The Z4 formalism in Numerical Relativity

Trieste, September 2003

NETWORK

Numerical Relativity

- Theoretical issues
- Discrete algorithms
- Computer science
- Visualization

Present Network

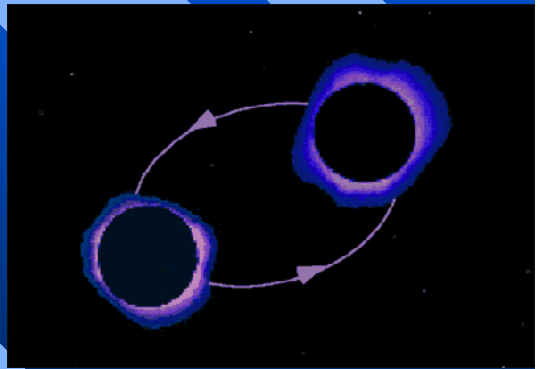
Proposed Network

- Data analysis

- Gravitational Wave Detection

NR Theoretical issues

- Structure of the field equations
- Initial data
- Boundary conditions
- Waveform extraction
- Perturbative (analytical) methods



- Matter modelling (Hydrodynamics)
- Microphysics

The Z4 system

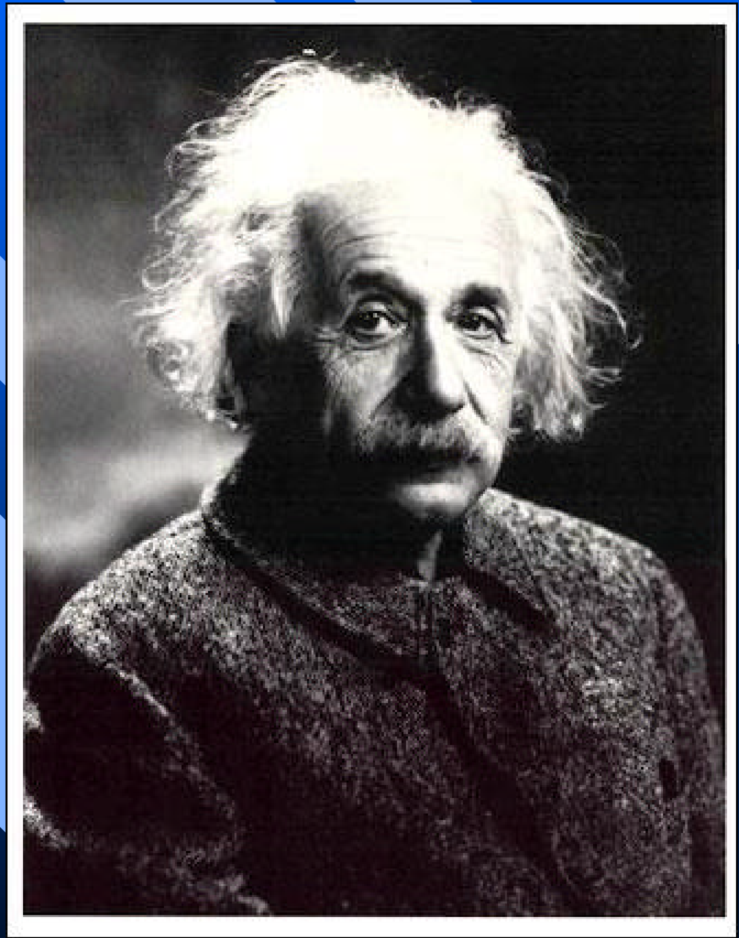
Physical Review D67, 104005 (2003)

10 Field equations

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi\,(T_{\mu\nu} - T/2\ g_{\mu\nu})$$

14 dynamical fields

$$g_{\mu\nu},\ Z_\mu$$



General covariance

Unknown fields

- Up to four coordinate conditions allowed
- Four kinematical fields
- Ten dynamical fields left

Field equations

- Six second order evolution equations on $g_{\mu\nu}$
- Four first order evolution equations on Z_m

