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Two-Dimensional 'Most Probable' States

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DECAYING TWO-DIMENSIONAL TURBULENCE — THE STATISTICAL MECHANICS OF "PATCHES" AND "POINTS"

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Background: Onsager (1949)

Joyce + Montgomery (1973)
Montgomery + Joyce (1974)
Pointin + Lundgren (1976)
Ting et al (1987)
Smith (1991)
Robert + Sommeria (1991), (1992)
Matthaeus et al (1991), (1991a)
Montgomery et al (1992), (1993)
Kuvshinov + Schep (2000)

2D NAVIER – STOKES TURBULENCE

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\nabla\left(\frac{p}{\rho}\right) + \nu \nabla^2 \underset{\sim}{v}$$

$$\underset{\sim}{v} = (v_x, v_y, 0), \qquad \frac{\partial}{\partial z} = 0$$

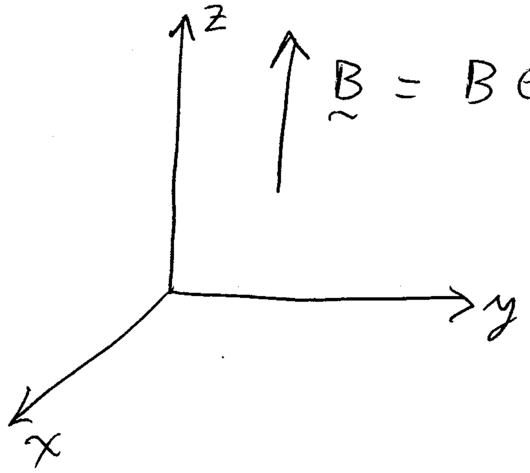
$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v} = (0, 0, \omega)$$

$$\frac{\partial \omega}{\partial t} + \underset{\sim}{v} \cdot \nabla \omega = \nu \nabla^2 \omega$$

$$\underset{\sim}{v} = \nabla \times \hat{e}_z \psi$$

$$\nabla^2 \psi = -\omega$$

GUIDING-CENTER PLASMA ANALOGUE



$$\underset{\sim}{B} = B\hat{e}_z \quad , \quad \frac{\partial}{\partial z} = 0$$

$$\underset{\sim}{v} = \frac{c\,\underset{\sim}{E} \times \underset{\sim}{B}}{B^2}$$

$$\underset{\sim}{E} = -\nabla \Phi = (E_x, E_y, 0)$$

$$\underset{\sim}{v} = (v_x, v_y, 0)$$

$$\nabla^2 \Phi = -4\pi\rho = -4\pi \frac{e}{\ell} (n_i - n_e)$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla\right) n_{e,i} = 0$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla\right) \rho = D \nabla^2 \rho$$

HYPOTHESIS (TURBULENCE WILL DECAY TO A MAXIMUM-ENTROPY STATE, SUITABLY DEFINED)

$$S = \ln W$$

$$W\,(\text{Boltzmann}) = \prod_j \prod_i \frac{(N^j)!}{(N_i^j!)} \quad (\text{"points"})$$

$$W\,(\text{Lynden-Bell}) = \prod_j \prod_i \frac{(N^j)!}{(N_i^j!)\left(M_i - \sum_{l=1}^{q} N_i^l\right)!}$$

"patches"

N^j = total no. of vortices of type j

N_i^j = total no. of vortices of type j in cell i

M_i = no. of possible "patch" sites in cell i

q = no. of "levels" or types of patch

MAXIMIZE "S" SUBJECT TO CONSTRAINTS, USING LAGRANGE MULTIPLIERS. USE STIRLING'S APPROX.

⇒ E = TOTAL ENERGY ≅ CONST.

TOTAL NO. OF POINTS OR PATCHES OF A GIVEN TYPE = CONST.

⇒ GET: "MOST PROBABLE" SET OF OCCUPATION NUMBERS N_i^j

⇒ PASS TO MEAN-FIELD LIMIT (MANY VORTICES, SMALL CELLS, SMALLER PATCH SIZES, SMALL VORTEX STRENGTHS), TO GET "MOST PROBABLE" CONTINUUM VORTICITY DISTRIBUTIONS.

⇒ STUFF "MOST PROBABLE" VORTICITIES INTO POISSON'S EQUATION AND DEMAND SELF-CONSISTENCY.

RESULT:

FOR POINTS:

$$\nabla^2 \psi = -\omega = -e^{-\alpha_+ - \beta\psi} + e^{-\alpha_- + \beta\psi}$$

(two types)

FOR PATCHES:

$$\nabla^2 \psi = -\omega = -\sum_{j=1}^{q} \frac{M}{\Delta} K_j \frac{e^{\alpha_j - \beta K_j \psi}}{\sum_{\ell=0}^{q} e^{\alpha_\ell - \beta K_\ell \psi}}$$

(q TYPES)

ASSUME +, − SYMMETRY:

$$\boxed{\nabla^2 \psi = 2 e^{\alpha} \sinh \beta\psi,} \qquad \beta < 0$$

(points, two types)

$$\boxed{\nabla^2 \psi = \frac{M}{\Delta} e^{\alpha} \frac{2 \sinh \beta\psi}{1 + e^{\alpha + \beta\psi} + e^{\alpha - \beta\psi}}}$$

(Δ / M = PATCH SIZE, TWO TYPES + VACUUM)

