Novel Finite-differencing Techniques for Numerical Relativity:

Application to black-hole excision

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OUTLINE

- Introduction
- Summation by parts
- Outer boundary conditions
- Inner boundaries–excision
- Interfaces
- Applications
  - Cubic Excision
  - Spherical Excision
- Summary

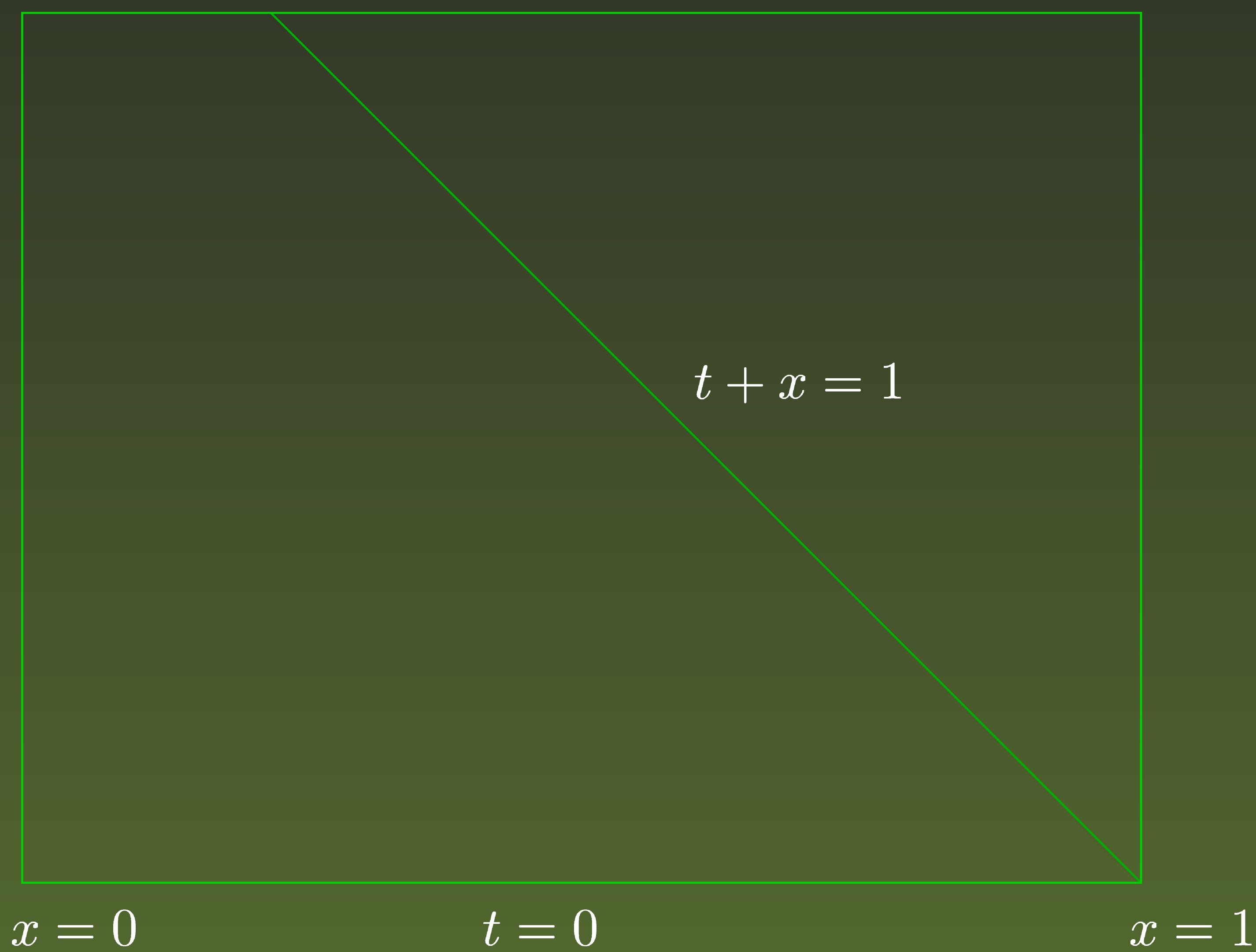
Introduction

One way wave equation in $[0,1]$:

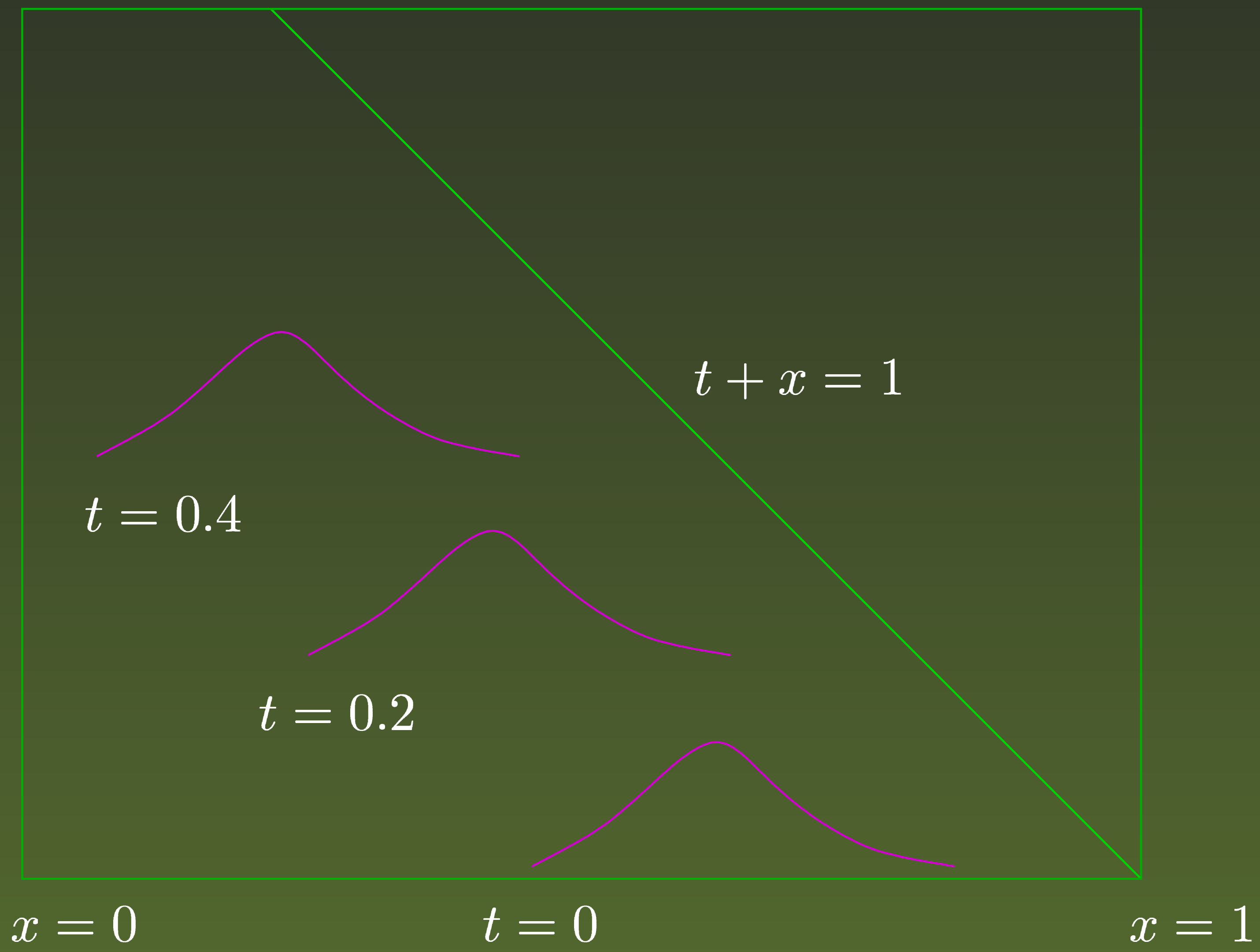
$$\partial_t u = \partial_x u \qquad \text{wave traveling to the left}$$

$$u(t,x) = \hat{u}(t+x) \qquad \text{is a solution}$$

Introduction



Introduction



Introduction

One way wave equation in $[0,1]$:

$$\partial_t u = \partial_x u$$

Initial-Boundary Value Problem.

$$u(t=0,x) = f(x) \qquad u(t,x=1) = g(t)$$

Energy inequality

$$\mathcal{E} := \int_0^1 u^2 dx$$

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$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \int_0^1 2uu_t dx = \int_0^1 2uu_x dx = \int_0^1 (u^2)_x dx \\ &= u(t, x=1)^2 - u(t, x=0)^2 \leq g(t)^2\end{aligned}$$

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$$\begin{aligned}\mathcal{E}(t) &\leq \mathcal{E}(0) + \int_0^t g(t)^2 dt \\ &\leq \int_0^1 f(x)^2 + \int_0^t g(t)^2 dt\end{aligned}$$

Energy inequality

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- Existence

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The energy inequality implies:

- Uniqueness
- Existence
- Stability

Energy Inequality

Let U be a vector valued function in a region Ω satisfying

$$U_t = A^i \partial_i U + BU$$

With A a matrix which is symmetric with respect to some scalar product $< \cdot, \cdot >$ and with all its eigenvalues real, then the initial-boundary value problem:

$$U(t=0,x) = F(x), \qquad P^+(U(t,1) - G(t)) = 0$$

Is well posed with respect to the energy:

$$\mathcal{E} := \int_\Omega < U, U > dV$$

Where P^+ is the projector to the positive eigenvalue space of $A^i n_i$, with n_i the normal to the boundary.

Energy Inequality

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_\Omega \{< U, A^i \partial_i U > + < A^i \partial_i U, U >\} dV \\
&= \int_\Omega \{\partial_i < U, A^i U > - < U, (\partial_i A^i) U >\} dV \quad \text{Symmetry + Leibniz} \\
&\leq \int_{\partial\Omega} < U.A^i n_i U > dS + ||(\partial_i A^i)||_\infty \mathcal{E} \quad \text{integration by parts} \\
&\leq \int_{\partial\Omega} < P^+, A^i n_i P^+ U > + ||(\partial_i A^i)||_\infty \mathcal{E}
\end{aligned}
$$

Energy Inequality

Energy inequality is based on:

1. Symmetry of A w.r.t. a scalar product (Symmetric hyperbolicity)
2. Integration by parts
3. Leibniz's rule
4. Boundary condition implementation

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Want to imitate this numerically!

Numerical Implementation I

- Choose a symmetric hyperbolic formulation, or at least a strongly hyperbolic one.
- Choose correct boundary conditions.

Numerical Implementation: SBP

- The finite-difference operator has to satisfy the Summation By Parts property:

$$\sum_{i=0}^{N} \sigma_i \{ < V_i, DV_i > + < DV_i, V_i > \} = < V_N, V_N > - < V_0, V_0 > .$$

Numerical Implementation: SBP

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- This property depends on the integration region and on the order of accuracy of the operator.
- The operator D and the scalar product (given by the values of σ_i) are tied together.
- Pairs (Σ, D) are known for square grids in any dimension. [Strand]
- Are also known for square and cubic excision regions. [Gustaffsson, Kreiss, Oliger], [C-LSU]

Numerical Implementation: SBP

Example: in $[0,1]$

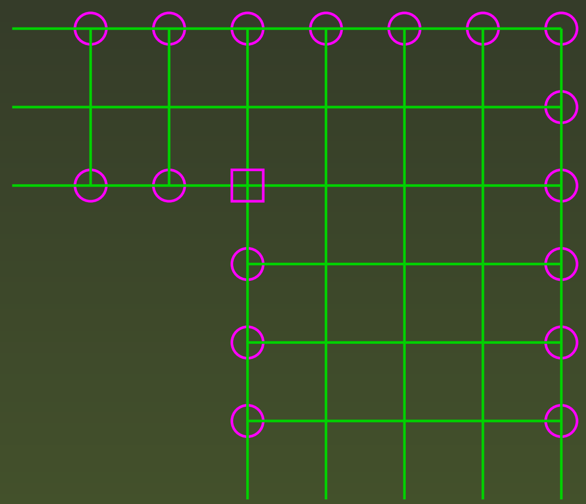
$$(Dv)_i = \begin{cases} \frac{v_{i+1}-v_{i-1}}{2\Delta x} & i \neq 0, N \\ \frac{v_N - v_{N-1}}{\Delta x} & i = N \\ \frac{v_1 - v_0}{\Delta x} & i = 0 \end{cases}$$

$$\sigma_i = \begin{cases} 1 & i \neq 0, N \\ \frac{1}{2} & i = 0, N \end{cases}$$

Second order accurate in the interior, first order at boundary points. Overall second order accurate.

