united nations educational, scientific and cultural organization
international atomic energy agency

the abdus salam international centre for theoretical physics

ICTP 40th Anniversary

SCHOOL ON SYNCHROTRON RADIATION AND APPLICATIONS
In memory of J.C. Fuggle & L. Fonda

19 April - 21 May 2004

Miramare - Trieste, Italy

1561/7

Synchrotron Radiation (Part 1 - 2 - 3 - 4)

Leonid Rivkin

Synchrotron Radiation
An Introduction

L. Rivkin

Swiss Light Source

Books

Helmut Wiedemann

- Synchrotron Radiation
  Springer-Verlag Berlin Heidelberg 2003

A. W. Chao, M. Tigner

- Handbook of Accelerator Physics and Engineering
  World Scientific 1999

A. A. Sokolov, I. M. Ternov

- Synchrotron Radiation
  Pergamon, Oxford 1968

CERN Accelerator School Proceedings

Synchrotron Radiation and Free Electron Lasers

- Grenoble, France, 22 - 27 April 1996
  (in particular A. Hofmann's lectures on synchrotron radiation)
  CERN Yellow Report 98-04

  http://cas.web.cern.ch/cas/CAS_Proceedings-DB.html

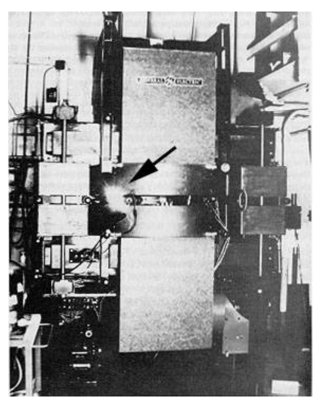
- Brunnen, Switzerland, 2 – 9 July 2003

  http://cas.web.cern.ch/cas/BRUNNEN/lectures.html

Crab Nebula
6000 light years away

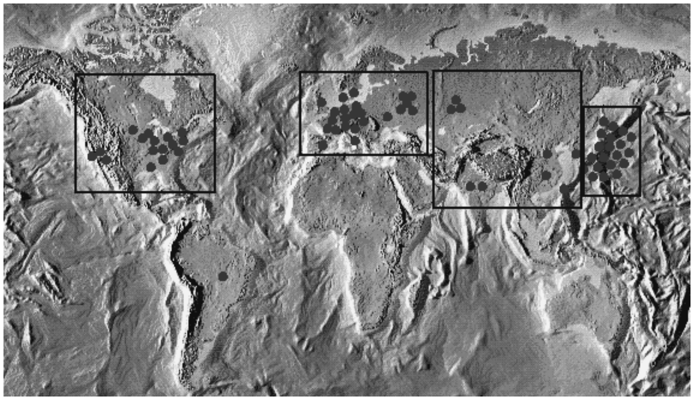
First light observed
1054 AD

GE Synchrotron
New York State



First light observed
1947

20 000 users world-wide

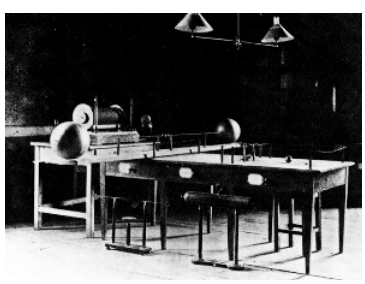


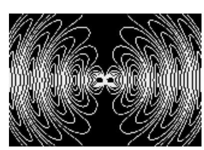
THEORETICAL UNDERSTANDING →

1873 Maxwell's equations

→ made evident that changing charge densities would result in electric fields that would radiate outward

1887 Heinrich Hertz demonstrated such waves:





….. this is of no use whatsoever !

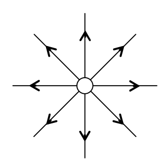
Maxwell equations (poetry)

War es ein Gott, der diese Zeichen schrieb
Die mit geheimnisvoll verborg'nem Trieb
Die Kräfte der Natur um mich enthüllen
Und mir das Herz mit stiller Freude füllen.
Ludwig Boltzman

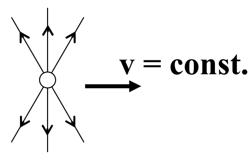
Was it a God whose inspiration
Led him to write these fine equations
Nature's fields to me he shows
And so my heart with pleasure glows.
translated by John P. Blewett

Why do they radiate?

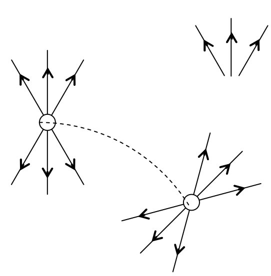
Charge at rest: Coulomb field



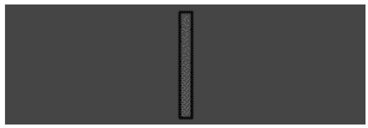
Uniformly moving charge



Accelerated charge



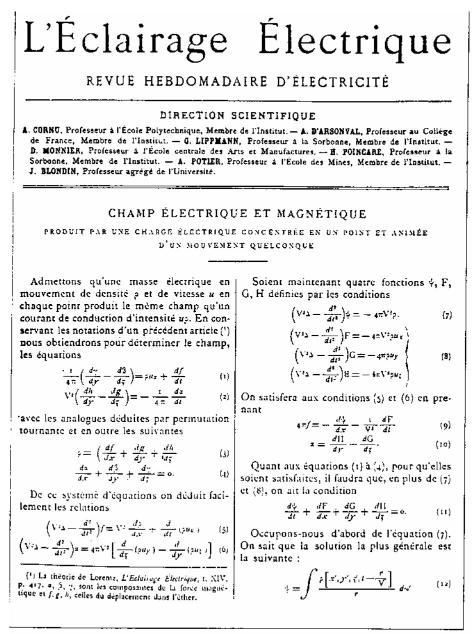
Bremstrahlung



1898 Liénard:

ELECTRIC AND MAGNETIC FIELDS PRODUCED BY A POINT CHARGE MOVING ON AN ARBITRARY PATH

(by means of retarded potentials)



