First order gravitational radiation reaction of a system of compact objects

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Abstract

We study the dynamical equations for a binary system of compact objects arising from the asymptotic structure of the corresponding isolated system. We make use of the center of mass frame at future null infinity and keep terms up to first order of gravitational radiation.

Contents

1 Introduction

1.1 Where are the gravitational waves?

Unless we are dealing with symmetric solutions, like plane waves, gravitational waves can only be given a meaningful definition when one is observing the system at very large distance from the concentration of masses. More precisely, since gravitational waves travel along null directions,

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it was constructed in the past the notion of future null infinity, with the intention, among other things, to describe the radiation fields.

Therefore, from a formal point of view, the real physical system consisting of compact objects that radiate gravitational waves, is modeled, in a relativistic theory of gravitation, by what is called an ***asymptotically flat spacetime at future null infinity***.

Imposing the vacuum Einstein equations to these spacetimes, provide a set of asymptotic conditions that constitute the ***structure equations of an asymptotically flat spacetime***.

It is our intention to extract all the available information from these equations.

Throughout the work we will use the GHP[GHP73] notation.

1.2 The problem of gauge

Since we are dealing with asymptotically flat spacetimes, the curvature goes to zero as one approaches null infinity. The asymptotic symmetry group however is not the 10-dimensional Poincaré group (that one finds in Minkowski spacetime) but the complicated ∞-dimensional BMS (Bondi-Metzner-Sachs) group[BvdBM62, Sac62].

This has as a consequence the gauge problem, since under these circumstances one has an ∞-parameter family of legitimate rest frames. Therefore in order to calculate, let us say, the quadrupole radiation formula, one should settle first the frame of reference.

We have solved this problem previously with the construction of a preferred set of sections at future null infinity, the so called ***nice sections***[Mor88, MD98, DM00] and also with the definition of ***intrinsic angular momentum*** and ***center of mass***[Mor00, Mor02, Mor03] of an isolated system.

This provides with a unique set of sections at future null infinity, which are used in the calculations.

1.3 System of compact objects

When dealing with a system of compact objects it is tempting to use the minimum number of degrees of freedom that provide a consistent model of the system. Such as for example is done in the post-Newtonian approximations. In those models one can use the notion of particles for each compact object.

In the past we have used some hybrid models, for the collision of black holes, with excellent agreement[MD96, Mor99] with full numeric calculations of the total energy radiated. More recently we have extended our calculations to include systems with orbital angular momentum[MP01, MPL02].

It is interesting to note that our models, for the case of non-head on collision, show a noticeable burst of radiation[MPL02]; which might be important for observation.

1.4 Dynamics of a system of compact objects

The quadrupole radiation formula can be derived in the weak field approximation; and it is found to be of order $O(\frac{1}{c^5})$[LL92]. Accordingly there is a post-Newtonian Lagrangian formulation of the dynamics of lower order[LL92].

This raises the question whether is it possible to derive dynamical equations of motion from the structure equations of an asymptotically flat spacetime, with disregard of the radiation fields.

So let us study below this question; but first, let us settle the notation.

1.5 The notation

Since we study the gravitational fields in the asymptotic region along null directions, it is useful to describe the geometry in terms of a null tetrad formalism; which has a one to one relation with a spinor dyad formalism.

One of the null vectors is taken as the generator of null geodesics reaching future null infinity. And another turns out to be asymptotically tangent to null infinity.

In this way the metric is expressed in terms of the null tetrad

$$g_{ab} = \ell_a n_b + n_a \ell_b - m_a \bar{m}_b - \bar{m}_a m_b; \tag{1}$$

which in terms of coordinates $(u, r, \zeta, \bar{\zeta})$ can be represented as

$$\ell^a = \left(\frac{\partial}{\partial r}\right)^a \qquad \Longleftrightarrow \qquad o^A o^{A'}, \tag{2}$$

$$m^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a \qquad \Longleftrightarrow \qquad o^A \iota^{A'}, \tag{3}$$

$$\bar{m}^a = \bar{\xi}^i \left(\frac{\partial}{\partial x^i}\right)^a \qquad \Longleftrightarrow \qquad \iota^A o^{A'}, \tag{4}$$

$$n^a = \left(\frac{\partial}{\partial u}\right)^a + U \left(\frac{\partial}{\partial r}\right)^a + X^i \left(\frac{\partial}{\partial x^i}\right)^a \qquad \Longleftrightarrow \qquad \iota^A \iota^{A'}; \tag{5}$$

where the index $i = 2, 3$, the complex stereographic coordinate is $\zeta = \frac{1}{2}(x^2 + i\, x^3)$; and where we have also indicated the relation between the null tetrad vectors and the spinor dyad (o^A, ι^B).