Numerical Implementation: SBP

Example: cubic excision



Numerical Implementation: Leibniz

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- Remedy 1: Choose the discretization in a careful way so that Leibniz's rule is not used in the discrete energy inequality.

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi) \to \frac{1}{\sqrt{g}}D_\mu(\sqrt{g}g^{\mu\nu}D_\nu\Phi)$$

$$A^i\partial_i U \quad \to \quad \frac{1}{2}(A^i D_i U + D_i(A^i U) - (D_i A^i)U)$$

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- Remedy 2: Use artificial dissipation (Kreiss-Oliger). They must also satisfy the corresponding SBP property.

Numerical Implementation: Boundary Conditions

$$
\begin{aligned}
U_t &= A^i \partial_i U \\
P^+ U &= G(t)
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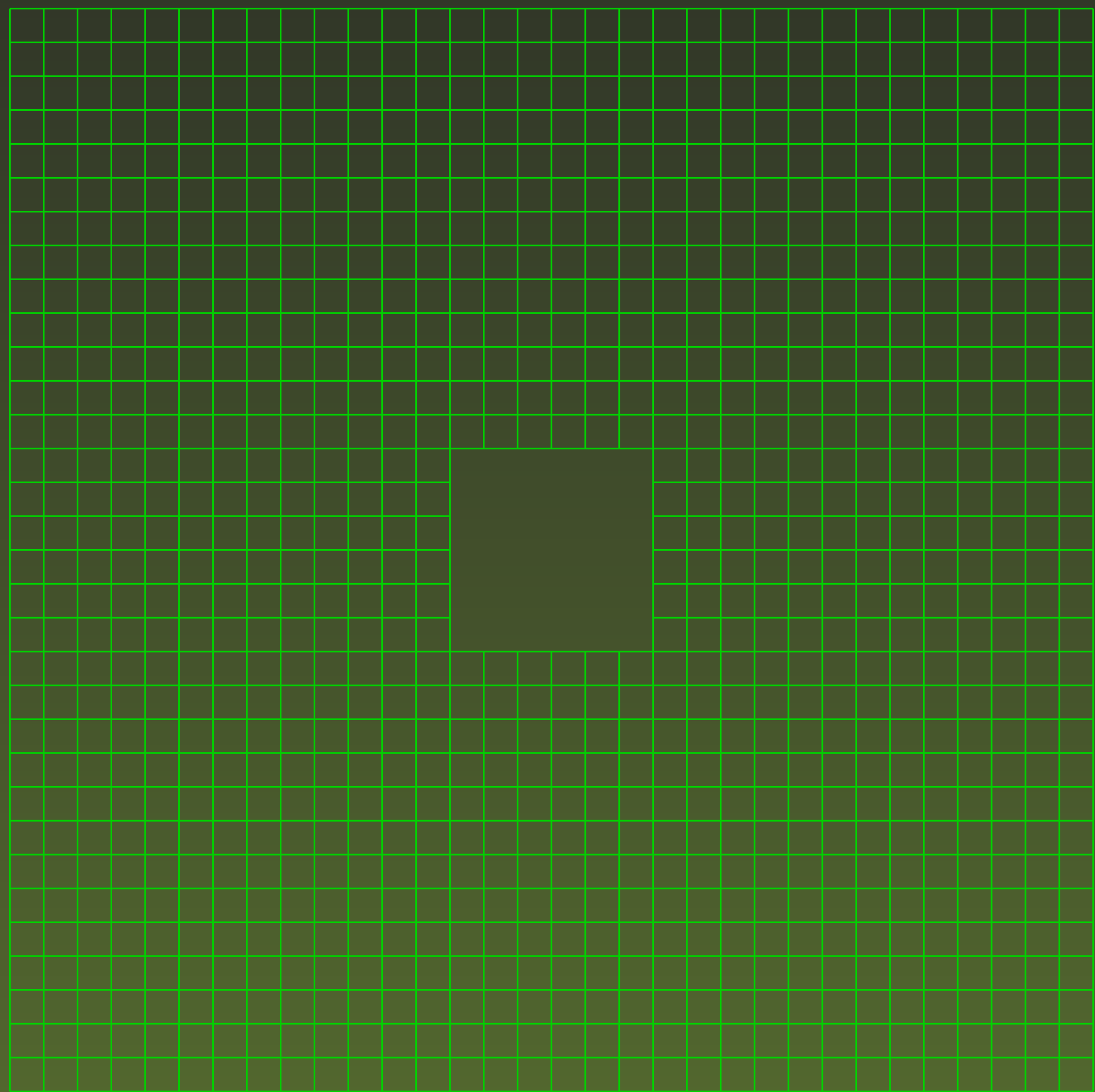
- Olsson (Projection method):

$$V_t = (I - P^+) A^i D_i V + P^+ G_t$$

- Carpenter et. al. (Penalty function method):

