Black hole initial data from a non-conformal decomposition

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I. INTRODUCTION

A method for constructing instantaneously Kerr-Schild initial data has been developed [1], and implemented for the case of a perturbed Schwarzschild geometry. The general case is problematic because

- The momentum and Hamiltonian constraints are inter-mixed;
- The construction of intial data starts at the horizon and goes outwards. Experience with the perturbed Schwarzschild solution suggests that the data near infinity will be singular.

Thus some variations of the initial data method were investigated, with the objective of finding a form that resembles Kerr-Schild, and is not York-Lichnerowicz.

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[1] N.T. Bishop, R. Isaacson, M. Maharaj & J. Winicour *Black hole data via a Kerr-Schild approach* Phys. Rev. D **57** 6113-6118 (1998).

II. ANSATZ

The following ansatz appears to satisfy the above criteria

$$\begin{aligned} K_{ij} &= 0 \\ \gamma_{ij} &= \frac{d_{ij} - 2V k_i k_j}{1 - 2V} \end{aligned} \tag{1}$$

where

- K_{ij} and γ_{ij} are, as usual, the extrinsic curvature and the intrinsic 3-metric
- d_{ij} is a Euclidean (flat) 3-metric
- k_i is a unit vector field
- V is a scalar field.

The data is time-symmetric, as this is the easiest way to satisfy the momentum constraints. Non-time-symmetric initial data is being investigated, but as yet there are no results to report.

It is useful to note here that

$$\gamma^{ij} = (1 - 2V)d^{ij} + 2V k^i k^j \tag{2}$$

with $k^i = d^{ij} k_j$, so that $1 = d^{ij} k_i k_j$.

III. THE HAMILTONIAN CONSTRAINT

The Hamiltonian constraint for (1) has been computed by Maple in the specific case of axial symmetry, i.e. $x^i = (r, \theta, \varphi)$, $d_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$, and with $V = V(r,\theta)$ and

$$k_i = \frac{\Phi_{,i}}{\sqrt{d^{ij}\Phi_{,i}\Phi_{,j}}}, \tag{3}$$

where $\Phi = \Phi(r,\theta)$. The only unknown is therefore V, which was found to satisfy the following elliptic equation

$$c_{11}V_{,rr} + c_{12}V_{,r\theta} + c_{22}V_{,\theta\theta} + c_{211}\left(V_{,r}\right)^2 + c_{212}V_{,r}V_{,\theta} + c_{222}\left(V_{,\theta}\right)^2 + c_1 V_{,r} + c_2 V_{,\theta} + c_0 V = 0 \tag{4}$$

The coefficients $c_{i\ldots}$ were computed by Maple. The script also computes

$$\Delta = c_{11}c_{22} - \frac{c_{12}^2}{4} = \frac{8}{(1-2V)r^2} \tag{5}$$

demonstrating that Eq. (4) is elliptic for $V < \frac{1}{2}$.

$$c11 := 2\,\frac{2\left(\frac{d}{dr}\phi\right)^2 r^2 - 2V\left(\frac{d}{d\theta}\phi\right)^2 + \left(\frac{d}{d\theta}\phi\right)^2}{\left(\left(\frac{d}{dr}\phi\right)^2 r^2 + \left(\frac{d}{d\theta}\phi\right)^2\right)(-1 + 2V)}$$

$$c12 := 4\,\frac{(2V + 1)\left(\frac{d}{dr}\phi\right)\left(\frac{d}{d\theta}\phi\right)}{\left(\left(\frac{d}{dr}\phi\right)^2 r^2 + \left(\frac{d}{d\theta}\phi\right)^2\right)(-1 + 2V)}$$

$$c22 := -2 \, \frac{-\left(\frac{d}{dr}\,\mathrm{phi}\right)^2 r^2 + 2\left(\frac{d}{dr}\,\mathrm{phi}\right)^2 r^2\, V - 2\left(\frac{d}{dth}\,\mathrm{phi}\right)^2}{r^2\left(\left(\frac{d}{dr}\,\mathrm{phi}\right)^2 r^2 + \left(\frac{d}{dth}\,\mathrm{phi}\right)^2\right)(-1 + 2\,V)}$$