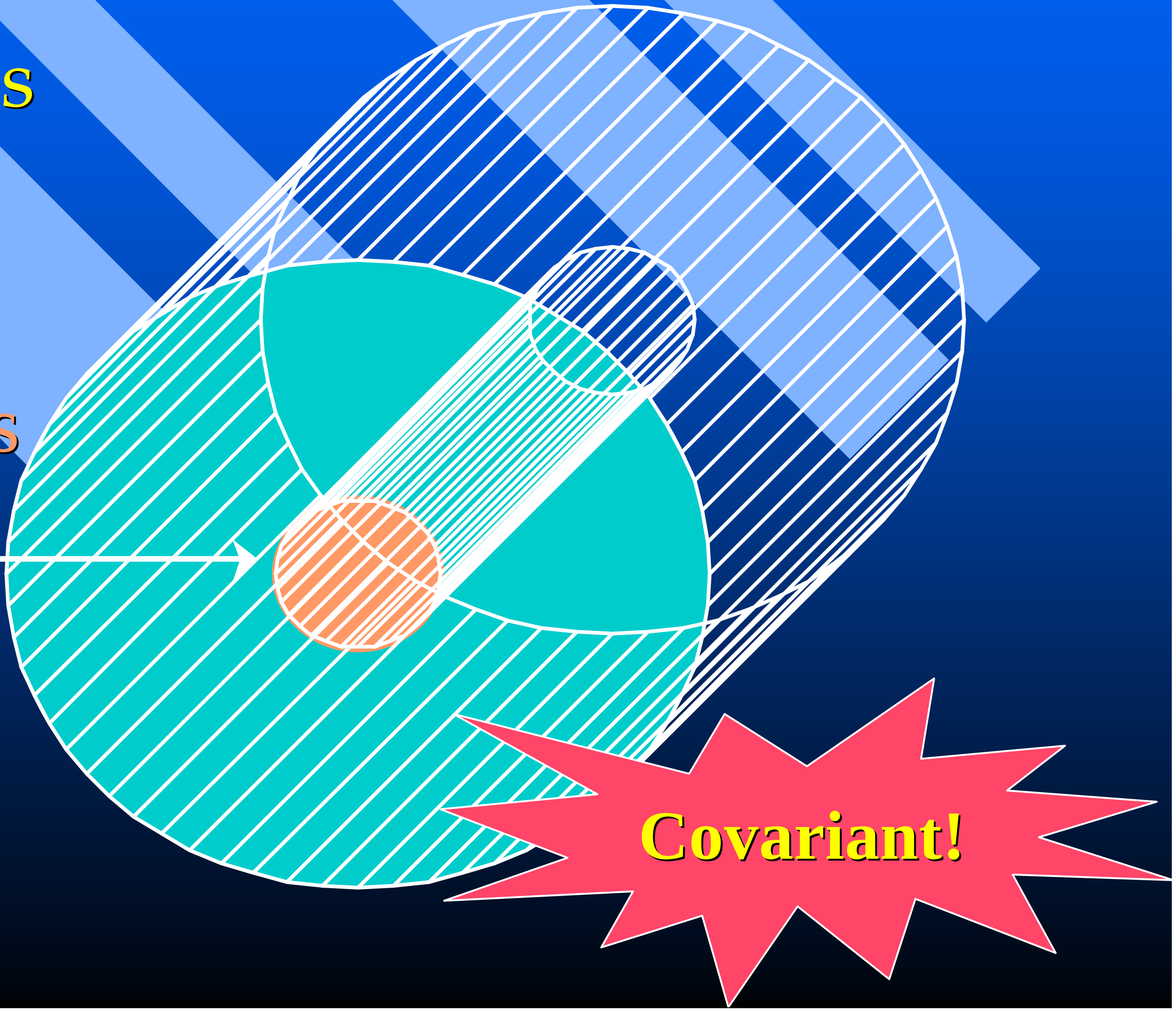
Recovering Einstein's solutions

Bianchi identities

$$ð\, Z_{\mu} + R_{\mu\nu} Z^{\nu} = 0$$

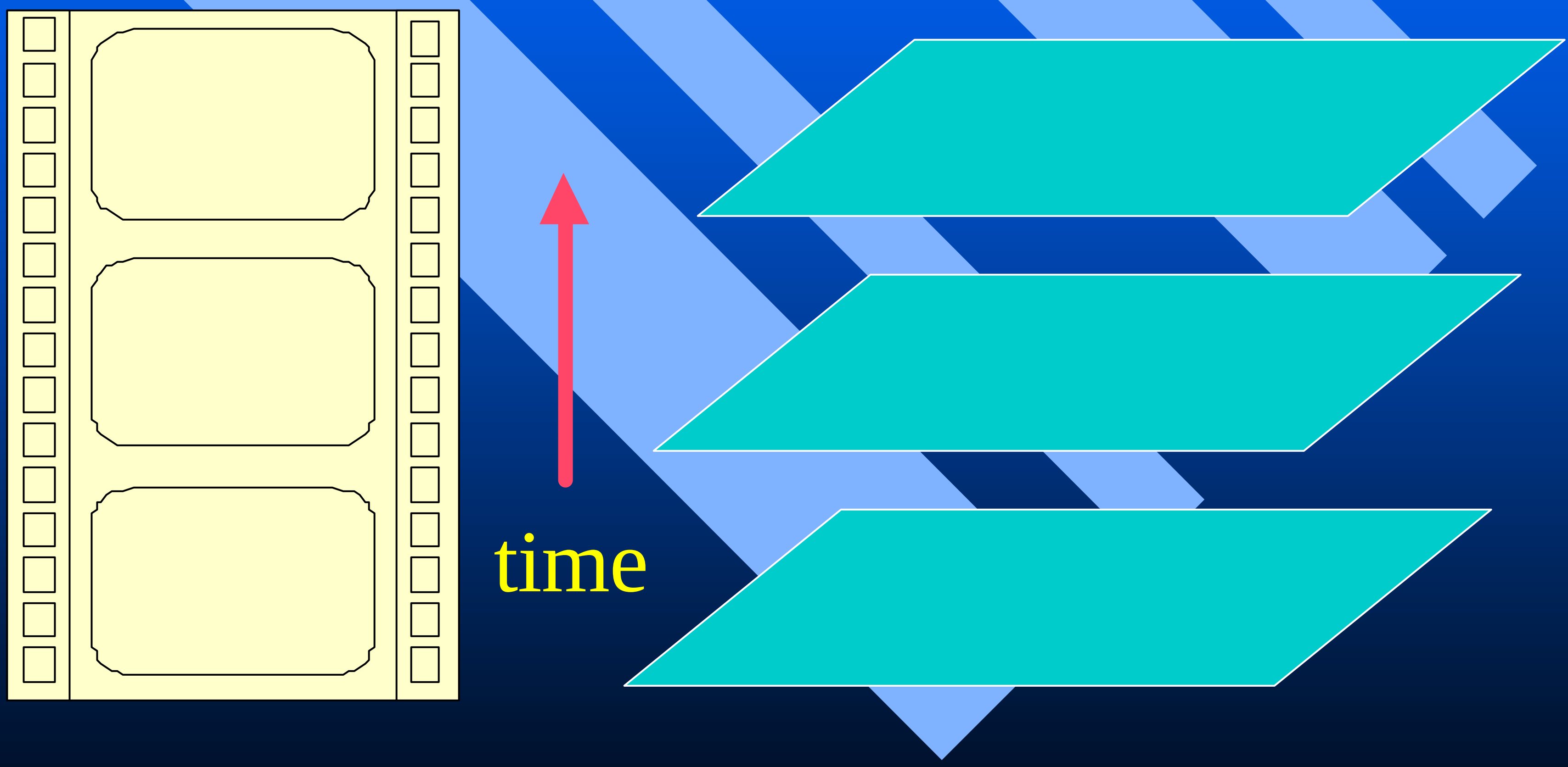
Einstein's solutions

$$Z_{\mu} = 0$$



Algebraic
error tracking

Time slicing of spacetime



3+1 decomposition

$$ds^2 = -\alpha^2\, dt^2 + \gamma_{ij}\, (dx^i + \beta^i\, dt)\, (dx^j + \beta^j\, dt)$$

$K_{ij} = -1/2\alpha\, (\partial_t - \mathcal{L}_\beta)\, \gamma_{ij}$ (extrinsic curvature)

ADM system (Yvonne Fourés-Bruhat, 1956)

- Matter terms decomposition

$$\tau \equiv 8\pi\, n_\mu n_\nu T^{\mu\nu} \quad S_k \equiv 8\pi\, n_\mu T^\mu{}_k \quad S_{ij} \equiv 8\pi\, T_{ij}$$

- Constraints

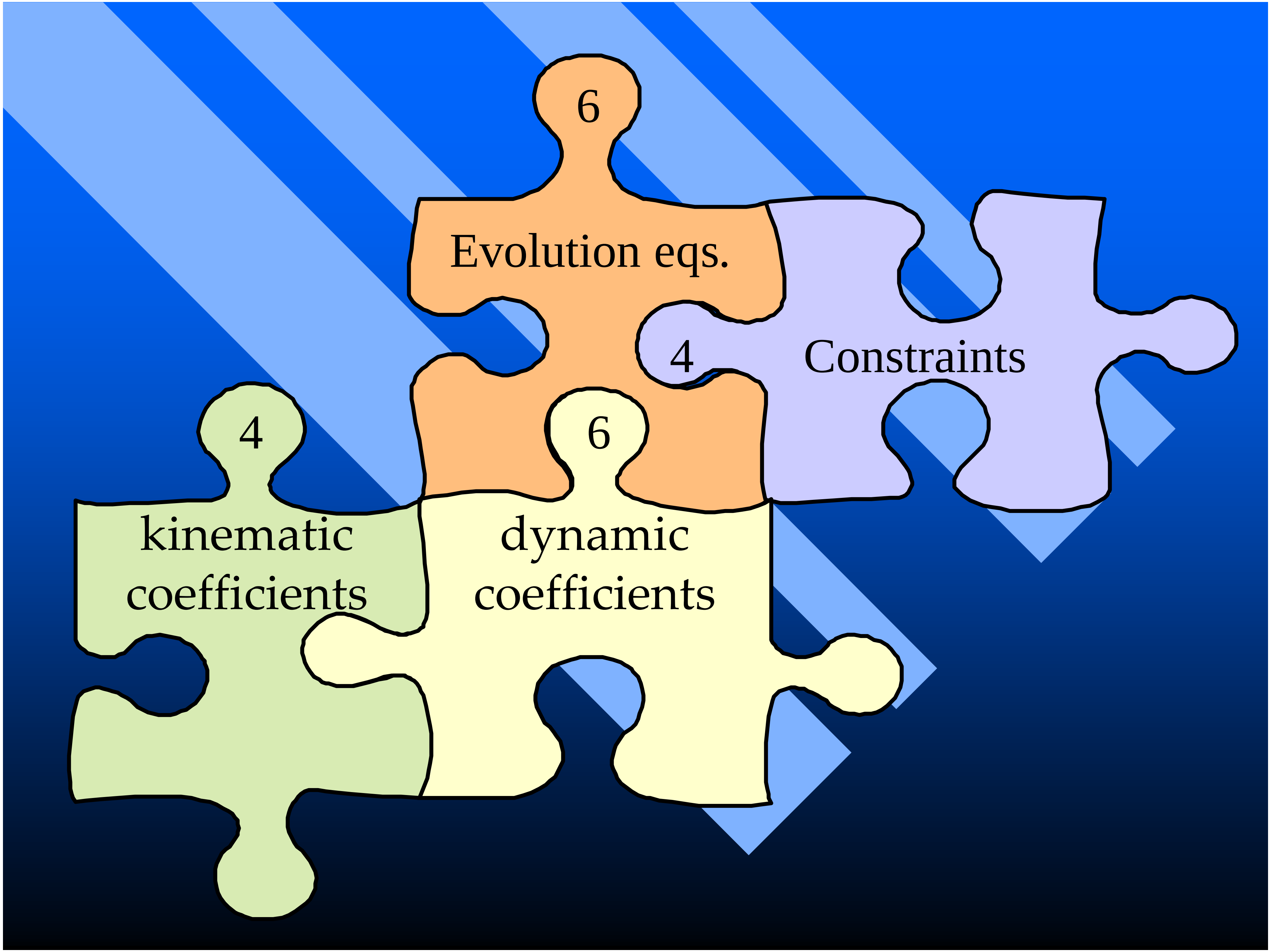
$$2\tau = {}^{(3)}R + (tr K)^2 - tr(K^2)$$