Then the connection is specified[GHP73] in terms the so called spin coefficients:

$$\left(\rho, \sigma, \tau, \kappa, \beta, \epsilon, \rho', \sigma', \tau', \kappa', \beta', \epsilon'\right); \tag{6}$$

while the curvature is expressed in terms of the spinor complex components:

$$\mathcal{R}\text{icci} \qquad \Longleftrightarrow \qquad (\Phi_{00}, \Phi_{01}, \Phi_{11}, \Phi_{02}, \Phi_{12}, \Phi_{22}, \Lambda), \tag{7}$$

$$\mathcal{W}\text{eyl} \qquad \Longleftrightarrow \qquad (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4). \tag{8}$$

Since in our case we are interested in vacuum solutions of the Einstein equations, all the components of the Ricci tensor vanish.

2 Asymptotic structure of radiating spacetimes in a general frame

The different set of geometric equations provide relations among the fields.

In particular, assuming vacuum Einstein equations and an asymptotic expansion for Ψ_0 of the form

$$\Psi_0 = \frac{\Psi_{00}}{r^5} + \frac{\Psi_{01}}{r^6} + \frac{\Psi_{02}}{r^7} + \frac{\Psi_{03}}{r^8} + O(\frac{1}{r^9}); \tag{9}$$

one can integrate all the equations involving r-derivatives. Then, from the vacuum Bianchi identities one can obtain evolution equations for the initial data:

$$(\Re(\Psi_{20}), \Psi_{10}, \Psi_{00}, \Psi_{01}, \Psi_{02}, \Psi_{03}); \tag{10}$$

where $\Re$ means real part, Ψ_{20} and Ψ_{10} are the leading order behavior of Ψ_2 and Ψ_1 respectively. These equations can in principle can be solved if the leading order of σ is known, along with an auxiliary function $V(u, \zeta, \bar{\zeta})$; which has the information of the set of sections being used. In our case, we use the center of mass sections, mentioned previously, which define the intrinsic retarded time τ; then V has the information

$$V = \frac{\partial u_B}{\partial \tau}, \tag{11}$$

where u_B is a Bondi retarded time. In other words, V has the information of the sections with respect to an inertial reference frame.

In the presence of gravitational radiation, the evolution equations of the asymptotic fields, up to the 36-pole term, are

$$\frac{\partial \Psi_{20}}{\partial u} = 3\,\frac{\partial V}{\partial u}\,\Psi_{20}\,V^{-1} + \eth_V\left(\Psi_{30}\right) + \Psi_{40}\,\sigma \tag{12}$$

$$\frac{\partial \Psi_{10}}{\partial u} = 3\,\frac{\partial V}{\partial u}\,\Psi_{10}\,V^{-1} + \eth_V\left(\Psi_{20}\right) + 2\,\Psi_{30}\,\sigma \tag{13}$$

$$\frac{\partial \Psi_{00}}{\partial u} = 3\,\frac{\partial V}{\partial u}\,\Psi_{00}\,V^{-1} + \eth_V\left(\Psi_{10}\right) + 3\,\Psi_{20}\,\sigma \tag{14}$$

$$\frac{\partial \Psi_{01}}{\partial u} = 4\,\frac{\partial V}{\partial u}\,\Psi_{01}\,V^{-1} - \bar{\eth}_V\left(\eth_V\left(\Psi_{00}\right)\right) - 4\,\bar{\eth}_V\left(\Psi_{10}\right)\,\sigma - 4\,\bar{\eth}_V\left(\sigma\right)\,\Psi_{10} \tag{15}$$

