Singular Perturbation and Scale Hierarchy in Plasma Flows

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Singular perturbations on nonlinear systems, by providing a robust mechanism for creating patterns at new characteristic scales, could be the underlying cause of physical phenomena whose description requires an interacting hierarchy of scales. This attribute is the defining label for "complexity". By investigating the passage from a single fluid magnetohydrodynamics to a two-fluid plasma model, we delineate a possible theoretical route to complexity.

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I. INTRODUCTION

Simultaneous existence, and interaction of structures on disparate scales are the defining characteristic of complexity; the scale hierarchy is likely to prevent the system from being analyzed in terms of noninteracting independent elements.

Interactions of different scales are created by "nonlinearity". We can see the role of nonlinear terms by Fourier analysis. For example, suppose that dynamical variables $u(x)$ and $v(x)$ appear as a nonlinear term $u \cdot v$ in the governing equation. Then, the Fourier components $\hat{u}(k)e^{ikx}$ and $\hat{v}(k')e^{ik'x}$ of both fields yield

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$\hat{u}(k)\hat{v}(k')e^{i(k+k')x} + \hat{u}(k)\overline{\hat{v}(k')}e^{i(k-k')x}$ in the equation. If the independent variable x represents a coordinate, $1/k$ gives the length scale of variation in the field u. We, thus, see that the nonlinear term produce coupling with other scales $1/(k \pm k')$, which may have disparate scales from k and k'.

Note 1 (non-Hermitian system) *When the variable v is a known function, then $u \cdot v$ is a linear term. If v is not a constant, however, the Fourier transform still produce scale couplings (we call, then, k is a bad quantum number). In a linear system, there is a possibility of using an appropriate set of functions (not necessarily sinusoidal functions) to decompose the fields into non-interacting "modes". This is the notion of "spectral resolution". Due to von Neumann's theorem, a "self-adjoint (Hermitian) operator" can be completely resolved into real spectra. However, general non-Hermitian system cannot assume such resolution. In a sense that infinite mode interactions may occur, a non-Hermitian linear system resembles nonlinear systems.*

It is the "singular perturbation" that puts anchor on unlimited scale conversions induced by nonlinearity. A singular perturbation appears as a term with a higher-order derivatives multiplied by a small "scale parameter" ϵ. From a mathematical view-point, the nature of differential equations is determined by the terms that includes the highest-order derivatives. However, in standard understanding of physicists, terms multiplied with a small coefficient may be negligible in "normal situation". Then, its contribution may be just perturbation (minor).

A singular perturbation has an intrinsic scale ℓ. The ϵ is called a scale parameter, because it is the ratio of ℓ and L, the latter is the scale of observer's interest. Smallness of $\epsilon = \ell/L$, thus, means that the role of such a term is assumed to be out of the scope of the observer. However, "abnormal situation" occurs, when the system itself highlights the scale ℓ. Such an autonomous selection of scales, obstructing observer's prejudice, is the role of nonlinearity –the observer may no longer ignore the effects stemming from the scale hierarchy of ℓ.

The viscosity term multiplied by the Reynolds number (ratio of the length scale where the viscosity dominates and the L) is the most simple, but profound example of

singular perturbation. The ideal (inviscid, incompressible, constant density) model of flow obeys the Euler equation

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla p. \quad \nabla \cdot \boldsymbol{V} = 0. \qquad (1)$$

Adding a small viscosity term (singular perturbation), we obtain the Navier-Stokes (NS) equation Euler equation

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \epsilon \Delta \boldsymbol{V} - \nabla p. \quad \nabla \cdot \boldsymbol{V} = 0, \qquad (2)$$

where ϵ is a positive number. For the Euler equation (1), we have the conservation of energy; assuming that the flow is confined in a bounded volume Ω, i.e., $\boldsymbol{n} \cdot \boldsymbol{V} = 0$ on the boundary $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal vector onto $\partial\Omega$),

$$E(t) := \|\boldsymbol{V}(t)\|^2 := \int_\Omega |\boldsymbol{V}(\boldsymbol{x}, t)|^2 \, dx = \text{constant.} \qquad (3)$$

On the other hand, for the NS equation (2), the viscosity dissipates the energy; assuming the adherent boundary condition $\boldsymbol{V} = 0$ on $\partial\Omega$), we obtain

$$E(t) = \|\boldsymbol{V}(t)\|^2 = E(0) - \epsilon \int_0^t \|\nabla \times \boldsymbol{V}(t)\|^2 \, dt, \qquad (4)$$

which shows monotone decrease of the energy.

Figure shows the typical history of $E(t)$ in the NS system with various ϵ. It seems that $E(t)$ for $\epsilon \to 0$ will not approach to the conservation law that is predicted in the Euler system. The reason why the effect of the viscosity does not vanish in the limit of $\epsilon \to 0$ is clear –when ϵ is reduced, the size of vortices created in the flow also decrease, resulting in larger $|\nabla \times \boldsymbol{V}|$.

The cooperation of nonlinear effects and singular perturbation creates an autonomous regulation of the scale hierarchy. Self-organization of structures in nonlinear systems must be understood on the axis of disparate scales.

II. SINGULAR PERTURBATION

A. Scale invariance of ideal model

The role of singular perturbation is better highlighted by comparing it with ideal models that have scale-invariant nature.

Let us consider the simplest model of ideal flow in one dimensional space (cf. (1)):

$$\partial_t u + u \partial_x u = 0. \qquad (5)$$

This equation is "scale invariant" in the following sense. Let L and U be (arbitrary) representative values of the length and velocity, respectively. We "normalize" the variables as

$$\hat{x} = x/L, \quad \hat{u} = u/U, \quad \hat{t} = t/(L/U). \qquad (6)$$

In terms of these dimension-less variables, (5) reads

$$\partial_{\hat{t}} \hat{u} + \hat{u} \partial_{\hat{x}} \hat{u} = 0. \qquad (7)$$

Since (7) is a universal form of the governing equation, which is independent to the arbitrary scales L and U, the equation (5) has "scale invariance" with respect to the normalization (6).

The following "singular solutions" are reflecting the scale-less nature of the equation (5). Let us try to find a propagating solution of the form of $u(x, t) = \varphi(x - ct)$. Plugging this expression in (5), we obtain

$$-c\varphi' + \varphi\varphi' = 0 \quad \Leftrightarrow \quad \frac{1}{2}[(\varphi - c)^2]' = 0. \qquad (8)$$

The general solution to (8) must satisfy

$$(\varphi - c)^2 = a \quad \text{(constant)},$$

which allows any number switching between

$$\varphi = c \pm \sqrt{a},$$

causing discontinuities (Figure 1).

Note 2 (Riemann Problem) *Let us consider initial value problem for (5) with a discontinuous initial condition*

$$u(x, 0) = \begin{cases} -1 & (x < 0) \\ +1 & (x > 0). \end{cases} \qquad (9)$$

We immediately find a stationary solution

$$u(x, t) = \begin{cases} -1 & (x < 0) \\ +1 & (x > 0), \end{cases} \qquad (10)$$

which shows that the discontinuity at $x = 0$ can persist. There are other kind of solutions. Because the governing equation (5) and the initial condition (9) are invariant for transformation $x \to \alpha x$ and $t \to \alpha t$, we may expect that a solution in the form of $u(x, t) = f(x/t)$. Indeed, we have a solution such as

$$u(x, t) = \begin{cases} -1 & (x < -t) \\ x/t & (-t \le x \le +t) \\ +1 & (x > +t), \end{cases} \qquad (11)$$

which describes "expansion waves".

Combination of these two types of solutions yields infinite number of solutions: For $t = T \ (> 0)$ we take (10), and then, assuming that $t' = t - T = 0$ is the initial time for the expansion wave (11).

B. Non-ideal effect with intrinsic scale

The scale invariance or scale insensitivity (allowing discontinuity) of some class of equations is destroyed by addition of terms that is multiplied by scaling factors. The

"Reynolds number" scaling the viscous dissipation is the most well-known example of such scaling factors.

In a real system, there is a finite viscosity effect represented by a higher-order derivatives (cf. (2)). With a positive constant ϵ, we consider

$$\partial_t u + u \partial_x u = \epsilon \partial_x^2 u. \qquad (12)$$

Because of the new term on the right-hand side, (12) is no longer scale invariant –a different scale L changes ϵ.

The second-order derivative in (12) inhibits any discontinuity. It is shown that (12) has a smooth unique solution $u_\epsilon(x, t)$ (so-called "viscosity solution").