$$V_t = A^i D_i V - \frac{2\lambda}{\Delta x} P^+ (V - G)$$

Numerical Implementation: Excision



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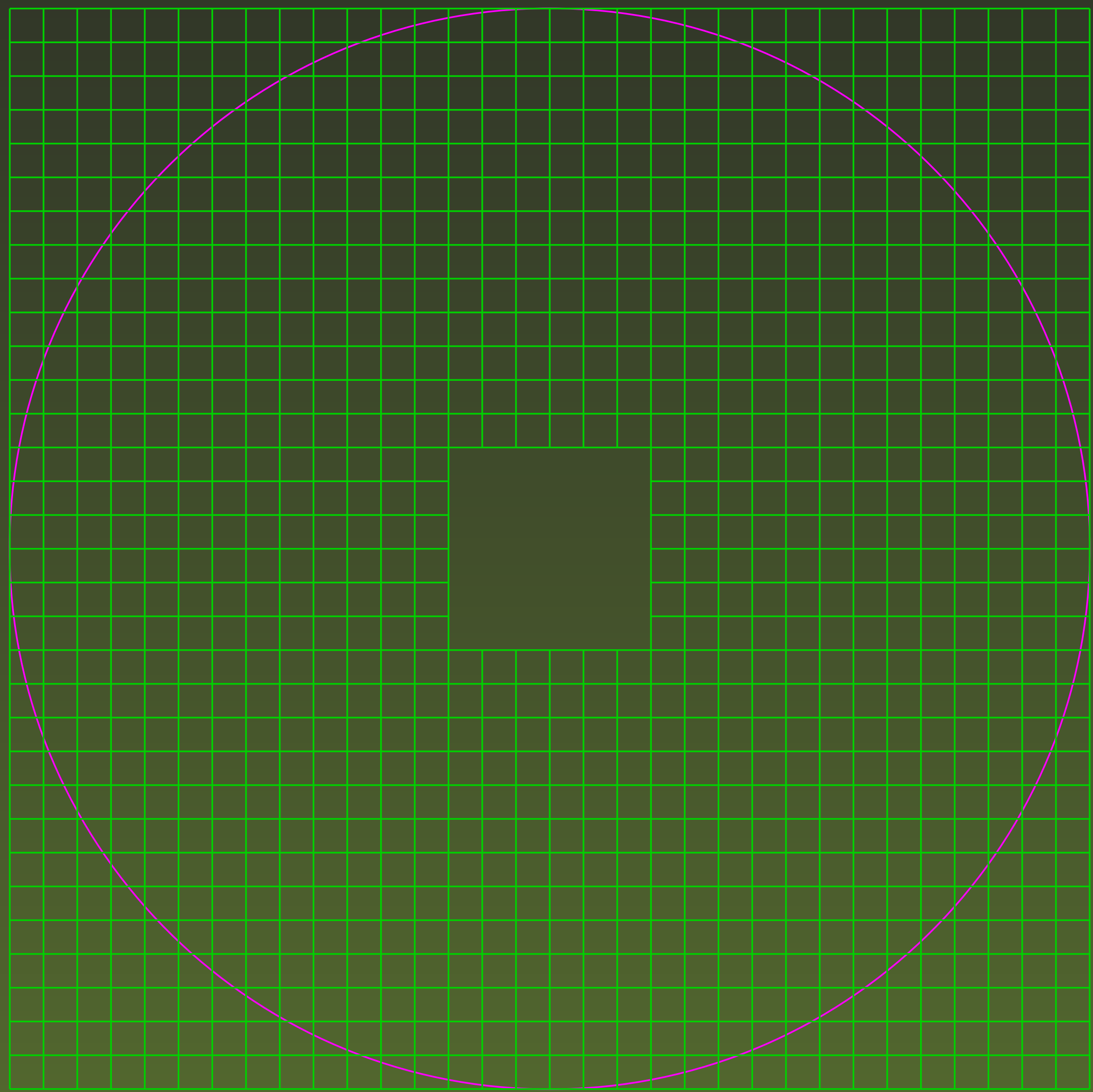

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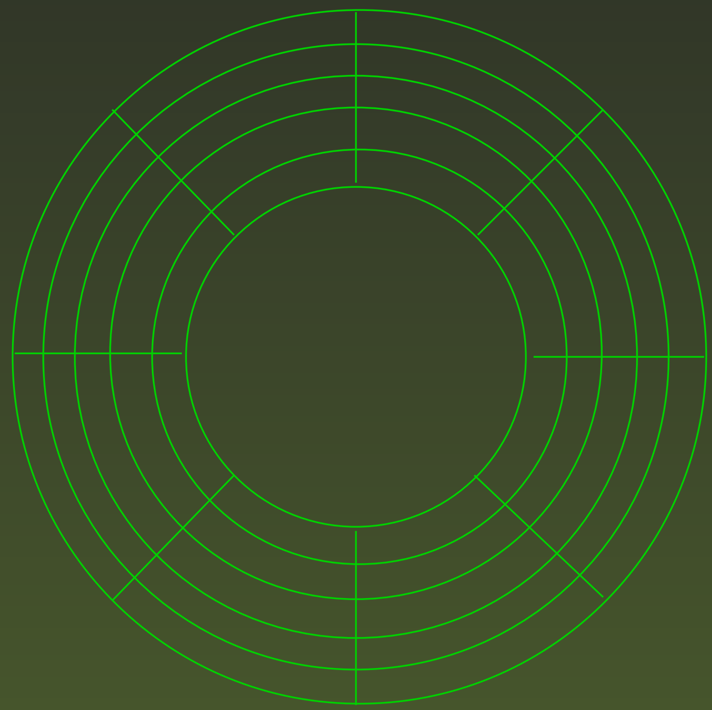