$$S_i = \nabla_k \,(K^k{}_i - tr K\, \delta^k{}_i)$$

- Evolution equations

$$(\partial_t - \mathcal{L}_\beta)\, K_{ij} = -\nabla_i d_j\, \alpha + \alpha\, [{}^{(3)}R_{ij} + tr K\, K_{ij} - 2\, K^2{}_{ij}$$

$$- S_{ij} + ½\, (tr S - \tau)\, \gamma_{ij}\,]$$

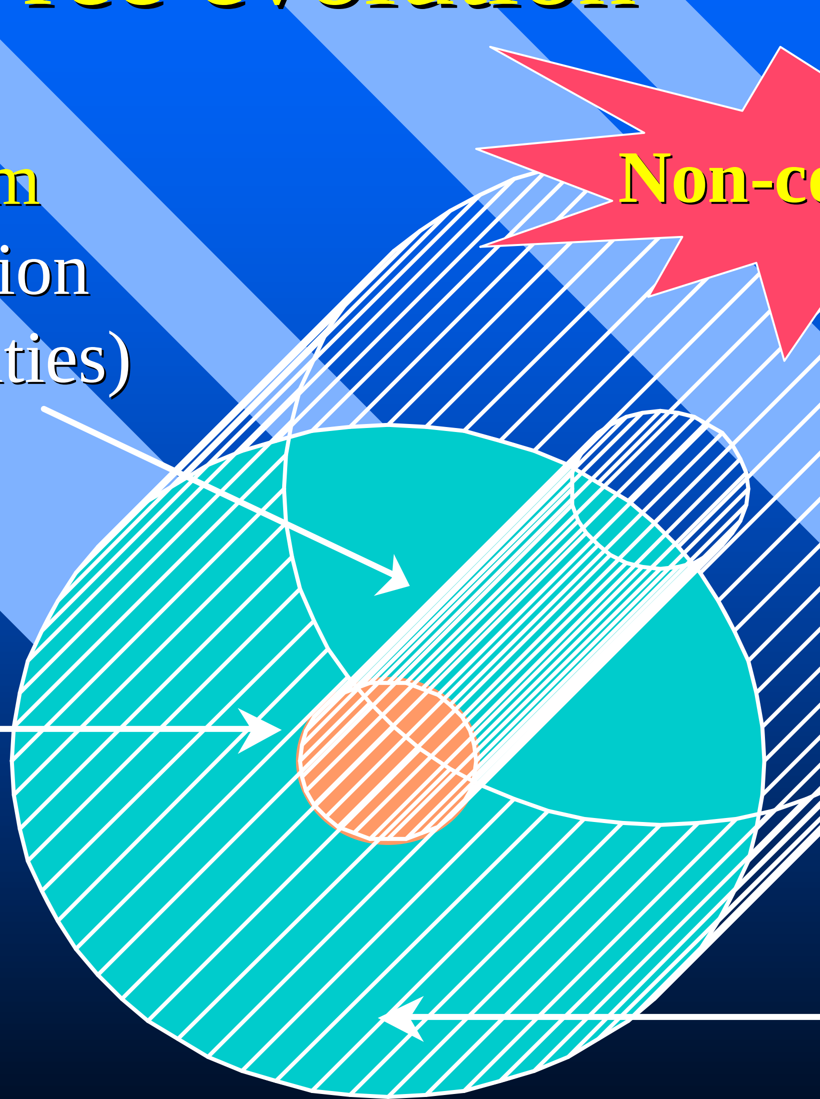

6
Evolution eqs.
4
Constraints
4
6
kinematic
coefficients
dynamic
coefficients

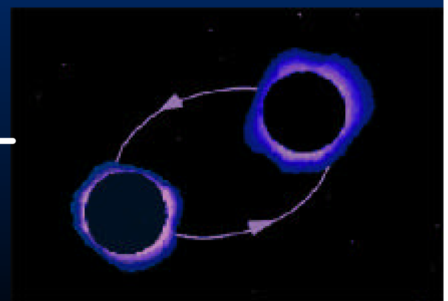
Free evolution

Subsidiary system
(constraints evolution
from Bianchi identities)

Constrained
Initial data

Unconstrained
Initial data





Z4 evolution equations

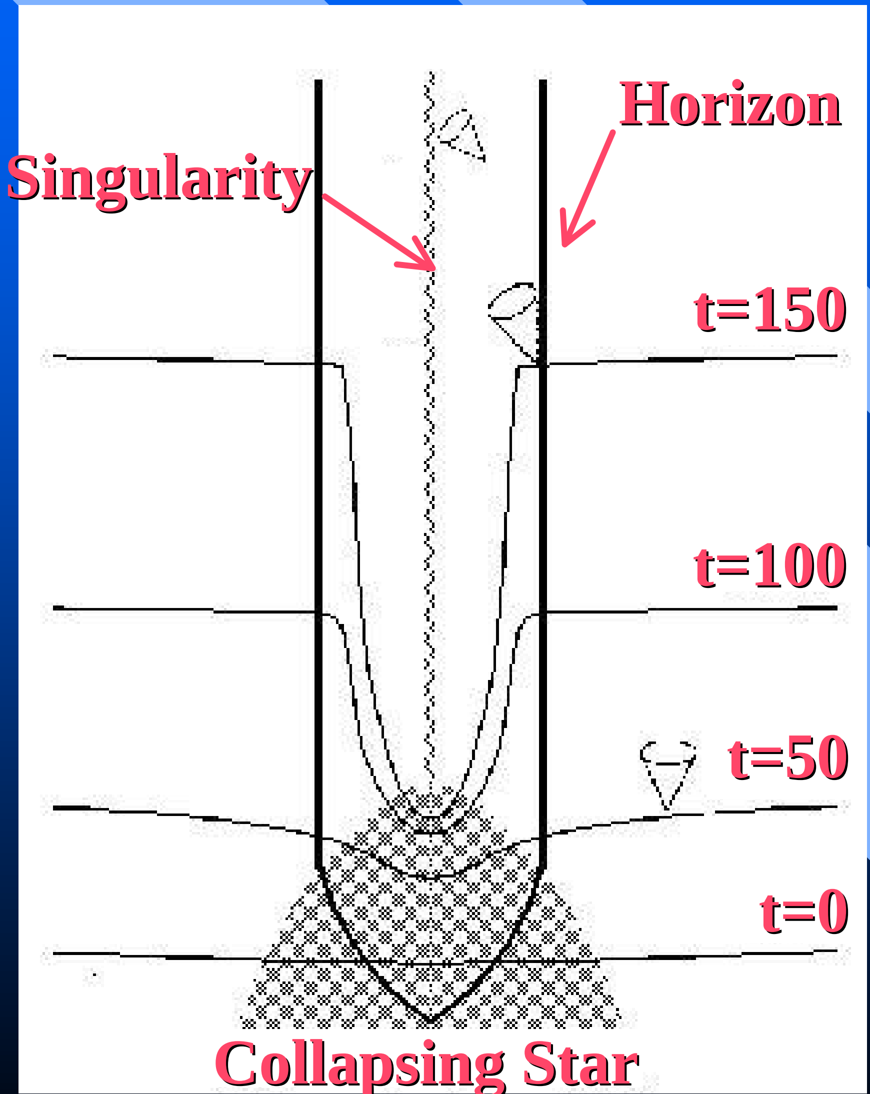
- $(\partial_t - \mathcal{L}_\beta)\, K_{ij} = - \nabla_i d_j\, \alpha + \alpha\, [\, {}^{(3)}R_{ij} + \nabla_i Z_j + \nabla_j Z_i - 2\, K^2{}_{ij} + (\mathit{tr}K - 2\Theta)\, K_{ij} - S_{ij} + \frac{1}{2}\, (\mathit{tr}S - \tau)\, \gamma_{ij}\,]$

- $(\partial_t - \mathcal{L}_\beta)\, Z_i = \alpha\, [\nabla_k\, (K^k{}_i - \mathit{tr}K\, \delta^k{}_i) - 2\, K_i{}^k\, Z_k + \partial_i\, \Theta - \Theta\, \alpha_i/\alpha - S_i\,]$

- $(\partial_t - \mathcal{L}_\beta)\, \Theta = \alpha/2\, [{}^{(3)}R + (\mathit{tr}K - 2\Theta)\, \mathit{tr}K - \mathit{tr}(K^2) + 2\, \nabla_k Z^k - 2\, Z^k\, \alpha_k/\alpha - 2\tau]$