L'Éclairage Électrique

REVUE HEBDOMADAIRE D'ÉLECTRICITÉ

DIRECTION SCIENTIFIQUE

A. CORNU, Professeur à l'École Polytechnique, Membre de l'Institut. — **A. D'ARSONVAL**, Professeur au Collège de France, Membre de l'Institut. — **G. LIPPMANN**, Professeur à la Sorbonne, Membre de l'Institut. — **D. MONNIER**, Professeur à l'École centrale des Arts et Manufactures. — **H. POINCARÉ**, Professeur à la Sorbonne, Membre de l'Institut. — **A. POTIER**, Professeur à l'École des Mines, Membre de l'Institut. — **J. BLONDIN**, Professeur agrégé de l'Université.

CHAMP ÉLECTRIQUE ET MAGNÉTIQUE

PRODUIT PAR UNE CHARGE ÉLECTRIQUE CONCENTRÉE EN UN POINT ET ANIMÉE D'UN MOUVEMENT QUELCONQUE

Admettons qu'une masse électrique en mouvement de densité ρ et de vitesse u en chaque point produit le même champ qu'un courant de conduction d'intensité $u\rho$. En conservant les notations d'un précédent article (1) nous obtiendrons pour déterminer le champ, les équations

$$\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)=\rho u_x+\frac{df}{dt} \qquad (1)$$

$$V^2\left(\frac{dh}{dy}-\frac{dg}{dz}\right)=-\frac{1}{4\pi}\frac{d\alpha}{dt} \qquad (2)$$

avec les analogues déduites par permutation tournante et en outre les suivantes

$$\rho=\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right) \qquad (3)$$

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \qquad (4)$$

De ce système d'équations on déduit facilement les relations

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)f=V^2\frac{d\rho}{dx}+\frac{d}{dt}(\rho u_x) \qquad (5)$$

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\alpha=4\pi V^2\left[\frac{d}{dz}(\rho u_y)-\frac{d}{dy}(\rho u_z)\right] \qquad (6)$$

(1) La théorie de Lorentz, *L'Éclairage Électrique*, t. XIV, p. 417. α, β, γ, sont les composantes de la force magnétique et f, g, h, celles du déplacement dans l'éther.

Soient maintenant quatre fonctions ψ, F, G, H définies par les conditions

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\psi=-4\pi V^2\rho. \qquad (7)$$

$$\left.\begin{aligned}\left(V^2\Delta-\frac{d^2}{dt^2}\right)F&=-4\pi V^2\rho u_x\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)G&=-4\pi\rho u_y\\ \left(V^2\Delta-\frac{d^2}{dt^2}\right)H&=-4\pi V^2\rho u_z\end{aligned}\right\} \qquad (8)$$

On satisfera aux conditions (5) et (6) en prenant

$$4\pi f=-\frac{d\psi}{dx}-\frac{1}{V^2}\frac{dF}{dt} \qquad (9)$$

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz}. \qquad (10)$$

Quant aux équations (1) à (4), pour qu'elles soient satisfaites, il faudra que, en plus de (7) et (8), on ait la condition

$$\frac{d\psi}{dt}+\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0. \qquad (11)$$

Occupons-nous d'abord de l'équation (7). On sait que la solution la plus générale est la suivante :

$$\psi=\int\frac{\rho\left[x',y',z',t-\frac{r}{V}\right]}{r}\,d\omega' \qquad (12)$$

Fig. 1. First page of Liénard's 1898 paper.

Liénard-Wiechert potentials

$$\varphi(t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[r\left(1 - \vec{\mathbf{n}}\cdot\vec{\beta}\right)\right]_{ret}} \qquad \vec{\mathbf{A}}(t) = \frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{\vec{\mathbf{v}}}{r\left(1 - \vec{\mathbf{n}}\cdot\vec{\beta}\right)}\right]_{ret}$$

and the electromagnetic fields:

$$\nabla\cdot\vec{\mathbf{A}} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \qquad \text{(Lorentz gauge)}$$

$$\vec{\mathbf{B}} = \nabla\times\vec{\mathbf{A}}$$

$$\vec{\mathbf{E}} = -\nabla\varphi - \frac{\partial\vec{\mathbf{A}}}{\partial t}$$

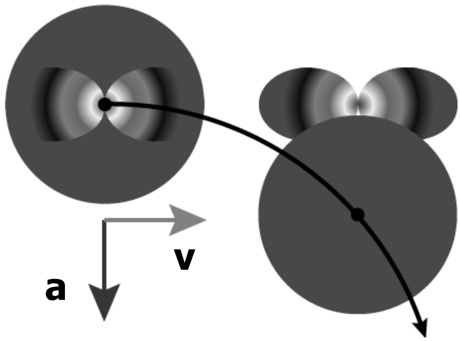
Fields of a moving charge

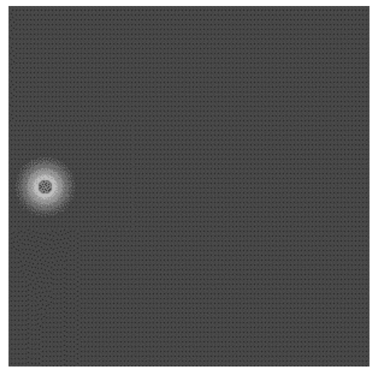
$$\vec{\mathbf{E}}(t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{\mathbf{n}} - \vec{\beta}}{\left(1 - \vec{\mathbf{n}}\cdot\vec{\beta}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}^2}}\right]_{ret} +$$