The function define by the limit

$$u_0(x,t) = \lim_{\epsilon \to 0} u_\epsilon(x,t)$$

satisfies (5). Moreover, it "selects" a physically plausible solution from the set of non-unique solutions to the ideal model (5). This is the so-called "entropy condition" for shock waves; The sustainment of discontinuity in (10) cannot occur if the solution is constructed as the limit of $\lim_{\epsilon \to 0} u_\epsilon(x,t)$.

C. Singular perturbation

Generally the physics controlling the microscopic scale appears as a singular perturbation to the macroscopic equations of motion, i.e, it enters through a term containing higher order derivatives multiplying a small coefficient (ϵ).

Here we make some general remarks. Let us call the purely macroscopic model as the "0-model" and the more comprehensive microscopic model as the "ϵ-model". If a solution f_ϵ of our ϵ-model converges to f_0 that satisfies the 0-model when $\lim \epsilon \to 0$, the singular perturbation has negligible effect on this solution and the 0-model would be quite adequate. The "ϵ-model" may still be useful in weeding out any unphysical f_0 that does not follow as a limit of f_ϵ. The "entropy solution" in the theory of shocks is a well-known example.

The situation changes drastically if f_ϵ is singular (divergent) as $\lim \epsilon \to 0$. Now the 0-model is not aware of important "physical" solutions that can be constructed in the ϵ-model. The ϵ-model may have a far richer physical content; the singular perturbation, then, is the harbinger of new complexity. And we are likely to run into phenomena which defy understanding in a purely macroscopic model.

III. BELTRAMI FIELDS

A. Singular perturbation in Hall MHD

The physical system we choose is the simplest two-fluid magneto-plasma with flows, the HMHD. In its general form, it constitutes our ϵ-model with both the macroscopic and microscopic scales (the latter introduced as a singular perturbation) [8, 17]. For negligible (or a a specific kind of) flows, the system degenerates into standard magnetohydrodynamics (MHD) with a single relevant scale (macroscopic) corresponding to the 0-model in our argument.

Within the framework of this model we will compare and contrast the virtues of two contending mechanisms for the creation of short-scale fields. Both these models, the relative recent double Beltrami relaxation mode [8, 17], and the well-known "current sheet" model of Parker [12], have been invoked to explain the heating of the solar corona. The former relies on the dissipation of the short scale during the relaxation process [7]. It describes a "pattern" generated by the cooperation of the nonlinearity (convective type) and the dispersion (singular perturbation due to the two-fluid effect).

For simplicity, we consider a quasineutral plasma with singly charged ions. Neglecting its small inertia, the electron equation of motion is

$$\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B} + \frac{1}{en}\nabla p_e = 0, \qquad (13)$$

where $\boldsymbol{V}_e$ and p_e are, respectively, the electron flow velocity and pressure, $\boldsymbol{E}$ ($\boldsymbol{B}$) is the electric (magnetic) field, $-e$ is the electron charge, and n is the number density. The ion velocity $\boldsymbol{V}$ obeys

$$\frac{\partial}{\partial t}\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \frac{e}{M}\left(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}\right) - \frac{1}{Mn}\nabla p_i, \quad (14)$$

where M is the ion mass ($M \gg$ electron mass), and p_i is the ion pressure. We can eliminate $\boldsymbol{E}$ and $\boldsymbol{V}_e$ using $\boldsymbol{V}_e = \boldsymbol{V} - \boldsymbol{j}/(en)$, $\boldsymbol{j} = \mu_0^{-1}\nabla \times \boldsymbol{B}$ ($\boldsymbol{j}$ is the electric current), and $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \nabla\phi$, where $\boldsymbol{A}$ (ϕ) is the vector (scalar) potential. We assume barotropic relations to write $n^{-1}\nabla p_j = \nabla\Pi_j$ ($j = e, i$).

Choosing an arbitrary length scale L_0 and representative magnetic field B_0 and density n_0, we normalize variables as

$$\boldsymbol{x} = L_0\widehat{\boldsymbol{x}}, \quad \boldsymbol{B} = B_0\widehat{\boldsymbol{B}}, \quad n = n_0\widehat{n}, \quad t = (L_0/V_A)\widehat{t},$$
$$p = (B_0^2/\mu_0)\widehat{p}, \quad \phi = (L_0 B_0 V_A)\widehat{\phi}, \quad \boldsymbol{V} = V_A\widehat{\boldsymbol{V}},$$

where $V_A = B_0/\sqrt{\mu_0 M n_0}$ is the Alfvén speed. Equations (13) and (14) transform to

$$\frac{\partial}{\partial \widehat{t}}\widehat{\boldsymbol{A}} = \left(\widehat{\boldsymbol{V}} - \frac{\varepsilon}{\widehat{n}}\widehat{\nabla} \times \widehat{\boldsymbol{B}}\right) \times \widehat{\boldsymbol{B}} - \widehat{\nabla}\left(\widehat{\phi} - \varepsilon\widehat{\Pi}_e\right), (15)$$

$$\frac{\partial}{\partial \widehat{t}}(\varepsilon\widehat{\boldsymbol{V}} + \widehat{\boldsymbol{A}}) = \widehat{\boldsymbol{V}} \times \left(\widehat{\boldsymbol{B}} + \varepsilon\widehat{\nabla} \times \widehat{\boldsymbol{V}}\right)$$
$$- \widehat{\nabla}\left(\varepsilon\widehat{V}^2/2 + \varepsilon\widehat{\Pi}_i + \widehat{\phi}\right) (16)$$

where the scaling coefficient $\varepsilon = \delta_i/L_0$ is the ratio of the intrinsic scale, the ion skin depth

$$\delta_i = \frac{c}{\omega_{pi}} = \sqrt{\frac{M}{\mu_0 n e^2}},$$

to the macroscopic scale.

In what follows, we remove ˆ to simplify notation. In the present purpose, constant density incompressible model is sufficient, so we assume $n = 1$ and $\Pi_j = p_j$ $(j = e, i)$.

We will assume that ε is much less than unity implying that system of equations (15) and (16) constitutes our ϵ-model. Notice that in the $\lim \epsilon \to 0$, the terms of order unity, and the terms of order ϵ respectively reproduce the Induction and the Lorentz force equations of MHD. The standard form is obtained by first taking the curl of the order unity equation and then writing them in terms of fields (rather than the potentials)

$$\partial_t \boldsymbol{B} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{V} = 0, \qquad (17)$$
$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = -\nabla(p + B^2/2), \qquad (18)$$

where the incompressibility closure $\nabla \cdot \boldsymbol{V} = 0$ has been used. The $\lim \epsilon \to 0$ MHD system is clearly scale-less (that is it has no scale apart from the macroscopic system size) and represents the 0-model for the current investigation.

B. Beltrami solutions –linear structure built in nonlinear system

The two-fluid model can be cast in a symmetric system of vortex transport equations; taking the curl of (15) and (16), we obtain

$$\partial_t \boldsymbol{\Omega}_j = \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\Omega}_j) \quad (j = 1, 2), \qquad (19)$$

where a pair of "vorticities" and their corresponding flows are

$$\begin{cases} \boldsymbol{\Omega}_1 = \boldsymbol{B}, \\ \boldsymbol{U}_1 = \boldsymbol{V} - \varepsilon \nabla \times \boldsymbol{B}, \end{cases} \quad \begin{cases} \boldsymbol{\Omega}_2 = \boldsymbol{B} + \varepsilon \nabla \times \boldsymbol{V}, \\ \boldsymbol{U}_2 = \boldsymbol{V}. \end{cases}$$

We note that the distinctness as well as coupling of the two vorticities ($\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$) is induced by the singular perturbation scaled by ε. For $\varepsilon = 0$, the sets ($\boldsymbol{\Omega}_j$, $\boldsymbol{U}_j$) become degenerate, and the system (19) reduces into a single vortex dynamics, the "0-model"

Here, we explore a "linear structure" built in the nonlinear dynamics equations (19), that is the "kernel" of the generator (right-hand side) of the evolution equation (19). General stationary solutions (null set) of the ϵ-model (19) are characterized by the equations

$$\boldsymbol{U}_j \times \boldsymbol{\Omega}_j = \nabla \phi_j \quad (\exists \phi_j; \ j = 1, 2).$$