$$\begin{aligned}
\frac{\partial \Psi_{02}}{\partial u} = {} & \frac{5}{2}\,\frac{\partial \bar{\sigma}}{\partial u}\,\Psi_{00}\,\sigma - \frac{5}{2}\,\frac{\partial V}{\partial u}\,\Psi_{00}\,\sigma\,\bar{\sigma}\,V^{-1} + 5\,\frac{\partial V}{\partial u}\,\Psi_{02}\,V^{-1} + \frac{1}{2}\,\eth_V\left(\eth_V\left(\Psi_{00}\right)\right)\,\bar{\sigma} \\
& + \frac{5}{2}\,\eth_V\left(\eth_V\left(\bar{\sigma}\right)\right)\,\Psi_{00} + 3\,\eth_V\left(\Psi_{00}\right)\,\eth_V\left(\bar{\sigma}\right) + \frac{9}{2}\,\eth_V\left(\Psi_{10}\right)\,\sigma\,\bar{\sigma} + 2\,\eth_V\left(\sigma\right)\,\Psi_{10}\,\bar{\sigma} \\
& + 12\,\eth_V\left(\bar{\sigma}\right)\,\Psi_{10}\,\sigma + 3\,\eth_V\left(V\right)\,\bar{\eth}_V\left(V\right)\,\Psi_{01}\,V^{-2} - \frac{1}{2}\,\bar{\eth}_V\left(\eth_V\left(\Psi_{01}\right)\right) \\
& - 3\,\bar{\eth}_V\left(\eth_V\left(V\right)\right)\,\Psi_{01}\,V^{-1} + \frac{5}{2}\,\eth_V\left(\bar{\eth}_V\left(\Psi_{00}\right)\right)\,\sigma + 5\,\bar{\eth}_V\left(\Psi_{00}\right)\,\bar{\eth}_V\left(\sigma\right) \\
& - 5\,\Psi_{00}\,\Psi_{20} - \frac{5}{2}\,\Psi_{00}\,\bar{\Psi}_{20} - \frac{3}{2}\,\Psi_{01}\,V^{2} + 5\,\Psi_{10}^{2} + \frac{15}{2}\,\Psi_{20}\,\sigma^{2}\,\bar{\sigma}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{\partial \Psi_{03}}{\partial u} = {} & 2\,\frac{\partial \bar{\sigma}}{\partial u}\,\Psi_{01}\,\sigma - 2\,\frac{\partial V}{\partial u}\,\Psi_{01}\,\sigma\,\bar{\sigma}\,V^{-1} + 6\,\frac{\partial V}{\partial u}\,\Psi_{03}\,V^{-1} + \frac{1}{3}\,\eth_V\left(\eth_V\left(\Psi_{01}\right)\right)\,\bar{\sigma} \\
& + 2\,\eth_V\left(\eth_V\left(\bar{\sigma}\right)\right)\,\Psi_{01} - \frac{14}{3}\,\eth_V\left(\Psi_{00}\right)\,\bar{\eth}_V\left(\sigma\right)\,\bar{\sigma} - \eth_V\left(\Psi_{00}\right)\,\bar{\eth}_V\left(\bar{\sigma}\right)\,\sigma - \eth_V\left(\Psi_{00}\right)\,\bar{\Psi}_{10} \\
& + \frac{7}{3}\,\eth_V\left(\Psi_{01}\right)\,\eth_V\left(\bar{\sigma}\right) - \frac{5}{3}\,\eth_V\left(\sigma\right)\,\bar{\eth}_V\left(\Psi_{00}\right)\,\bar{\sigma} - \frac{50}{3}\,\eth_V\left(\bar{\sigma}\right)\,\bar{\eth}_V\left(\Psi_{00}\right)\,\sigma \\
& - \frac{50}{3}\,\eth_V\left(\bar{\sigma}\right)\,\bar{\eth}_V\left(\sigma\right)\,\Psi_{00} - \frac{70}{3}\,\eth_V\left(V\right)\,\bar{\eth}_V\left(V\right)\,\Psi_{00}\,\sigma\,\bar{\sigma}\,V^{-2} \\
& + \frac{14}{3}\,\eth_V\left(V\right)\,\bar{\eth}_V\left(V\right)\,\Psi_{02}\,V^{-2} - \frac{14}{3}\,\bar{\eth}_V\left(\eth_V\left(\Psi_{00}\right)\right)\,\sigma\,\bar{\sigma} \\
& - \frac{1}{3}\,\bar{\eth}_V\left(\eth_V\left(\Psi_{02}\right)\right) - \frac{5}{3}\,\bar{\eth}_V\left(\eth_V\left(\sigma\right)\right)\,\Psi_{00}\,\bar{\sigma} - 5\,\bar{\eth}_V\left(\eth_V\left(\bar{\sigma}\right)\right)\,\Psi_{00}\,\sigma \\
& + \frac{70}{3}\,\bar{\eth}_V\left(\eth_V\left(V\right)\right)\,\Psi_{00}\,\sigma\,\bar{\sigma}\,V^{-1} - \frac{14}{3}\,\bar{\eth}_V\left(\eth_V\left(V\right)\right)\,\Psi_{02}\,V^{-1} \\
& + \bar{\eth}_V\left(\bar{\eth}_V\left(\Psi_{01}\right)\right)\,\sigma - \frac{25}{3}\,\bar{\eth}_V\left(\Psi_{00}\right)\,\Psi_{10} + 3\,\bar{\eth}_V\left(\Psi_{01}\right)\,\bar{\eth}_V\left(\sigma\right) \\
& + \frac{10}{3}\,\bar{\eth}_V\left(\Psi_{10}\right)\,\Psi_{00} - 12\,\bar{\eth}_V\left(\Psi_{10}\right)\,\sigma^{2}\,\bar{\sigma} - 24\,\bar{\eth}_V\left(\sigma\right)\,\Psi_{10}\,\sigma\,\bar{\sigma} - 4\,\bar{\eth}_V\left(\bar{\sigma}\right)\,\Psi_{10}\,\sigma^{2} \\
& + \frac{35}{3}\,\Psi_{00}\,\sigma\,\bar{\sigma}\,V^{2} - 5\,\Psi_{01}\,\Psi_{20} - 3\,\Psi_{01}\,\bar{\Psi}_{20} - \frac{7}{3}\,\Psi_{02}\,V^{2} - 4\,\Psi_{10}\,\bar{\Psi}_{10}\,\sigma
\end{aligned} \tag{17}$$