$$\mathrm{Delta} := -\frac{8}{(-1 + 2\,V)\, r^2}$$

IV. THE SCHWARZSCHILD CASE

In the case of a single Schwarzschild black hole, we set $\Phi = 1/r$ and the metric (1) is

$$ds^2 = dr^2 + \frac{r^2}{1-2V}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{6}$$

which we construct from the standard isotropic form

$$ds^2 = \left(1 + \frac{m}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\varphi^2\right) \tag{7}$$

by means of the coordinate transformation

$$\left(1 + \frac{m}{2\bar{r}}\right)^2 d\bar{r} = dr. \tag{8}$$

Thus,

$$r = \bar{r} + m\ln\bar{r} - \frac{m^2}{4\bar{r}} + C. \tag{9}$$

For convenience, we fix C so that the horizon in both coordinate systems is at $r = \bar{r} = \frac{m}{2}$. Then

$$r = \bar{r} + m\ln\frac{2\bar{r}}{m} - \frac{m^2}{4\bar{r}} + \frac{m}{2}. \tag{10}$$

Thus the metric (6) is recovered with

$$V(r) = \frac{1}{2}\left(1 - \left(1 + \frac{m}{2\bar{r}}\right)^{-4}\frac{r^2}{\bar{r}^2}\right) \tag{11}$$

with $\bar{r}$ defining r implicitly by Eq. (10).

On the horizon, $r = (\bar{r}) = \frac{m}{2}$, so

$$V = \frac{15}{32}. \tag{12}$$

Note that the horizon is also defined by the condition that it is a stationary surface, i.e.

$$\frac{\partial}{\partial r}\left(\frac{r^2}{1 - 2V(r)}\right) = 0. \tag{13}$$

For large r, we apply the binomial expansion, and use Eq. (10), in Eq. (11) to find

$$V(r) = \frac{m}{2\bar{r}}\left(1 - 2\ln\frac{2\bar{r}}{m}\right). \tag{14}$$

Since, *to first order*, $r = \bar{r}$, this can be rewritten as

$$V(r) = \frac{m}{2r}\left(1 - 2\ln\frac{2r}{m}\right). \tag{15}$$

V. PROGRESS TOWARDS A NUMERICAL SOLUTION

A Cactus programme solves Eq. (4) numerically, using a general nonlinear elliptic solver TAT Jacobi

- In the Schwarzschild case, the programme converges to the expected analytic solution. The input data was

  Boundary conditions:

$$\begin{aligned} V &= \frac{15}{32} \qquad r = \frac{1}{2} \\ V &= \frac{1}{2r_0}\left(1 - 2\ln 2r_0\right) \qquad r = r_0 \gg 1 \\ \frac{\partial V}{\partial \theta} &= 0 \qquad \theta = 0, \pi \end{aligned} \tag{16}$$

  Φ:

$$\Phi = \frac{1}{r} \tag{17}$$

  Initial V:

$$V = \frac{15}{32} + \left(\frac{r - \frac{1}{2}}{r_0 - \frac{1}{2}}\right)\left(\frac{1}{2r_0}\left(1 - 2\ln 2r_0\right) - \frac{15}{32}\right) \tag{18}$$

- The perturbed Schwarzschild case is discussed below
- The case of two equal Schwarzschild black holes is being investigated.

VI. PERTURBED SCHWARZSCHILD

Again we choose $\Phi = 1/r$, and write

$$V(r,\theta) = V_S(r) + \epsilon W(r,\theta). \tag{19}$$

We found that, to first order in ϵ, Eq. (4) reduces to a linear partial differential equation in $W(r,\theta)$. Further this equation is **separable** with solution

$$W(r,\theta) = \Sigma_1^\infty a_n w_n(r) P_n(\cos\theta) \tag{20}$$

where P_n is a Legendre polynomial and we define $w_n(r)$ to be the solution of

$$b_2 \frac{d^2 w_n(r)}{dr^2} + b_1 \frac{d w_n(r)}{dr} + b_{0(n)} = 0 \tag{21}$$