The "Beltrami conditions" impose more restrictive relations:

$$\boldsymbol{U}_j = \mu_j \boldsymbol{\Omega}_j \quad (j = 1, 2), \qquad (20)$$

implying the alignment of the vorticities with the corresponding flows [8]. Writing $a = 1/\mu_1$ and $b = 1/\mu_2$,

and assuming that a and b are constants, the Beltrami conditions (20) read as a system of "linear equations"

$$\boldsymbol{B} = a(\boldsymbol{V} - \varepsilon \nabla \times \boldsymbol{B}), \qquad (21)$$
$$\boldsymbol{B} + \varepsilon \nabla \times \boldsymbol{V} = b\boldsymbol{V}. \qquad (22)$$

Combining (21) and (22) yields ($\boldsymbol{u} = \boldsymbol{B}$ or $\boldsymbol{V}$) [8]

$$\varepsilon^2 \nabla \times (\nabla \times \boldsymbol{u}) + \varepsilon(a^{-1} - b)\nabla \times \boldsymbol{u} + (1 - b/a)\boldsymbol{u} = 0. \qquad (23)$$

Defining "Beltrami fields" by

$$\nabla \times \boldsymbol{G}_\pm = \lambda_\pm \boldsymbol{G}_\pm,$$
$$\lambda_\pm = \frac{1}{2\varepsilon}\left[(b - a^{-1}) \pm \sqrt{(b - a^{-1})^2 - 4(1 - b/a)}\right],$$

the general solution of (23) is given by the linear combination $\boldsymbol{u} = c_+\boldsymbol{G}_+ + c_-\boldsymbol{G}_-$. In view of (21), we obtain

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-, \qquad (24)$$
$$\boldsymbol{V} = \left(a^{-1} + \varepsilon\lambda_+\right)C_+\boldsymbol{G}_+ + \left(a^{-1} + \varepsilon\lambda_-\right)C_-\boldsymbol{G}_-, \qquad (25)$$

where $C_\pm$ are arbitrary constants. The fields (24) and (25) are called "double Beltrami (DB) fields".

It is shown that the totality of the Beltrami fields (eigenfunctions of the curl operator) span the Hilbert space of solenoidal fields [15]. A DB field, which is an exact stationary solution to the nonlinear evolution equations (19), contains only two components of the eigenfunctions, implying that it is a condensate state. Unlike linear dynamical systems, such as the Schrödinger equation, any further superposition of eigenfunctions make the state nonequilibrium –the nonlinear generator will spread the spectrum over all eigenfunctions

It is interesting to note that Beltrami fields, with appropriate boundary and flux conditions defines an invariant measure of the nonlinear vortex transport in the Hilbert space of vortices [5]. We can consider a canonical statistical distribution on this invariant measure [5, 9]..

On the other hand, condensation into a single eigenfunction is just the so-called "Taylor relaxed state" [13] (with a generalization including a magnetic-field-aligned flow) that is a stationary solution to the 0-model, ideal MHD.

In this section, we have introduced the DB fields from a rather formal considerations. They are particular stationary solutions characterized by two scalars $C_\pm$. These parameters will be related to "constants of motion" of the evolution equation in the latter discussion on "self-organization"; see Sec. IV. Before discussing the physical scenario to convince realization (self-organization) of the DB fields in two-fluid plasmas, we shall use them as particular solutions to reveal the scale separation caused by the nonlinearity and the singular perturbation.

Remark 1 (linear structure in nolinear system)
Some nonlinear systems are "integrable" in a sense that a linear structure built in the nonlinearity can produce

infinite number of constants of motion that are complete to construct general solution. Such a field is the so-called "soliton". The motion of a soliton is described by a commutation law of two Hermitian operators L and H, called "Lax pair", i.e., the evolution equation can be cast in the form of $\partial_t L + i[H, L] = 0$. This equation implies that the "observable" L is conserved for the wave generated by the Hamiltonian H. Hence, the spectral resolution of L will give the total set of constants of motion.

C. Scale separation

The parameter λ_+ (λ_-), being the eigenvalue of the curl operator, characterizes the reciprocal of the length scale on which $\boldsymbol{G}_+$ ($\boldsymbol{G}_-$) changes significantly. As the "Beltrami parameters" a and b vary, $\lambda_\pm$ can range from real to complex values of arbitrary magnitude.

To view the solution and the associated scale lengths as explicit functions of the small parameter ε, let $|\lambda_-| = O(1)$ so that $\boldsymbol{G}_-$ varies on the system size. The terms of order ε in (21) and (22) must, then, be negligible for the $\boldsymbol{G}_-$–parts of $\boldsymbol{B}$ and $\boldsymbol{V}$ dictating $a \approx b$ to have a significant large-scale component in the solution. Consequently the inverse of the second scale, $\lambda_+ \approx (a - a^{-1})/\varepsilon$. Barring the case $a \approx b \approx 1$ (Alfvénic flows –the normalized flow speed of order unity), we observe $\lim_{\varepsilon\to 0} |\lambda_+| = \infty$, i.e, the short scale shrinks to zero. Writing $b/a = 1 + \delta$ [$\delta = O(\varepsilon)$], we can approximate

$$\lambda_- \approx \frac{\delta}{\varepsilon}\left(\frac{1}{a} - a\right)^{-1}, \quad \lambda_+ \approx -\frac{1}{\varepsilon}\left(\frac{1}{a} - a\right). \quad (26)$$

Since in the limit $\varepsilon \to 0$, the current ($\nabla \times \boldsymbol{B}$) and the vorticity ($\nabla \times \boldsymbol{V}$) diverge in the small-scale ($\boldsymbol{G}_+$), the 0-model cannot ever capture the essence of the ϵ-model. The divergence of these small-scale components implies that the resistive and viscous dissipations can be very large even when resistivity and viscosity coefficient are relatively small. More over, the jittery magnetic fields in the length scale of the ion skin depth can produce a large chaos-induced dissipations [10].

Let us now derive the condition for which the singular part of the solution vanishes (i.e., $C_+ = 0$). To do this we will relate $C_\pm$ with the energy E and the helicity H of the fields [17], and express this condition as a relation between E and H. Here we assume that $\boldsymbol{B}$ and $\boldsymbol{V}$ are confined in a simply connected domain (normal components vanish on the boundary). The resulting orthogonality $\int \boldsymbol{G}_- \cdot \boldsymbol{G}_+ \, dx = 0$ (integral is taken over the total domain) [15] helps simplify the analysis. We can evaluate

$$\begin{aligned} E &\equiv \int (B^2 + V^2) \, dx \\ &= \left[1 + \left(a^{-1} + \varepsilon\lambda_-\right)^2\right] C_-^2 \end{aligned}$$

$$+ \left[1 + \left(a^{-1} + \varepsilon\lambda_+\right)^2\right] C_+^2,$$

$$H \equiv \int \boldsymbol{A} \cdot \boldsymbol{B} \, dx = \frac{C_-^2}{\lambda_-} + \frac{C_+^2}{\lambda_+},$$

and solve these equations for $C_\pm$,

$$\begin{aligned} C_-^2 &= -\frac{\lambda_-}{D}\left\{E - \left[1 + \left(a^{-1} + \varepsilon\lambda_+\right)^2\right]\lambda_+ H\right\} \\ &\to \Lambda_- H, \end{aligned}$$

$$\begin{aligned} C_+^2 &= \frac{\lambda_+}{D}\left\{E - \left[1 + \left(a^{-1} + \varepsilon\lambda_-\right)^2\right]\lambda_- H\right\} \\ &\to \frac{1}{(1 + a^2)}\left[E - \left(1 + a^{-2}\right)\Lambda_- H\right], \end{aligned}$$

where $D \equiv b(b + a^{-1})(\lambda_+ - \lambda_-)$, $\Lambda_- = \lim_{\varepsilon\to 0} \lambda_-$ [see (26)], and the limit is for $\varepsilon \to 0$. If E and H satisfy the relation

$$E = \left(1 + a^{-2}\right)\Lambda_- H, \quad (27)$$

the singular component vanishes ($\lim_{\varepsilon\to 0} C_+ = 0$), and the solution converges to the "relaxed state"

$$\boldsymbol{B} = C_-\boldsymbol{G}_-, \quad \boldsymbol{V} = \boldsymbol{B}/a.$$

The energy E satisfying (27) is the "minimum energy" accessible for given helicity H and cross helicity $K \equiv \int \boldsymbol{V} \cdot \boldsymbol{B} \, dx$ [13] The single Beltrami parameter a is determined by $a + a^{-1} = E/K$.

These relations clearly show that the singular (small scale) part of the double Beltrami field, which can produce a large resistive and viscous dissipations [7], disappears as the field relaxes into the final minimum energy state for given helicity and cross helicity.