$\Theta \equiv n_\mu Z^\mu = \alpha\, Z^0$

Harmonic slicing



- β^i ‘arbitrary’ (space coordinates free)
- Harmonic time

$$\eth\, x^0 = 0$$

$$(\partial_t - \mathcal{L}_\beta)\, \sqrt{\gamma}/\alpha = 0$$

- Collapse of the lapse

$$\alpha \sim \sqrt{\gamma}$$

Singularity avoidance

Generalized harmonic slicings

- 3+1 covariance:

$$t' = f(t) \qquad x' = g(x,t)$$

- (3+1)-covariant generalization:

$$(\partial_t - \mathcal{L}_\beta)\, \mathit{ln}\, \alpha = -\, \alpha\, (\, f\, \mathit{tr}K - \lambda\, \Theta\,)$$

Hyperbolicity analysis

Kreiss & Ortiz, gr-qc/0106085

- Linearized Fourier modes:

$$\gamma_{ij} = \delta_{ij} + 2 \exp(i\, \omega \cdot x)\, \gamma_{ij}(\omega,t)$$

$$a = 1 + \exp(i\, \omega \cdot x)\, a\,(\omega,t)$$

$$K_{ij} = i\, \omega \exp(i\, \omega \cdot x)\, K_{ij}(\omega,t)$$

$$\Theta = i\, \omega \exp(i\, \omega \cdot x)\, \Theta(\omega,t)$$

$$Z_k = i\, \omega \exp(i\, \omega \cdot x)\, Z_k(\omega,t)$$

- $u = (a\;, \gamma_{ij}, K_{ij}, \Theta\;, Z_k)$ (17 fields, no shift)

Characteristic Matrix & Eigenfields

$$\partial_t\, u = - i\, \omega\, A \cdot u$$

- 3 standing eigenfields:

  $a - f\, tr\, \gamma + \lambda\, (\, tr\, \gamma - \gamma^{nn} - Z^n\,), \quad \gamma^{n}{}_{\perp} + Z_{\perp}$

- 12 light cone eigenfields (speed ±1):

  $K_{\perp\perp} \pm \gamma_{\perp\perp} \qquad K^{n}{}_{\perp} \pm Z_{\perp} \qquad \Theta \pm (\, tr\, \gamma - \gamma^{nn} - Z^n\,)$

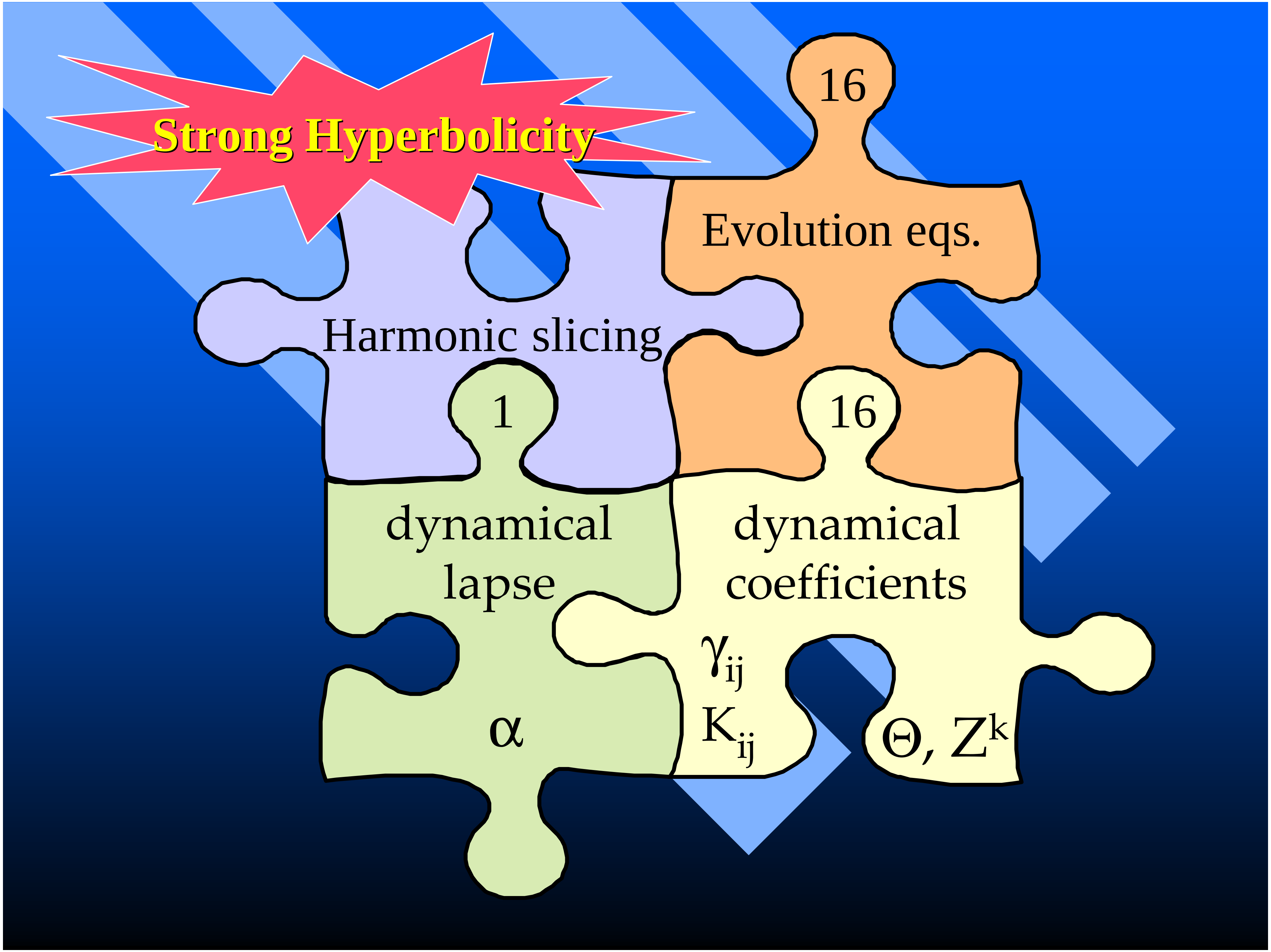
- Gauge eigenfields (speed $\pm\sqrt{f}$):