$$\frac{q}{4\pi\varepsilon_0 c}\left[\frac{\vec{\mathbf{n}}\times\left[\left(\vec{\mathbf{n}} - \vec{\beta}\right)\times\boxed{\dot{\vec{\beta}}}\right]}{\left(1 - \vec{\mathbf{n}}\cdot\vec{\beta}\right)^3\gamma^2}\cdot\boxed{\frac{\mathbf{1}}{\mathbf{r}}}\right]_{ret}$$

$$\vec{\mathbf{B}}(t) = \frac{1}{c}\left[\vec{\mathbf{n}}\times\vec{\mathbf{E}}\right]$$

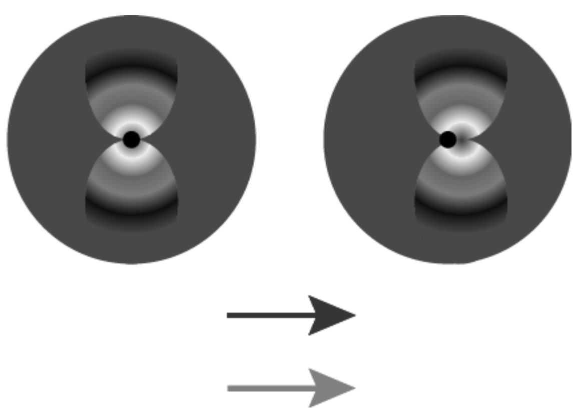
Transverse acceleration





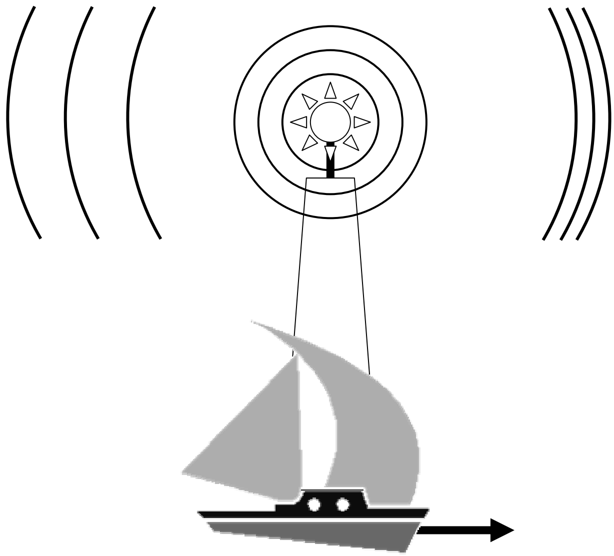
Radiation field quickly separates itself from the Coulomb field

Longitudinal acceleration



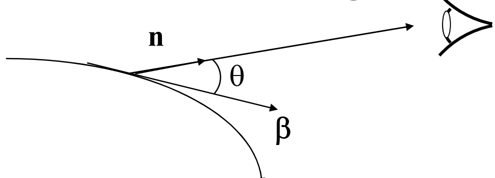
Radiation field cannot separate itself from the Coulomb field

Moving Source of Waves



Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta})\, T_{emit}$$

The wavelength is shortened by the same factor

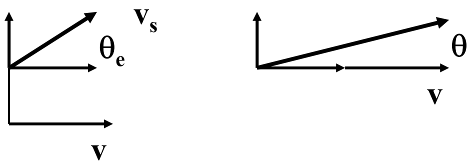
$$\lambda_{obs} = (1 - \beta \cos\theta)\, \lambda_{emit}$$

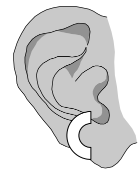
in ultra-relativistic case, looking along a tangent to the trajectory

$$\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}$$ **since** $$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}$$

Angular Collimation

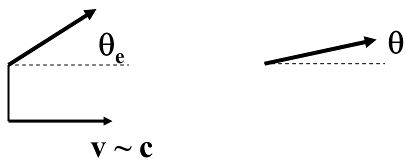
Galileo: sound waves v_s = 331 m/s

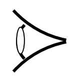




$$\theta = \frac{v_{s\perp}}{v_{s\|} + \mathbf{v}} = \frac{v_{s\perp}}{v_{s\|}} \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}} \approx \theta_e \cdot \frac{1}{1 + \frac{\mathbf{v}}{v_s}}$$

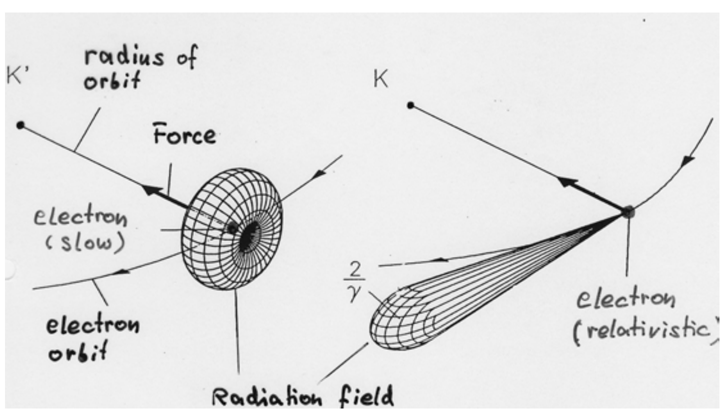
Lorentz: speed of light $c = 3 \cdot 10^8$ m/s