D. Cauchy characteristics

We now turn our attention to the mechanisms that create small scales by analyzing the Cauchy characteristics of the "0-model". This analysis will also highlight the difference between the double Beltrami model of two-scale vortices and the "current sheet" model of Parker.

The "0-model" consists of the ideal MHD equations (17)-(18) together with the incompressibility condition $\nabla \cdot \boldsymbol{V} = 0$. With $\varphi(\boldsymbol{x}, t)$ denoting the eikonal function, ∂_t and ∇ are replaced by $\partial_t\varphi$ and $\nabla\varphi$, respectively, and we obtain a system of homogeneous algebraic equations in seven variables ($\boldsymbol{B}, \boldsymbol{V}, p$) ($\partial_t\varphi$ and $\nabla\varphi$ appear as coefficients). The characteristic equation, demanding the solvability of the homogeneous equation, reads

$$\begin{aligned} (\nabla\varphi)^2(\partial_t\varphi + \boldsymbol{V} \cdot \nabla\varphi)[\partial_t\varphi + (\boldsymbol{V} + \boldsymbol{B}) \cdot \nabla\varphi]^2 \\ \times [\partial_t\varphi + (\boldsymbol{V} - \boldsymbol{B}) \cdot \nabla\varphi]^2 = 0. \quad (28) \end{aligned}$$

The propagating eikonals represent hyperbolic parts of the system – the characteristics (rays) are the streamlines of the flow $\boldsymbol{V}$, as well as those of the Doppler-shifted Alfvén waves propagating in the directions of $\boldsymbol{V} \pm \boldsymbol{B}$

The incompressibility closure is sufficient in the present purpose of analysis. For the standard MHD characteristics, see [14]. . Eliminating the short time scales of waves (and possible instabilities), we may analyze structures that can persist for longer times. Setting $\partial_t = 0$ in (17) and (18), we can formulate a static model, in fact, a quasi-static model allowing slow variation of parameters or boundary conditions:

$$\nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = 0, \tag{29}$$
$$(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} - (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla p = 0, \tag{30}$$
$$\nabla \cdot \boldsymbol{V} = 0. \tag{31}$$

The corresponding characteristic equation is obtained by setting $\partial_t = 0$ in (28);

$$(\nabla \varphi)^2 (\boldsymbol{V} \cdot \nabla \varphi)[(\boldsymbol{B} - \boldsymbol{V}) \cdot \nabla \varphi]^2 [(\boldsymbol{B} + \boldsymbol{V}) \cdot \nabla \varphi]^2 = 0. \tag{32}$$

The eikonal $\varphi(\boldsymbol{x})$ gives the characteristics that are the projections, onto the coordinate space, of the rays given by (28).

E. Nonlinear dispersive interference

Using the characteristics, we may construct the Cauchy solutions that represent the "propagation" of Cauchy data (initial conditions) along the characteristics (static structures may be regarded as standing waves). Here we encounter two essential problems:

1. Non-integrability (chaos): The characteristic ordinary differential equations (ray equations) may not be integrable.

2. Nonlinearity: The characteristics are determined by the fields themselves (unknown variables).

The first problem does not exist in two-dimensional spaces, where streamlines of divergence-free fields are always integrable. Let us begin with this simple case. If the problem were linear, the construction of the Cauchy solution is straightforward –the general solution is given by the superposition of d'Alembert solutions representing the propagation along given integrable rays. Multiple characteristics may cause intersections, resulting in a pattern produced by the "interference" of superposed different waves. In a nonlinear system, however, a simple superposition fails to describe the intersection of different waves.

The nonlinear two-dimensional analysis can be most simply done following the Grad-Shafranov recipe based on the Clebsch representations of incompressible fields. Assuming $\partial_z = 0$ in the x-y-z coordinates, we set $\boldsymbol{B} = \nabla \psi \times \boldsymbol{e}_z + B_z \boldsymbol{e}_z$ and $\boldsymbol{V} = \nabla \phi \times \boldsymbol{e}_z + V_z \boldsymbol{e}_z$ ($\boldsymbol{e}_z = \nabla z$). Substituting these representations in (17) and (18), and setting $\partial_t = 0$, we find that solvability constraints demand

$$\phi = \phi(\psi), \ B_z = B_z(\psi), \ V_z = V_z(\psi), \ p = p(\psi), \tag{33}$$

and a generalized Grad-Shafranov equation

$$-\Delta[\psi - \phi(\psi)] = F'(\psi), \tag{34}$$

where $F(\psi) = B_z(\psi)^2/2 - V_z(\psi)^2/2 + p(\psi)$ is the Cauchy data

Taking $\phi = 0$ and $V_z = 0$ (no flow), (34) reduces into the well-known Grad-Shafranov equation; see [1]. In the two-fluid model, the equilibrium equation becomes a coupled two elliptic PDEs for ψ and ϕ, because the Hall effect raises the order of the system. .

Since $F(\psi)$ is a combination of "arbitrary" functions of ψ, it may contain any arbitrary small scale. Parker's model of current sheets can, then, be represented by "wrinkles" in $[B_z(\psi)^2/2]'$ (force-free current), which may be produced by merging flux tubes.

We notice that the Cauchy solutions given by (33)-(34) are rather limited. Under the condition $\phi = \phi(\psi)$, the rays degenerate into a single direction that parallels the contour of ψ everywhere. This means that the two-dimensional projections of $\boldsymbol{V}$ and $\boldsymbol{B}$ (and hence, $\boldsymbol{V} \pm \boldsymbol{B}$) are parallel. This restriction, limiting the perpendicular components of the flow, does not allow intersections of characteristics. One must ask, then, if this restricted class of mathematical solutions exhausts the possibilities available to a physical system, or have we omitted other viable alternative solutions?

The scope of Cauchy solutions (underlying Parker's model of current sheets) becomes even narrower when we consider general non-integrable characteristics in three-dimensional systems. When the characteristic curves (magnetic field lines) are embedded densely in a space, inhomogeneous Cauchy data leads to pathology, and the homogeneous Cauchy data yields only the relatively trivial Taylor relaxed state (with a parallel flow).

More general solutions with intersecting rays, may be found if we allow singularities. The essential nature of the "nonlinear interference" produced by ray crossing can be seen in shocks. Since the rays are functions of the fields (variables $\boldsymbol{B}$ and $\boldsymbol{V}$), changes in the ray directions necessarily imply inhomogeneity of the fields. On the contrary, fields must be constant along the rays. One way out of this dilemma is to allow appropriate discontinuities at the intersecting points of the rays. For the elementary shock problem, the imposition of Rankine-Hugoniot conditions restores consistency. Generally, discontinuities (shocks) may not intersect, and hence, the discontinuous solutions with perpendicular flows (intersecting rays) are still limited.

The restriction of constructing self-consistent solutions is much relaxed in a system with a higher degree of freedom. A singular perturbation, represented by higher-order derivatives, introduces non-local effects that help to remove the inconsistency produced by ray crossing in the models without the singular term. By introducing dispersion, the singular perturbation present in the two-fluid (ϵ-) model heals discontinuities in MHD,the "0-model". The cooperation of nonlinearity and dispersion determines a

specific structure with a characteristic length scale (soliton is an analogy)

If $\boldsymbol{\Omega}_j$ and $\boldsymbol{U}_j$ ($j = 1, 2$) were independent, the system (19) is linear hyperbolic. The Beltrami conditions, then, yield two independent vortices that may have arbitrary length scales. In relating $\boldsymbol{\Omega}_j$ and $\boldsymbol{U}_j$ with $\boldsymbol{B}$ and $\boldsymbol{V}$, however, the system becomes nonlinear and dispersive (derivatives are included). For the general solvability of the double Beltrami equations (21)-(22), see [15, 16]. The small-scale part ($\boldsymbol{G}_+$) of the double Beltrami field is the pattern created by this mechanism. This structure is, thus, singular (infinitely oscillating) in the limit $\varepsilon \to 0$; see (26). Ignoring the flow, or assuming only parallel flows (in two dimension, the x-y components of $\boldsymbol{B}$ and $\boldsymbol{V}$ are parallel) one would fail to predict this structure, because the singularity is an expression of ray crossing.