$\sqrt{f}\, [tr K + (2 - \lambda)/(f - 1)\ \ \Theta\,]$

$\pm\, [a\ + (2f - \lambda)/(f - 1)\ \ (tr\, \gamma - \gamma^{nn} - Z^n)]$

f > 0

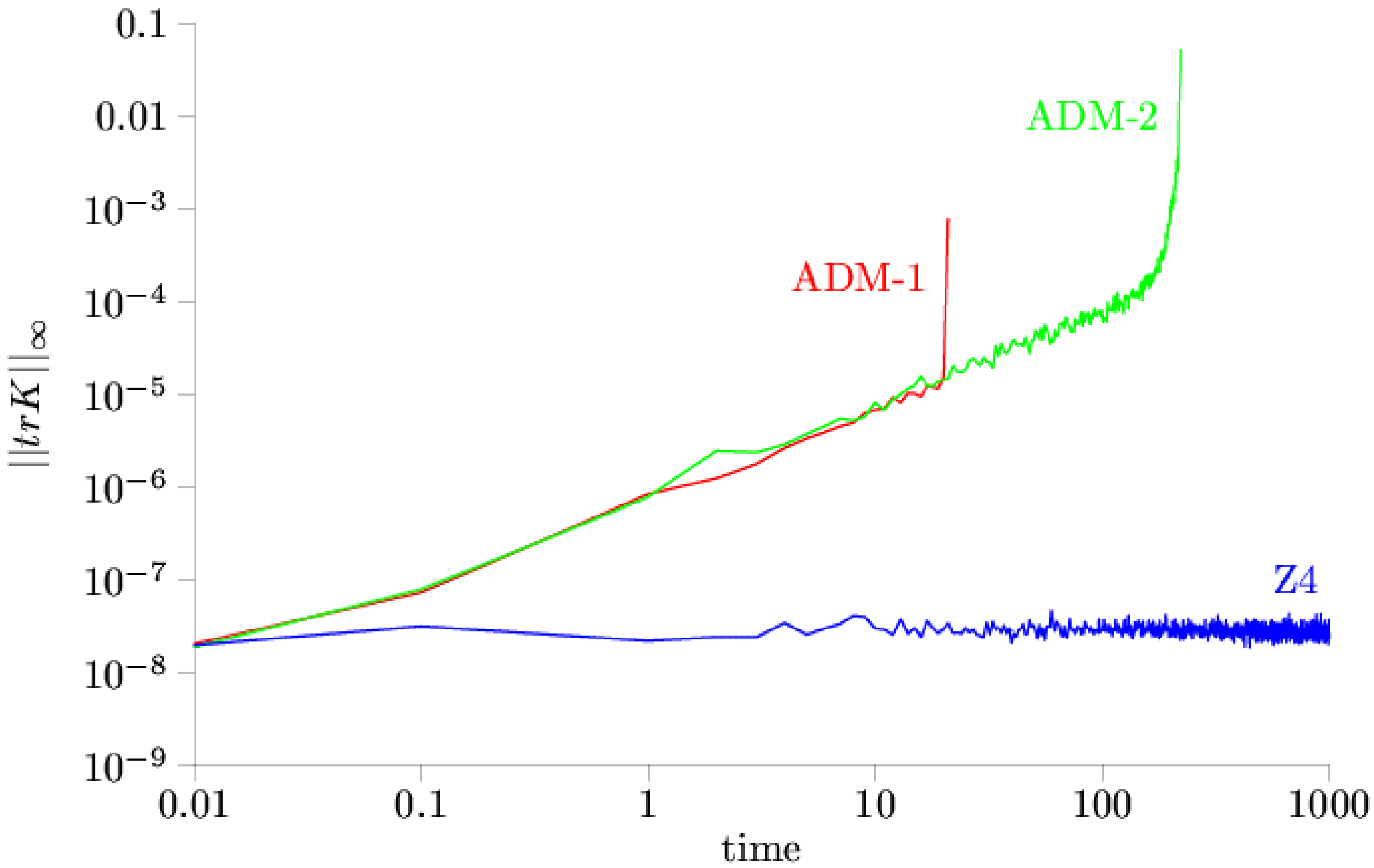
λ = 2 if f = 1


Strong Hyperbolicity
16
Evolution eqs.
Harmonic slicing
1
16
dynamical
lapse
α
dynamical
coefficients
γ_{ij}
K_{ij}
Θ, Z^k

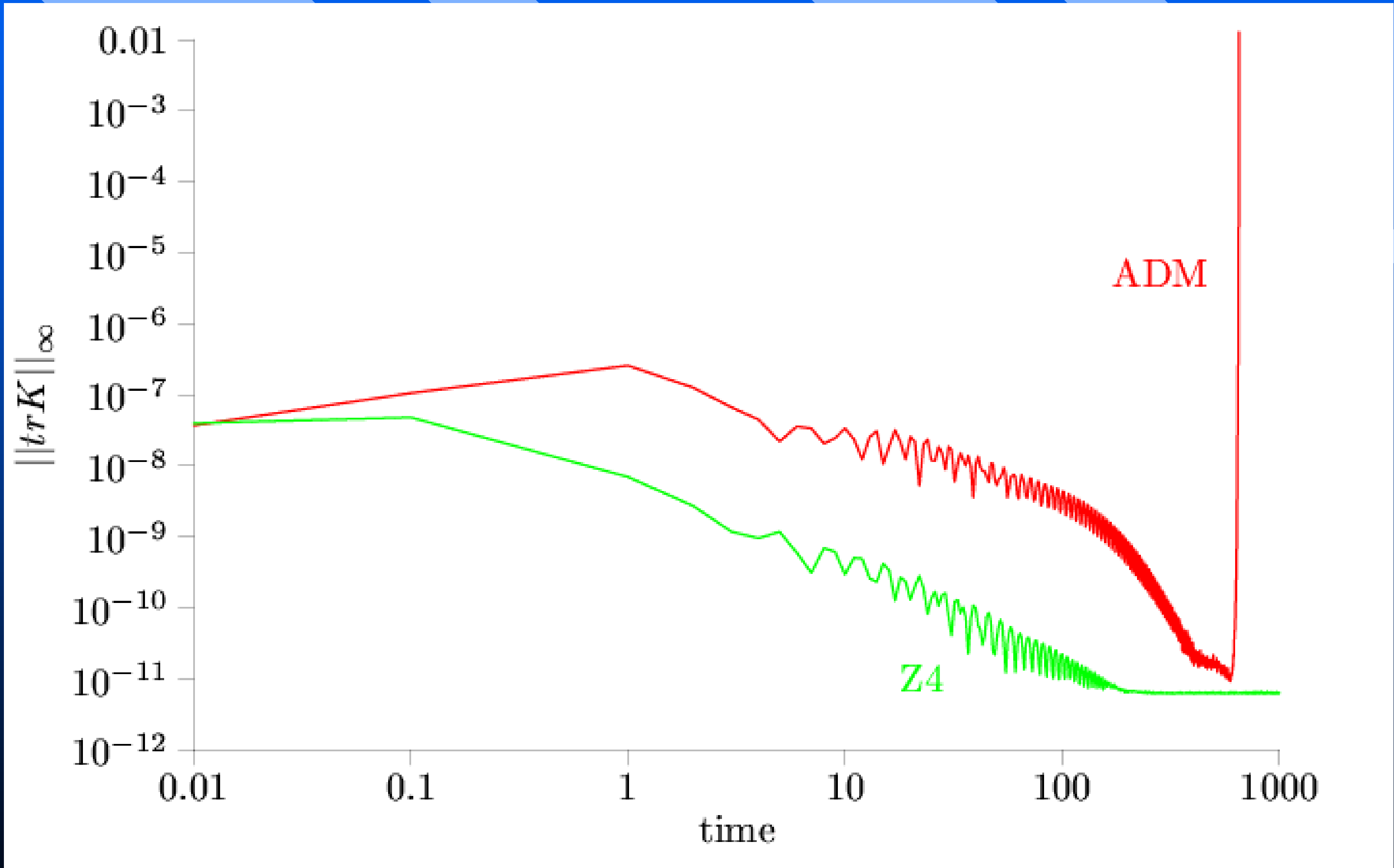
Robust stability test

- Random values the dynamical fields
- Full 3D code with small data (linear regime)
- Finite differencing: Method of lines
  - Standard centered 2nd order in space
  - Standard 3rd order Runge-Kutta in time
- Periodic boundary conditions (space 3-Torus)

Strong vs Weak Hyperbolicity



ICN results



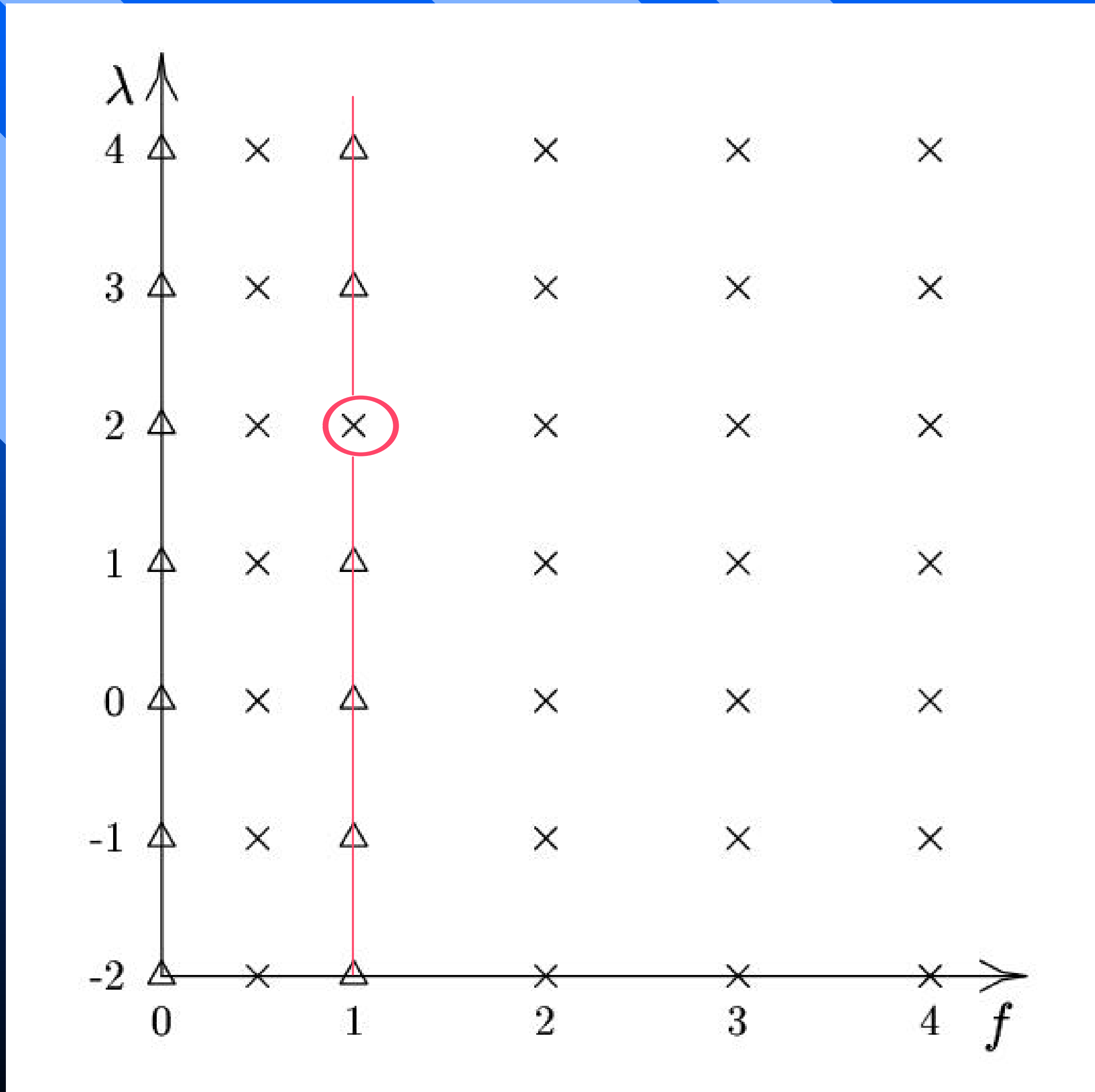
0.01
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
$||trK||_\infty$
ADM
Z4
0.01
0.1
1
10
100
1000
time

