Numerical simulations of turbulent dynamos

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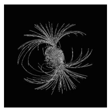
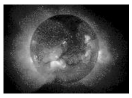
Magnetic fields in astrophysics

- Earth and planets.

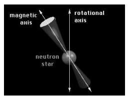


- Interstellar medium.

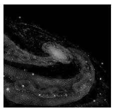
- Sun and stars.



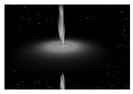
- Neutron stars and pulsars.



- Galaxies.



- Accretion disks.



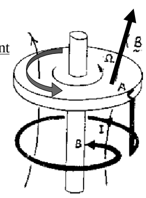
What is a dynamo?

- Dissipative timescale of a magnetic field ⟹ $\tau_\eta \approx \frac{L^2}{\eta}$
- For the Earth: $\begin{cases} R_{earth} \approx 6.10^8 \ cm \\ \eta_{core} \approx 2.10^5 \ \frac{cm^2}{s} \end{cases}$ ⟹ $\tau_\eta \approx 2.10^5 \ years$
- Astrophysical magnetic fields seem to live much longer than their τ_η ⟹ Magnetic fields need to be re-generated
- Dynamo mechanism ⟹ Efficient conversion from kinetic energy into magnetic energy

Homopolar dynamo

- It is probably the simplest dynamo. It consists of a rotating disk connected to a wire at points A & B.
- The current I induces a vertical magnetic field **B** which in turn creates an electromotive force. Under proper conditions, this EMF can cause the exponential growth of **B**.

Hall current



$$L\frac{dI}{dt} + RI = \frac{\omega M}{2\pi} I$$

- The inclusion of a Hall current in the disk, also induces a magnetic field which can cause further enhancement of this dynamo (Heintzmann 1983).

MHD & Hall-MHD

- In a two-fluid description (electrons and protons), the equation of motion for the electrons (neglecting their inertia) becomes

$$\vec{E} + \tfrac{1}{c}\vec{u}_e \times \vec{B} = \eta \vec{J}$$

- Therefore, the Hall-MHD equations in units of L_0 and U_0 are

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u}\bullet\nabla)\vec{u} - \frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\nabla\times\vec{B})\times\vec{B} + \nu\nabla^2\vec{u}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}_e\times\vec{B}) + \eta\nabla^2\vec{B}, \qquad \nabla\bullet\vec{u} = 0 = \nabla\bullet\vec{B}$$

$$where \qquad \vec{u}_e = \vec{u} - \varepsilon\,\nabla\times\vec{B}$$

- The relevance of the Hall term is measured by the dimensionless parameter

$$\varepsilon = \frac{c}{w_{pi}L_0} = \sqrt{\frac{m_i c^2}{4\pi e^2 n L_0^2}}$$

Kinematic dynamo

- If we assume **B** to be negligibly small, we can decouple the MHD equations. We can first integrate the Navier-Stokes equation

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u}\bullet\nabla)\vec{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} \quad , \qquad \nabla\bullet\vec{u} = 0$$

- Once $\vec{u}(\vec{x},t)$ is known, we integrate the induction equation to obtain $\vec{B}(\vec{x},t)$

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) + \eta\nabla^2\vec{B}, \qquad \nabla\bullet\vec{B} = 0$$

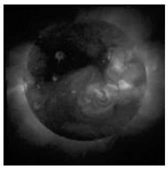
- This approximation is known as kinematic dynamo. Note that the induction equation is linear. For stationary flows, we have dynamo whenever

$$\vec{B}(\vec{x},t) = \vec{B}_0(\vec{x})e^{\gamma t} \quad , \qquad \gamma > 0$$

What kind of permanent flows are present in astrophysical objects ?

Rotation

- Rotation is ubiquitous among astrophysical objects.



- There is plenty of kinetic energy in most rotating objects to drive dynamos and generate magnetic fields (Omega effect). However, for axisymmetric cases.

$$\vec{B} = \nabla \times (\hat{\phi} A) + \hat{\phi} B_\phi \quad , \qquad \vec{u} = w r \sin\theta \, \hat{\phi}$$

- The induction equation reduces to

$$\frac{\partial B_\phi}{\partial t} - \eta \nabla^2 B_\phi = \hat{\phi} \bullet \nabla \times (\vec{u} \times \nabla \times (\hat{\phi} A))$$

$$\frac{\partial A}{\partial t} - \eta \nabla^2 A = 0$$

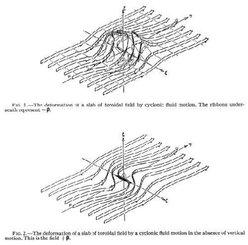
Growth of toroidal field requires the poloidal component to be non zero.