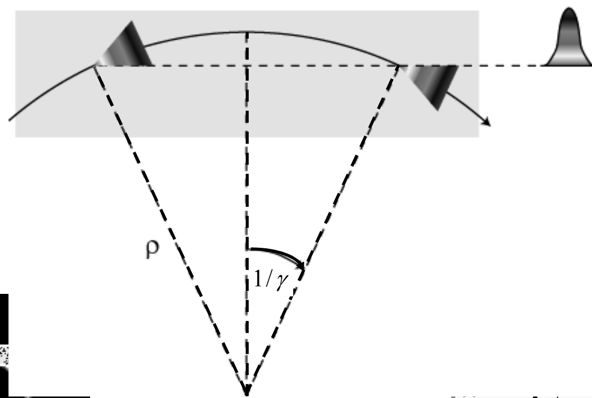




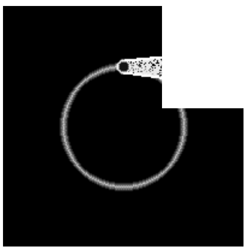
$$\theta = \frac{1}{\gamma} \cdot \theta_e$$

Radiation is emitted into a narrow cone





$$\frac{2\rho}{\gamma c}\cdot\frac{1}{2\gamma^2}$$



Δt $\gamma^3\omega_0$

Typical frequency of synchrotron light

Due to extreme collimation of light

- observer sees only a small portion of electron trajectory **(a few mm)**

$$l \sim \frac{2\rho}{\gamma}$$

- Pulse length: difference in times it takes an electron and a photon to cover this distance

$$\Delta t \sim \frac{l}{\beta c} - \frac{l}{c} = \frac{l}{\beta c}(1-\beta)$$

Synchrotron radiation power

Power emitted is proportional to: $\boldsymbol{P \propto E^2 B^2}$

$$P_{\mathrm{SR}} = \frac{cC_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2}$$

$$P_{\mathrm{SR}} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5}\left[\frac{\mathrm{m}}{\mathrm{GeV}^3}\right]$$

$$\alpha = \frac{1}{137}$$

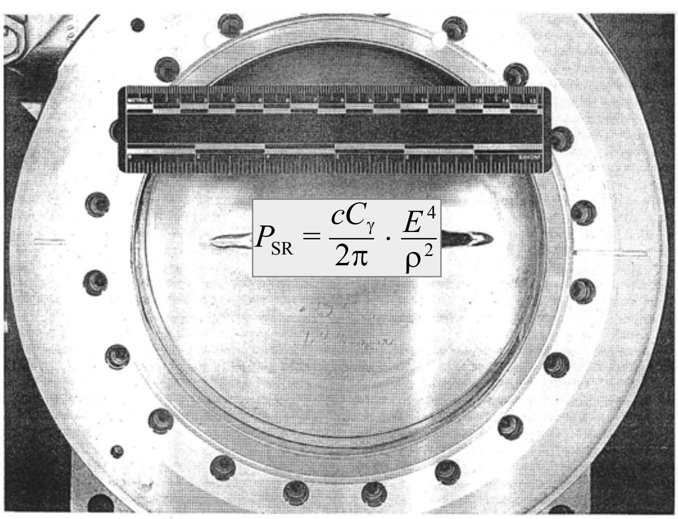
$$\hbar c = 197\ \mathrm{Mev} \cdot \mathrm{fm}$$

Energy loss per turn:

$$U_0 = C_\gamma \cdot \frac{E^4}{\rho}$$

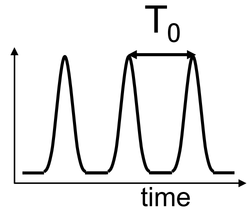
$$U_0 = \frac{4\pi}{3}\alpha\hbar c \frac{\gamma^4}{\rho}$$

The power is all too real!



ig. 12. Damaged X-ray ring front end gate valve. The power incident on the valve was approximately 1 kW for a duration estimated to 2–10 min and drilled a hole through the valve plate.

Spectrum of synchrotron radiation



- Synchrotron light comes in a series of flashes every T_0 (revolution period)

- the spectrum consists of harmonics of $$\omega_0 = \frac{1}{T_0}$$

- flashes are extremely short: harmonics reach up to very high frequencies $$\omega_{typ} \cong \gamma^3 \omega_0$$

$$\omega_0 \sim 1 \text{ MHz}$$
$$\gamma \sim 4000$$
$$\omega_{typ} \sim 10^{16} \text{ Hz}!$$

- At high frequencies the individual harmonics overlap

continuous spectrum !

