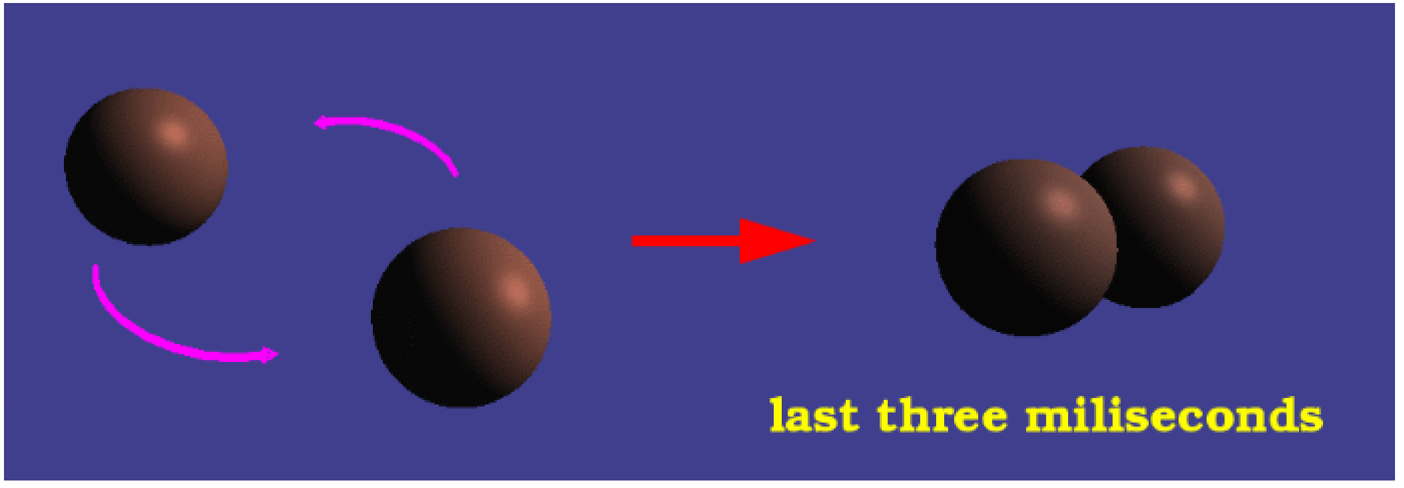
General Relativistic Numerical Simulation on Coalescing Binary Neutron Stars

Ken-ichi Oohara
Niigata University

in collaboration with
Mari Kawamura (Niigata Univ.)
Takashi Nakamura (Kyoto Univ.)

Introduction

- We are construction computer codes on 3D numerical relativity.
- The main target is to study the evolution of coalescing binary neutron stars and the radiation of gravitational waves from the merger.



Basic Equations

- **We write the 4D metric as**

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

α : **the lapse function**

β^i : **the shift vector**

γ_{ij} : **the intrinsic metric of 3-space**

- **Hamiltonian constraint**

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho_H$$

- **Momentum constraint**

$$D_j(K^j{}_i - \delta^j{}_i K) = 8\pi J_i$$

The evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\partial_t K_{ij} = \alpha \left[R_{ij} - 8\pi \left\{ S_{ij} + \frac{1}{2}(\rho_H - S^k{}_k) \right\} \right] - D_i D_j \alpha$$
$$+ \alpha (K K_{ij} - 2 K_{ik} K^k{}_j) + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k + \beta^k D_k K_{ij}$$

R_{ij} : the 3-D Ricci tensor, $R = \gamma^{ij} R_{ij}$

K_{ij} : the extrinsic curvature $K = \gamma^{ij} K_{ij}$

ρ_H : the energy density

J_i : the momentum density

S_{ij} : the stress tensor

measured by the observer moving along the line normal to the spacelike hypersurface of t = const.

General Relativistic Hydrodynamics Equations

- **We assume that the matter is the perfect fluid,**

$$T_{\mu\nu} = (\rho + \rho\varepsilon + P)u_\mu u_\nu + P g_{\mu\nu}$$

where

ρ : **the energy density**

ε : **the specific internal energy (per unit mass)**

P : **the pressure**

measured by the observer moving with the fluid

u_μ : **the 4-velocity of the fluid**

$$\rho_H = n^\mu n^\nu T_{\mu\nu},\ \ J_i = -h_i{}^\mu n^\nu T_{\mu\nu},\ \ S_{ij} = h_i{}^\mu h_j{}^\nu T_{\mu\nu}$$

where n_μ **is the unit timelike normal vector**

and $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

GR Hydrodynamics Equations (2)

- The relativistic hydrodynamics equations are obtained from $\nabla_\mu (\rho u^\mu) = 0$ and $\nabla_\mu T^{\mu\nu} = 0.$

- $\rho_N = \sqrt{\gamma}\alpha u^0 \rho \,, \quad u_i^N = \dfrac{J_i}{\alpha u^0 \rho}, \quad V^i = \dfrac{u^i}{u^0} = \dfrac{\alpha J^i}{P + \rho_H} - \beta^i$

✷ $\partial_t \rho_N + \partial_l (\rho_N V^l) = 0$

✷ $$\partial_t (\rho_N u_i^N) + \partial_l (\rho_N u_i^N V^l) = -\sqrt{\gamma}\alpha \partial_i P - \sqrt{\gamma}(P + \rho_H)\partial_i \alpha + \frac{\sqrt{\gamma}\alpha J^k J^l}{2(P + \rho_H)} \partial_i \gamma_{kl} + \sqrt{\gamma} J_l \partial_i \beta^l$$

✷ $\partial_t (\rho_N \varepsilon) + \partial_l (\rho_N \varepsilon V^l) = -P \partial_\mu (\sqrt{\gamma}\alpha u^\mu)$

Evolution Equations of the Metric

- **To solve the evolution of the metric tensor, we define the following variables:**

$$\phi = \gamma^{\frac{1}{12}}$$

$$\tilde{\gamma}_{ij} = \phi^{-4}\gamma_{ij}$$

$$\tilde{F}^{i} = \tilde{\gamma}^{ij}{}_{,j}$$

$$\hat{K}_{ij} = \phi^{-4}(K_{ij})^{\mathrm{STF}}$$

$$K = \gamma^{ij}K_{ij}$$

where $(K_{ij})^{\mathrm{STF}} = \frac{1}{2}\left(K_{ij} + K_{ji} - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}K_{kl}\right)$

- $\partial_t \phi - \beta^l \partial_l \phi = -\frac{\phi}{6}(\alpha K - \partial_l \beta^l)$
- $\partial_t \tilde{\gamma}_{ij} - \beta^l \partial_l \tilde{\gamma}_{ij} = -2\left[\alpha \widehat{K}_{ij} - (\tilde{\gamma}_{il}\partial_j \beta^l)^{\mathrm{STF}}\right]$
- $\partial_t \widetilde{F}^i - \beta^l \partial_l \widetilde{F}^i = 2\partial_j(\alpha \widehat{K}^{ij}) - \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i - \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k$
- $\partial_t \widehat{K}_{ij} - \beta^l \partial_l \widehat{K}_{ij} = \phi^{-4}\left[\alpha(R_{ij}^{\mathrm{STF}} - 8\pi S_{ij}^{\mathrm{STF}}) - (D_i D_j \alpha)^{\mathrm{STF}}\right]$

  $+\alpha(K \widehat{K}_{ij} - 2\widehat{K}_{il}\widehat{K}^l{}_j) + \widehat{K}_{il}\partial_j \beta^l + \widehat{K}_{jl}\partial_i \beta^l - \frac{2}{3}\widehat{K}_{ij}\partial_l \beta^l$
- $\partial_t K - \beta^l \partial_l K = \alpha\left[\widehat{K}_{ij}\widehat{K}^{ij} + \frac{1}{3}K^2 + 4\pi(\rho_H + S^i{}_i)\right] - D^i D_i \alpha$

- **Several kinds of the reformulation of the Einstein equation in numerical relativity have been proposed.**
- **Our formulation is the simplest one.**
- **The motivation to use the formulation is that we encountered numerical instability and suppose that numerical errors in the second derivatives of the metric tensor needed to calculate the Ricci tensor are likely to cause the instability.**
- **We decided to compute $\tilde{F}^i$ as an independent variable and calculate Ricci tensor using it.**

Coordinate Conditions

- **The choice of the shift β^i and the lapse function α is important because**
  - **the stability of the code depends on them**

  and
  - **it is intimately related to the extraction of physically relevant information such as gravitational radiation.**
- **The condition**

$$\widetilde{D}_j(\partial_t \tilde{\gamma}^{ij}) = 0$$

produces the minimal distortion shift vector, but gives an equation too complicated to be solved numerically.