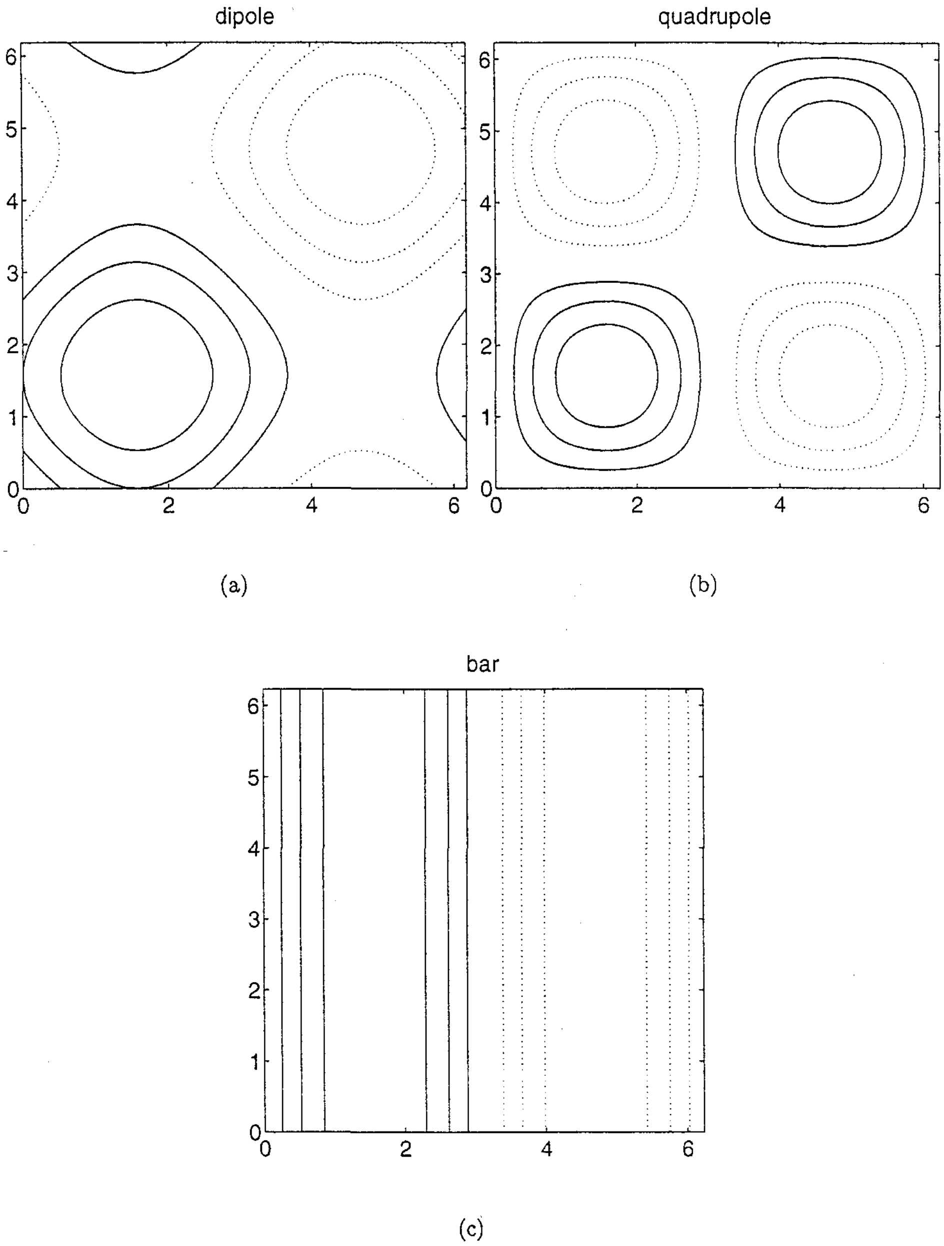


FIG. 2: Contours of constant stream function ψ for the solutions of Eq. (5). Negative values are shown as broken lines, throughout.