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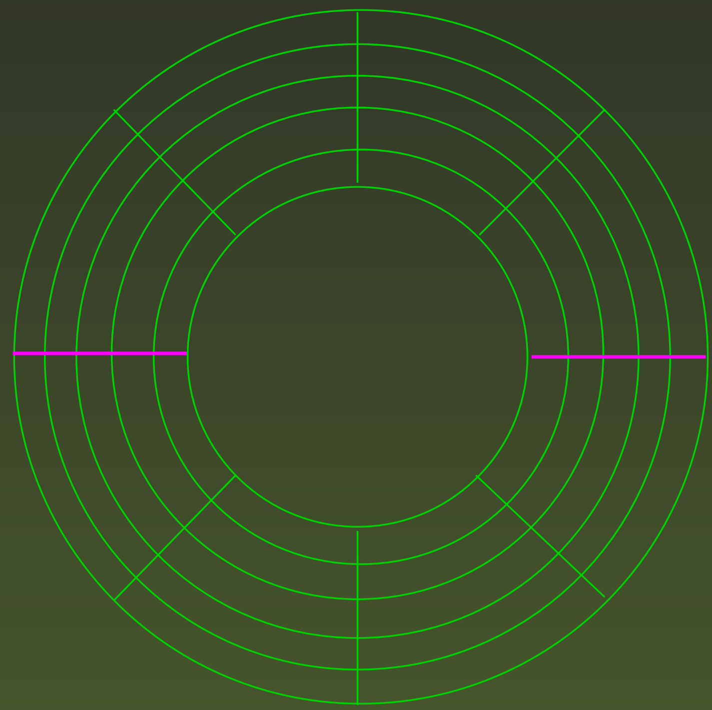

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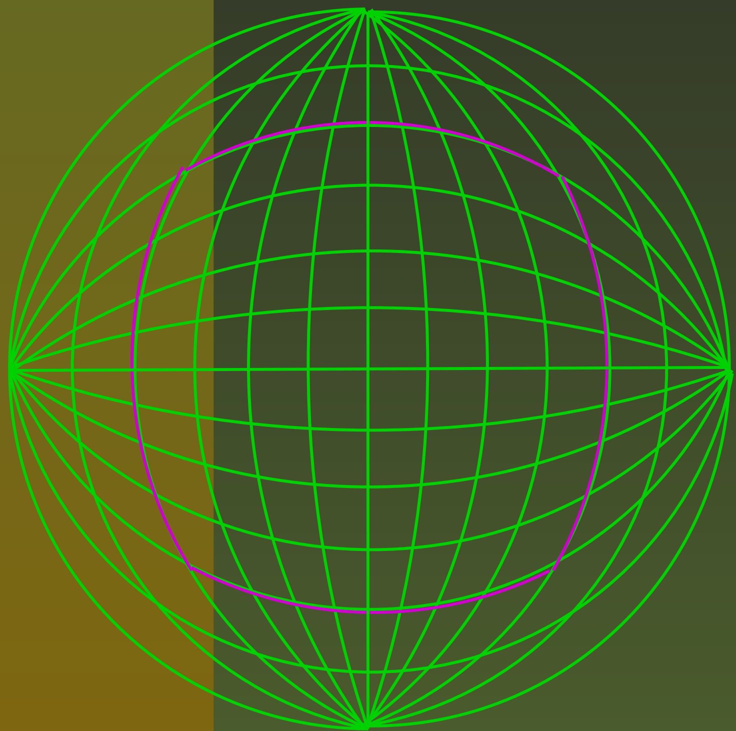
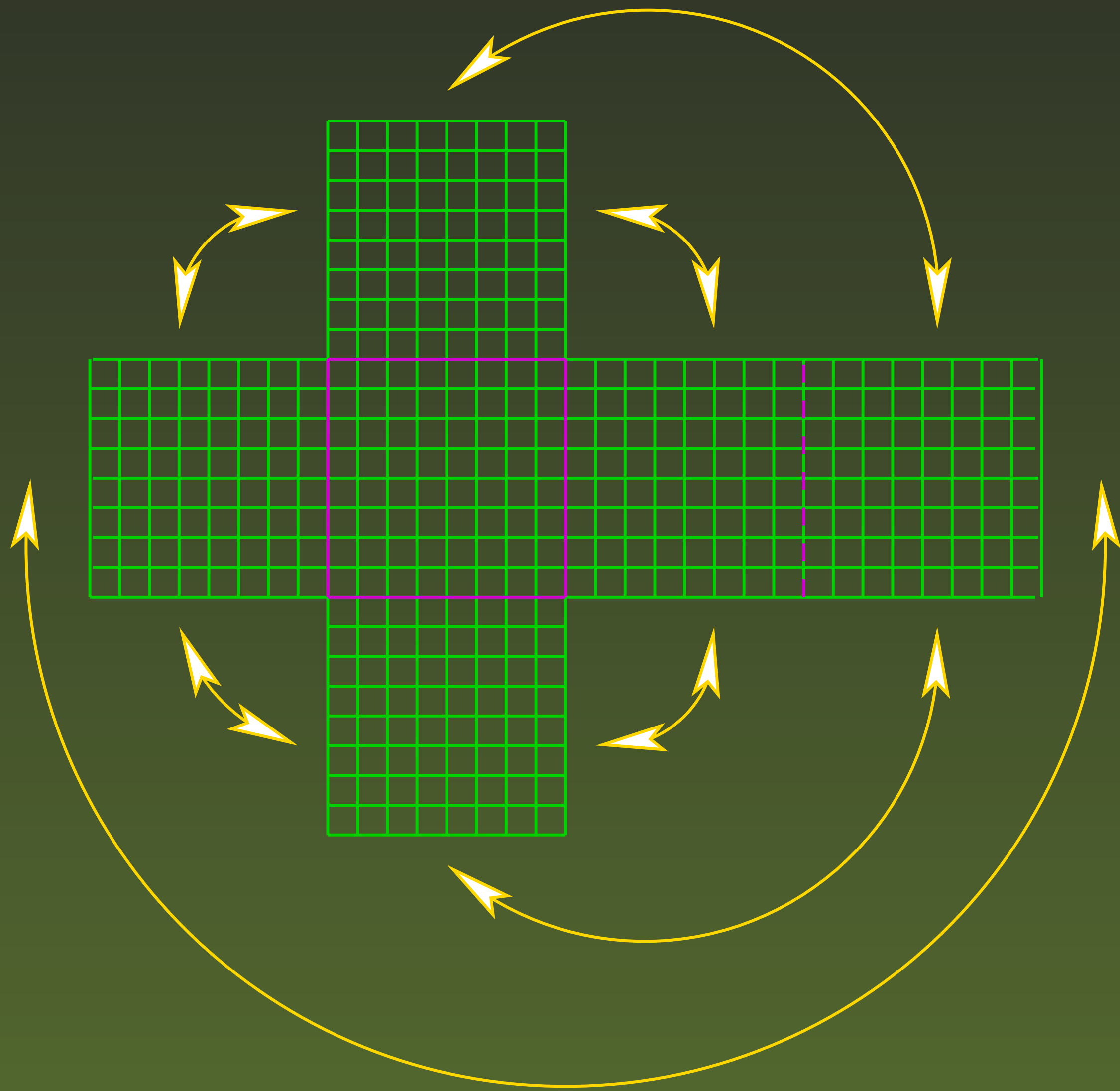
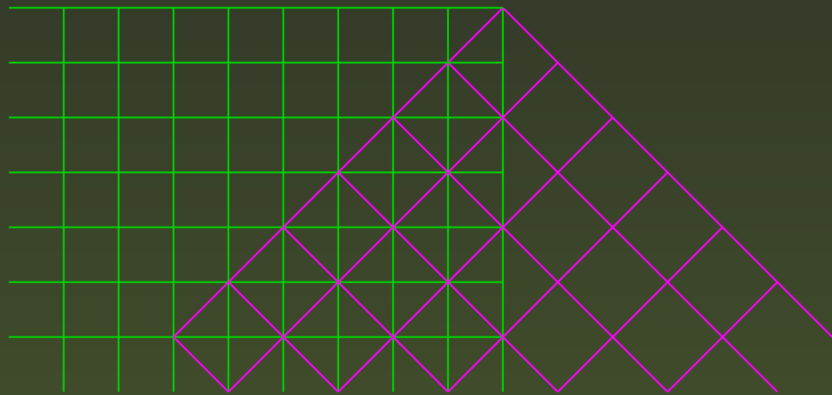