which expresses Cowling´s anti-dynamo theorem:

“An axisymmetric magnetic field cannot be mantained via dynamo action”

Convection is also important

- Convective motions involve smaller scales.
- Flows rising to the surface carry magnetic fields, which will be predominantly toroidal (if the Omega-effect succeeded).
- The Coriolis force produces helicoidal or cyclonic motions, which will tilt these toroidal fieldlines and return the poloidal component part of its energy (Parker 1955).
- The role of the small scales on the generation of the large scale magnetic field can be modeled through an electric field

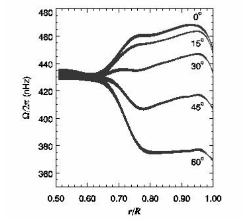


$$\vec{E} = \vec{\alpha} \bullet \vec{B}$$

which acts on the induction equation. This is called the alpha effect.

The solar tachocline

- Helioseismic observations show the rotation profiles at different latitudes and depths.
- The radial gradient concentrates in a narrow layer at 0.70-0.75 solar radii, called tachocline.
- The core of the Sun (below the convective region) rotates like a solid body.



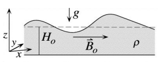
- Since the Omega-effect is concentrated in the tachocline, we developed a model for the dynamics of this region, using the shallow water approximation.

The SMHD equations

$$\frac{\partial}{\partial t}(1+h) = -\nabla \cdot [(1+h)\vec{u}]$$

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u} \cdot \nabla)\vec{u} + (\vec{B} \cdot \nabla)\vec{B} - gH\nabla(1+h) + \nu \nabla^2 \vec{u} + f \, \vec{u} \times \hat{z}$$

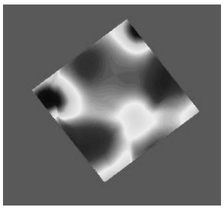
$$\frac{\partial \vec{B}}{\partial t} = -(\vec{u} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{u} + \eta \nabla^2 \vec{B}$$



$$\vec{u} = u_x(x,y)\hat{x} + u_y(x,y)\hat{y}$$
$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$
$$\nabla = \hat{x}\partial_x + \hat{y}\partial_y$$

$$u_z = -z \nabla \cdot \vec{u}$$
$$B_z = -z \nabla \cdot \vec{B}$$
$$\nabla(p + \frac{B^2}{2}) = gH\nabla(1+h)$$

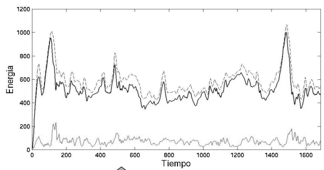
Numerical simulations



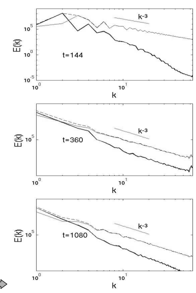
Surface *h(x,y,t)*

- Two dimensional box with periodic boundary conditions.
- Pseudo-spectral scheme for spatial derivatives.
- Fully dealiased using 2/3.
- Second order Runge-Kutta for temporal integration.
- The code satisfies energy balance with high accuracy.

Shallow water turbulence

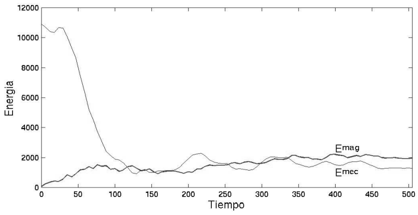


Kinetic and potential energy vs. time.



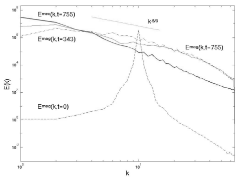
Energy power spectra at different times.

Amplification of magnetic energy



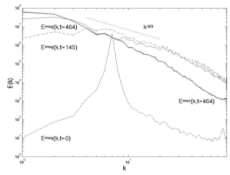
Mechanic and magnetic energies vs. time.
Parameters: ks=7, Pr=1 y N=256

Macroscopic magnetic field

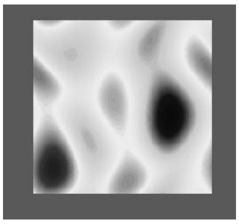


Simulation with Pr=1 y N=256

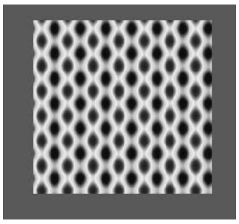
Simulation with Pr=10 y N=192



Time evolution

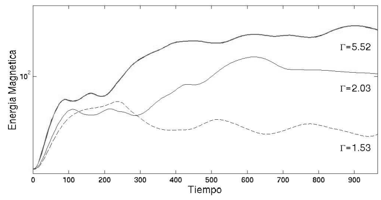


Vorticity



Electric current density

Dynamo efficiency



Simulations with different intensities of helicity.
Parameters: k_s=7, Pr=1000 y N=128