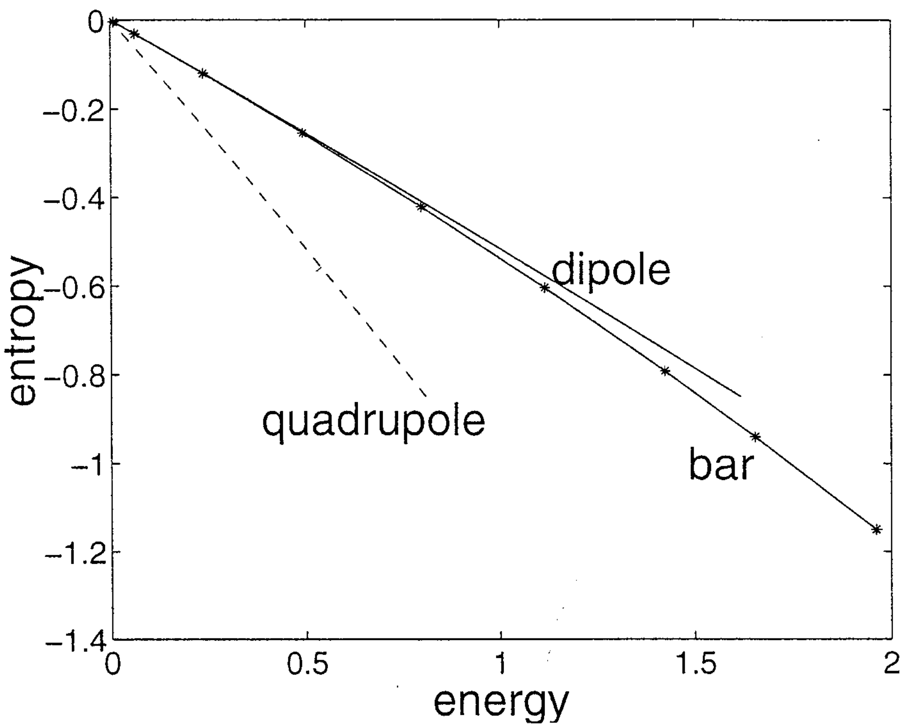


FIG. 3: Entropy vs. energy for the solutions of Eq. (5) for unit positive and negative vorticity flux, computed for the "point" discretization.

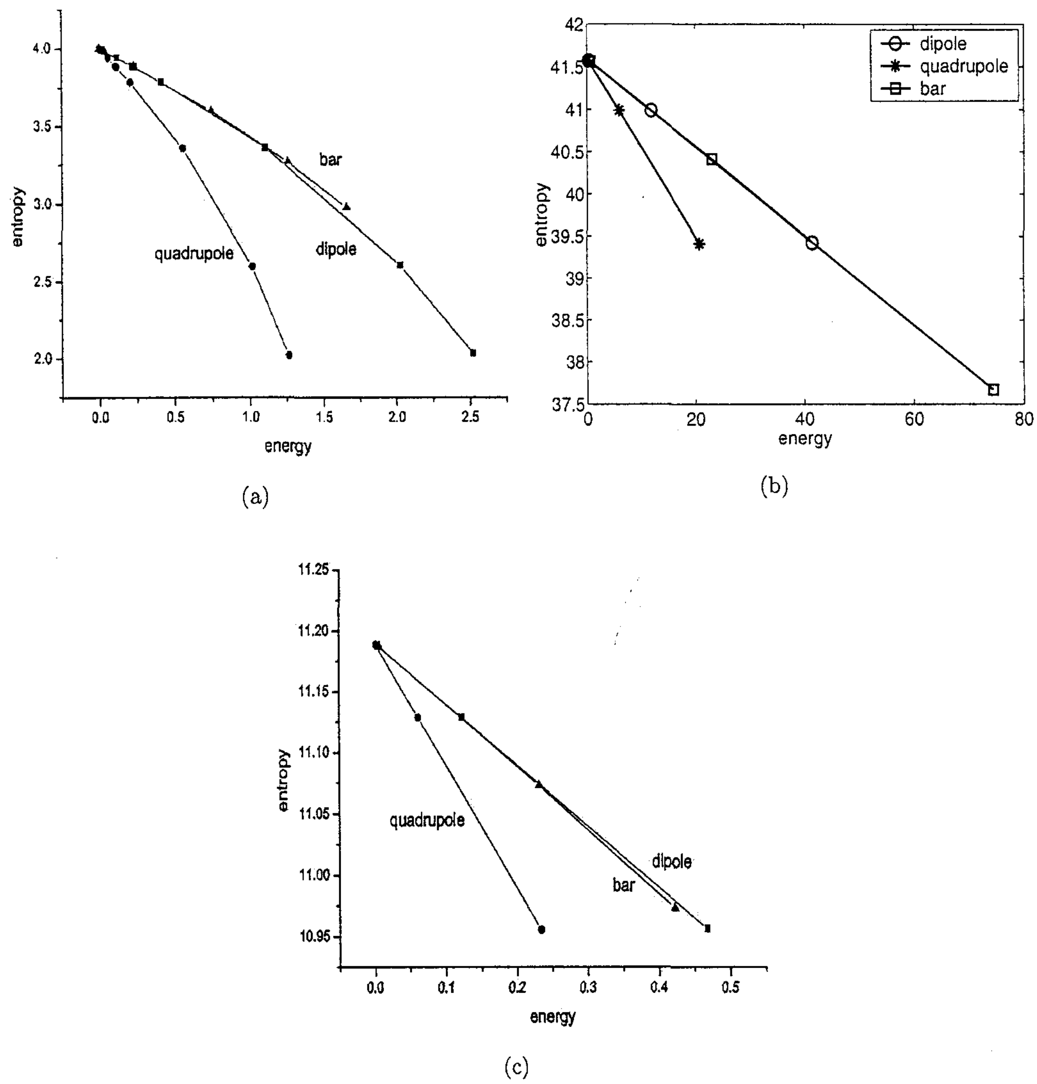


FIG. 4: Entropy vs. energy at unit vorticity fluxes for the solutions of Eq. (8): (a) with a relatively large patch size ($M/\Delta = 3.7814$); (b) with a somewhat smaller patch size ($M/\Delta = 25$); (c) with a still smaller patch size ($M/\Delta = 100$). Note that in (c) the dipole solution has become slightly more probable than the bar.