Shift Vector

- Instead we use the *pseudo-minimal distortion condition*;

$$\partial_j(\partial_t \tilde{\gamma}^{ij}) = (\partial_t \tilde{F}^i) = 0$$

- The shift vector β^i obeys

$$\nabla^2 \beta^i + \frac{1}{3}\partial_i \partial_j \beta^j = 2\partial_j(\alpha \hat{K}^{ij}) - h^{jk}\partial_j \partial_k \beta^i - \frac{1}{3} h^{ij}\partial_j \partial_k \beta^k$$

where $h^{ij} = \tilde{\gamma}^{ij} - \delta^{ij}$

- In this condition, we can simply set

$\tilde{F}^i = 0$ since $\partial_t \tilde{F}^i = \tilde{F}^i(t=0) = 0.$

Lapse Function

- **As for the slicing condition, we choose the maximal slicing, $K = 0$, which yields**

$$D^i D_i \alpha = \alpha \left[\widehat{K}_{ij} \widehat{K}^{ij} + 4\pi(\rho_H + S^i{}_i) \right]$$

The Conformal Factor ϕ

- **The evolution of ϕ is given by**

$$\partial_t \phi - \beta^l \partial_l \phi = -\frac{\phi}{6}(\alpha K - \partial_l \beta^l)$$

- **From the Hamiltonian constraint, it obeys**

$$\tilde{\Delta}\phi = -\frac{\phi^5}{8}(16\pi\rho_H + \widehat{K}_{ij}\widehat{K}^{ij} - \phi^{-4}\tilde{R})$$

- **We first make ϕ evolve using the former equation, then calculate the right-hand side of the latter one and solve it.**

The Ricci Tensor

- **We must take special care to calculate the Ricci tensor.**

$$R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}$$

$$R^{\phi}_{ij} = -2\phi^{-1}(\tilde{D}_j \tilde{D}_i \phi + \tilde{\gamma}_{ij} \tilde{\Delta}\phi) + 2\phi^{-2}\left[3(\tilde{D}_i\phi)(\tilde{D}_j\phi) - \tilde{\gamma}_{ij}(\tilde{D}_k\phi)(\tilde{D}^k\phi)\right]$$

$$\tilde{R}_{ij} = \frac{1}{2}\left[\tilde{\gamma}^{kl}(\tilde{\gamma}_{li,jk} + \tilde{\gamma}_{lj,ik} - \tilde{\gamma}_{ij,kl}) + \tilde{\gamma}^{kl}{}_{,l}(\tilde{\gamma}_{ki,j} + \tilde{\gamma}_{kj,i} - \tilde{\gamma}_{ij,k})\right] - \tilde{\Gamma}^{k}_{li}\tilde{\Gamma}^{l}_{jk}$$

$$= \frac{1}{2}(\tilde{F}_{i,j} + \tilde{F}_{j,i} - \tilde{\gamma}^{kl}{}_{,j}\tilde{\gamma}_{il,k} - \tilde{\gamma}^{kl}{}_{,i}\tilde{\gamma}_{jl,k} - \tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl})$$

$$+ \tilde{F}^{k}(\tilde{\gamma}_{ki,j} + \tilde{\gamma}_{kj,i} - \tilde{\gamma}_{ij,k}) - \tilde{\Gamma}^{k}_{li}\tilde{\Gamma}^{l}_{jk}$$

$$= -\frac{1}{2}(\tilde{\gamma}^{kl}{}_{,j}\tilde{\gamma}_{il,k} + \tilde{\gamma}^{kl}{}_{,i}\tilde{\gamma}_{jl,k} + \tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl}) - \tilde{\Gamma}^{k}_{li}\tilde{\Gamma}^{l}_{jk} \quad \text{if } \tilde{F}^{i} = 0$$

The Ricci Tensor (2)

- $\tilde{\gamma}^{kl}\tilde{\gamma}_{li,jk}$ **includes terms such as** $\dfrac{\partial^2 h_{xx}}{\partial x \partial y}$ $\quad (h_{kl} = \tilde{\gamma}_{kl} - \delta_{kl})$
  whose accuracy in numerical calculation is not so good.

- **The term** $\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} = \dfrac{\partial^2 h_{ij}}{\partial x^2} + \dfrac{\partial^2 h_{ij}}{\partial y^2} + \dfrac{\partial^2 h_{ij}}{\partial z^2} + h^{kl}\dfrac{\partial^2 h_{ij}}{\partial x^k \partial x^l}$

  includes the second derivatives, but inaccuracies in $h^{kl}h_{ij,kl}$ **is not so serious, if both** h^{kl} **and** $h_{ij,kl}$ **are small.**

Boundary Conditions on the Metric

- **We apply outgoing conditions for $\tilde{\gamma}_{ij}$ and $\widehat{K}_{ij}$;**

$$Q(t, x^i) = \frac{H(\alpha t - \phi^2 r)}{r} , \qquad r^2 = x^2 + y^2 + z^2$$

$$\partial_t H + c^i \partial_i H = 0 , \qquad c^i = \frac{\alpha x^i}{r\phi^2}$$

$$\partial_t Q + c^i \partial_i Q = -\frac{\alpha}{r\phi^2} Q$$

Numerical Methods

- **We solve the hydrodynamics equations using van Leer's scheme with the minmod limiter.**
- **The evolution equations of the metric are solved using the CIP (Cubic-Interpolated Propagation) method.**
- **To achieve second-order accuracy both in time as well as in space, we use a two-step predictor-corrector scheme.**

$$\partial_t \boldsymbol{Q} = \boldsymbol{F}$$

$$\boldsymbol{Q}(t + \tfrac{1}{2}\Delta t) = \boldsymbol{Q}(t) + \tfrac{1}{2}\Delta t \cdot \boldsymbol{F}(t)$$

$$\boldsymbol{Q}(t + \Delta t) = \boldsymbol{Q}(t) + \Delta t \cdot \boldsymbol{F}(t + \tfrac{1}{2}\Delta t)$$

Elliptic Equations

- $\nabla^2 \phi = S_\phi$
- $\nabla^2 \alpha = S_\alpha$
- $\nabla^2 \beta^i + \frac{1}{3}\partial_i \partial_k \beta^k = S_\beta^i$

They are solved with a pre-conditioned conjugate gradient method.

The most CPU hours are consumed for solving these equations.

- **50% for β**
- **~10% for each of α and ϕ**

Code Tests (Oscillation of a Spherical Star)

- **Put a spherical star of**

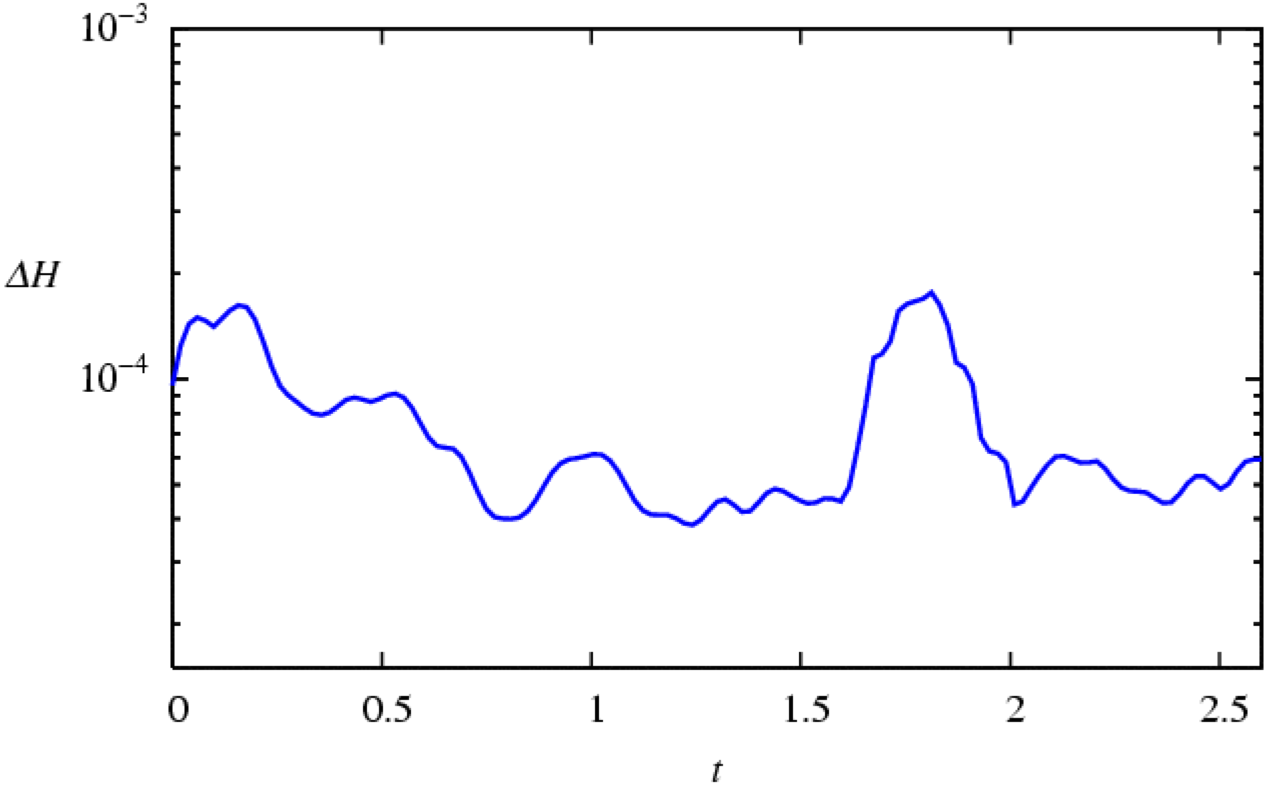
$$M = 1.5\,M_\odot\ ,\quad R = 7.7\,M_\odot$$