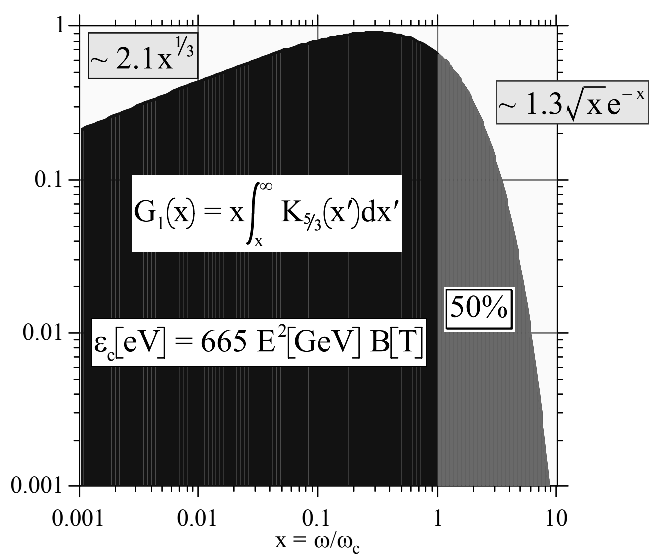
$$\frac{dP}{d\omega} = \frac{P_{tot}}{\omega_c} S\left(\frac{\omega}{\omega_c}\right)$$

$$S(x) = \frac{9\sqrt{3}}{8\pi} x \int_x^\infty K_{5/3}(x')dx'$$

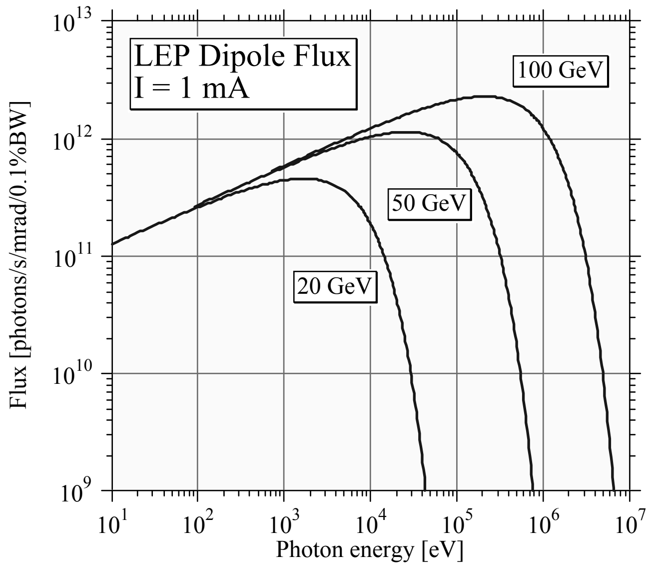
$$\int_0^\infty S(x')dx' = 1$$

$$P_{tot} = \frac{2}{3} \hbar c^2 \alpha \frac{\gamma^4}{\rho^2}$$

$$\omega_c = \frac{3}{2} \frac{c\gamma^3}{\rho}$$



Synchrotron radiation flux for different LEP energies



Flux from a dipole magnet:

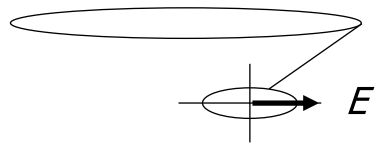
$$\mathrm{Flux}\left[\frac{\mathrm{photons}}{\mathrm{s}\cdot\mathrm{mrad}\cdot 0.1\%\mathrm{BW}}\right] = 2.46\cdot 10^{13}\,\mathrm{E[GeV]}\;\mathrm{I[A]}\,G_1(x)$$

Power density at the peak:

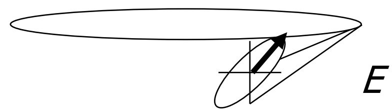
$$\frac{P_{tot}}{\omega_c} = \frac{4}{9}\,\alpha\hbar c\frac{\gamma}{\rho}$$

Polarisation

Synchrotron radiation observed in the plane of the particle orbit is horizontally polarized, i.e. the electric field vector is horizontal



Observed out of the horizontal plane, the radiation is elliptically polarized

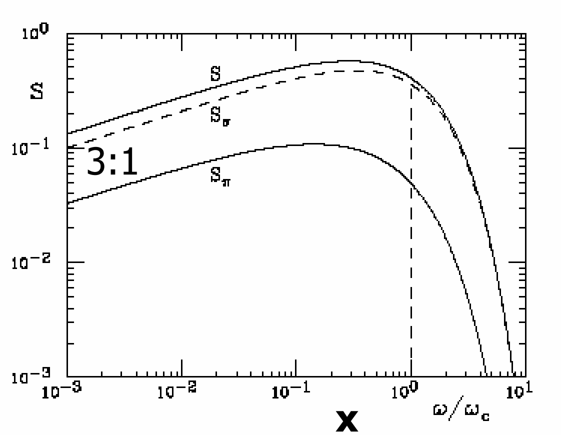


Polarisation: spectral distribution

$$\frac{dP}{d\omega} = \frac{P_{tot}}{\omega_c} S(x) = \frac{P_{tot}}{\omega_c} [S_\sigma(x) + S_\pi(x)]$$

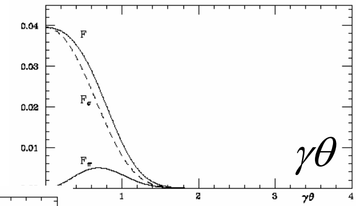
$$S_\sigma = \frac{7}{8} S$$

$$S_\pi = \frac{1}{8} S$$

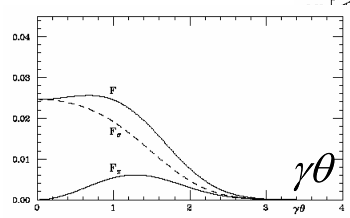


Angular divergence of radiation

•at the critical frequency



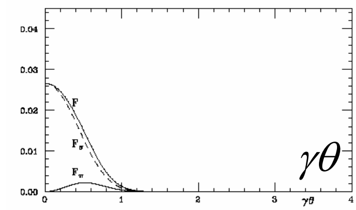
•well below



$\omega = 0.2\,\omega_c$

•well above

$\omega = 2\,\omega_c$



Angular divergence of radiation

The rms opening angle R'

- **at the critical frequency:** $\omega = \omega_c \qquad R' \approx \frac{0.54}{\gamma}$

- **well below** $\omega \ll \omega_c \qquad R' \approx \frac{1}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/3} \approx 0.4\left(\frac{\lambda}{\rho}\right)^{1/3}$

  independent of γ !