Exploring gauge parameter space

$\lambda = 2$

if

$f = 1$



$f > 0$

(partial) Symmetry Breaking

- Original system $(a\ , \gamma_{ij}, K_{ij}, \Theta\ , Z_k)$

Dynamical Fields recombination

$$\mathit{K}_{ij} \equiv K_{ij} - n/2\ \Theta\ \gamma_{ij}$$

+ Supressing extra quantity Θ

- Familly of reduced systems $(a\ , \gamma_{ij}, \mathit{K}_{ij}, Z_k)$

Z3 evolution systems

- $(\partial_t - \mathcal{L}_\beta)\, \gamma_{ij} = -2\alpha\, K_{ij}$
- $(\partial_t - \mathcal{L}_\beta)\, K_{ij} = -\nabla_i d_j\, \alpha + \alpha\, [\, {}^{(3)}R_{ij} + \nabla_i Z_j + \nabla_j Z_i$
  $+\, trK\, K_{ij} - 2\, K^2{}_{ij} - S_{ij} + \tfrac{1}{2}\, (trS - \tau)\, \gamma_{ij}\,]$

$- n\ \alpha/4\ \gamma_{ij}\, [{}^{(3)}R + 2\, \nabla Z + (trK)^2 - tr(K^2) - 2\ \alpha_k/\alpha\ Z^k - 2\tau]$

- $(\partial_t - \mathcal{L}_\beta)\, Z_i = \alpha\, [\nabla_k\, (K^k{}_i - trK\, \delta^k{}_i) - 2\, K_i{}^k\, Z_k - S_i\,]$

Quasiequivalent to
Phys. Rev. D66, 084013 (2002)

Roadmap to BSSN system

(quasiequivalence)

- Select n = 4/3 from the Z3 familly
- Perform conformal decomposition:

$$^*\gamma_{ij} \equiv e^{-4\phi}\, \gamma_{ij} \qquad det(^*\gamma) = 1$$

$$K \equiv \gamma^{ij} K_{ij}$$

$$^*A_{ij} \equiv e^{-4\phi}\, (K_{ij} - K/3\, \gamma_{ij}) \qquad tr\, ^*A = 0$$

- Translate to BSSN additional quantities

$$2\, Z_i = \Gamma_i + {}^*\gamma_{ik}\, (\partial_j {}^*\gamma^{kj})$$

Collapsing Gowdy waves

- Cosmological solution (vacuum)

$$ds^2 = t^{-1/2} e^{Q/2} (-dt^2 + dz^2) + t\,(e^{P} dx^2 + e^{-P} dy^2)$$

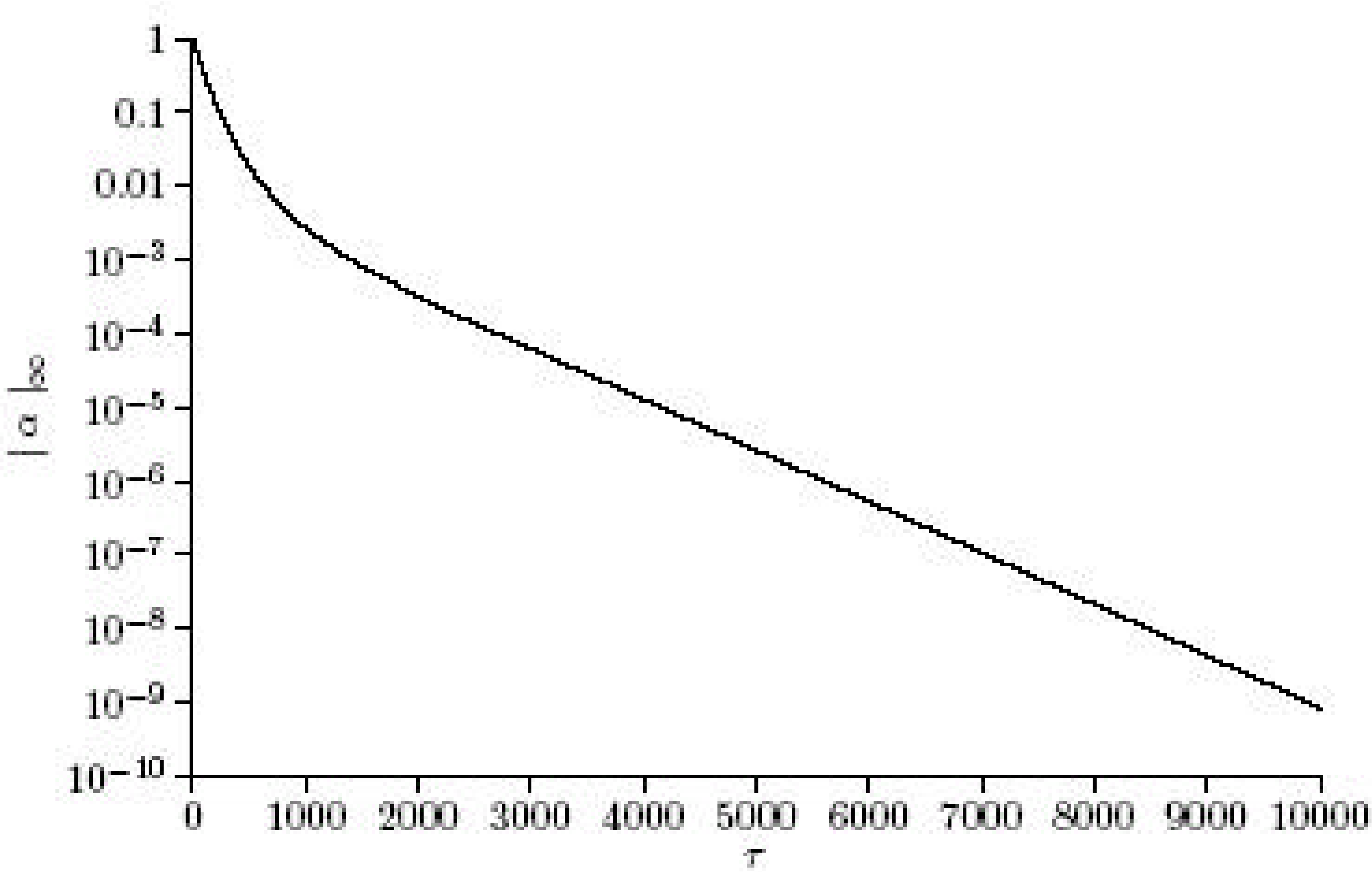
$P(t,z)$, $Q(t,z)$ periodic in z (pp wave)