$$\rho_c = 2.1\times 10^{-3}\,M_\odot^{-2}$$

$$P = (\Gamma - 1)\rho\varepsilon\ ,\quad \Gamma = 2$$

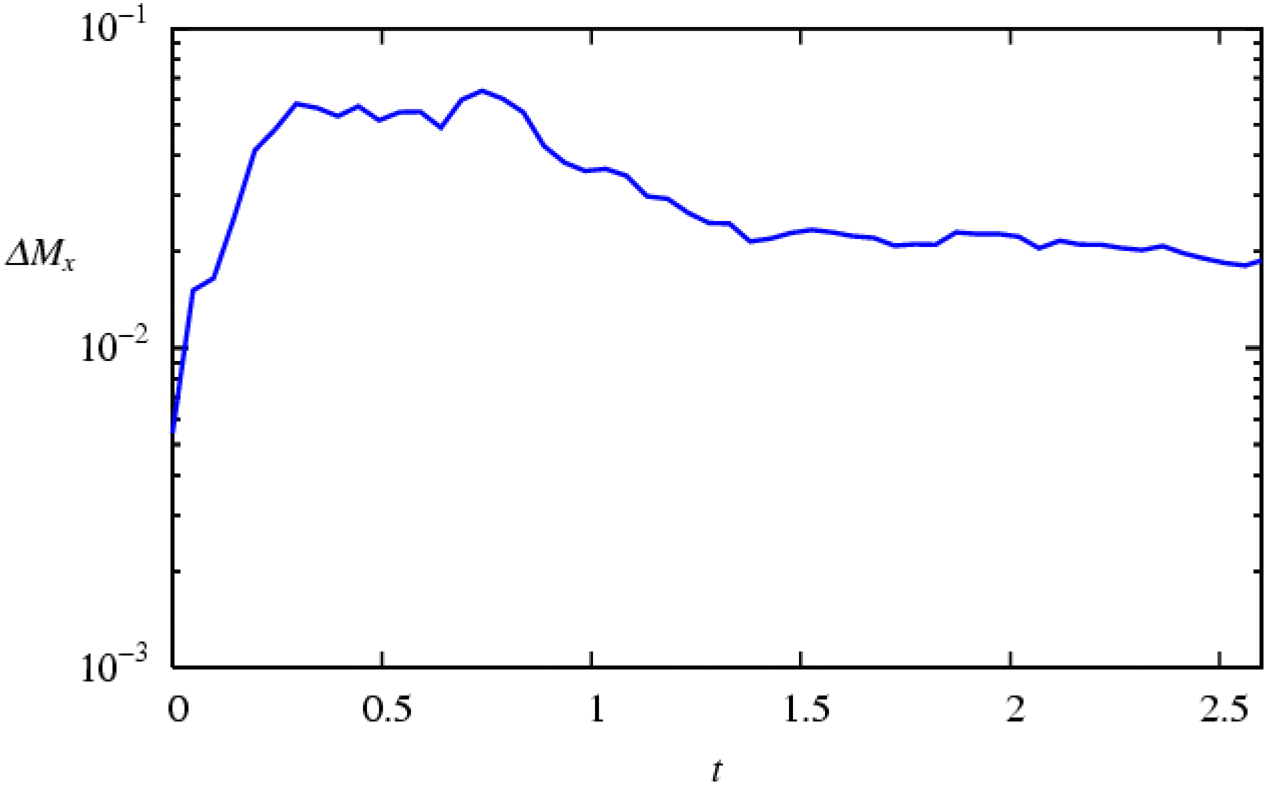
- **Reduce the pressure and the star begin to oscillate.**

Hamiltonian Constraint



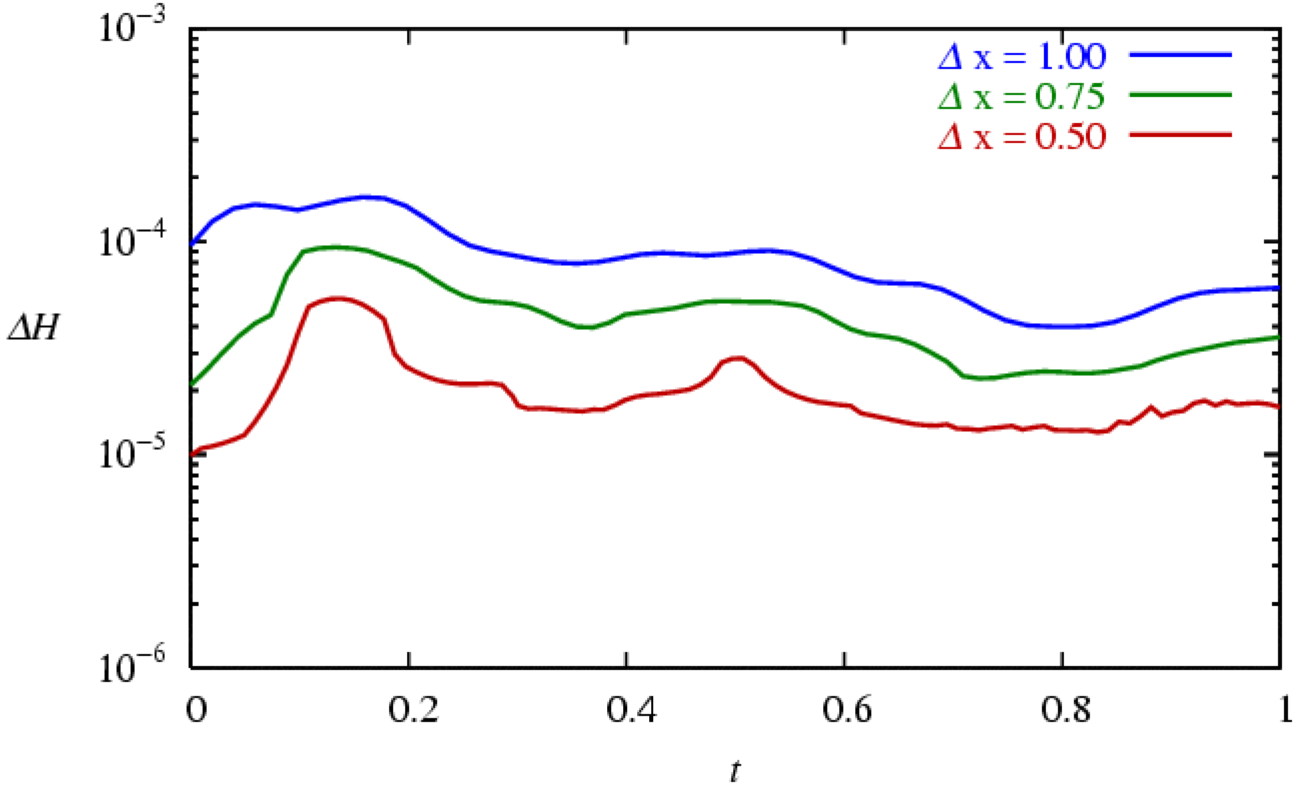
$$\Delta H = \frac{\left\| R + K^2 - K_{ij}K^{ij} - 16\pi\rho_H \right\|}{16\pi \left\| \rho_H \right\|}$$