It is remarkable that the elementary double Beltrami solution exists in any three dimensional geometry. This is due to the assumption that Beltrami parameters μ_j (a and b) are constant. The divergence-free conditions on $\boldsymbol{\Omega}_j$ and $\boldsymbol{U}_j$ demand $\boldsymbol{U}_j \cdot \nabla \mu_j = 0$ ($j = 1, 2$) implying that the Beltrami parameters are the required Cauchy data –they are assumed to be constant, and hence, the double Beltrami fields are robust in chaotic characteristics. This is in marked contrast to Parker's model that considers wrinkles of Cauchy data as the origin of small scales. In the double Beltrami field, the inhomogeneity stems from nonlinear dispersive interference produced by the intersections of the ideal characteristic curves.

Note 3 (singularities in ideal MHD) *The ideal MHD equations (0-model) has complicated nonlinearity causing rather serious singularities. Singularities are due to the "hyperbolic" part of the generator, which allows fields to have discontinuities or divergence. In the linearized equations, they appear as a type of $\boldsymbol{b} \cdot \nabla$ operators with some (given) vector $\boldsymbol{b}$, and they are translated into $k_\parallel$ ($\parallel$ means the direction of $\boldsymbol{b}$) in the dispersion relations. The Alfvén singularity associated with the continuous spectrum of Alfvén waves is due to such a term reflecting the hyperbolicity of the generator. In the framework of stationary nonlinear equations ($\partial_t = 0$, or $\partial_t = -C\partial_x$ modeling propagating solutions), the hyperbolicity of the MHD generator brings about a mathematical difficulty if the space dimension is two or three. The two dimensional equilibrium equation (34) may be written in a variational form $\delta_\psi L = 0$ with a Lagrangian*

$$L = \int \left[\frac{1}{2}(1 - M^2(\psi))|\nabla\psi|^2 - F(\psi) \right] \, dx,$$

where $M(\psi) = \phi'(\psi)$ is the Alfvén-Mach number. If $M = 1$ occurs in the domain, the convexity of L is destroyed. The resultant singularity corresponds to the Alfvén singularity. If the compressibility is included to the MHD equations, the structure of the generator becomes even more complicated –the M and F in the Lagrangian L includes $\nabla\psi$, and the equation $\delta_\psi L = 0$ may switch between elliptic and hyperbolic, implying creation of shocks.

These difficulties are primarily due to the degeneracy of the two essential hyperbolicity $\boldsymbol{B} \cdot \nabla$ and $\boldsymbol{V} \cdot \nabla$ in the static nonlinear equations –only parallel flows $\boldsymbol{B} \parallel \boldsymbol{V}$ are allowed in ideal MHD equilibrium. In the two-fluid MHD, this degeneracy is resolved. In the two dimension case, the hyperbolic parts are immediately integrated by Casimir invariants (Cauchy data associated with Clebsch potentials), and the remaining part of the equations become two regular elliptic equations.

IV. SELF-ORGANIZATION

A. Constants of motion

Relaxation is a general notion of dynamical process where stresses, inhomogeneities, or inconsistencies are gradually removed to reach stable equilibriums. It is, therefore, a temporally irreversible process associated with some finite "dissipation". In ideal (for example, Hamiltonian) dynamics, initial conditions are the essential restrictions determining the history of each particular evolution. If all data in the initial state are perfectly preserved in a future state, the evolution is temporally reversible. Irreversibility of evolution is, thus, due to some "coarse graining" of the dynamical law, i.e., reduction of the degree of freedom that we "observe"

The notion of "open system" means that we observe only a subsystem that may interact with external world. Then, we must model the interaction in terms of rather small number of parameters, such as an equilibrium temperature (canonical ensemble), or chemical potentials (grand canonical ensemble)..

In chaotic systems, the relation between the initial state and a future state becomes totally un-describable. Even then, only a few information contained in the initial condition may play an important role to characterize some aspects of future states. For example, the energy (Hamiltonian H) of the initial state is conserved if the system is autonomous ($\partial_t H = 0$). This conservation law says that the future state must not have a different energy –the system is restricted in an ensemble of states with the given same energy (so-called energy shell in the phase space).

If the number of such restrictions, posed by conservation laws, is smaller, the system may relax more. The relaxation proceeds with "homogenizing" the initial data –the "mixing paradigm"; if future states are not sensitive (in some average sense) to particular values of certain initial data, these data are homogenized.

To formulate the problem in a more mathematical shape, let us consider an abstract transport equation such as

$$\partial_t u + \boldsymbol{U} \cdot \nabla u = 0, \tag{35}$$

where $u(\boldsymbol{X}, t)$ is a density distribution function ($\boldsymbol{X}$ is the phase space coordinate). The "flow" $\boldsymbol{U}(\boldsymbol{X}, t)$ on the phase space must be incompressible ($\nabla \cdot \boldsymbol{U} = 0$j, when $u(\boldsymbol{X}, t)$ is constant along the stream lines of $\boldsymbol{U}$ that are determined by solving the "characteristic ODE":

$$\frac{d}{dt}\boldsymbol{X} = \boldsymbol{U}(\boldsymbol{X}, t). \tag{36}$$

Each streamline represents the orbit of individual particle.

For a Hamiltonian system, the phase space coordinates are the canonical pair $\boldsymbol{X} = {}^t(\boldsymbol{x}, \boldsymbol{p})$ ($\boldsymbol{x}$ is the position and $\boldsymbol{p}$ is the canonical momentum). The phase space flow can be written as

$$\boldsymbol{U} = \begin{pmatrix} \partial_{\boldsymbol{p}} H \\ -\partial_{\boldsymbol{x}} H \end{pmatrix},$$

with a Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p}, t)$. Then, (35) reads as the Liouville equation

$$\partial_t u + \{H, u\} = 0, \tag{37}$$

where $\{a, b\} = (\partial_{\boldsymbol{p}} a) \cdot (\partial_{\boldsymbol{x}} b) - (\partial_{\boldsymbol{p}} b) \cdot (\partial_{\boldsymbol{x}} a)$ is the "Poisson bracket".

Let ν be the dimension of the phase space. If we have $\nu + 1$ of constants of motion ($\psi_j(\boldsymbol{X}, t)$ ($j = 0, \cdots, \nu$)), then the general solution of (35) is given by

$$u(\boldsymbol{X}, t) = f(\psi_0, \cdots, \psi_\nu). \tag{38}$$

Apparently, a constant number C is a constant of motion, we need ν of nontrivial constants of motion. When (36) is "integrable", we can find the total set of constants of motion.

When we have a smaller number of constants of motion (then, we say that the system is "non-integrable" or "chaotic"), we have only "special solutions" in the form of

$$u(\boldsymbol{X}, t) = f(\psi_0, \cdots, \psi_M) \quad (M < \nu). \tag{39}$$

In general, nonlinear systems are not integrable. In a non-integrable system, some constants of motion plays an even more important role in characterizing the system. It is because the constants of motion are derived from the "symmetry" of the system, not from analyzing individual dynamics for specific initial conditions. Such "a priori" conservation laws may deduce universal understanding of the complexity in a chaotic system

In a "dissipative" system, we may study how the constancy of these quantities are destroyed –So called "selective dissipation arguments" is a generalization of such a priori estimates..

B. Statistical mechanical picture of self-organization

Statistical mechanical approach is the most natural consideration to utilize a small number of constants of motion in developing a "coarse grained" picture of chaotic systems. The Gibbs distribution function is derived from only two constants of motion in a many-body system, i.e., the total number N of particles and the total energy E. Assuming that there are no other constants (constraints) of motion, i.e., invoking the "ergodic hypothesis", one can find the most probable state (in the sense of Kinchin's axiom) by maximizing the Shannon entropy

$$S = -\int f(\boldsymbol{X}) \log f(\boldsymbol{X}) dX,$$

where $f(\boldsymbol{X})$ is the probability density in the phase space. The variational principle is

$$\delta \left[S + \alpha \int f \, dX + \beta \int f \mathcal{H} \, dX \right] = 0,$$

where $\mathcal{H}$ is the Hamiltonian. The maximizer of S is given by

$$f(\boldsymbol{X}) = Z^{-1} \exp[-\beta \mathcal{H}(\boldsymbol{X})] \quad (Z = 1 + \alpha). \tag{40}$$

If we have another constant of motion G, the ensemble of possible states (where we maximize S) becomes smaller. This allows the system to sustain a structure avoiding the "thermal death" of ultimate relaxation toward the Gibbs distribution. Let a macroscopic constant G be given in terms of a microscopic quantity $\mathcal{G}(\boldsymbol{X})$ as

$$G = N \int f(\boldsymbol{X}) \mathcal{G}(\boldsymbol{X}) \, dX.$$

Then, the maximum entropy state becomes

$$f(\boldsymbol{X}) = Z^{-1} \exp[-\beta \mathcal{H}(\boldsymbol{X}) - \gamma \mathcal{G}(\boldsymbol{X})]. \tag{41}$$

In comparison with (40), this state may have a richer structure. Let us see some examples.