Mean field theory

- It provides a quantitative expresion for the coefficient alpha. The first assumption is that there is a scale separation between the large scale magnetic field being generated and the small scale convective motions, i.e

$$\vec{B} \rightarrow \vec{B} + \vec{b} \quad , \quad \vec{u} \rightarrow \vec{U} + \vec{u} \quad , \quad <\vec{b}> = 0 = <\vec{u}>$$

where <...> is an average over small scales. To compute the evolution of the mean field, we average the induction equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{U} \times \vec{B}) + \nabla \times <\vec{u} \times \vec{b}> \quad , \quad \nabla \bullet \vec{B} = 0$$

- The extra term can be interpreted as an electromotive force exerted by small scale motions

$$\varepsilon_{EMF} = <\vec{u} \times \vec{b}>$$

- We still need to obtain an expresion for electromotive force, and that requires some assumptions (Steenbeck, Krause & Radler 1966).

Mean field theory

- Let us substract the averaged equation from the general induction equation

$$\frac{\partial \vec{b}}{\partial t} = \underbrace{\nabla \times (\vec{U} \times \vec{b})}_{[1]} + \nabla \times (\vec{u} \times \vec{B}) + \nabla \times \underbrace{(\vec{u} \times \vec{b} - <\vec{u} \times \vec{b}>)}_{[2]} \quad , \quad \nabla \bullet \vec{b} = 0$$

[1] Can be removed with a Galilean transformation.
[2] It´s a departure from average of a second order quantity (FOSA).

- Let us further assume that this system evolves in a typical correlation time of these small scale convective motions. Therefore

$$\varepsilon_{EMF} = \tau <\vec{u} \times \nabla \times (\vec{u} \times \vec{B})> = \vec{\alpha} \bullet \vec{B} - \vec{\beta} \bullet \nabla \times \vec{B}$$

where we neglected the gradient of the large scale magnetic field.

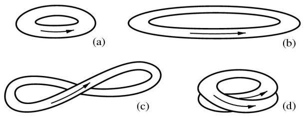
- For an isotropic state of these small scale flows, these tensors become

$$\alpha_{ij} = -\frac{\tau}{3} <\vec{u} \bullet \nabla \times \vec{u}> \delta_{ij} \qquad \beta_{ij} = \frac{\tau}{2} <\vec{u} \bullet \vec{u}> \delta_{ij}$$

The kinetic helicity of convective flows is important for dynamo activity.

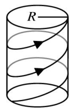
MHD dynamos

- Stretch-twist-fold (Vainshtein-Zeldovich 1972)



- Ponomarenko (1973)

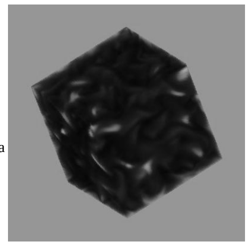
$$U = \begin{cases} (0, \omega r, V) & r < R \\ 0 & r > R \end{cases}$$



- 3D helicoidal flows

Turbulent dynamos

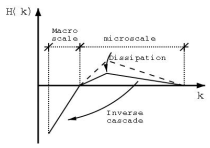
- We developed a 3D code to integrate the equations of incompressible MHD with and without the Hall effect.
- The boundary conditions are periodic, the spatial derivatives are computed with a pseudo-spectral scheme and the time forwarding is performed with Runge-Kutta
- We first perform a HD run (the magnetic field is set to zero), with a helical forcing at small wavenumbers.

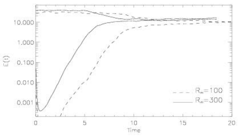


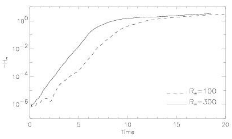
- We integrate until a turbulent stationary regime is established, and add a weak magnetic seed at large wavenumbers. The movie shows the growth of magnetic energy in time as well as its spatial distribution.

Energy and helicity

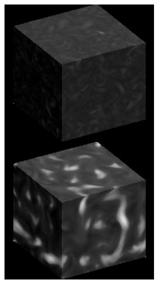
- Transfer from kinetic to magnetic energy.
- Initially, magnetic energy grows exponentially fast, as expected for a kinematic dynamo.
- The efficiency is higher at larger Reynolds.
- Magnetic helicity grows from zero to finite values because of the preferential dissipation of helicity of one sign.