where now

$$\Psi_{30} = \eth_V \sigma' - \bar{\eth}_V \rho', \tag{18}$$

$$\Psi_{40} = \dot{\sigma}' - 2\frac{\dot{V}}{V}\sigma' - \bar{\eth}_V^2 \frac{\dot{V}}{V}, \tag{19}$$

and the spin coefficients σ' and ρ' are

$$\sigma' = \frac{\dot{V}}{V}\bar{\sigma} - \dot{\bar{\sigma}}, \tag{20}$$

$$\rho' = \frac{1}{2}K_V; \tag{21}$$

with

$$K_V = \frac{2}{V}\,\bar{\eth}_V\eth_V\,V - \frac{2}{V^2}\,\eth_V V\;\bar{\eth}_V V + V^2. \tag{22}$$

3 Expansion of a radiating spacetime in terms of the strength of the radiation (The λ expansion)

One important aspect of general relativity is that for many interesting astrophysical systems the gravitational energy radiated is weak.

Also most of the difficulties that arise at future null infinity in the construction of appropriate reference frames[MD98] comes from the presence of gravitational radiation.

It is therefore tempting to expand the structure equations of an asymptotically flat spacetime in terms of the strength of the gravitational radiation.

In a Bondi system the gravitational energy flux can be given in terms of the time derivative of the shear, namely

$$\dot{\Psi} = \dot{\sigma}_B\,\dot{\bar{\sigma}}_B. \tag{23}$$

We will use the smallness parameter λ and assign to the flux the order $O(\lambda^2)$.

Every field at scri will be expanded as a series with different powers of λ; that is for a field $f(u,\zeta,\bar{\zeta})$ we will write

$$f(u,\zeta,\bar{\zeta}) = f_0(\zeta,\bar{\zeta}) + \lambda\,f_1(u,\zeta,\bar{\zeta}) + \lambda^2\,f_2(u,\zeta,\bar{\zeta}) + O(\lambda^3).$$

The idea is to carry out this expansion on the 'center of mass frame' defined in terms of 'nice sections'.

It is important to realize that if we call α the supertranslation, from a reference Bondi section, that satisfies the nice section equation, then the scalar V appearing above would be related to α by

$$V = \dot{\alpha}; \tag{24}$$

but from the nice section equation one can see that actually $\alpha = O(\lambda^2)$ if $\alpha = 0$ is a nice section. This implies that the center of mass family of sections coincide with the initial Bondi frame up to order $O(\lambda)$.

It is also deduced that

$$V = 1 + \lambda^2\,V_2(u,\zeta,\bar{\zeta}) + O(\lambda^2). \tag{25}$$

Let us also note that

$$\Psi_4^0 = \Psi_3^0 = O(\lambda), \tag{26}$$

$$\dot{V} = O(\lambda^2); \tag{27}$$

and

$$K_V = 1 + O(\lambda^2). \tag{28}$$

The evolution equations in first order of the radiation field, that is, $O(\lambda)$, become

$$\frac{\partial\Psi_{20}}{\partial u} = \eth\left(\Psi_{30}\right) \tag{29}$$

$$\frac{\partial\Psi_{10}}{\partial u} = \eth\left(\Psi_{20}\right) \tag{30}$$

$$\frac{\partial\Psi_{00}}{\partial u} = \eth\left(\Psi_{10}\right) + 3\,\Psi_{20}\,\sigma \tag{31}$$

$$\frac{\partial\Psi_{01}}{\partial u} = -\bar\eth\left(\eth\left(\Psi_{00}\right)\right) - 4\,\bar\eth\left(\Psi_{10}\right)\,\sigma - 4\,\bar\eth\left(\sigma\right)\,\Psi_{10} \tag{32}$$