Example 1 (non-neutral plasma trap) *Let us consider a single-species non-neutral plasma in a homogeneous magnetic field. In the cylindrical coordinates r-θ-z, the magnetic field is represented written as $\boldsymbol{B} = B e_z$ ($B = $ constant). For a single particle motion in an axisymmetric system, the canonical angular momentum $p_\theta = m r v_\theta + q r A_\theta$ is a constant of motion (A_θ is the θ component of the vector potential $\boldsymbol{A}$). Let us "assume" that the sum of the angular momentum over all particles are conserved, i.e.,*

$$G = \int p_\theta f \, dX$$

is constant. Then, the maximum entropy state (41) becomes

$$f = Z^{-1} \exp[-\beta(\mathcal{H} - \omega p_\theta)], \tag{42}$$

$(\beta\omega = \gamma$: Lagrange multiplier). The number density $n(\boldsymbol{x})$ is related to the scalar potential $\phi(\boldsymbol{x})$ through the Poisson equation

$$-\Delta\phi = \epsilon_0^{-1}qn = \epsilon_0^{-1}q\int f(\boldsymbol{x},\boldsymbol{p})\,dp, \qquad (43)$$

where q is the charge of the particle. This ϕ is included in the Hamiltonian H, and hence, (43) is a nonlinear equation.

Writing the velocity $\boldsymbol{V} = \tilde{\boldsymbol{V}} + \boldsymbol{U}$ with $\boldsymbol{U} = r\omega\boldsymbol{e}_\theta$ (rotation with an angular velocity ω), we observe

$$\tilde{\mathcal{H}} := \mathcal{H} - \omega p_\theta = \frac{m}{2}\tilde{v}^2 - \frac{m}{2}(r\omega)^2 - qr\omega A_\theta + q\phi. \quad (44)$$

The function $\mathcal{H} - \omega p_\theta$ is just the canonical transform of the Hamiltonian H in the rotating frame with angular velocity ω, and hence, the distribution (42) is Gibbs' thermal equilibrium in the rotating frame.

For a constant density n, the potential determined by (43) is $\phi = -(qn/4\epsilon_0)r^2$. Since $\boldsymbol{A} = (rB/2)\boldsymbol{e}_\theta$ for the homogeneous magnetic field

Here we consider a low density plasma, so the currents in the plasma is too weak to change the magnetic field generated by external coil system., the three terms, excepting the first one, on the right-hand side of (44) are proportional to r^2. Hence, an appropriate vale of ω may make all of them cancel. Then, the $\tilde{H}$ becomes independent to the coordinates, and (42) gives the hoped-fore constant density n. One can easily verify that such an ω is given by solving

$$\omega^2 + \omega_c\omega + \frac{1}{2}\omega_p^2 = 0 \qquad (45)$$

$(\omega_c = qB/m,\ \omega_p^2 = nq^2/m\epsilon_0$

If $\omega_p/\omega_c \ll 1$, we obtain two solutions $\omega \approx \omega_D,\ \ \omega_c + \omega_D$, where $\omega_D = \omega_p^2/(2\omega_c)$ (diocotron frequency) is the $\boldsymbol{E}\times\boldsymbol{B}$ drift frequency in the self-electric field $E = qnr/2\epsilon_0$. .

Here, we must note that the conservation of the macroscopic "canonical" angular momentum G is NOT warranted mathematically. In a macroscopic system where particle collisions reads the system relax into an equilibrium, the symmetry in θ is no longer valid. Instead, the macroscopic "mechanical" angular momentum $G' = \int rmv_\theta f\,dX$ is the well known constant of motion of the macroscopic (fluid) system. The use of G' in the place of G to determine the maximum entropy state fails to obtain the rotating thermal equilibrium (omitting the term $-qr\omega A_\theta$ in (44), we lose the term $omega_c\omega$ in (45), and then, we have no real solution ω). Implication of these observations is that the constancy of G, possibly achieved with avoiding macroscopic θ inhomogeneity in the relaxation process, is the key to obtain a confinement of non-neutral plasma in the magnetic field. The G is rather fragile, so experimentalists must carefully optimize the system to keep it constant.

Example 2 (Harris sheet, Bennet pinch) *Let us consider a neutral plasma in a slab geometry that is homogeneous in the coordinate y. The magnetic field has only the z component. The vector potential is $\boldsymbol{A} = A_y(x)\boldsymbol{e}_y$. For a single particle, the canonical momentum $p_y = my' - qA_y(x)$ is conserved. Assuming that the macroscopic canonical momentum in y direction is conserved, as well as the total energy, the maximum entropy state (41) reads*

$$f = Z^{-1}\exp[-\beta(\mathcal{H} - Up_y)]. \qquad (46)$$

$(\beta U = \gamma$: Lagrange multiplier). Since

$$\begin{aligned}\tilde{\mathcal{H}} &= \mathcal{H} - Up_y \\ &= \frac{m}{2}\tilde{v}^2 - \frac{m}{2}U^2 - qUA_y + q\phi,\end{aligned}$$

$\tilde{\mathcal{H}}$ is the Galilean transform of the Hamiltonian in the inertial frame of velocity $\boldsymbol{U} = U\boldsymbol{e}_y$. Hence, (46) reads

$$f \propto e^{-\beta q(\phi - UA_y(x))}e^{-\beta\tilde{v}^2/2m}, \qquad (47)$$

which is the drift Maxwellian with the velocity $\boldsymbol{U}$.

Calculating the currents in each spices using (47), and solving $-\Delta\boldsymbol{A} = \mu_0\boldsymbol{j}$ ($\boldsymbol{j}$ is the total current), we obtain the vector potential $A_y(x)\boldsymbol{e}_y$ that must be consistent to the A_y in (47). The self consistent solution is the well-known Harris sheet.

Similar calculations in cylindrical geometry with z-homogeneity, we obtain the Bennet pinch equilibrium.

C. Self-organization in function space

To consider a continuous medium and to discuss self-organization of fields (member of functions spaces with infinite dimension), we pick up the general "homogenizing" property of the relaxation, which we have developed invoking the statistical mechanical framework

For statistical mechanics in function spaces, see [5, 9]. The creation of Beltrami fields can be described by maximizing an entropy defined on a functions space with restricting the ensemble by helicity conservation law., and formulate variational principles in the sense of Dirichlet's principle for diffusion equations.

Let us start with a most simple example. The projection of the kinetic phase space $\boldsymbol{x}$-$\boldsymbol{p}$ onto the coordinate space is the coarse-graining of a many particle system. The homogenization of the density $n(x,t)$ (we consider one dimensional system for simplicity) is described by the "diffusion equation"

$$\partial_t n = D\partial_x^2 n \qquad (48)$$

where D (diffusion coefficient) is assume to be a positive number. Assuming that particles are confined in an interval $(0, 1)$, we impose boundary conditions

$$\partial_x n(0,t) = \partial_x n(1,t) = 0 \quad (\forall t \geq 0). \qquad (49)$$

Multiplying the solution $n(x,t)$ on both sides of (48), and integrating over $(0,1)$, we obtain, using (49),

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 n(x,t)^2\,dx &= 2\int_0^1 (\partial_t n)n\,dx \\
&= 2\int_0^1 \partial_x\left(D\partial_x n\right)n\,dx \\
&= -2\int_0^1 D(\partial_x n)^2\,dx \leq 0. \quad (50)
\end{aligned}
$$

Let us call the integral $\int_0^1 n(x,t)^2\,dx$ the "energy integral" for the equation (48)

The integral $\|n\|^2 = \int_0^1 n(x,t)^2\,dx$ is the norm of the Hilbert space $L^2(0,1)$.. Because the right-hand side of (50) can become 0 only when n is homogeneous in x, we find that the energy integral decreases monotonically until the final relaxed state

$$
\lim_{t\to\infty} n(x,t) = n_\infty \ (= \text{constant})
$$

is achieved. By the conservation law of the total number of particles, we can calculate $n_\infty = \int_0^1 n(x,0)\,dx$. The relaxed state is, thus, determined by only one constant of motion (particle number).