- Harmonic slicing

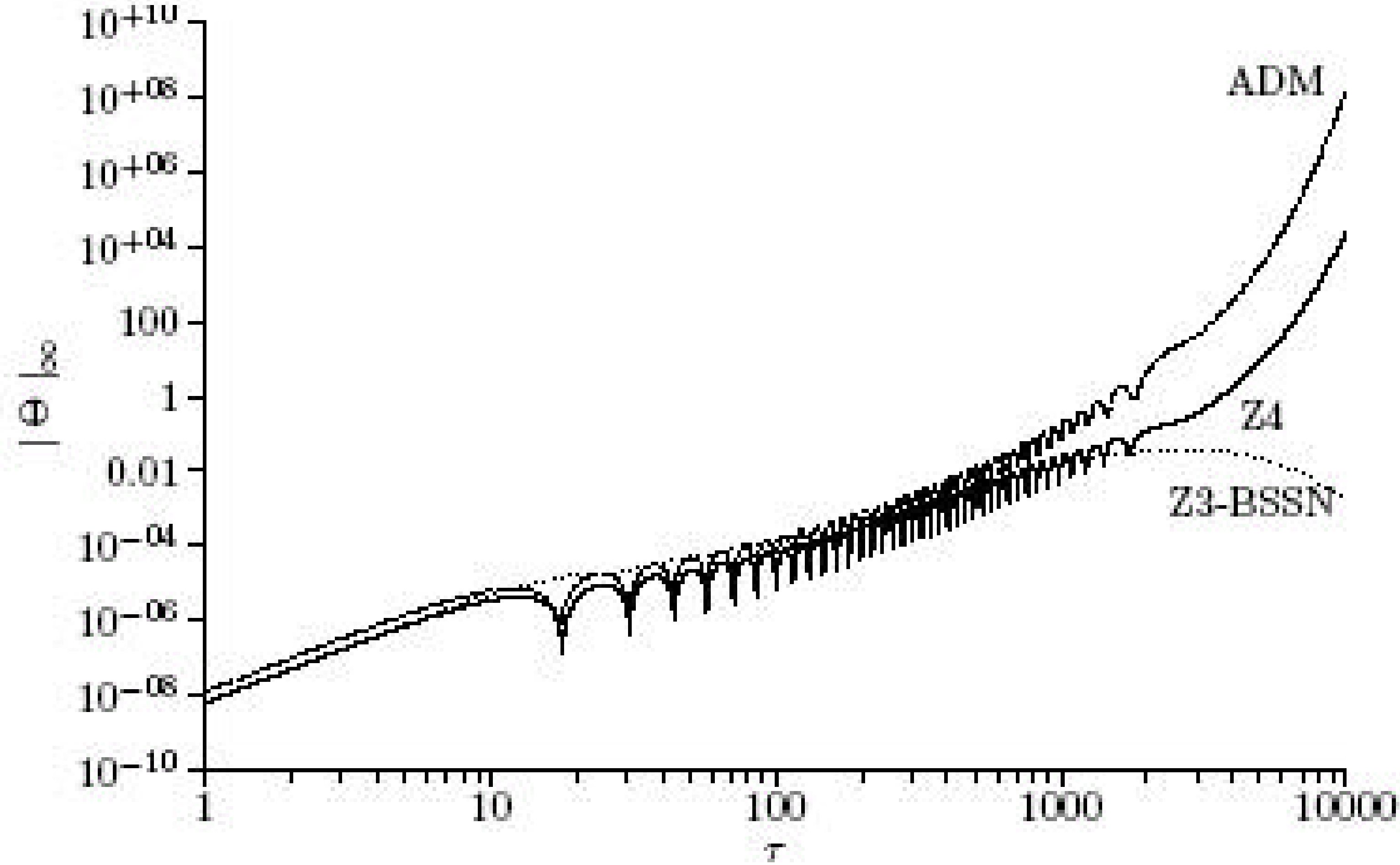
$$t = t_0 \exp(-t/t_0)$$

- Test model for collapse of a 3-torus (periodic boundaries)

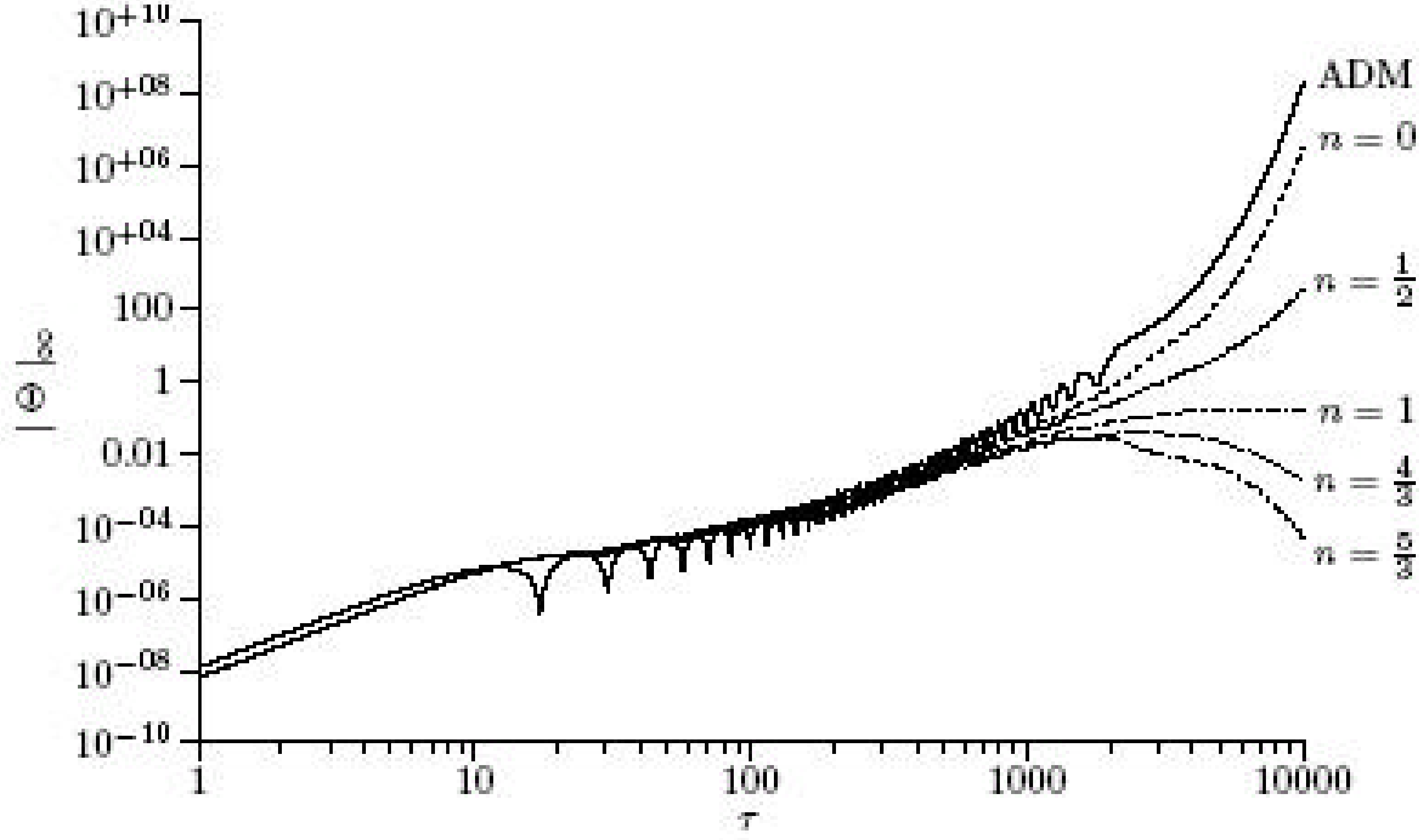
Lapse collapse (Harmonic slicing)



Oscillation & Collapse



Z3 parameter space: n



First order version of Z4

gr-qc/0307067

- 1rst order variables

$$(\alpha \, , \gamma_{ij} \, , K_{ij} \, , \Theta \, , Z_k \, , A_k \, , D_{kij})$$

$$A_k \equiv \partial_k (ln\alpha) \qquad\qquad D_{kij} \equiv \tfrac{1}{2}\, \partial_k \, \gamma_{ij}$$

more constraints!