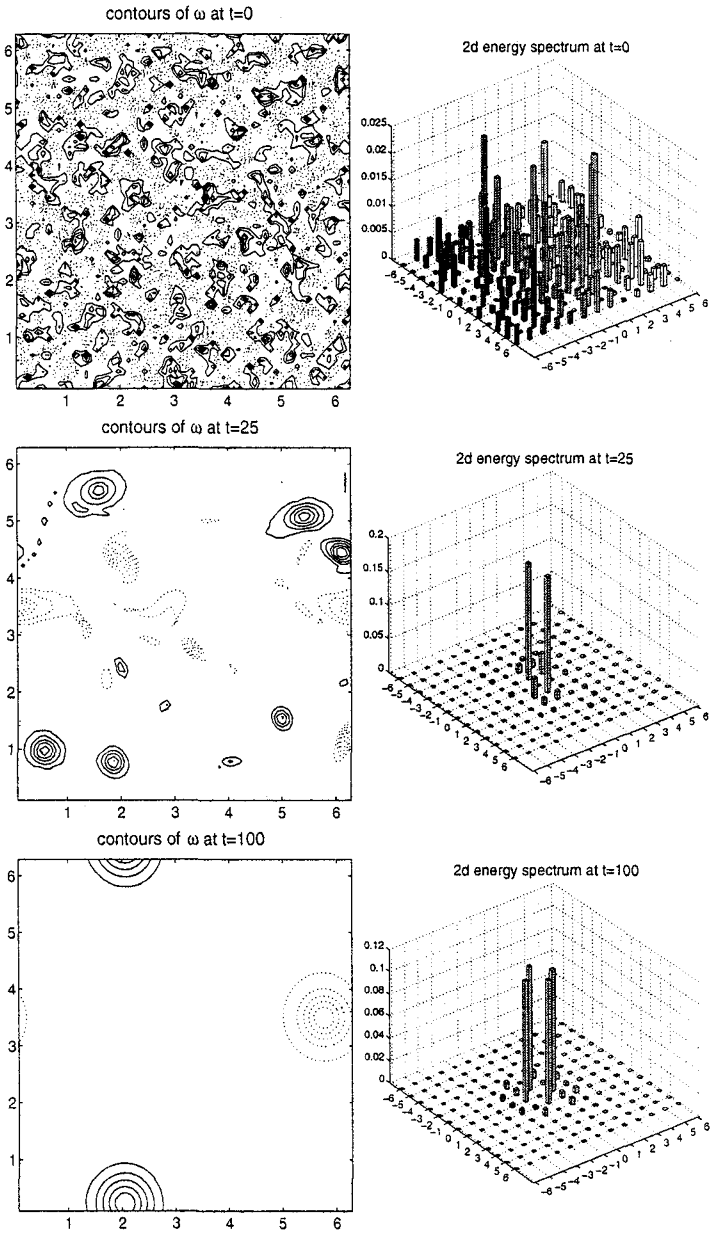


FIG. 5: Equally spaced contours of constant vorticity at three different times (left column) and corresponding modal energies at the lower values of k (right column), during the evolution of the McWilliams/Matthaeus initial conditions. These have no flat patches of vorticity initially, even approximately.

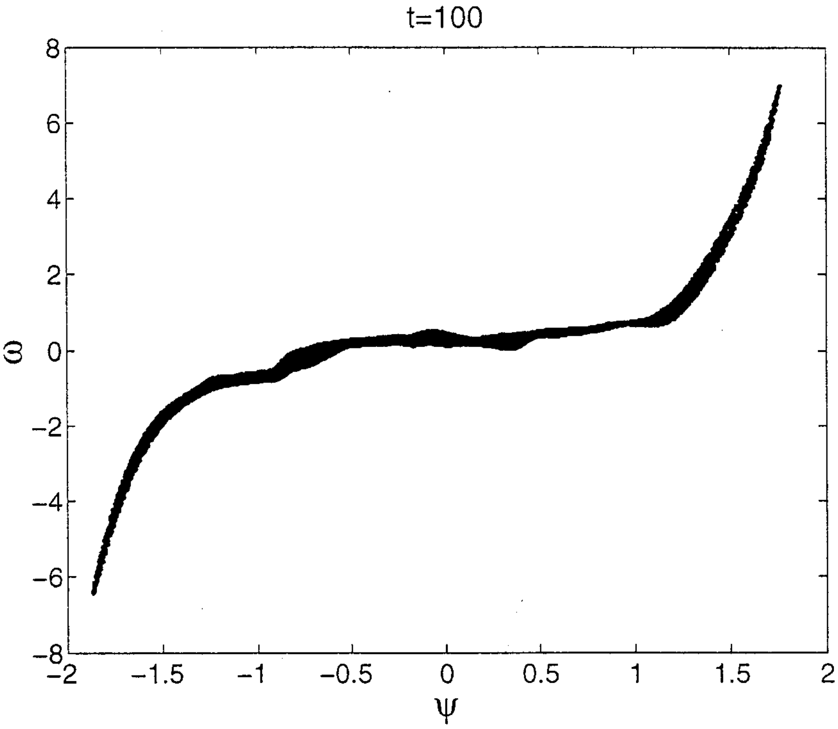


FIG. 6: The $\omega - \psi$ scatter plot for the run shown in Fig. 5 (which is close to Fig. 1(a)).