This simple example is a linear system, so the realization into the homogeneous distribution was shown as a rigorous mathematical theorem. However, the final relaxed state is rather trivial –the distribution is totally homogenized. In a nonlinear system, we may expect richer structures are created through relaxation, and they may sustain for an appreciably long time. Such self-organization of structures may no longer be rigorously proved for arbitrary given initial conditions. What we will be able to prove is the existence of a "relaxed state" under the ansatz that a relaxed state is the minimizer of the energy integral (norm of the function space), implying the most homogeneous state, under some dynamical constraints imposed by conservation laws.

In the next section, we will test this concept in two-fluid MHD, and show that the DB fields are such relaxed states.

D. Self-organization of DB fields

Before discussing the self-organization in plasmas, we must note that self-organization is not a general tendency of plasmas. Since many of the relaxed states in recent literature follow from seemingly standard variational principles, one could draw the erroneous conclusion that self-organization occurs universally in any plasma. We shall presently show that it is not so; self-organization, in fact, may occur only under rather restrictive conditions –Not all variational principles are well-posed, and even when they are, not all solutions to variational principles lead to stable equilibriums (even to equilibriums) –an essential minimum qualification for "relaxed states".

Here, we repeat the two-fluid MHD equations in the form of vortex transport equations (see Subsec. III B)

$$
\frac{\partial}{\partial t}\boldsymbol{\omega}_j - \nabla\times(\boldsymbol{U}_j\times\boldsymbol{\omega}_j) = 0 \quad (j=1,2) \quad (51)
$$

in terms of a pair of generalized vorticities and the corresponding flows

$$
\begin{cases} \boldsymbol{\omega}_1 = \boldsymbol{B}, \\ \boldsymbol{U}_1 = \boldsymbol{V} - \nabla\times\boldsymbol{B}, \end{cases} \qquad \begin{cases} \boldsymbol{\omega}_2 = \boldsymbol{B} + \nabla\times\boldsymbol{V}, \\ \boldsymbol{U}_2 = \boldsymbol{V}. \end{cases}
$$

Here, we have set $\varepsilon = 1$ choosing $L = \delta_i$.

One can easily verify, under appropriate boundary conditions (here, we assume periodic boundary conditions for simplicity), the invariance of the three integrals

$$
E = \int_\Omega (|\boldsymbol{V}|^2 + |\boldsymbol{B}|^2)\,dx, \quad (52)
$$

$$
H_1 = \int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,dx, \quad (53)
$$

$$
H_2 = \int_\Omega (\boldsymbol{A}+\boldsymbol{V})\cdot(\boldsymbol{B}+\nabla\times\boldsymbol{V})\,dx, \quad (54)
$$

representing the energy (the incompressibility condition eliminates the thermal energy), the magnetic helicity, and the helicity of the generalized vorticity.

The target functional (TF) we minimize to characterize the relaxed state is the norm of the function space of vortices $\boldsymbol{\Omega}_j$ $(j=1,2)$:

$$
\begin{aligned}
F &= \|\boldsymbol{\Omega}_1\|^2 + \|\boldsymbol{\Omega}_2\|^2 \\
&= \int |\nabla\times\boldsymbol{A}|^2 + |\nabla\times(\boldsymbol{V}+\boldsymbol{A})|^2\,dx, \quad (55)
\end{aligned}
$$

which is the "canonical enstrophy" of the two-fluid system.

This functional is a measure of dissipation in the controlling dynamics (an almost ideal system is very different from a dissipation dominated system such as a diffusion equation for which the dissipation determines the structure of the relaxed states). Further F is also be a measure of turbulence –the relaxed state should be as free of turbulence as possible

Note that F is a hybrid functional combining the magnetic and fluid aspects of the plasma and can be equivalently thought of as the energy associated with the generalized magnetic field seen by the ions. This choice may be seen either as an extrapolation to three-dimensional fluids of the two-dimensional fluid dynamics where the fluid enstrophy serves as the TF or equivalently as the magneto-fluid reincarnation of the one-fluid Taylor model..

A well-posed variational principle results when we minimize F keeping E, H_1, and H_2 constant. The general solution for arbitrary values of the three invariants is not necessarily an equilibrium solution. We find that stable equilibrium solutions result when the three invariants obey a certain relationship, i.e, on well-defined surfaces

in the three space spanned by the invariants. The parameter space for the existence of stable solutions is still very large. Physically it implies that starting from arbitrary values, one of the invariants must adjust (in this case H_2, the most fragile one) as F is minimized so that it can find the value dictated by the original values of E and H_1. If H_2, for instance, is initially far away from its desired final value, the system may not find its way into a relaxed state. These ideas will become clearer in later parts of the paper.

The minimization of F keeping E, H_1, and H_2 constant is carried out through the variation

$$\delta(F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2) = 0,$$

where μ_0, μ_1, and μ_2 are Lagrange multipliers (constants). This variational principle is well-posed because of the "coercive" term F that makes the functional $F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2$ to be convex having a unique minimizer (we will see examples of ill-posed variational principles in the next section, where the notion of "coercivity" will become clear). Calculating independent variations in $\boldsymbol{A}$ and $\boldsymbol{V}$, and after a manipulation, we obtain

$$\nabla \times \nabla \times \boldsymbol{V} - \mu_0 \boldsymbol{V} - \mu_2 \nabla \times \boldsymbol{V} = -\nabla \times \boldsymbol{B} + \mu_2 \boldsymbol{B},$$
$$\tag{56}$$

$$\mu_0 \boldsymbol{V} = \mu_0 \nabla \times \boldsymbol{B} + \mu_1 \boldsymbol{B}. \tag{57}$$

The general solution will be a "triple Beltrami field" -a linear combination of three Beltrami functions;

$$\boldsymbol{B} = \sum_{j=1}^{3} C_j \boldsymbol{G}_j, \quad \boldsymbol{V} = \sum_{j=1}^{3} D_j \boldsymbol{G}_j, \tag{58}$$

where (57) demands $D_j = [\lambda_j - (\mu_1/\mu_0)] C_j \equiv d_j C_j$.

Since the general triple Beltrami field (58) may not even be an equilibrium state, we must strive to find a sub-class which span the double Beltrami states -known to be stable equilibriums for eigenvalues (λ_1 and λ_2) sufficiently small that the resulting magnetic and flow shear do not drive kink and Kelvin-Helmholtz instabilities. We know from the toy model that this may be achieved by the adjustment of the constraints; the generalized helicity H_2, containing the higher order derivatives, will be affected the most.

Because the relation between $\boldsymbol{V}$ and $\boldsymbol{B}$ includes λ_j ($j = 1, 2, 3$) and μ_ℓ ($\ell = 0, 1, 2$), the algebra for carrying out the adjustment process is rather complicated[11]. Let us consider a simpler case when the system has no harmonic (external) field. Then, the eigenfunctions $\boldsymbol{G}_j$ are orthogonal[15], and we obtain

$$\begin{pmatrix} E \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 1 + d_1^2 & 1 + d_2^2 & 1 + d_3^2 \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \\ X_1 & X_2 & X_3 \end{pmatrix} \begin{pmatrix} C_1^2 \\ C_2^2 \\ C_3^2 \end{pmatrix}, \tag{59}$$

where $X_j = (1 + \lambda_j d_j)^2/\lambda_j$. Solving (59) for C_1 and C_2 with $C_3 = 0$, we obtain a relation among E, H_1,

and H_2. Denoting the matrix on the right-hand side of (15) by M and writing $L = M^{-1}$, $\mu_1' = -L_{3,2}/L_{3,1}$, and $\mu_2' = -L_{3,3}/L_{3,1}$, the equation for $C_3^2 = 0$ reads as

$$E - \mu_1' H_1 - \mu_2' H_2 = 0,$$

which determines the adjusted value of H_2 for a prescribed E and H_1. This "adjustment" is formally equivalent to solving

$$\delta(E - \mu_1' H_1 - \mu_2' H_2) = 0, \tag{60}$$

because all functionals are symmetric bilinear forms. The variational principle (60) yields the double Beltrami equation (23). We have finally obtained the self-organized relaxed states through a mathematically well-posed variational principle.

We must emphasize that the adjustment condition of the last paragraph is not a variational principle to "minimize" either E (ill-posed, because H_2 is "fragile") or H_1 or H_2 (not lower bounded). It is the adjustment of the ideal constants of motion to yield a minimizer of F on the corresponding hyper-surface (ensemble) where the state vector is restricted.

E. Self-organization and topology –remarks on ill-posed variational principle

Finding a suitable TF is the essential input for the theory of self-organization. It follows from the preceding discussion that the TF cannot be one of the constants of motion. This point is the essential difference of the present theory from the so-called selective dissipation arguments. The TF is the "norm" measuring the complexity in the relevant topology of the function space. Mathematically, it is a "coercive" form to formulate the well-posed variational principle.