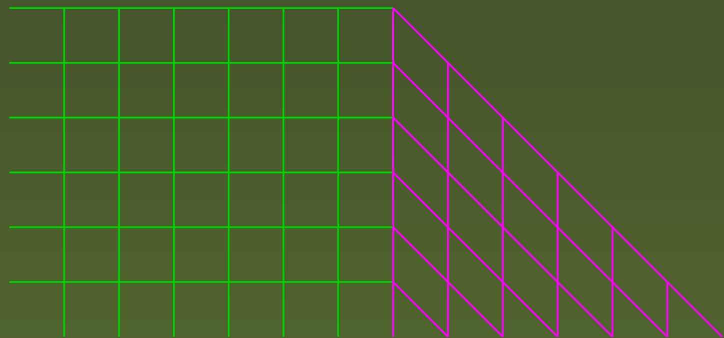


Multi-grids

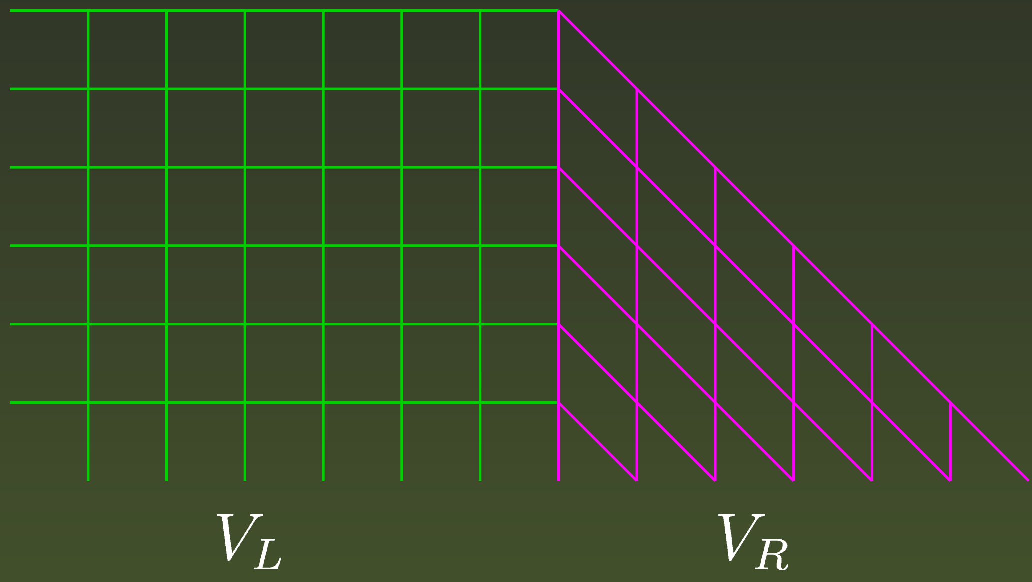
- Overlapping Grids



- Touching Grids



Touching Grids



- Carpenter et. al. penalty method:

$$
\begin{aligned}
\partial_t V_L &= A^i D_i V_L - \frac{2\lambda}{\Delta x} P_R^{+}(V_L - V_R) \\
\partial_t V_R &= A^i D_i V_R - \frac{2\lambda}{\Delta x} P_L^{+}(V_R - V_L)
\end{aligned}
$$