Momentum Constraint

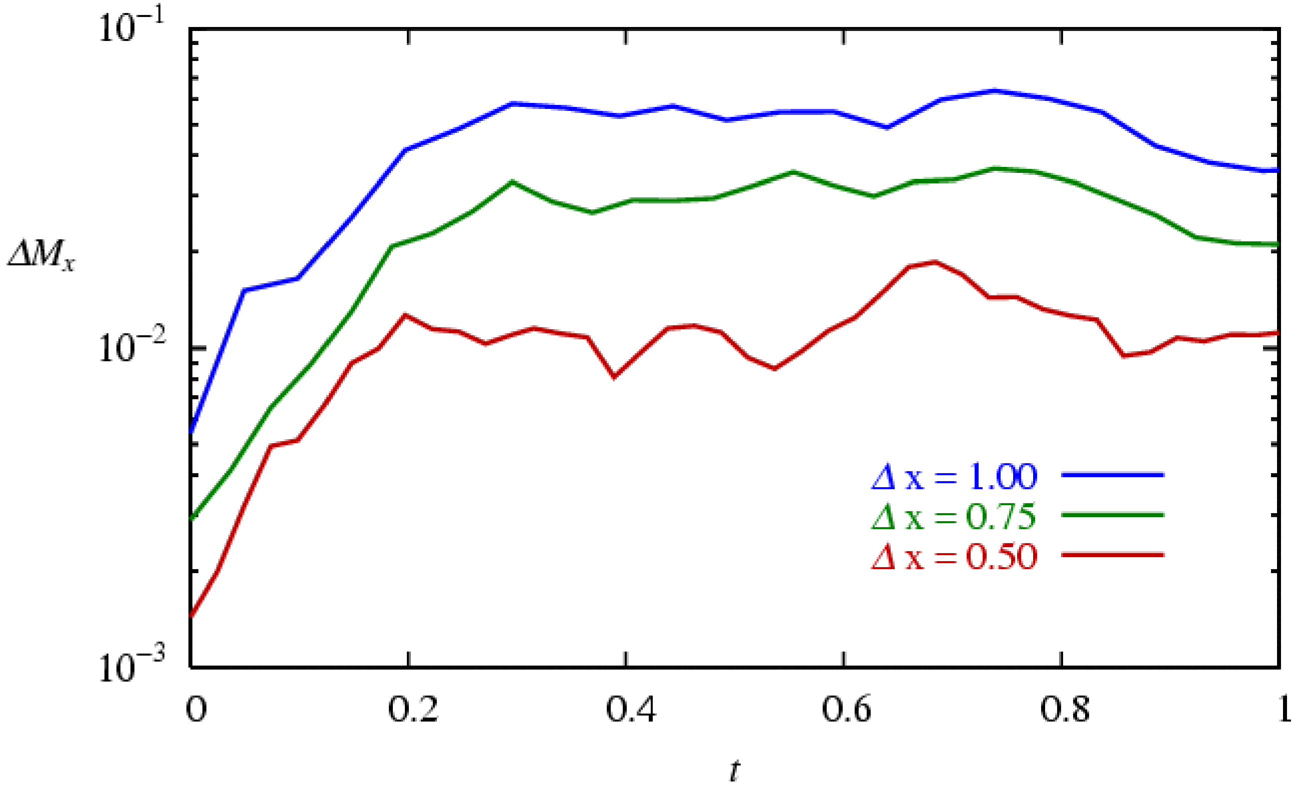


$$\Delta M_i = \frac{\left\| D_j (K^j{}_i - \delta^j{}_i K) - 8\pi J_i \right\|}{8\pi \sqrt{\left\| J^2 \right\|}}$$

Convergence Tests (1)

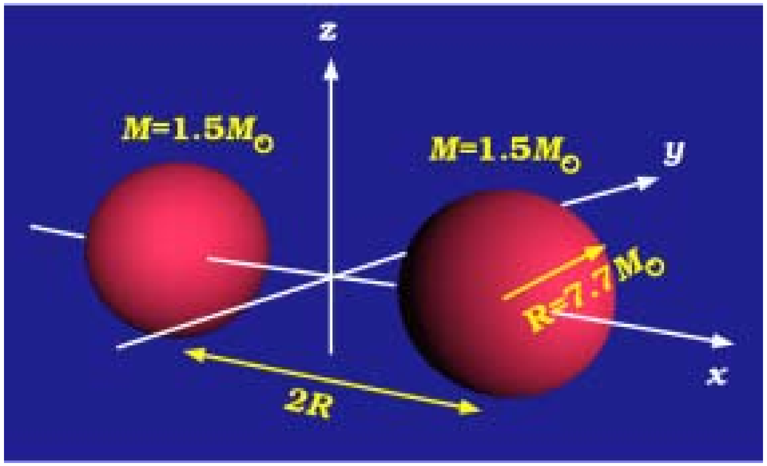


Convergence Tests (2)