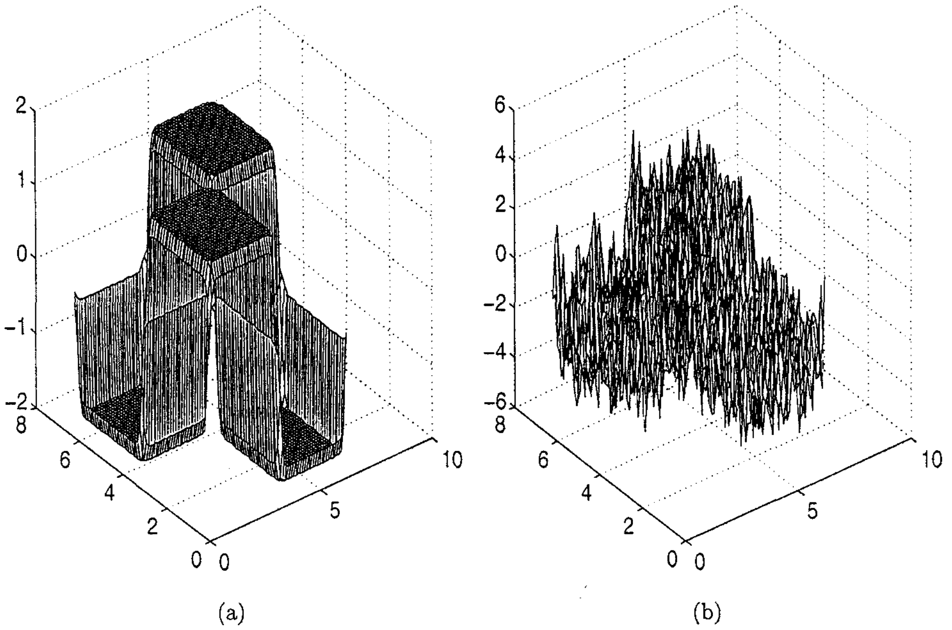


FIG. 7: The initial vorticity field as a function of x and y for a run intended to exhibit patch characteristics. (a) without random noise, (b) has had substantial random noise added to the vorticity field.

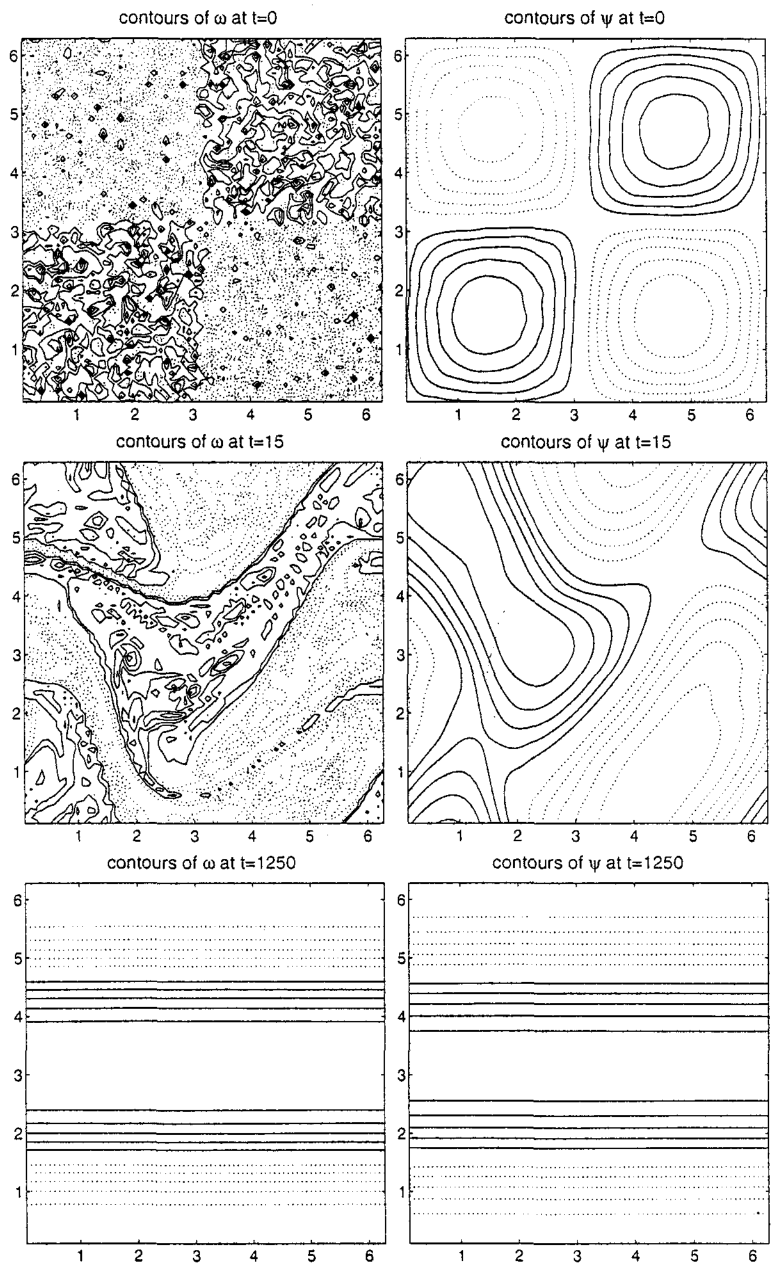


FIG. 8: Contours of constant vorticity (left column) and constant stream function (right column) at three different times for the run originating from the initial vorticity distribution (with noise) shown in Fig. 7(b).