$$\begin{aligned}\frac{\partial\Psi_{02}}{\partial u} &= \frac{1}{2}\eth\left(\eth\left(\Psi_{00}\right)\right)\bar\sigma + \frac{5}{2}\eth\left(\eth\left(\bar\sigma\right)\right)\,\Psi_{00} + 3\,\eth\left(\Psi_{00}\right)\,\eth\left(\bar\sigma\right) - \frac{1}{2}\bar\eth\left(\eth\left(\Psi_{01}\right)\right)\\ &+ \frac{5}{2}\bar\eth\left(\bar\eth\left(\Psi_{00}\right)\right)\sigma + 5\,\bar\eth\left(\Psi_{00}\right)\,\bar\eth\left(\sigma\right) - 5\,\Psi_{00}\,\Psi_{20} - \frac{5}{2}\Psi_{00}\,\bar\Psi_{20} - \frac{3}{2}\Psi_{01} + 5\,\Psi_{10}^2\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial\Psi_{03}}{\partial u} &= \frac{1}{3}\eth\left(\eth\left(\Psi_{01}\right)\right)\bar\sigma + 2\,\eth\left(\eth\left(\bar\sigma\right)\right)\,\Psi_{01} - \eth\left(\Psi_{00}\right)\,\bar\Psi_{10}\\ &+ \frac{7}{3}\eth\left(\Psi_{01}\right)\,\eth\left(\bar\sigma\right) - \frac{1}{3}\bar\eth\left(\eth\left(\Psi_{02}\right)\right) + \bar\eth\left(\bar\eth\left(\Psi_{01}\right)\right)\sigma - \frac{25}{3}\bar\eth\left(\Psi_{00}\right)\,\Psi_{10}\\ &+ 3\,\bar\eth\left(\Psi_{01}\right)\,\bar\eth\left(\sigma\right) + \frac{10}{3}\bar\eth\left(\Psi_{10}\right)\,\Psi_{00} - 5\,\Psi_{01}\,\Psi_{20} - 3\,\Psi_{01}\,\bar\Psi_{20} - \frac{7}{3}\Psi_{02} - 4\,\Psi_{10}\,\bar\Psi_{10}\,\text{(}\end{aligned} \tag{34}$$

4 Application to a system of compact objects and post-linear expansion

4.1 The γ-expansion

Consider two very separated compact objects with non-relativistic relative velocities. Then in first order, in the asymptotic region, they will contribute to the geometry with a superposition of two Schwarzschild geometries.

More precisely, let γ represent a small parameter which one can think to be proportional to the gravitational constant; then for each asymptotic field f, one can make a double expansion of the form

$$f = \gamma f_{10} + \sum_{i=2,j=1} f_{i,j}\,\gamma^i\,\lambda^j; \tag{35}$$

where the first term has information of the linear solution.

It is generally assumed in this expansion that

$$\frac{\partial O(\gamma^n)}{\partial u} = O(\gamma^{n+1}); \tag{36}$$

with just few exceptions to this rule.

4.2 Evolution equations disregarding gravitational radiation

In order to study whether there are equation of motion for the compact objects, coming from the asymptotic structure equations, in which the radiation field has been neglected, one should concentrate on the equations:

$$
\begin{aligned}
\frac{\partial \Psi_{20}}{\partial u} &= 0 && (37)\\
\frac{\partial \Psi_{10}}{\partial u} &= \eth\left(\Psi_{20}\right) && (38)\\
\frac{\partial \Psi_{00}}{\partial u} &= \eth\left(\Psi_{10}\right) && (39)\\
\frac{\partial \Psi_{01}}{\partial u} &= -\bar{\eth}\left(\eth\left(\Psi_{00}\right)\right) && (40)\\
\frac{\partial \Psi_{02}}{\partial u} &= -\frac{1}{2}\bar{\eth}\left(\eth\left(\Psi_{01}\right)\right) - 5\,\Psi_{00}\,\Psi_{20} - \frac{5}{2}\,\Psi_{00}\,\bar{\Psi}_{20} - \frac{3}{2}\,\Psi_{01} + 5\,\Psi_{10}^2 && (41)\\
\frac{\partial \Psi_{03}}{\partial u} &= -\eth\left(\Psi_{00}\right)\,\bar{\Psi}_{10} - \frac{1}{3}\bar{\eth}\left(\eth\left(\Psi_{02}\right)\right) - \frac{25}{3}\bar{\eth}\left(\Psi_{00}\right)\,\Psi_{10}\\
&\quad + \frac{10}{3}\bar{\eth}\left(\Psi_{10}\right)\,\Psi_{00} - 5\,\Psi_{01}\,\Psi_{20} - 3\,\Psi_{01}\,\bar{\Psi}_{20} - \frac{7}{3}\,\Psi_{02} && (42)
\end{aligned}
$$

4.3 Decomposition in terms of each compact object

The first order in γ data for each compact object '$\mathtt{a}$' is just given by

$$\Psi_{20\mathtt{a}} = -m_{\mathtt{a}}, \tag{43}$$

where $m_{\mathtt{a}}$ is the mass of the object '$\mathtt{a}$' and where the expression is in terms of its intrinsic comoving frame; which depends on its time $\tau_{\mathtt{a}}$.