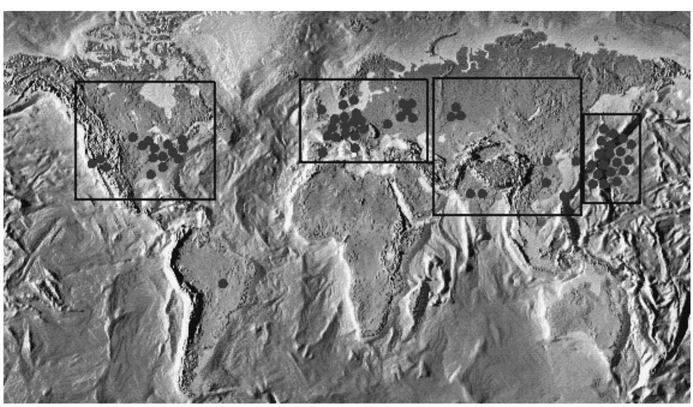
- **well above** $\omega \gg \omega_c \qquad R' \approx \frac{0.6}{\gamma}\left(\frac{\omega_c}{\omega}\right)^{1/2}$

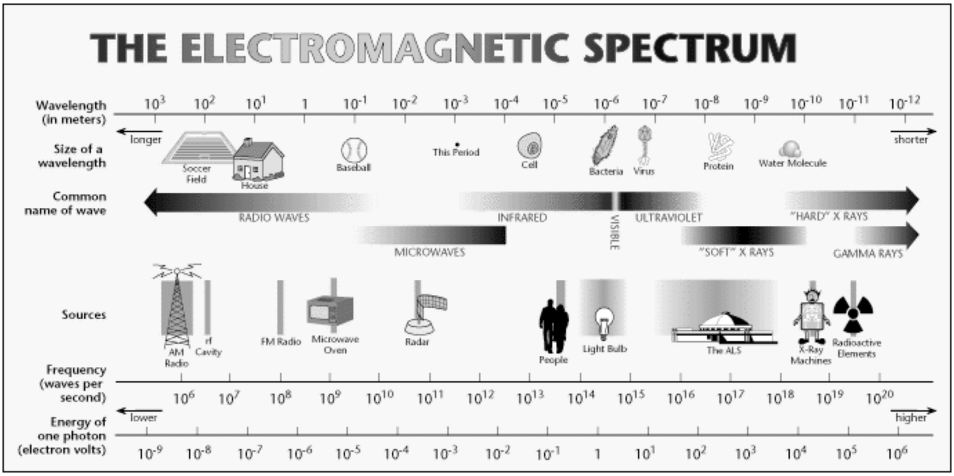
Synchrotron Radiation
Sources and properties

L. Rivkin

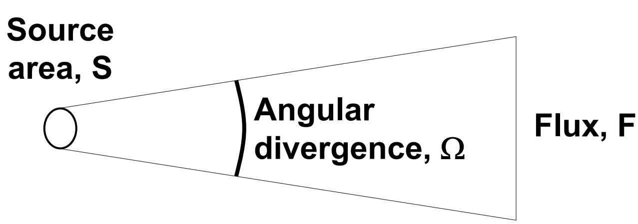
Swiss Light Source

20 000 users world-wide



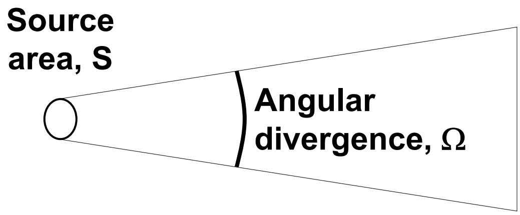


The “brightness” of a light source:



$$\text{Brightness} = \text{constant} \times \frac{F}{S \times \Omega}$$

The electron beam "emittance":



The brightness depends on the geometry of the source, i.e., on the electron beam emittance

Emittance = S x Ω

WHAT DO USERS EXPECT FROM A HIGH PERFORMANCE LIGHT SOURCE ?

- **PROPER PHOTON ENERGY FOR THEIR EXPERIMENTS**
- **BRILLIANCE** ⟶ $$\mathrm{B} = \frac{\Phi}{(2\pi)^2 \Sigma_x \Sigma_{x'} \Sigma_y \Sigma_{y'}}$$
- **STABILITY**

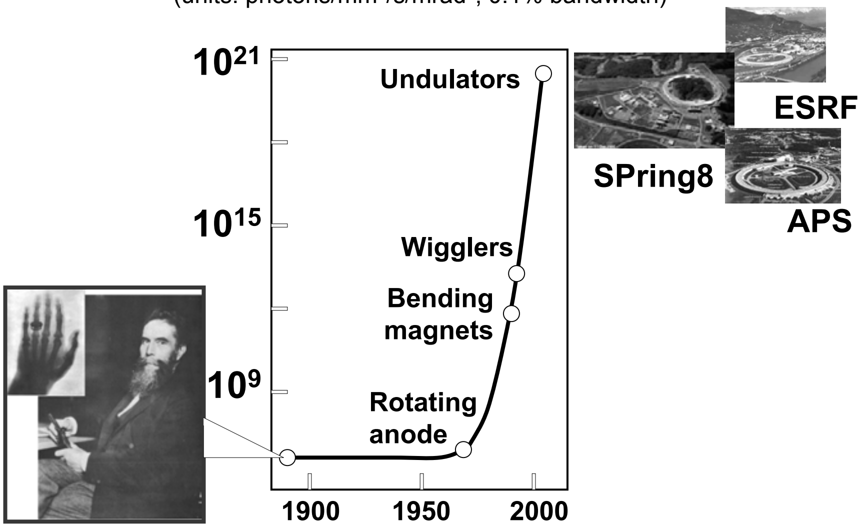
FIGURE OF MERIT

$$\Sigma^2 = \sigma_e^2 + \sigma_\gamma^2 \qquad \Sigma_x \Sigma_{x'} \approx \sigma_x \sigma_x^{'} \sim \varepsilon_x$$