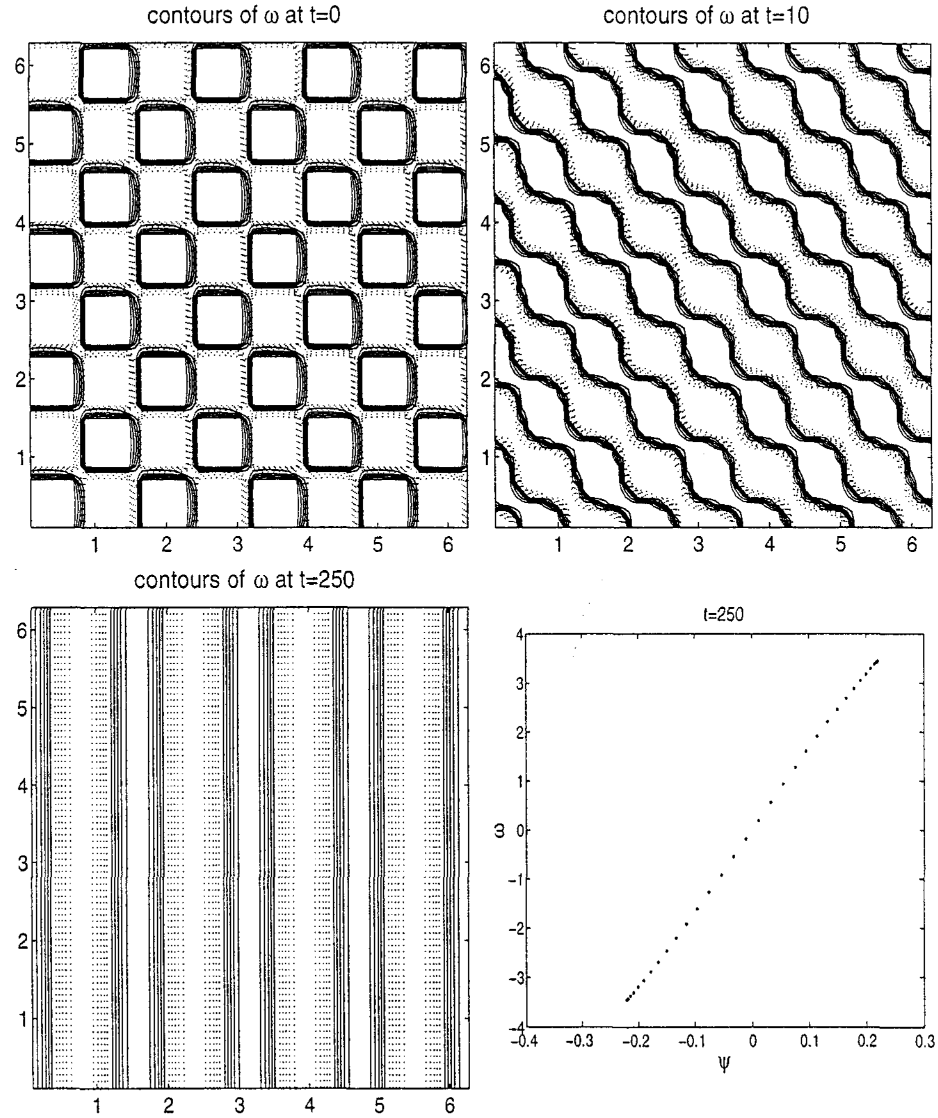


FIG. 13: Time evolution of a 64-pole initial condition, triggered only by round-off error, into a bar state (the scatter plot of final state is close to Fig. 1(c)). No noise beyond round-off error has been added to the initial condition.