For compact objects with no additional structure; that is, no intrinsic angular momentum or multipole structure; one takes $\Psi_{10\mathtt{a}} = 0$ and $\Psi_{00\mathtt{a}} = 0$.

In terms of the center of mass frame, one needs to transform these fields by the corresponding translation and boost. Let $y_{\mathtt{a}}(\tau_{\mathtt{a}})$ be such translation of the compact object '$\mathtt{a}$'; and let us use tilde to denote the fields for the center of mass frame.

The relation between a regular dyad $(\hat{\tilde{o}}^A, \hat{\tilde{\iota}}^A)$ at future null infinity, associated to the frame which defines de retarded time $\tilde{u}$, and another one $(\hat{o}^A, \hat{\iota}^A)$ associated to the frame which defines de retarded time τ, with the relation $\tilde{u} = \alpha(\tau, \zeta, \bar{\zeta})$ is given by

$$
\begin{aligned}
\hat{\tilde{o}}^A &= \frac{1}{\sqrt{\dot{\alpha}}}\left(\hat{o}^A - \frac{\eth_{\dot{\alpha}}\alpha}{\dot{\alpha}}\,\hat{\iota}^A\right), && (44)\\
\hat{\tilde{\iota}}^A &= \frac{1}{\sqrt{\dot{\alpha}}}\,\hat{\iota}^A; && (45)
\end{aligned}
$$

where we are using $\dot{\alpha} \equiv \frac{\partial \alpha}{\partial \tau}$.

Using $\alpha = y_{\mathtt{a}}$, and $V_{\mathtt{a}} \equiv \frac{\partial y_{\mathtt{a}}}{\partial \tau_{\mathtt{a}}}$, one obtains from these equations:

$$\tilde{\Psi}_{20\mathtt{a}} = \frac{\Psi_{20\mathtt{a}}}{V_{\mathtt{a}}^3}, \tag{46}$$

$$\tilde{\Psi}_{10\mathtt{a}} = -3\frac{\eth_{V_{\mathtt{a}}} y_{\mathtt{a}}}{V_{\mathtt{a}}^4}\Psi_{20\mathtt{a}}, \tag{47}$$

$$\tilde{\Psi}_{00\mathtt{a}} = 6\frac{(\eth_{V_{\mathtt{a}}} y_{\mathtt{a}})^2}{V_{\mathtt{a}}^5}\Psi_{20\mathtt{a}}. \tag{48}$$

The total first order in γ Weyl components, in the center of mass frame, come from the sum of terms like these for each compact object.

4.4 Simplifying notation

Since Ψ_{20} has information on the mass aspect, Ψ_{10} on the angular momentum and Ψ_0 on the quadrupole and higher moments; let us introduce the following simplifying notation:

$$P \equiv -\Psi_{20}, \tag{49}$$
$$J \equiv \Psi_{10}, \tag{50}$$
$$Q \equiv \Psi_{00}, \tag{51}$$
$$Q_1 \equiv \Psi_{01}, \tag{52}$$
$$Q_2 \equiv \Psi_{02}, \tag{53}$$
$$Q_3 \equiv \Psi_{03}. \tag{54}$$

Then for the two body problem, one has in particular

$$P = P_{\mathtt{a}} + P_{\mathtt{b}} + P_{\mathtt{ab}} = \frac{m_{\mathtt{a}}}{V_{\mathtt{a}}^3} + \frac{m_{\mathtt{b}}}{V_{\mathtt{b}}^3} + P_{\mathtt{ab}}, \tag{55}$$
$$J = J_{\mathtt{a}} + J_{\mathtt{b}} + J_{\mathtt{ab}} = 3\frac{\eth_{V_{\mathtt{b}}} y_{\mathtt{a}}}{V_{\mathtt{a}}^4} m_{\mathtt{a}} + 3\frac{\eth_{V_{\mathtt{b}}} y_{\mathtt{b}}}{V_{\mathtt{b}}^4} m_{\mathtt{b}} + J_{\mathtt{ab}}, \tag{56}$$
$$Q = Q_{\mathtt{a}} + Q_{\mathtt{b}} + Q_{\mathtt{ab}} = -6\frac{(\eth_{V_{\mathtt{a}}} y_{\mathtt{a}})^2}{V_{\mathtt{a}}^5} m_{\mathtt{a}} - 6\frac{(\eth_{V_{\mathtt{b}}} y_{\mathtt{b}})^2}{V_{\mathtt{b}}^5} m_{\mathtt{b}} + Q_{\mathtt{ab}}; \tag{57}$$

which must satisfy the evolution equations above.