- supplementary evolution equations

$$\partial_t \, D_{kij} + \partial_k \, [\alpha \, K_{ij} \,] = 0$$

$$\partial_t \, A_k + \partial_k \, [\, \alpha \, (f \, trK - \lambda \, \Theta) \,] = 0$$

Ordering ambiguity

- $\partial_r D_{sij} \neq \partial_s D_{rij}$ more constraints!

- Ricci tensor decomposition

$$ {}^{(3)}R_{ij} = \partial_k \Gamma^k{}_{ij} - \partial_{(i} D_{j)k}{}^k + \ldots \qquad \text{(standard)} $$

$$ {}^{(3)}R_{ij} = - \partial_k D^k{}_{ij} + \partial_{(i} \Gamma_{j)k}{}^k + \ldots \qquad \text{(deDonder-Fock)} $$

- Ordering parameter ζ one more parameter!

$$ {}^{(\zeta)}R_{ij} \equiv (1+\zeta)/2 \; {}^{(\text{standard})}R_{ij} + (1-\zeta)/2 \; {}^{(\text{Fock})}R_{ij} $$

Three main options

- $\zeta = -1$: - Everybody is using it
  - Closer to wave equation
- $\zeta = 0$: - Textbook alternative
  - $\partial_{[r} D_{s]ij}$ disappears
- $\zeta = +1$: - Geometrical interpretation of transverse traceless eigenfields
  - Difficult boundary problem

Hyperbolicity conditions

- Strong Hyperbolicity (for all ζ)

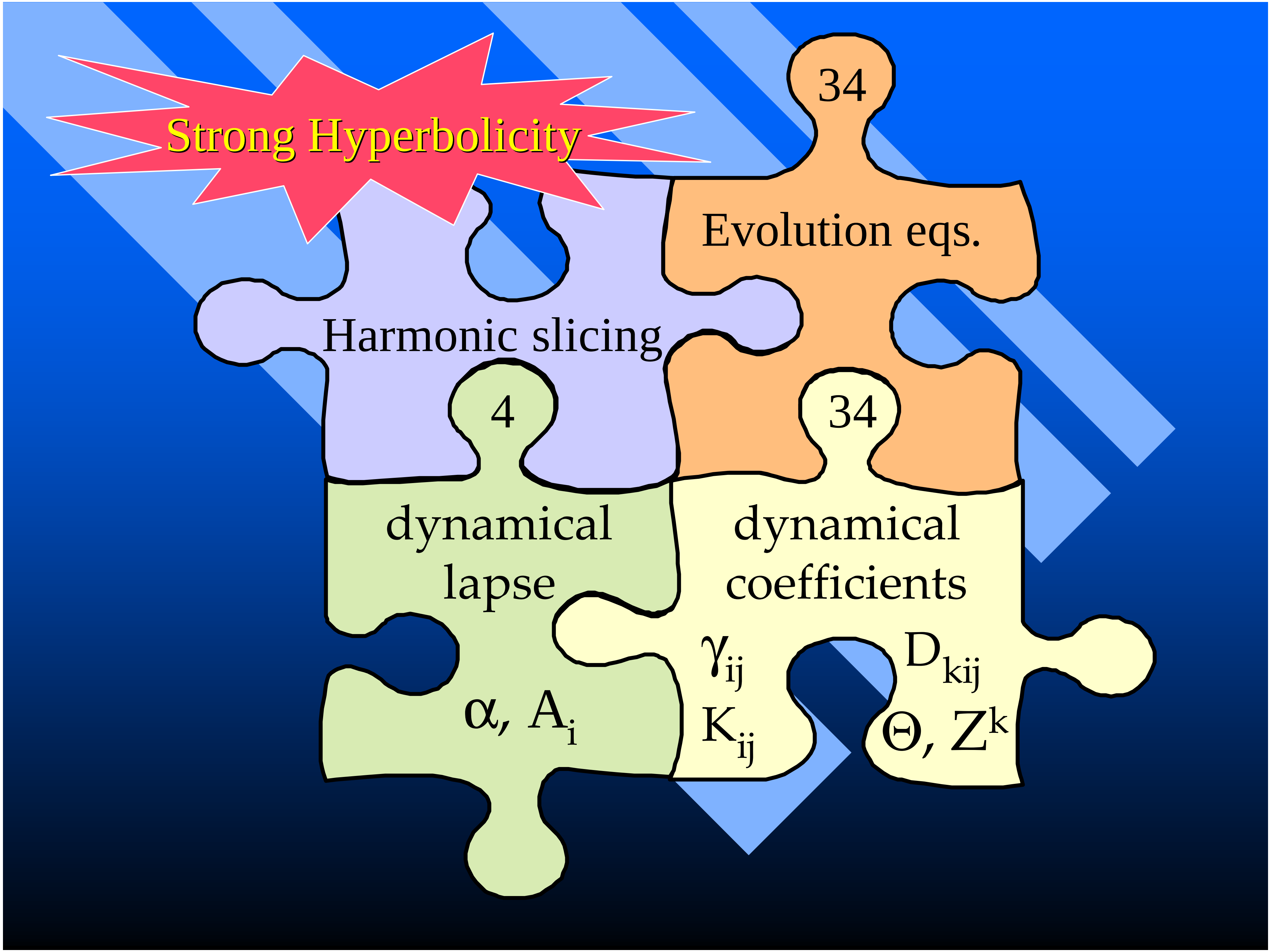
  $f > 0, \qquad \lambda = 2 \text{ if } f = 1$

- Symmetric Hyperbolicity for

  $f = 1 , \lambda = 2$ (harmonic slicing)

  and

  $\zeta = -1$ (deDonder-Fock decomposition)


Strong Hyperbolicity
34
Evolution eqs.
Harmonic slicing
4
34
dynamical
lapse
α, A_i
dynamical
coefficients
γ_{ij}
K_{ij}
D_{kij}
Θ, Z^k

(full) Symmetry Breaking

- Original system $(a\ ,\gamma_{ij}, K_{ij}, A_k, D_{kij}, \Theta, Z_k)$

Dynamical Fields recombination

$$\mathcal{K}_{ij} \equiv K_{ij} - \ n/2\ \Theta\ \gamma_{ij}$$

$$d_{kij} \equiv 2D_{kij} + \eta\ \gamma_{k(i}\ Z_{j)} + \chi\ Z_k\ \gamma_{ij}$$

+ Supressing extra quantities Θ, Z_k

- Familly of reduced systems $(a\ ,\gamma_{ij}, \mathcal{K}_{ij}, A_k, d_{kij})$

5-parameter familly of KST systems

(dynamical gauge version)

- $\partial_t \, ln\alpha = - f \, \alpha \, trK \qquad \partial_t \, \gamma_{ij} = - 2\alpha \, K_{ij}$
- $\partial_t K_{ij} = - \nabla_{(i}[\alpha \, A_{j)}] + \alpha \, [{}^{(\zeta)}R_{ij} + trK \, K_{ij} - 2 \, K^2_{ij} - S_{ij}$
  $+ ½ \, (trS - \tau) \, \gamma_{ij} \,] - n \, \alpha/4 \, \gamma_{ij} \, [{}^{(\zeta)}R + (trK)^2 - tr(K^2) - 2\tau]$
- $\partial_t A_k + \partial_k [\, \alpha \, f \, trK \,] = 0$
- $\partial_t d_{kij} = 2 \, \partial_k (\alpha \, K_{ij}) + \chi \, \alpha \, \gamma_{ij} \, (\partial_r K^r{}_k - \partial_k \, trK)$
  $+ \eta/2 \, \alpha \, [\gamma_{ki} \, (\partial_r K^r{}_j - \partial_j \, trK) + \gamma_{kj} \, (\partial_r K^r{}_i - \partial_i \, trK)] + \ldots$

Conclusions

- General covariant framework
- Algebraic constraints
- Constraint violation allowed
- Strongly hyperbolic systems (1st & 2nd order)
- Z3-BSSN recovered (partial symmetry breaking)
- Z3-BonaMassó recovered (“)
- KST recovered (full symmetry breaking)

Why to use it?

Evolution equations

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi\,(T_{\mu\nu} - T/2\; g_{\mu\nu})$$

Subsidiary system (Bianchi identities)

ð $Z_\mu + R_{\mu\nu} Z^\nu = 0$ no 'adjustment' needed

Constraint-preserving
Boundary conditions