It would appear that the DB equations may be "derived" by minimizing the total energy E keeping H_1 and H_2 constant (we call VP2). Two questions arise: 1) What could be the possible significance of a minimum energy state? 2) Since E, H_1 and H_2 are on an equal a priori footing, all being the exact constants of ideal dynamics, what dictates the choice of using E as the target functional (TF) for minimization -remember that the target functional E_m for VP1 is not a constant of motion.

If one could define an appropriate "free energy" in a statistical mechanical framework, a minimum energy state (as distinct from a minimum potential energy state) could acquire an important connotation in relaxation theory. Several authors[5, 9] have constructed such a statistical mechanics in a Hilbert space of single-fluid MHD. The success of the procedure is contingent upon finding a suitable "free energy" function which is bounded from below. When the theory is applied to the current system, we find that the free energy is not bounded in the energy norm because of the ion-flow helicity term (a part of the invariant H_2). Thus the minimum free energy route is not available for relaxed states to emerge in

a magneto-fluid. This characteristic of the ion-flow helicity will render VP2 to be a mathematically ill-posed problem; a demonstration will be given soon.

To understand the essence of the second question we remind that we are dealing with a near ideal but not a perfect ideal system. The dissipation, however small, is the primary agent for leading the system toward relaxation. Simulations of the single fluid system reveal that during the relaxation process (as the system evolves from an initial state to a relaxed state) the magnetic energy E_m goes through a severe adjustment (decrease) while H_1, the collisionless constant of motion, changes only mildly[4]. This is totally consistent with the demands of the variational principle; the system is driven to equilibrium by changes (even drastic) in the value of TF. Different responses to dissipation have given rise to the notion of "selective dissipation" which may be used to order the "fragility" of the invariants[2]. The most fragile (suffering the most from dissipation) invariant could, then, be used as TF, because it is "least invariant" among invariants. This criterion, however, would pick H_2 as TF because of the higher derivatives in the ion- flow helicity contribution. A variational principle like VP2 with E as TF (and H_1, H_2 as constraints) will be physically unsound and will turn out to be mathematically ill-posed.

One cannot get out of this difficulty by simply switching the roles of E and H_2; the latter, though more fragile than the former, is unsuitable as TF because it is not bounded below. In the two fluid dynamics, therefore, the invocation of "selective dissipation", by itself, does not lead to a well-posed variational problem; some additional input is sorely needed.

Here, we examine a simple example of ill-posed variational principle, and show how we can overcome the pathology, which will highlight the mathematical aspect of the variational principle we have developed in the previous subsection. Let us consider two functionals

$$\mathcal{G}(u) = \int_\Omega |\nabla u(\boldsymbol{x})|^2 dx, \quad \mathcal{H}(u) = \int_\Omega |u(\boldsymbol{x})|^2 dx$$

with $u = 0$ on $\partial\Omega$ where Ω is a bounded domain in $\mathbf{R}^N$. First, we seek for a minimizer of $\mathcal{G}(u)$ with the constraint $\mathcal{H}(u) = 1$. This is a well-posed problem; The minimizer is found from the variational principle,

$$\delta[\mathcal{G}(u) - \lambda\mathcal{H}(u)] = 0 \qquad (61)$$

where λ is a Lagrange multiplier. The Euler-Lagrange equation $-\Delta u = \lambda u$ with the above-mentioned boundary condition constitutes an eigenvalue problem. We can easily show that the eigenvalue $\lambda > 0$. Let λ_j be an eigenvalue and φ_j be the corresponding normalized eigenfunction ($\|\varphi_j\|^2 \equiv \int_\Omega |\varphi_j|^2 \ dx = 1$). With setting $u = a\varphi_j$, and demanding $\mathcal{H}(u) = 1$, we obtain $a = 1$ and $\mathcal{G}(u) = \lambda_j$. The smallest λ_j, then, yields the minimum $\mathcal{G}(u)$.

The complementary problem of finding a minimizer of $\mathcal{H}(u)$ with the restriction $\mathcal{G}(u) = 1$ turns out to

be ill-posed. The inherent pathology of the problem is exposed when we set up the variational principle $\delta[\mathcal{H}(u) - \mu\mathcal{G}(u)] = 0$ (μ is a Lagrange multiplier), and analyze the Euler-Lagrange equation $-\Delta u = \mu^{-1}u$. Let $\mu^{-1} = \lambda_j$ (an eigenvalue of $-\Delta$), and $u = a\varphi_j$. The condition $\mathcal{G}(u) = 1$ yields $a = \lambda_j^{-1/2}$, and $\mathcal{H}(u) = 1/\lambda_j$. Hence, the minimum of $\mathcal{H}(u)$ is achieved by the largest eigenvalue that is unbounded, viz., $\inf \mathcal{H}(u) = 0$ and the minimizer $\lim_{\lambda_j \to \infty} \lambda_j^{-1/2}\varphi_j = 0$ is nothing but the minimizer of $\mathcal{H}(u)$ without any restriction. The constraint $\mathcal{G}(u) = 1$ plays no role in this minimization problem[6].

These examples demonstrate that unless the constraints are less fragile (fragility being determined by the number of derivatives) than TF[6], the constrained minimization is meaningless at best. The variational principle simply does not "see" the constraint because an infinitesimal perturbation with a small length scale can contribute any required value to the constraint.

If the second problem is to be properly posed, i.e, for it to lead to a nontrivial manifestation of constrained minimization, the mathematical requirement is the existence of a higher order (coercive) TF. Generalizing our toy model let us affect the minimization of the higher order functional

$$\mathcal{F}(u) = \int_\Omega |\Delta u|^2 \ dx \qquad (62)$$

with restrictions $\mathcal{G}(u) = g$ and $\mathcal{H}(u) = h$ and with the additional boundary condition $-\Delta u = 0$ on $\partial\Omega$. Now the well-posed variational principle

$$\delta[\mathcal{F}(u) - \mu_1\mathcal{G}(u) - \mu_2\mathcal{H}(u)] = 0 \qquad (63)$$

yields the Euler-Lagrange equation

$$(-\Delta - \lambda_+)(-\Delta - \lambda_-)u = 0$$

with $\lambda_\pm = (\mu_1 \pm \sqrt{\mu_1^2 + 4\mu_2})/2$. The general solution is $u = C_+\varphi_+ + C_-\varphi_-$, where $\varphi_\pm$ are the eigenfunction of $-\Delta$ belonging to the eigenvalue $\lambda_\pm$. The set of double eigenfunction solutions extends the single eigenfunction solution of the previous problems. However by adjusting either g or h, the general solution can collapse to a single eigenfunction solution. If $g = \lambda_1^2 h$ (λ_1 is the smallest eigenvalue), we obtain $C_+^2 = h$ and $C_- = 0$, and the double eigenfunction solution degenerates into $u = h\varphi_1$. This solution gives the minimum of $\mathcal{F}(u)$ under the given constraints.

The addition of a coercive TF, and the subsequent procedure of adjustment, has accomplished two desirable objectives: 1) the pathology of the earlier ill-posed problem is removed, and 2) the direct influence of the added coercive term on the equation of motion is also removed. The desirability of the latter cannot be overemphasized: without this the theory would have become arbitrary and quite undependable. Notice that the matching condition $\mathcal{G}(u) - \mu\mathcal{H}(u) = 0$, converting the solution to a single eigenfunction of $-\Delta$, renders the larger problem entirely

equivalent to the original problem $\delta[\mathcal{G}(u) - \mu\mathcal{H}(u)] = 0$ (because both $\mathcal{G}(u)$ and $\mathcal{H}(u)$ are homogeneous quadratic forms) but with perfect mathematical rigor.

In this section, we have derived a well-posed variational principle that reproduces the double Beltrami states characterizing the relaxed states of a two-fluid plasma. The coercive target functional (F) is the measure of complexity in the relevant topology (norm) of the function space. This is not to be confused as selective dissipation of F; selective dissipation occurs in the fragile quantity (primarily H_2) through the adjusting process. The system can self-organize to a quiescent state, minimizing F, if the constants of motion are appropriately adjusted through a weakly dissipative process. This logical characterization of self-organization is naturally built in our formulation of the variational principle. We have shown that the required adjustment is not the minimization of energy. The conventional model of finding an energy minimizer constitutes an ill-posed problem. This mathematical observation has profound consequences suggesting that the conventional "energy" is not the principal variable whose minimization characterizes self-organization in vortex dynamics.

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