5 The finite multipolar expansion

Let us now consider a finite multipolar expansion; which in terms of the structure equations means that one takes into account the information up to a certain level of the expansion. For example for a 16-pole approximation (up to order $O(\lambda^0)$) one would consider the set of equations (37)-(42); which in terms of the simplifying notation, are:

$$\frac{\partial P}{\partial u} = 0 \tag{58}$$
$$\frac{\partial J}{\partial u} = -\eth\,(P) \tag{59}$$
$$\frac{\partial Q}{\partial u} = \eth\,(J) \tag{60}$$
$$\frac{\partial Q_1}{\partial u} = -\bar{\eth}\,(\eth\,(Q)) \tag{61}$$
$$\frac{\partial Q_2}{\partial u} = -\frac{1}{2}\bar{\eth}\,(\eth\,(Q_1)) + \frac{15}{2}\,Q\,P - \frac{3}{2}\,Q_1 + 5\,J^2 \tag{62}$$
$$\begin{aligned} 0 = {} & -\eth\,(Q)\,\bar{J} - \frac{1}{3}\bar{\eth}\,(\eth\,(Q_2)) - \frac{25}{3}\,\bar{\eth}\,(Q)\,J \\ & + \frac{10}{3}\,\bar{\eth}\,(J)\,Q + 8\,Q_1\,P - \frac{7}{3}\,Q_2; \end{aligned} \tag{63}$$

where we are considering the 16-pole approximation expansion, and therefore the 32-pole term involving Q_3 is neglected.

Let us consider these equations up to order γ^2. Equation (58) states that P can not change with time; and implies the conservation of total momentum. Then, equation (59) involves the angular derivative of P; but in the center of mass frame, one has $\eth(P) = 0$; therefore also J can not change with time; which implies the conservation of total angular momentum..

Equation (60) sets the time variation of Q in terms of the angular derivatives of J; where in order to apply the $\eth$ operator, it is important to recall the relation with the intrinsic derivatives defined relative to each object, namely

$$\eth f = \frac{1}{V_{\mathsf{a}}}\left(\eth_{V_{\mathsf{a}}} f - \frac{\eth_{V_{\mathsf{a}}} y_{\mathsf{a}}}{V_{\mathsf{a}}}\frac{\partial f}{\partial \tau_{\mathsf{a}}}\right) + s\, f\, \eth_{V_{\mathsf{a}}}\left(\frac{1}{V_{\mathsf{a}}}\right); \tag{64}$$

where s is the spin weight of the quantity f.

The last three equations are analyzed in the inverse order.

From equation (63) one can deduce the angular dependence of Q_2; namely

$$Q_2 = Q_{2(\gamma)} + Q_{2(\gamma^2)}; \tag{65}$$

where

$$Q_{2(\gamma)} = \sum_m Q^m_{2(\gamma)}\, {}_2Y_{4m}; \tag{66}$$

and

$$Q_{2(\gamma^2)} = \sum_{l\neq 4,m} Q^{lm}_{2(\gamma^2)}\, {}_2Y_{lm}; \tag{67}$$

that is, the linear order in γ has only contribution from the spin spherical harmonics with $l = 4$, and the quadratic terms do not have contribution from these spin spherical harmonics.

This is important in equation (62) where one can deduce the angular dependence of Q_1. Similarly as above, one has

$$Q_1 = Q_{1(\gamma)} + Q_{1(\gamma^2)}; \tag{68}$$

where

$$Q_{1(\gamma)} = \sum_m Q^m_{1(\gamma)}\, {}_2Y_{3m}; \tag{69}$$

and

$$Q_{1(\gamma^2)} = \sum_{l\neq 3,m} Q^{lm}_{1(\gamma^2)}\, {}_2Y_{lm}. \tag{70}$$

Then from equation (61) one deduces that the ${}_2Y_{2m}$ components of Q_1 do not change with time; due to the fact that the right hand side is the total $\bar\eth$ derivative of a spin 3 quantity. Since the time derivative of the $l = 2$ components of Q_2 are order $O(\gamma^3)$, it is deduced that the $l = 2$ part of Q_1 comes only from the quantity:

$$q \equiv \frac{10}{3}\left(J^2 + \frac{3}{2}Q\,P\right). \tag{71}$$

The $l = 2$ components of q are the so called NP conserved quantities[NP68].