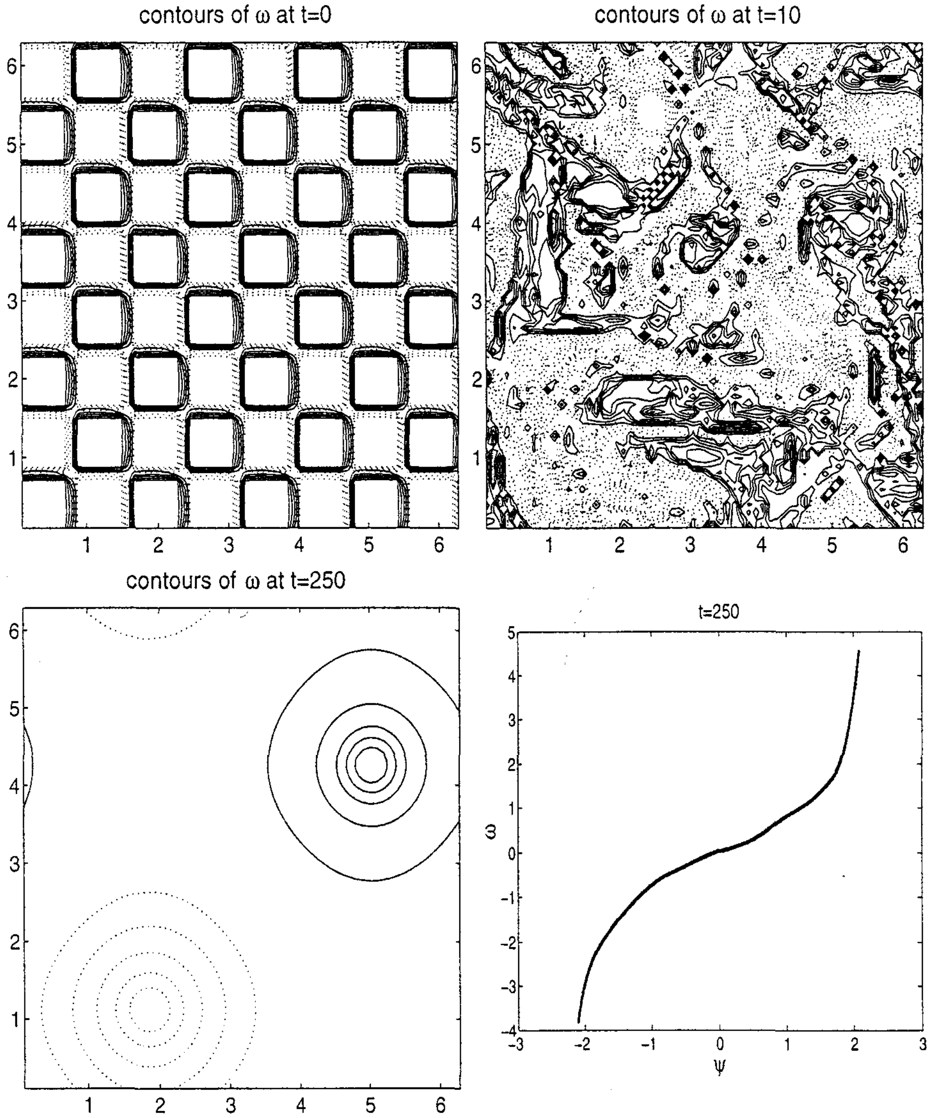


FIG. 14: Time evolution of the vorticity contours, starting from the same initial condition as in Fig. 13, but with a healthy addition of random noise (In this run, R_λ increases from 2036 initially to 11400 at the end.). The last panel is the late-time $\omega - \psi$ scatter plot for the resulting dipole (which is close to Fig. 1(a)).

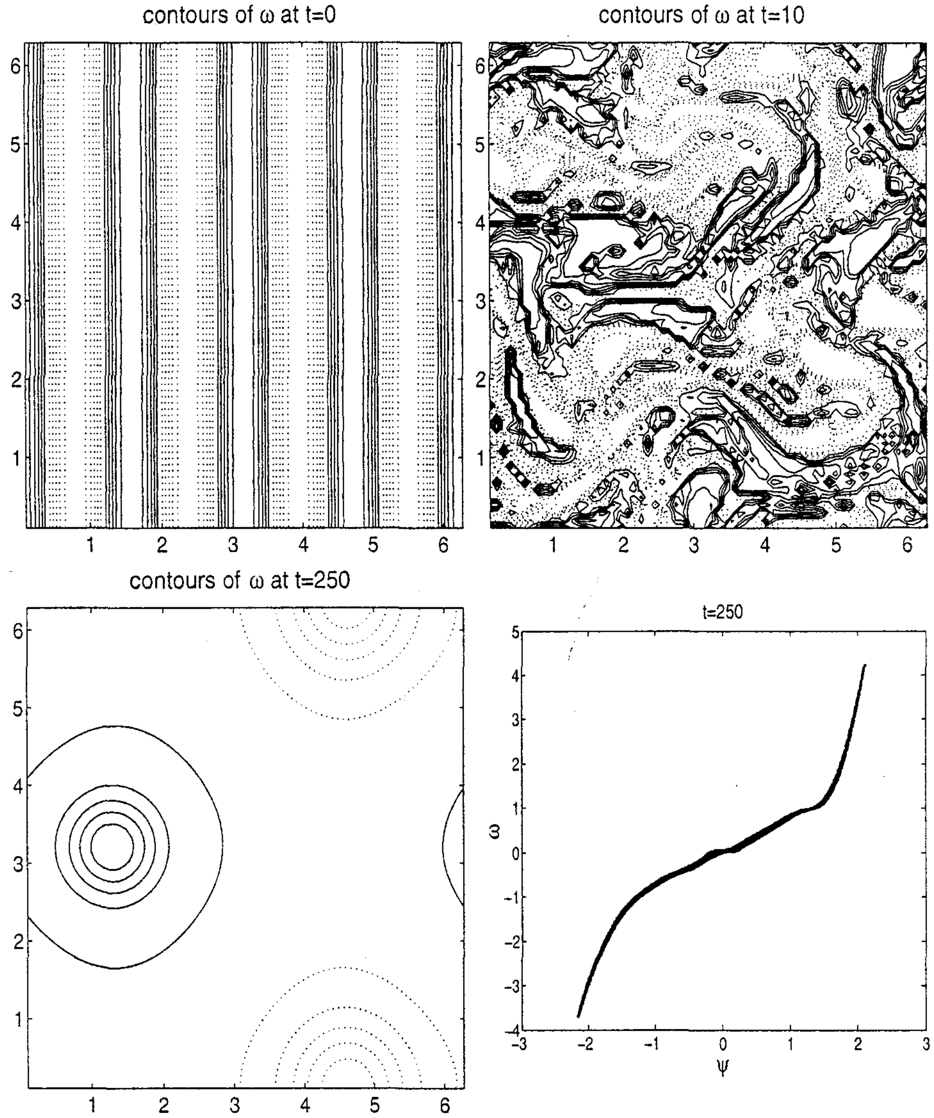


FIG. 15: Evolution of the one-dimensional "8-bar" initial condition, with random noise added initially (In this run, R_λ increases from 3228 initially to 11500 at the end.). The evolution is toward a dipole, now, as seen in the bottom panel (which is close to Fig. 1(a)).