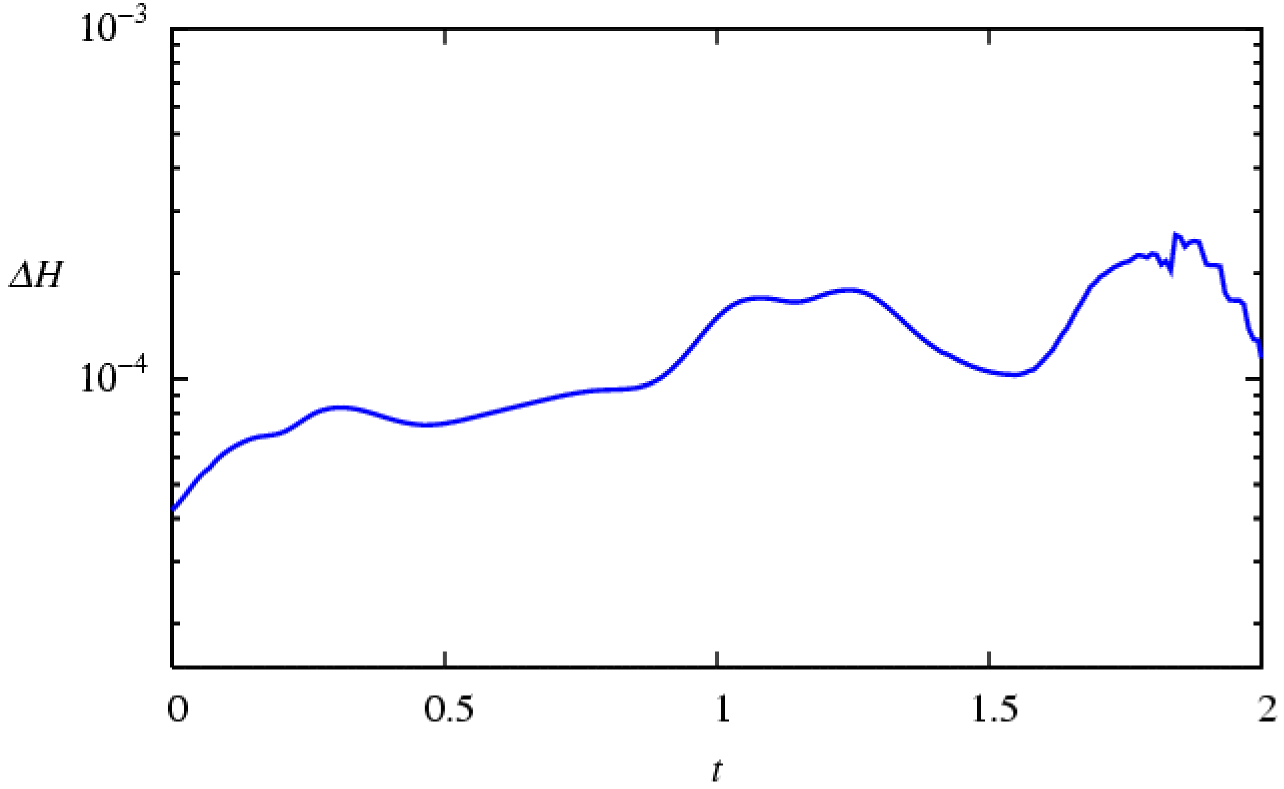


Coalescing Binary Neutron Stars

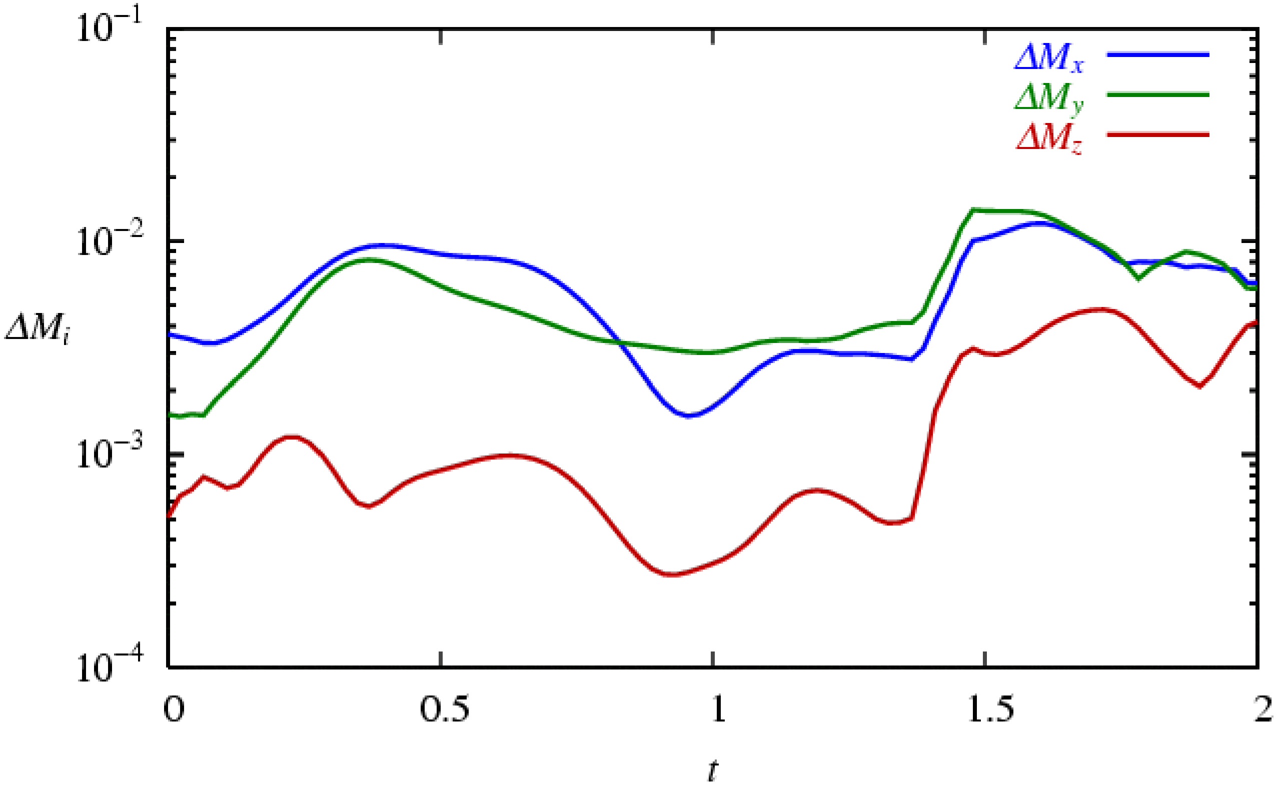
- Grid properties: $475 \times 475 \times 238$, $\Delta x = \Delta y = \Delta z = 1\,M_{\odot}$
- We assume the symmetry wrt the equatorial plane.
- Initial data:
  - 2 spherical stars of rest mass $M = 1.5\,M_{\odot}$
    radius $R = 7.7G\,M_{\odot}/c^2 = 11.6\text{km}$
    located at $x = \pm 2R,\ y = z = 0$
  - The initial rotational velocity is given so that the circulation of the system vanishes.



Hamiltonian Constraint



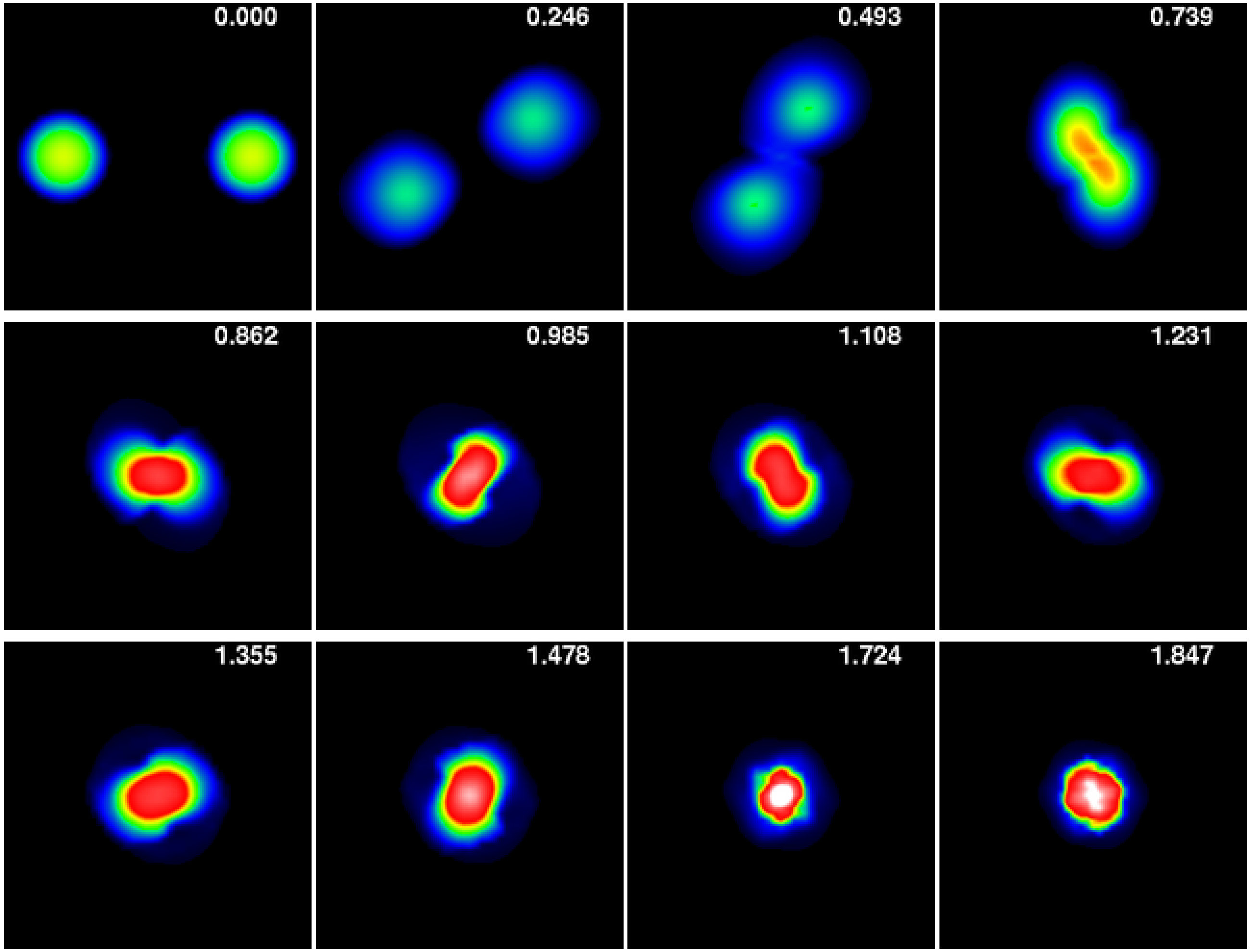
Momentum Constraint



Numerical Precision

- **We need not solve for K since we use the maximal slicing, but it is solved to monitor the numerical precision.**
  The value of K is kept as small as 0.1% of the typical value of K_{ij}, but it becomes large from the numerical boundary region in the final stage of simulation. It is due to a small reflection of GW at the numerical boundary.
- **Since we use the pseudo-minimal distortion condition, $\widetilde{F}^{i} = \tilde{\gamma}^{ij}{}_{,j}$ must be zero. It is satisfied within a few % of the derivative of $\tilde{\gamma}_{ij}$. The error grows in the same way as K in the final stage of simulation.**
- **The total ADM mass is conserved within 10% error.**

Coalescing Binary NS (density *xy*)



Gauge-Invariant Wave Form Extraction

- **Outside of the star, we consider the total spacetime metric $g_{\mu\upsilon}$ as a sum of a Scwarzschild metric and non-spherical perturbation parts:**

$$g_{\mu\nu} = g_{\mu\nu}^{(B)} + h_{\mu\nu}^{(e)} + h_{\mu\nu}^{(o)}$$

$$g_{\mu\nu}^{(B)} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2\theta \end{pmatrix}$$