Summary

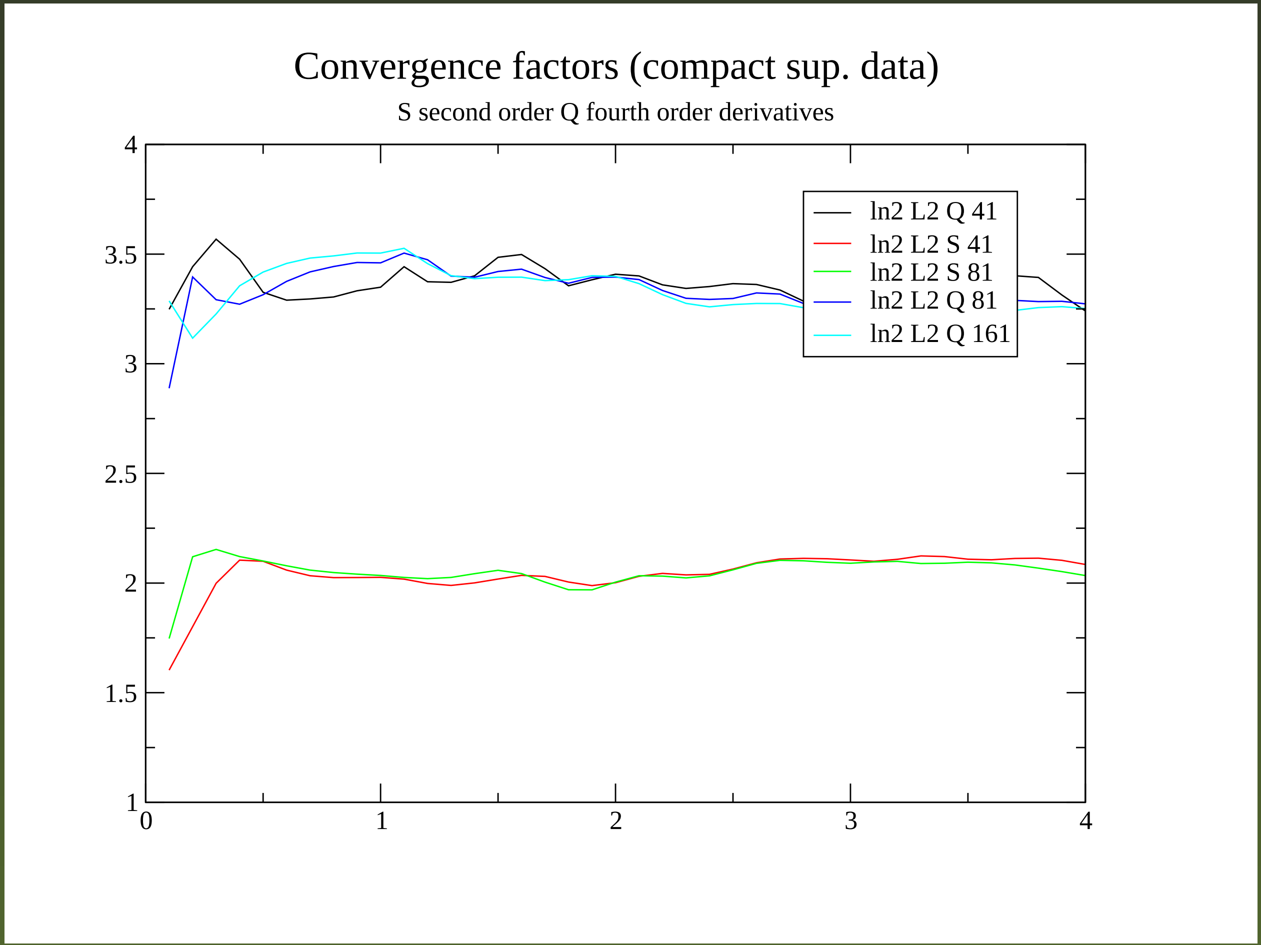
- Stress importance of symmetric hyperbolic systems
- Finite-difference operators satisfying SBP
- Stress strict-stability (do not use Leibniz)
- Dissipative operators (Kreiss-Oliger)
- Boundary treatment: Projection or Penalty
- Interface treatment: Penalty in multi-grids
- Runge-Kutta third or fourth order $\rightarrow$ stability

Example 1

- Cubic excision on a Kerr-Schild Static Black Hole
- No dissipation (Except for Maxwell).
- strict stability (w.r.t. physical energy)
- Olsson Projection at outer boundary
- Excised cube has to be VERY SMALL
- Not possible for rotating BH