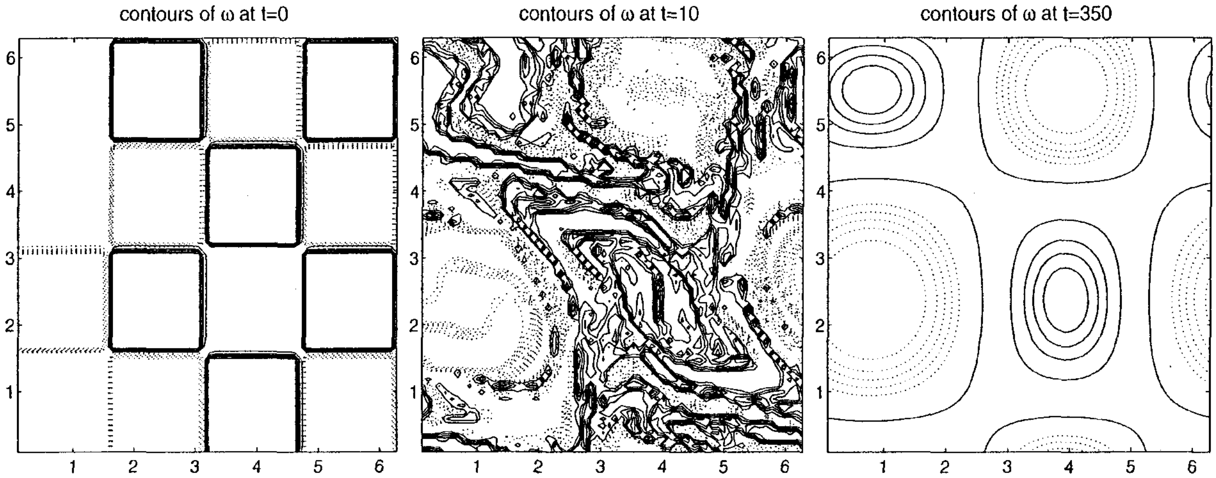


FIG. 21: A second run, whose late-time state is outside the patch/point classifications, that seems to have reached some metastable late-time state. Here, a large area of zero vorticity was created in the initial conditions by removing, asymmetrically, four pieces of a 16-pole patch solution.

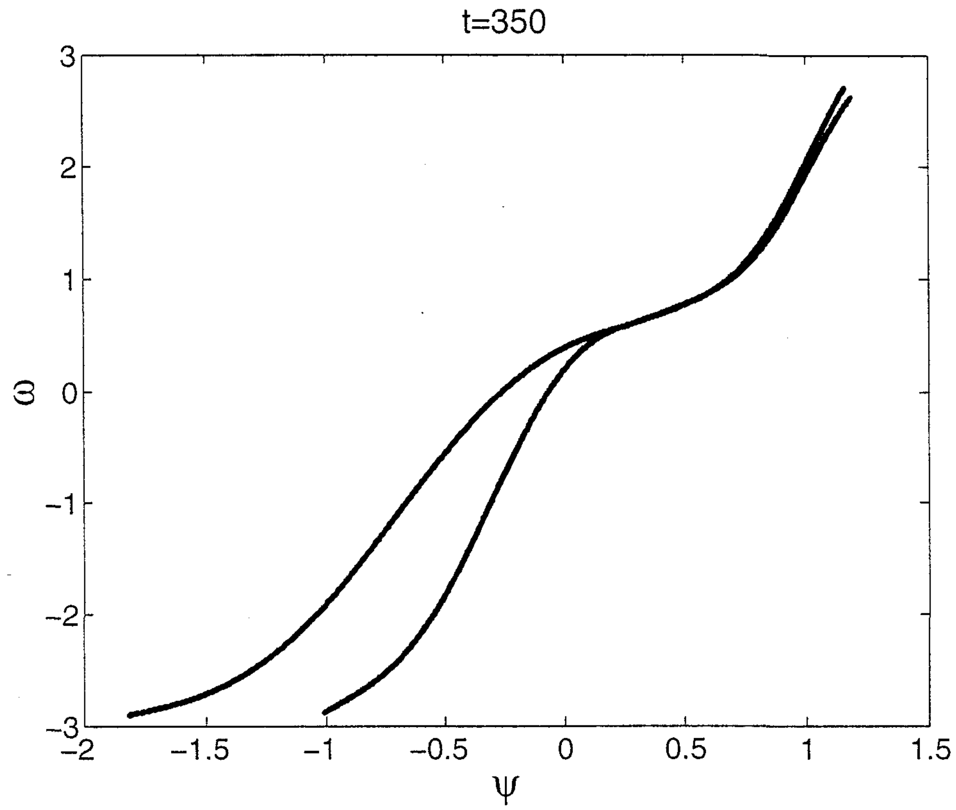


FIG. 22: The $\omega - \psi$ scatter plot of the late-time state achieved in Figs. 21, suggesting two independent co-existing sinh-Poisson states which have developed asymmetrically. This is for the evolution shown in Figs. 21.

CONCLUSIONS and PARADOXES

(1) The "point" version of the theory appears to give correct predictions most of the time.

(2) There are cases in which the "patch" prediction seems to be more accurate, though in some of them, random noise in sufficient quantity can bump them towards "point" final states.

(3) Why the "most probable" state should change as a consequence of patch size is not known.

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(4) There are MHD turbulent decay statistical mechanics waiting to be worked out, both in 2D and 3D. All make extraordinary numerical demands.