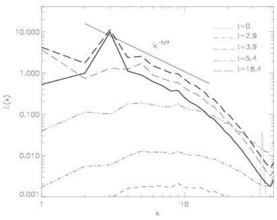






Energy power spectrum

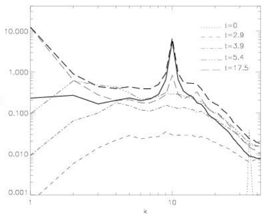




- The magnetic energy spectrum grows until it reaches equipartition at each scale.
- Full line is kinetic energy and dotted line is total energy.

Energy power spectrum

- The magnetic energy spectrum grows until it reaches equipartition at each scale.
- Full line is kinetic energy and dotted line is total energy.
- When the forcing is at intermediate scales, the preferential accumulation of magnetic energy at the largest scales can be observed.
- Furthermore, these magnetically dominated states are approximately force-free.



Hall dynamo

$$\varepsilon = \sqrt{\frac{m_i c^2}{4\pi e^2 n L_0^2}} \geq 1$$

- The Hall effect is expected to be relevant in a number of astrophysical plasmas ⇒
  - Dense molecular clouds (Wardle & Ng 1999)
  - Accretion disks (Balbus & Terquem 2001)
  - Neutron stars (Urpin & Yakovlev 1980)
  - Early universe (Tajima et al. 1992)
- However, it was not being properly included when considering dynamo activity. We studied the role of the Hall term on turbulent dynamos ⇒
  - MFT Hall-MHD (MGM 2002, ApJ 567, L81)
  - Dynamo waves (MGM 2003a, ApJ, 584, 1120)
  - 3D simulations (MGM 2003b, ApJ, 587, 472)
- We find that it can strongly affect the properties of the dynamo.

MFT on Hall dynamos

- We also use scale separation but adopt the reduced smoothing approximation (RSA, Blackman & Field 1999) rather than FOSA, since quadratic small-scale quantities are not necessarily small.

$$\vec{B} \to \vec{B} + \vec{b} + \vec{b}_0 \quad , \quad \vec{u} \to \vec{U} + \vec{u} + \vec{u}_0$$

where $\vec{u}_0 , \vec{b}_0$ are the small-scale equilibria in the absence of $\vec{U}_0 , \vec{B}_0$

- For the full MHD equations, the obtained alpha is (Frisch et al. 1976)

$$\alpha = -\frac{\tau}{3}(<\vec{u}_0 \bullet \nabla\times\vec{u}_0> - <\vec{b}_0 \bullet \nabla\times\vec{b}_0>)$$

which cancels out if the system reaches an alfvenic state $\vec{u}_0 = \pm\vec{b}_0$

- When the Hall effect is considered, we obtain

$$\alpha = -\frac{\tau}{3}(<\vec{u}_0^e \bullet \nabla\times\vec{u}_0^e> - <\vec{b}_0 \bullet \nabla\times\vec{b}_0> + <\vec{b}_0 \bullet \nabla\times\vec{\omega}_0^e>)$$

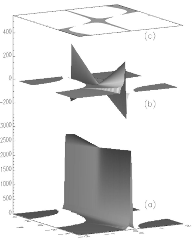
which does not vanish for alfvenic states.

MFT on Hall dynamos

- We assume the microscopic states $\vec{u}_0 , \vec{b}_0$ to be given by double-Beltrami stationary solutions of the Hall-MHD equations, i.e.

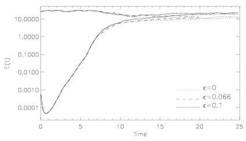
$$\vec{u}_0 - \varepsilon\nabla\times\vec{b}_0 = \frac{1}{a}\vec{b}_0 \quad , \quad \vec{b}_0 + \varepsilon\nabla\times\vec{u}_0 = d\vec{u}_0$$

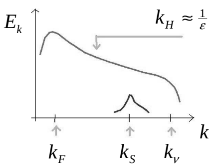
- As a function of parameters (a,d), the figure shows:
  - Region of large scale separation
  - The ratio $\alpha_{Hall} / \alpha_{MHD}$
  - The ratio u_0^e / u_0
- Regions of strong intensification of alpha can be be observed for $a \to 0$, which corresponds to cases where $u_0^e / u_0 >> 1$