(in the spherical coordinates)

$$h_{\mu\nu}^{(e)} = \sum_{l,m} \begin{pmatrix} N^2 H_{0lm} Y_{lm} & H_{1lm} Y_{lm} & h_{0lm}^{(e)} Y_{lm,\theta} & h_{0lm}^{(e)} Y_{lm,\phi} \\ * & A^2 H_{2lm} Y_{lm} & h_{1lm}^{(e)} Y_{lm,\theta} & h_{1lm}^{(e)} Y_{lm,\phi} \\ * & * & r^2 [K_{lm} + G_{lm} \partial_\theta^2] Y_{lm} & r^2 G_{lm} X_{lm} \\ * & * & * & h_{3lm}^{(e)} \end{pmatrix}$$

$$h_{\mu\nu}^{(o)} = \sum_{l,m} \begin{pmatrix} 0 & 0 & -h_{0lm}^{(o)} Y_{lm,\phi} / \sin\theta & h_{0lm}^{(o)} Y_{lm,\theta} \sin\theta \\ * & 0 & -h_{1lm}^{(o)} Y_{lm,\phi} \sin\theta & h_{1lm}^{(o)} Y_{lm,\theta} \sin\theta \\ * & * & \frac{1}{2} h_{2lm}^{(o)} X_{lm} / \sin\theta & -\frac{1}{2} h_{2lm}^{(o)} W_{lm} \sin\theta \\ * & * & * & -\frac{1}{2} h_{2lm}^{(o)} X_{lm} \sin\theta \end{pmatrix}$$

- **From the linearized theory about perturbations of the Schwarzschild spacetime, the gauge invariant quantities are give by**

$$\Psi_{lm}^{(o)}(t,r) = \sqrt{2\Lambda(\Lambda-2)}\,N^2\left[h_{1lm}^{(o)} + \frac{r^2}{2}\frac{\partial}{\partial r}\left(\frac{h_{2lm}^{(o)}}{r^2}\right)\right]$$

$$\Psi_{lm}^{(e)}(t,r) = -\sqrt{\frac{2(\Lambda-2)}{\Lambda}}\,\frac{4rN^2k_{2lm} + \Lambda r k_{1lm}}{(\Lambda+1-3N^2)}$$

$$\Lambda = l(l+1)$$

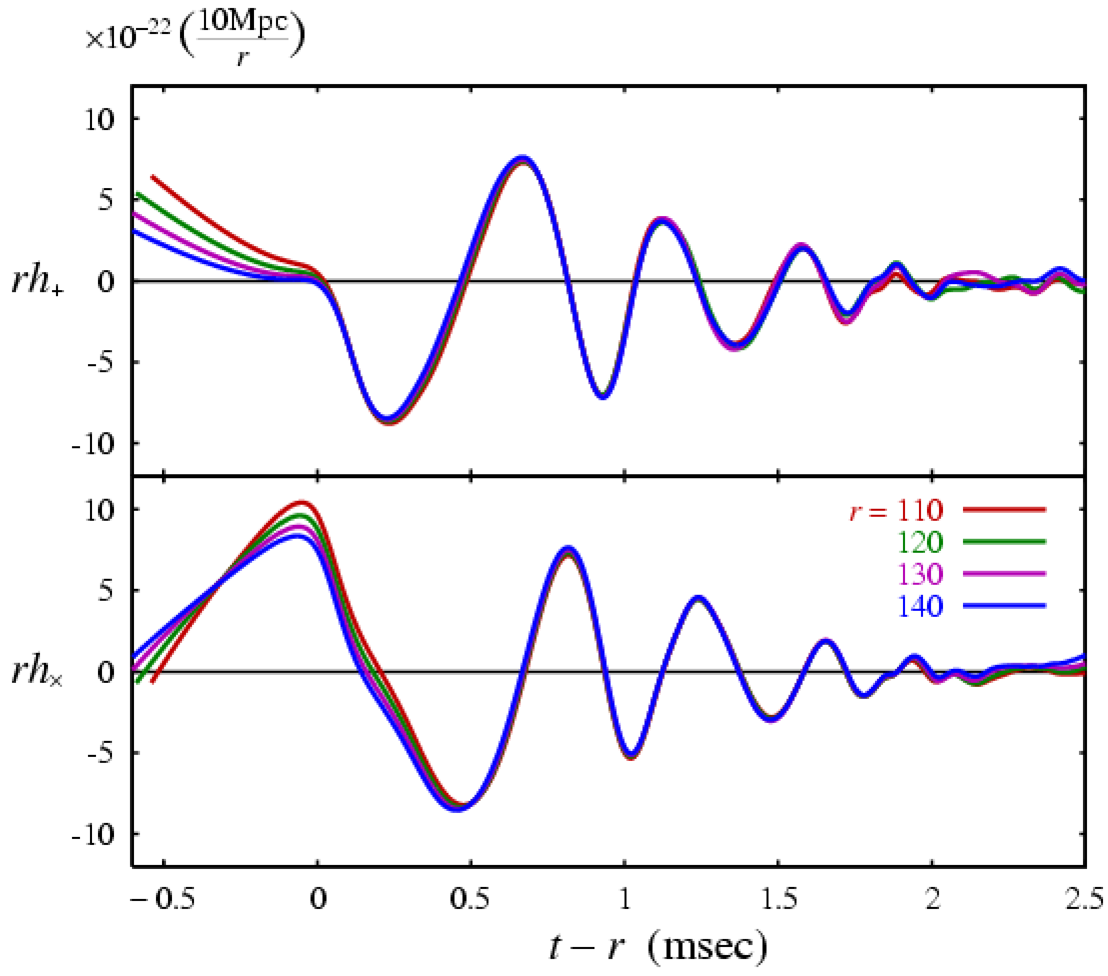
$$k_{1lm} = K_{lm} + N^2 r G_{lm,r} - 2N^2 h_{1lm}^{(o)}/r$$

$$k_{2lm} = \frac{H_{2lm}}{2N^2} - \frac{1}{\sqrt{N^2}}\frac{\partial}{\partial r}\left(\frac{r}{N^2}K_{lm}\right)$$

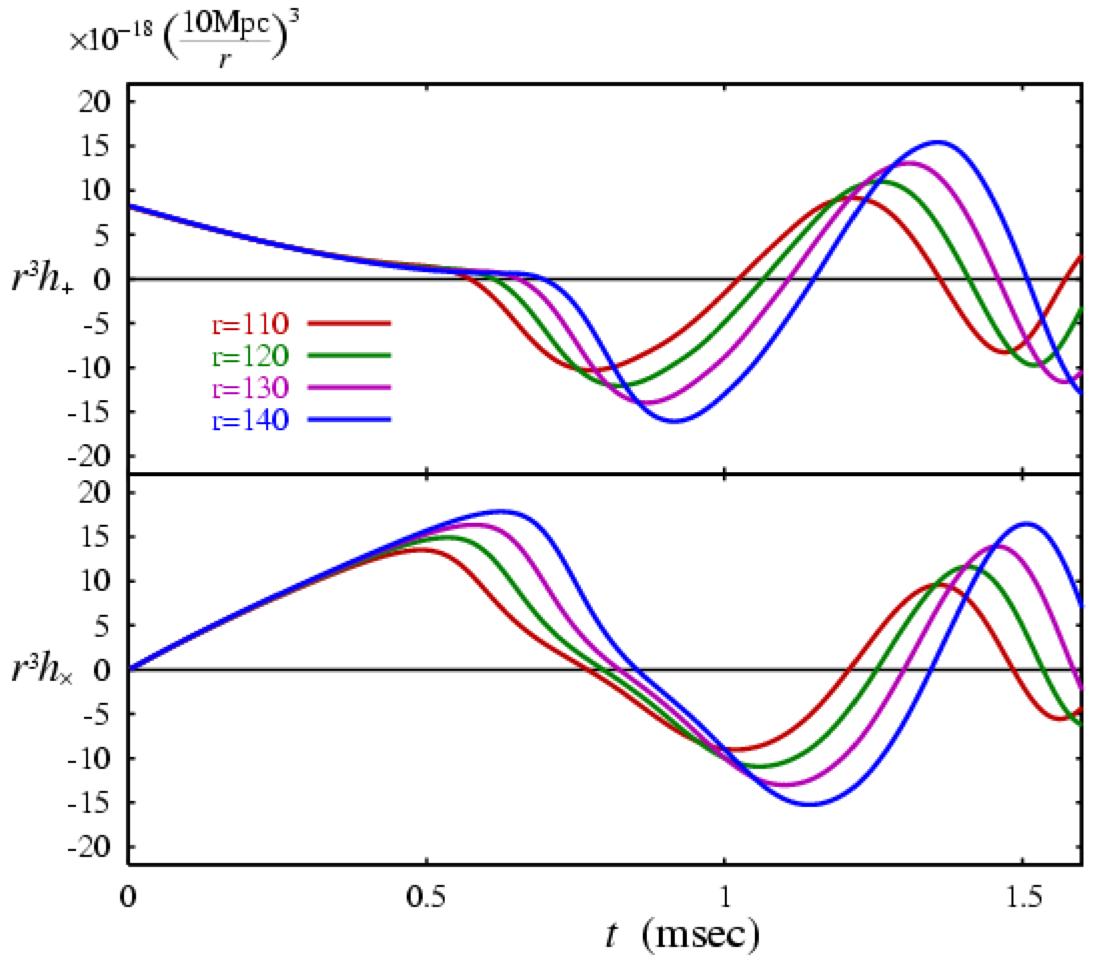
- $\Psi^{(o)}_{lm}$ and $\Psi^{(e)}_{lm}$ satisfy the Regge-Wheeler and the Zerilli equations, respectively.
- Two independent polarizations of the gravitational waves are given by