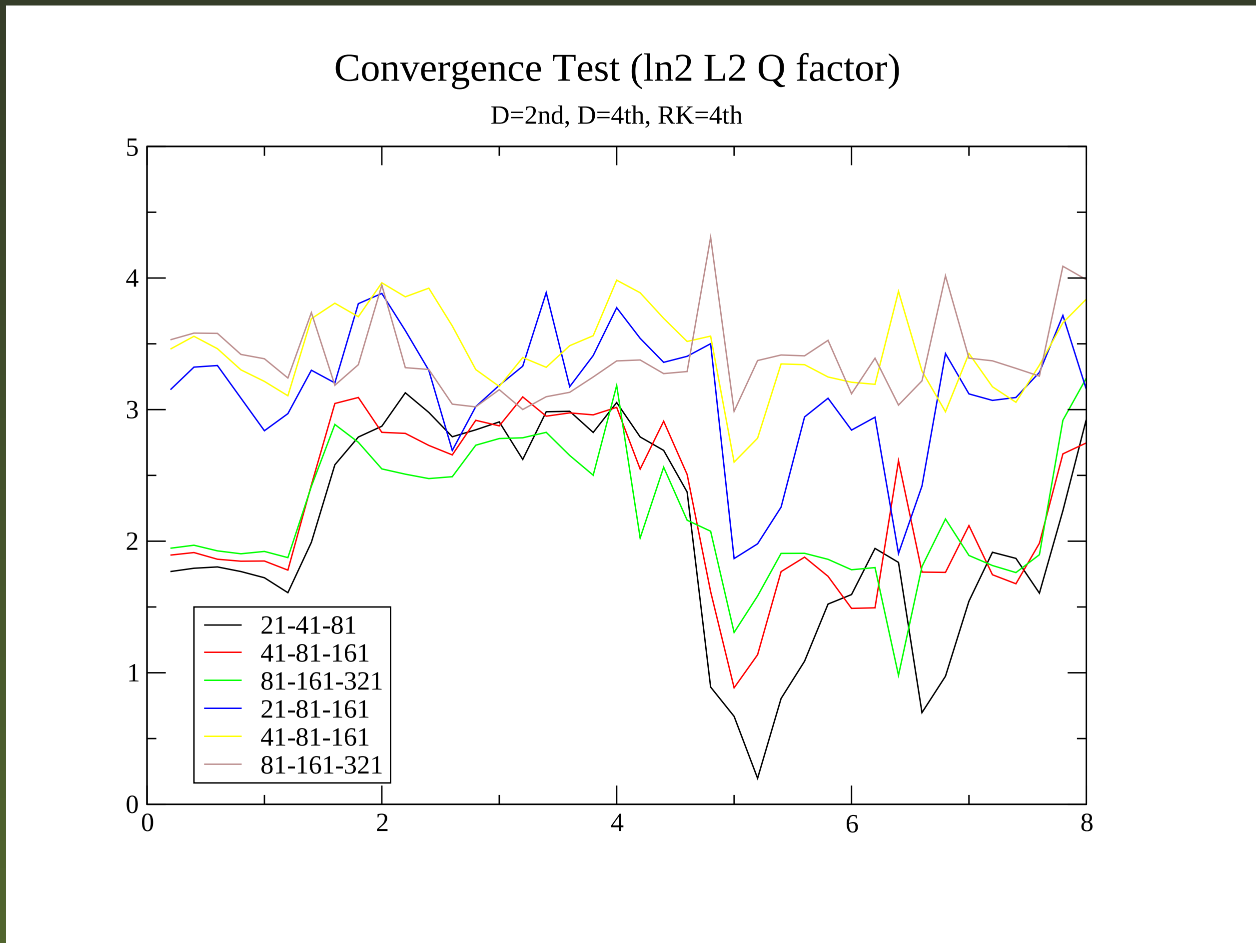
Example 2

- Wave equation on an annulus around a “2D Kerr-Schild Black Hole”
  - 2 touching grids
  - no-dissipation
  - strict stability
  - penalty method



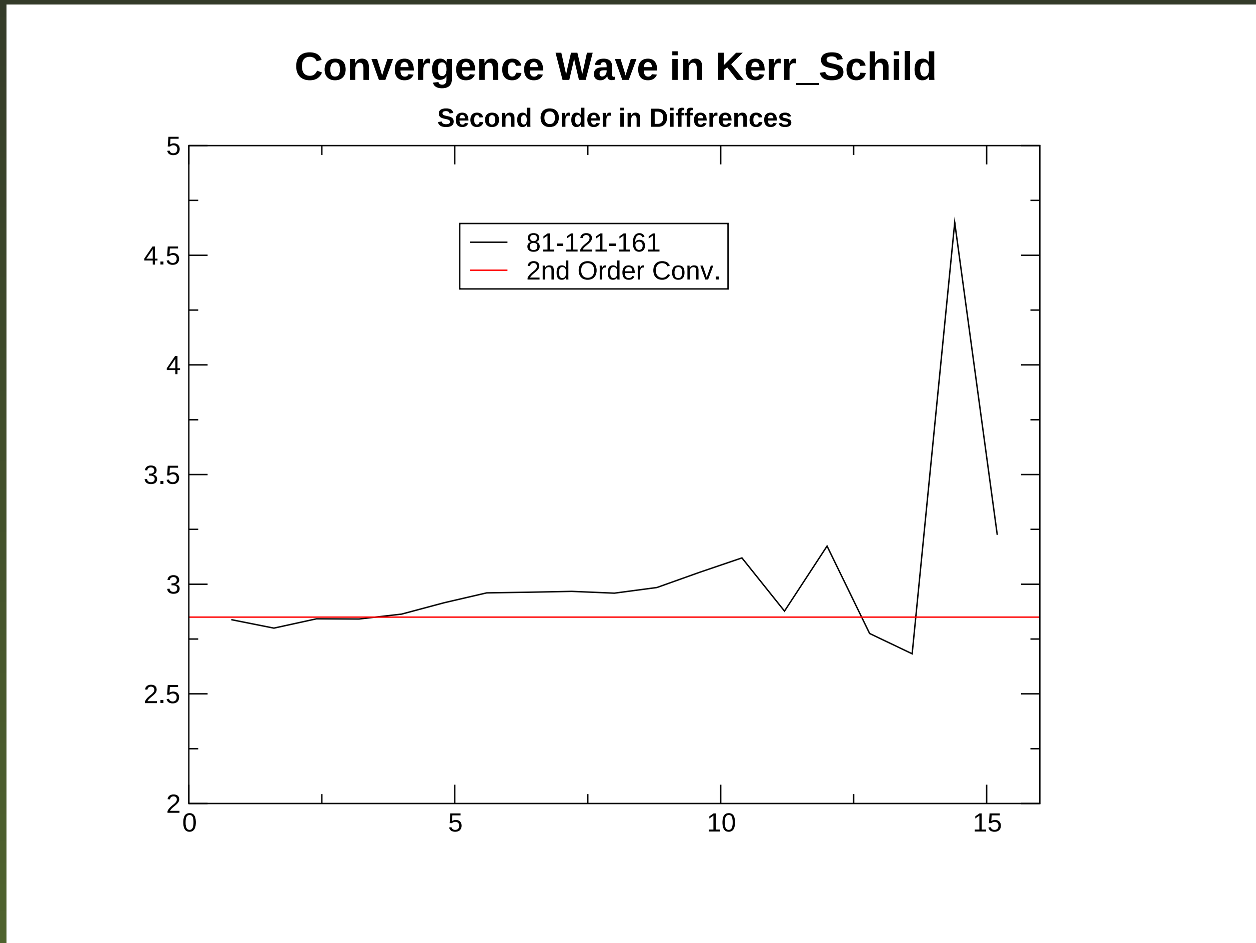
Example 3

- Wave equation on a sphere
  - 6 touching grids
  - no-dissipation
  - strict stability
  - penalty method



Example 4

- Wave equation on a 3D Kerr-Schild Black Hole
  - 6 touching grids
  - no-dissipation
  - strict stability
  - penalty method



TO THE MOVIES!