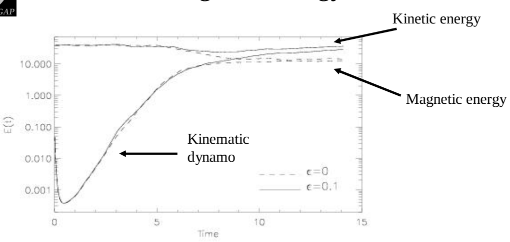
Hall-enhanced dynamos

- Magnetic and kinetic energy vs. time for different Hall intensities. Equipartition is obtained for all these cases.
- The dynamo enhancement is larger as epsilon is gradually increased. However, this behavior is not monotonic.
- We have runs for ε = 0.066, 0.1, 0.2, 0.5, 1. We classify these runs according to
  - $L_{Hall} > L_{force} > L_{seed}$: Hall is relevant at all scales *(massively-Hall dynamo)*
  - $L_{force} > L_{Hall} > L_{seed}$: Hall is relevant in the microscale *(microscale-Hall dynamo)*
  - $L_{force} > L_{seed} > L_{Hall}$: *Quasi-MHD dynamo?*





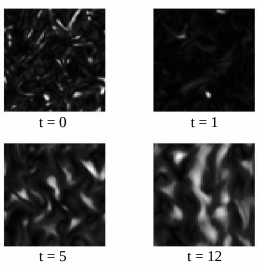
Magnetic energy



Magnetic energy as a function of time
• MHD: Equipartition between magnetic and kinetic energy
• Hall-MHD: Equipartition with twice the magnetic energy

Turbulent Hall dynamo

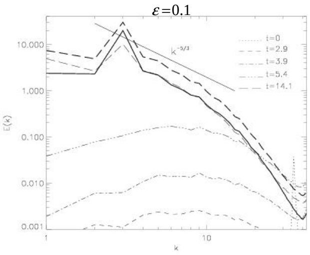
- Magnetic field intensity at different times (t=0,1,5 and 12) for $\varepsilon = 0.2$ are shown.
- The initial magnetic field is located at small scales. Both an enhancement of magnetic intensity and the formation of large scale patterns can be observed.
- We performed several runs with different values of the Hall intensity (epsilon).



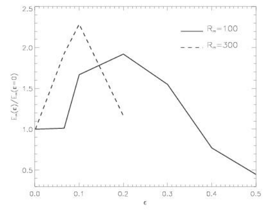
Energy power spectrum

- Kinetic and magnetic energy spectra for the case $\varepsilon = 0.1$
- Full line is kinetic energy and dotted line is total energy.
- The magnetic energy spectrum grows until it reaches equipartition at each scale.
- As far as we know, this are the first results for Hall-MHD turbulence. The energy spectrum is consistent with Kolmogorov's.

$\varepsilon = 0.1$



The Hall length scale



- *Massively Hall dynamo* ⟹ inhibits dynamo activity
- *Microscale Hall dynamo* ⟹ enhances dynamo activity
- As $\varepsilon \rightarrow 0$ ⟹ classical MHD dynamo
- At higher Reynolds, the dynamo is more efficient at smaller ε

Magnetic helicity in mean field

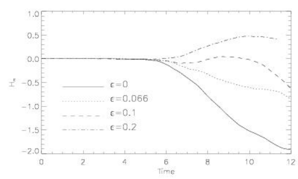
- From the mean field induction equation (MHD & Hall-MHD)

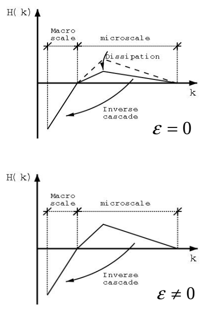
$$\frac{\partial \overline{H}}{\partial t} = 2\int\left(\alpha \overline{B}^2 - \eta_{eff}\,\overline{J}\cdot\overline{B}\right)d^3x$$

- Mean field helicity grows with the same sign as α
- Net magnetic helicity is conserved (except for difussion):
- The Alpha-effect creates opposite amounts of micro an macroscale helicity.
- At small scales, dissipation of microscale helicity takes place with

$$\left|\frac{\Delta h}{h}\right| \leq \sqrt{\frac{\Delta t}{\tau_d}}$$ *(Berger, 1999)*

Magnetic helicity





- As ε increases generation of net magnetic helicity decreases.
- Dissipation destroys short scale helicity.

Conclusions

- We performed a brief (and incomplete) overview of our understanding on the dynamo problem.
- We developed a mean field closure for the Hall MHD dynamo. We find that the Hall effect can enhance dynamo activity. This result is relevant to astrophysical systems such as accretion disks, neutron stars or molecular clouds, for which Hall is non-negligible.
- Our 3D simulations also show dynamo enhancement caused by Hall. We find that the enhancement takes place when the Hall scale remains smaller than the forcing scale.
- Incidentally, our simulations also show that both MHD and Hall-MHD stationary turbulent regimes display a Kolmogorov energy spectrum.