$$h_{+} - i h_{\times} = \frac{1}{\sqrt{2} r} \sum_{l,m} \left(\Psi^{(o)}_{lm}(t,r) + \Psi^{(e)}_{lm}(t,r) \right) {}_{-2}Y_{lm}$$

Wave Form



×10^-22 (10Mpc / r)
10
5
0
-5
-10
rh+
rh×
r = 110
120
130
140
−0.5
0
0.5
1
1.5
2
2.5
t − r (msec)



$\times 10^{-18} \left(\frac{10\text{Mpc}}{r}\right)^3$
$r^3 h_+$
$r^3 h_\times$
r=110
r=120
r=130
r=140
20
15
10
5
0
-5
-10
-15
-20
0
0.5
1
1.5
t (msec)

- h **includes a non-wave mode proportional to r^{-3}, which corresponds to the quadrupole part in the Newtonian potential.**
  It decreases fast as the merger of stars proceeds.
- **It becomes small for large r even before the merger.**
- **To eliminate the non-wave part,**

$$h(t-r) \equiv h_{+} - \mathrm{i}h_{\times} = \frac{F(t-r)}{r} + \frac{G(t)}{r^{3}}$$

- **Fourier components of** h **are written as**

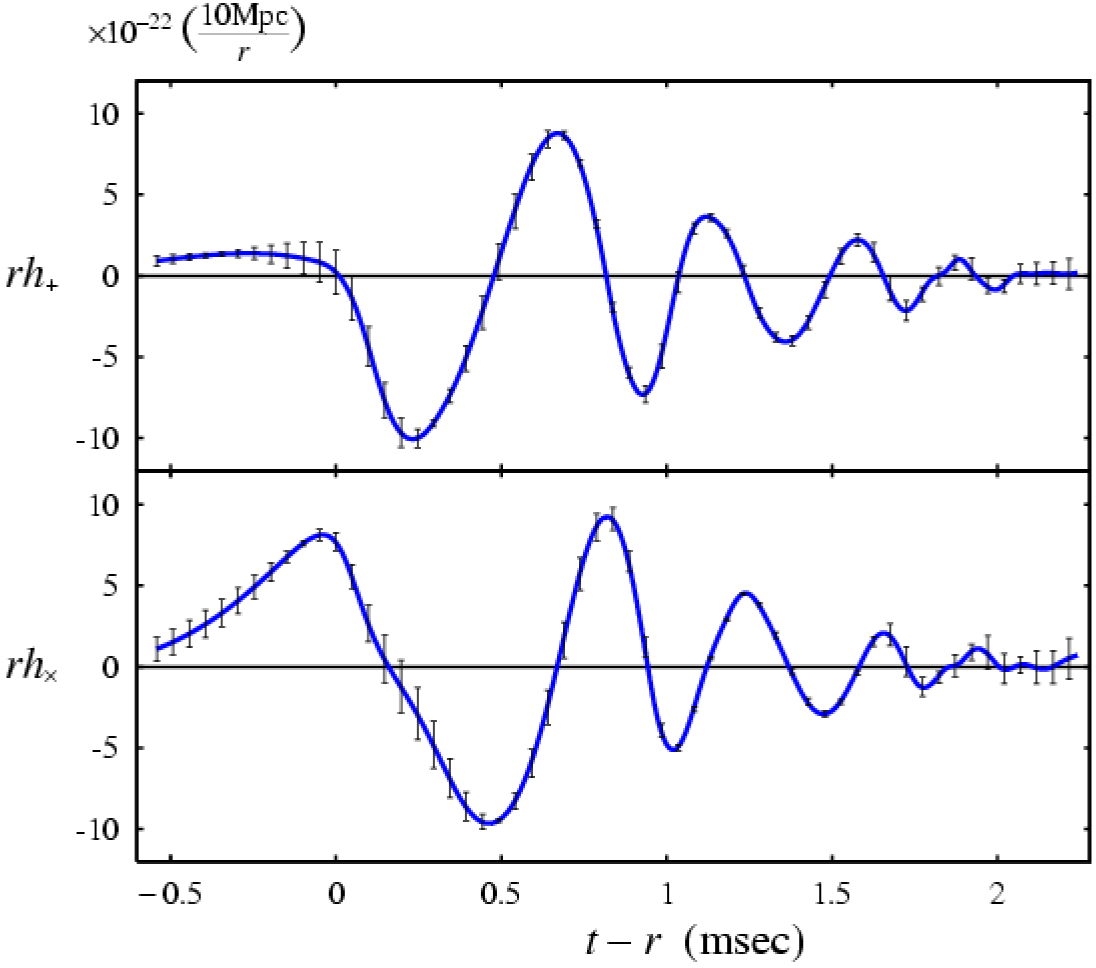
$$h_\omega(r) = \frac{e^{-i\omega r} F_\omega(r)}{r} + \frac{G_\omega(t)}{r^3}$$

$$F_\omega(r) = \frac{1}{2\pi}\int F(t,r)e^{-i\omega t}dt, \quad G_\omega(r) = \frac{1}{2\pi}\int G(t,r)e^{-i\omega t}dt$$

- **From the values** $h_\omega(r)$ **in** r_1 **and** r_2,

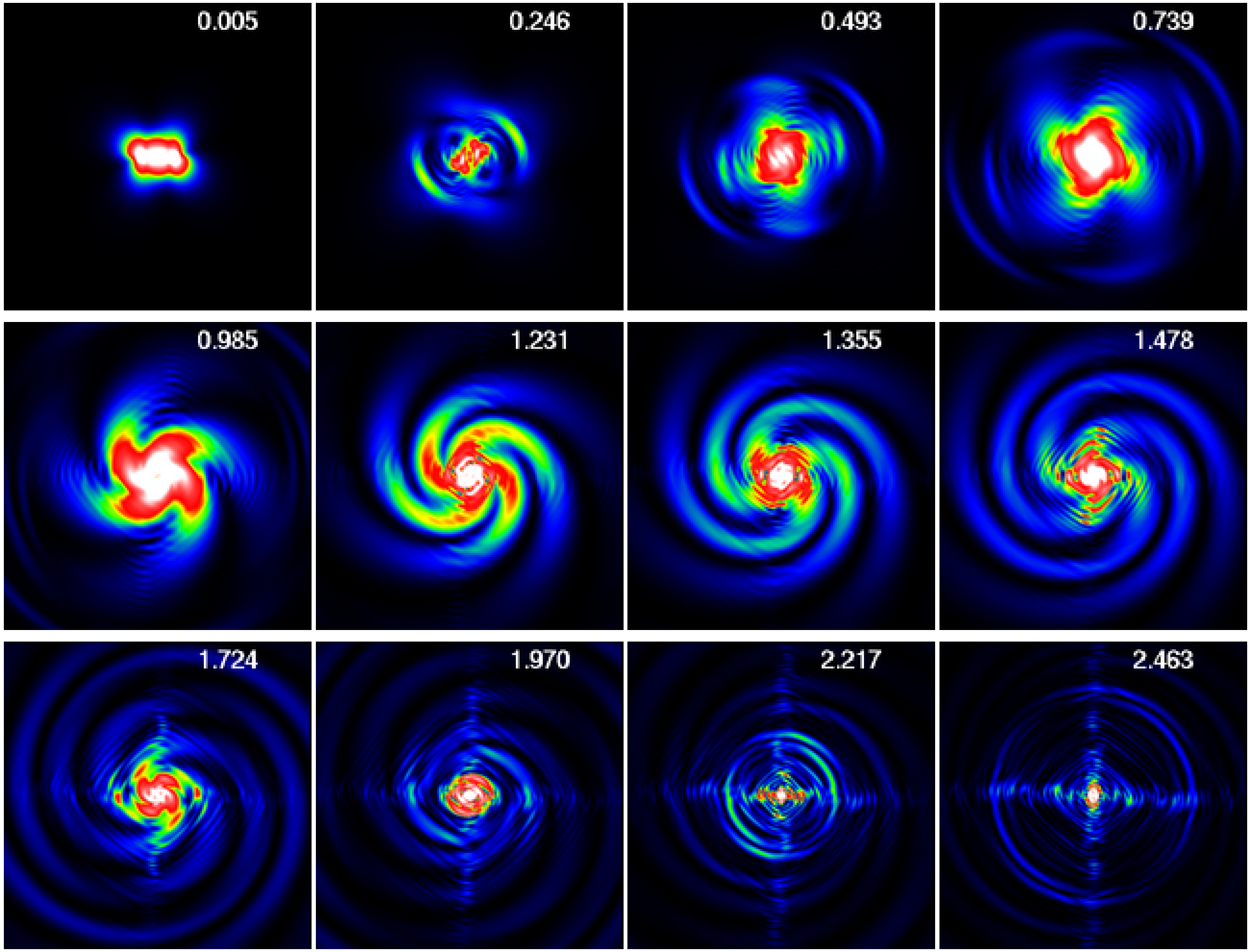
$$F_\omega(r) = \frac{r_2^3 h_\omega(r_2) - r_1^3 h_\omega(r_1)}{r_2^2 e^{-i\omega r_2} - r_1^2 e^{-i\omega r_1}}$$