Photon beam size (U): $\sigma_\gamma^{'} = \sqrt{\frac{\lambda}{L}} \qquad \sigma_\gamma = \frac{\sqrt{\lambda L}}{4\pi}$

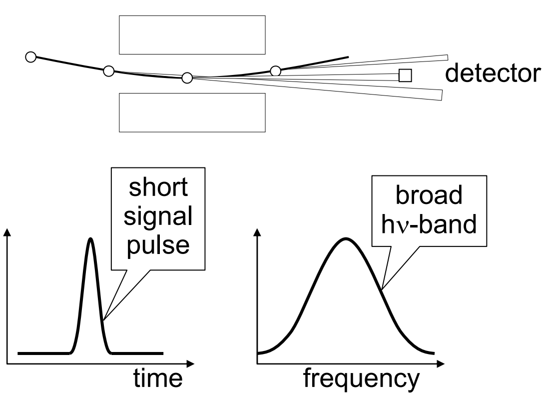
Steep rise in brightness/brilliance

(units: photons/mm^2/s/mrad^2, 0.1% bandwidth)



3 types of storage ring sources:

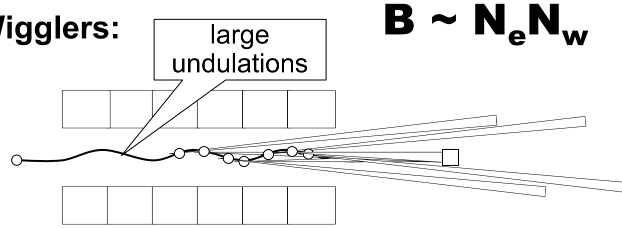
1. **Bending magnets:** $B \sim N_e$

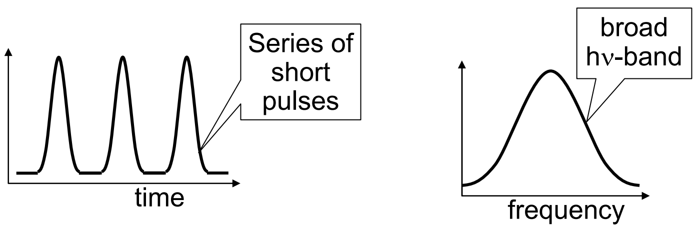


3 types of storage ring sources:

2. Wigglers:

$$B \sim N_e N_w \quad x10$$

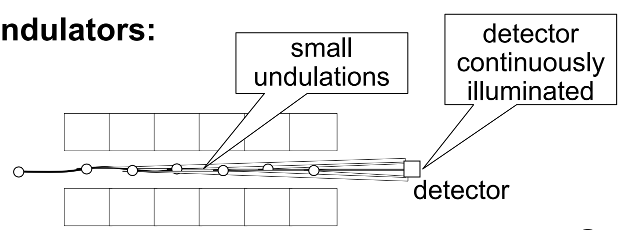




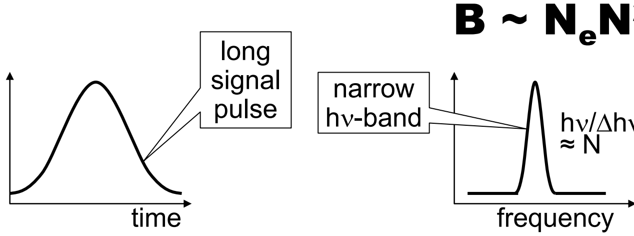
G. Margaritondo

3 types of storage ring sources:

3. Undulators:



$$B \sim N_e N^2_u \quad x10^3$$



G. Margaritondo

The three generations

1. First experiments

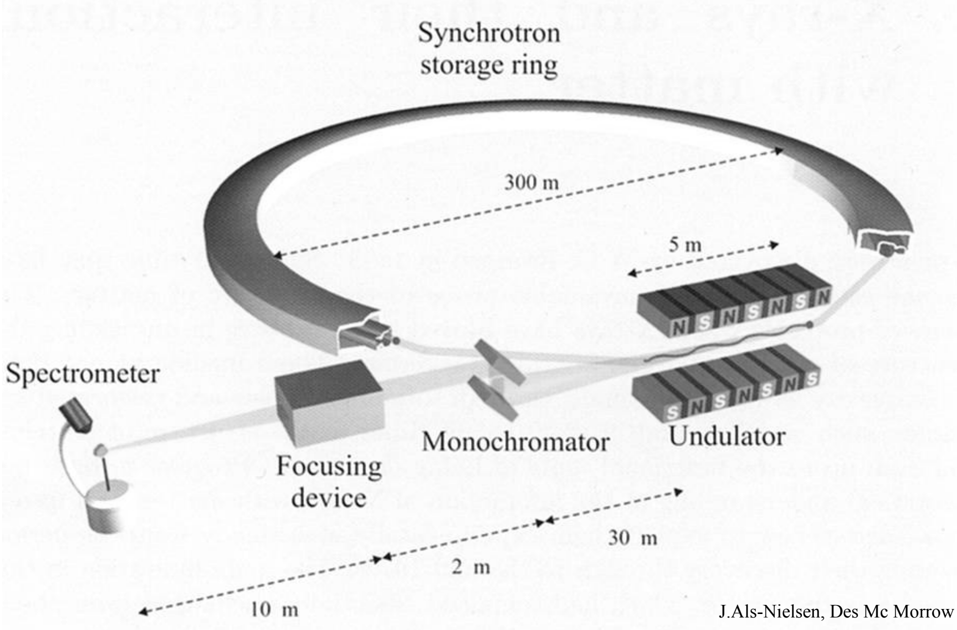
2. Basic phenomena, new methods
   tunability, flux
   photoeffect, X-rays