For the binary system case, the quantity q is:

$$q = -30 P_{\mathsf{a}}\, P_{\mathsf{b}}\left(\frac{\eth_{V_{\mathsf{a}}} y_{\mathsf{a}}}{V_{\mathsf{a}}} - \frac{\eth_{V_{\mathsf{b}}} y_{\mathsf{b}}}{V_{\mathsf{b}}}\right)^2. \tag{72}$$

Let us recall that the translations y_{a} are quantities with components $l = 1$ different from zero; therefore the factor between parenthesis is a function with spin weight 2 and contributes with $l = 2$ components. It is deduced then from the NP conservation law that the direction determined by $y_{\mathsf{a}} - y_{\mathsf{b}}$ must be a constant at this order of approximation.

In other words:

The equations of motion coming from the structure equations at the order γ^2 and λ^0, are only compatible with motion contained along a fixed line.

In particular this means that the dynamics for the binary system coming from the structure equations of an asymptotically flat spacetime, at this level of approximation, does not coincide with the dynamics coming from the post-Newtonian approximations.

It seems that the two approaches give geometrical models to the binary system problem from two nonintersecting sets in the space of approximate solutions of the vacuum Einstein equations.

6 The first order gravitational radiation reaction dynamical equations. (The $O(\lambda)$ expansion)

The 16-pole approximation, in order γ^2 and λ, using the simplifying notation, are:

$$\frac{\partial P}{\partial u} = \eth^2 (\dot{\bar{\sigma}}) \tag{73}$$

$$\frac{\partial J}{\partial u} = -\eth (P) \tag{74}$$

$$\frac{\partial Q}{\partial u} = \eth (J) - 3 P \sigma \tag{75}$$

$$\frac{\partial Q_1}{\partial u} = -\bar{\eth} (\eth (Q)) - 4 \bar{\eth} (J) \ \sigma - 4 \bar{\eth} (\sigma) \ J \tag{76}$$

$$\begin{aligned}\frac{\partial Q_2}{\partial u} = {} & \frac{1}{2} \eth (\eth (Q)) \ \bar{\sigma} + \frac{5}{2} \eth (\eth (\bar{\sigma})) \ Q + 3 \eth (Q) \ \eth (\bar{\sigma}) - \frac{1}{2} \bar{\eth} (\eth (Q_1)) \\ & + \frac{5}{2} \bar{\eth} \left(\bar{\eth} (Q)\right) \ \sigma + 5 \bar{\eth} (Q) \ \bar{\eth} (\sigma) + 5 Q P + \frac{5}{2} Q \bar{P} - \frac{3}{2} Q_1 + 5 J^2\end{aligned} \tag{77}$$

$$\begin{aligned}0 = {} & \frac{1}{3} \eth (\eth (Q_1)) \ \bar{\sigma} + 2 \eth (\eth (\bar{\sigma})) \ Q_1 - \eth (Q) \ \bar{J} \\ & + \frac{7}{3} \eth (Q_1) \ \eth (\bar{\sigma}) - \frac{1}{3} \bar{\eth} (\eth (Q_2)) + \bar{\eth} \left(\bar{\eth} (Q_1)\right) \ \sigma - \frac{25}{3} \bar{\eth} (Q) \ J \\ & + 3 \bar{\eth} (Q_1) \ \bar{\eth} (\sigma) + \frac{10}{3} \bar{\eth} (J) \ Q + 5 Q_1 P + 3 Q_1 \bar{P} - \frac{7}{3} Q_2 - 4 J \bar{J} \sigma.\end{aligned} \tag{78}$$

From the previous discussion, one can see that from the first equation one has

$$P = P_0 + P_1; \tag{79}$$

where P_0 is a constant of order λ^0; and P_1 is of order λ.

Therefore, from the third equation one deduces that, up to order λ,

$$\sigma = \frac{\eth J - \frac{\partial Q}{\partial u}}{3 P_0}. \tag{80}$$

This constitutes the order λ quadrupole radiation formula.

Introducing back this value of σ en the first two equations one obtains the first order back reaction for the mass aspect and angular momentum aspect. However it should be emphasized that the total momentum and angular momentum are conserved at this level of approximation.

The analysis of the NP conserved quantities is much more difficult in this case; since now one has additional terms with contributions with products involving Q and σ.

7 Comments

This three fold expansion; namely:

- post-linear expansion (The γ expansion.)
- strength of the radiation field expansion (The λ expansion.)
- finite multipolar expansion

provides with a model for an isolated system consisting of compact objects; and also provides with the necessary link between the free radiation data at future null infinity and the sources.

Our future goal is to extract as much information as possible from these equations with the intention to describe the dynamics of compact objects in GR.

In future work we plan to set the equations necessary to describe a binary system up to second order in λ.

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