- $\hat{h}_\omega(r) = \dfrac{e^{-i\omega r} F_\omega(r)}{r}$, $\quad h(t,r) = \int \hat{h}_\omega(r) e^{i\omega t} d\omega$



Wave forms along z-axis as a function of $t - r$. The curves are average of rh estimated at r=110 ~ 200 $M_\odot$ and error bar denote 2σ.

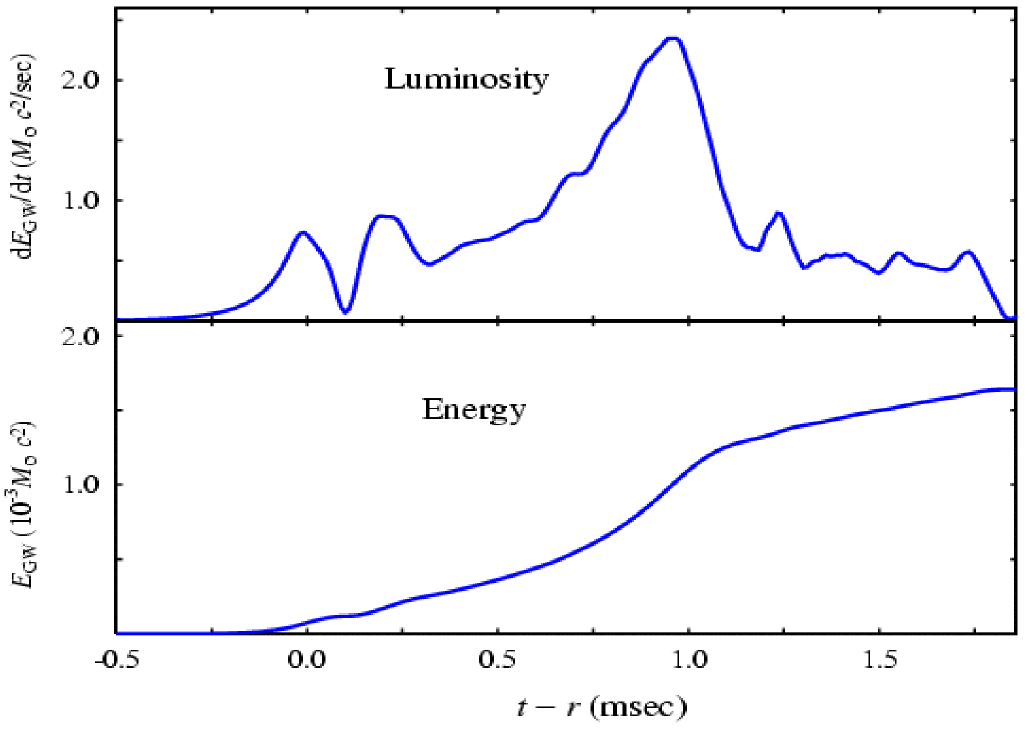
Coalescing Binary NS (energy of GW *xy*)



The `energy density of grav. waves' $r^2\rho_{\mathrm{GW}}$.

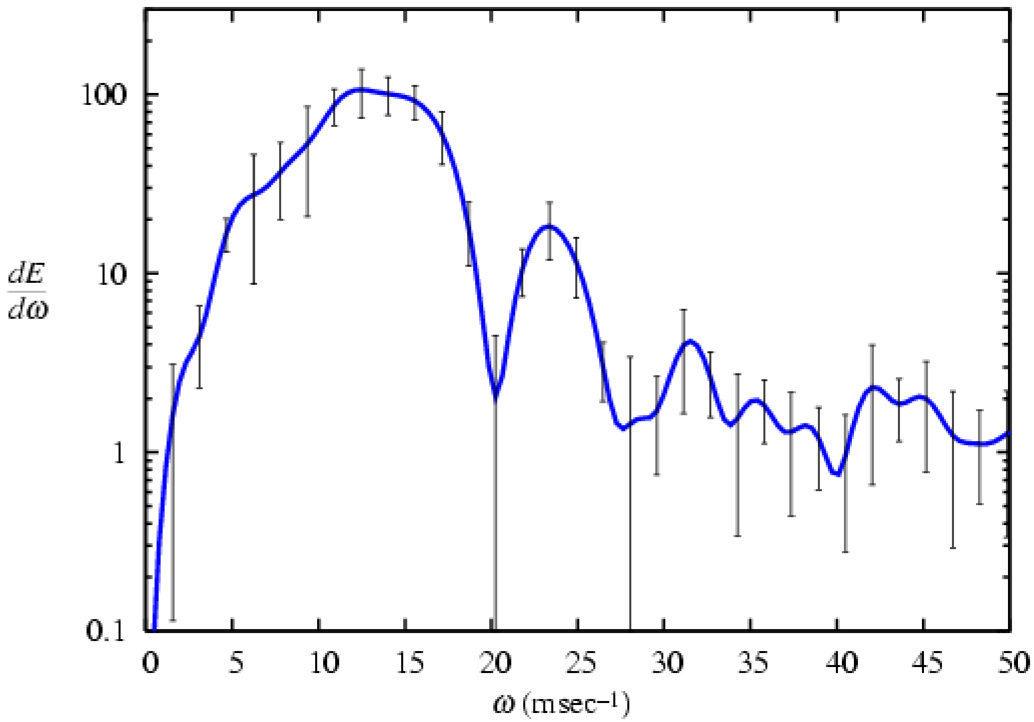
$$\rho_{\mathrm{GW}} = \frac{1}{32\pi}\sum_{ij}(\partial_t \tilde{\gamma}_{ij})^2$$

Energy of the Gravitational Waves



$$\frac{dE_{GW}}{dt} = \frac{1}{32\pi} \sum_{l,m} \left(\left| \partial_t \Psi_{lm}^{(o)} \right|^2 + \left| \partial_t \Psi_{lm}^{(e)} \right|^2 \right)$$

Energy Spectrum of the GW



The curves are average of $dE/d\omega$ estimated at r=110 ~ 200 $M_\odot$ and error bar denote 2σ .

$$\frac{dE_{\rm GW}}{d\omega} = \frac{1}{32\pi}\sum_{l,m}\omega^2\left(\left|\Psi^{(o)}_{lm\omega}\right|^2 + \left|\Psi^{(e)}_{lm\omega}\right|^2\right)$$

Concluding Remarks

- **We obtained a stable 3-D code for coalescing binary neutron stars.**
- **The initial data used here is not realistic, but we are ready to perform scientific simulation for coalescing binary neutron stars.**
- **Shibata and Uryu (2002) has presented results for it using different code. Comparison must be made soon.**