subject to

$$w_n(r = \frac{1}{2}) = 1, \quad \lim_{r\to\infty} w_n(r) = 0. \tag{22}$$

The coefficients b_2, b_1 and $b_{0(n)}$ are

$$b_2 = \; r^2 \left(-1 + 2\,V_S(r)\right)^2$$

$$b_1 = -\,r \;\left(-1 + 2\,V_S(r)\right)\,\left(3 - 6\,V_S(r) + 7\,r\,V_S'(r)\right)$$

$$b_{0(n)} = 1 - \frac{n(n+1)}{2} + (3n(n+1) - 8)\; V_S(r) - 2\; (3n(n+1) - 10)\; V_S(r)^2 \\ + 4\; (n(n+1) - 4)\; V_S(r)^3 + 7\, r^2\, V_S'(r)^2.$$

Is the above linearized solution conformally flat? We evaluated the York tensor

$$Y_{ijk} = R_{ij;k} - R_{ik;j} + \frac{1}{4}(R_{,j} g_{ik} - R_{,k} g_{ij}), \tag{27}$$

and found that, to first order in ϵ,

$$Y_{ijk} = 0 \text{ for } n = 1, \tag{28}$$

but that for $n > 1$ $Y_{ijk} \neq 0$; for example

$$Y_{112} = \frac{2 - n(n+1)}{4\, r^2}\, w_n(r) P_n'(\cos\theta)\epsilon. \tag{29}$$

Thus, our method does produce initial data that is not conformally flat.

VII. BOUNDARY, AND OTHER, DATA FOR TWO BLACK HOLES

Consider two equal Schwarzschild black holes, each of unit mass, situated at the origin and at $(2a, 0, 0)$, where $a > 1$. The scalar field Φ is freely choosable. Thus, it can be set so that it takes the Schwarzschild form in a neighbourhood of $r = \frac{1}{2}$, so

$$\frac{\partial}{\partial r}\left(\frac{r^2}{1-2V(r)}\right)_{r=\frac{1}{2}} = 0. \tag{30}$$

Boundary data for large r is obtained from Eq.(15) with $m = 2$

$$V(r) = \frac{1}{r}\left(1 - 2\ln r\right). \tag{31}$$

The 2-surface S defined by $r\cos\theta = a$ is a surface of symmetry. Also, the line L defined by

$$L = \{\frac{1}{2} < r < a, \theta = 0\} \cup \{\frac{1}{2} < r < r_{\max}, \theta = \pi\}$$

is an axis of symmetry. Therefore on S and L

$$\frac{\partial V}{\partial n} = 0, \tag{32}$$

where n is a normal to S or L.

Eq. (4) is singular at the point of symmetry (O_S). Let (ρ, α) be spherical coordinates with origin at O_S. Then trying

$$V = \rho^n f_n(\alpha), \tag{33}$$

we find that $f_n(\alpha)$ satisfies an equation of the form

$$\frac{d^2 f_n}{d\alpha^2} + a_n(\alpha)\frac{df_n}{d\alpha} + b_n(\alpha) f_n = 0. \tag{34}$$

The functions $a_n(\alpha)$ and $b_n(\alpha)$ are complicated, but

- Near $\alpha = 0$, Eq. (34) is Bessel's equation of order 0, with frequency that increases with n;
- Near $\alpha = \pi/2$, Eq. (34) is the harmonic equation, with frequency that increases with n.

Thus, we expect that values of n exist for which Eq. (34) has non-trivial solutions with $f_n'(0) = f_n'(\pi/2) = 0$ (which is necessary for consistency with Eq. (32)). Once a solution to Eq (34) has been constructed, the region around $\rho = 0$ can be excised from the computational domain and the solution used to provide boundary data at $\rho = \rho_0$, with ρ_0 small.

VIII. FURTHER WORK

- Convergence testing of the solution found by the elliptic solver in the perturbed Schwarzschild case
- A comparative evolution, in the perturbed Schwarzschild case, from initial data generated by (a) the York-Lichnerowicz decomposition, (a) the method described here, and (c) instantaneously Kerr-Schild initial data
- The two black hole problem