3. Brightness, coherence, time structure

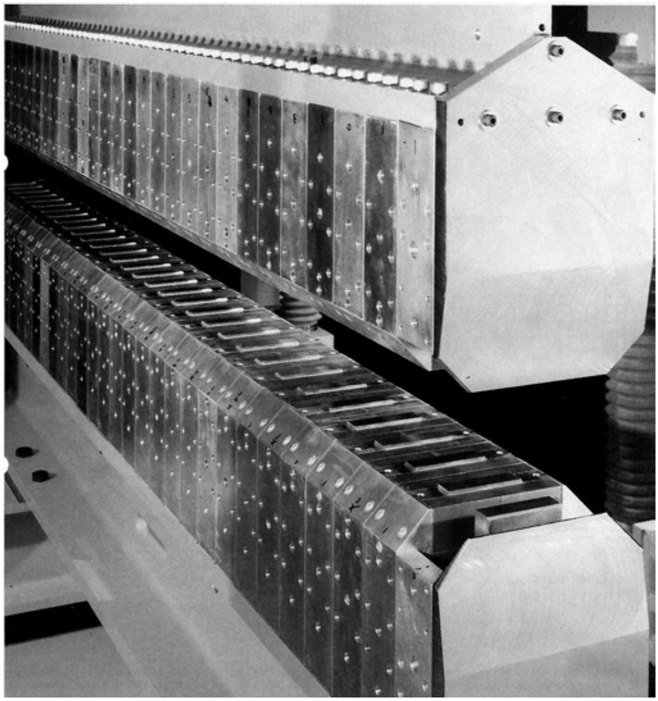
   From rings to linacs (ERLs) to X-ray FELs:
   new community
   new techniques

FIRST GENERATION

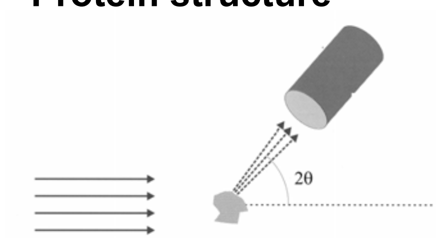
About 60 ring sources world-wide

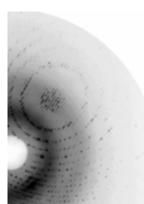


J.Als-Nielsen, Des Mc Morrow

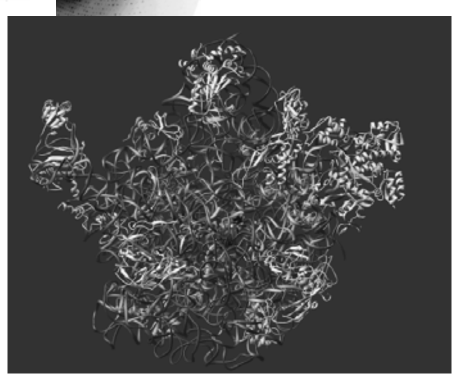


Protein structure

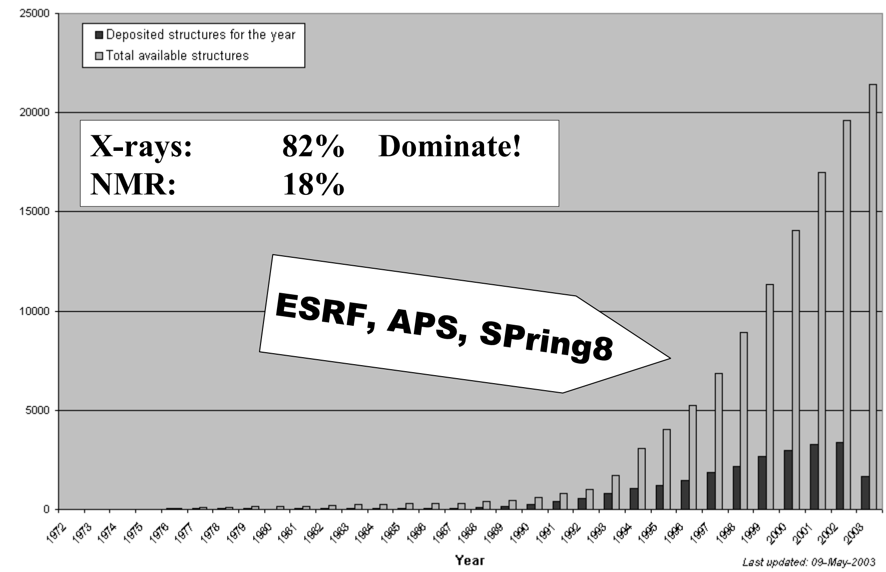




Diffraction pattern



Protein Data Bank



THERE IS AN INCREASING NEED

FOR HIGHER PHOTON ENERGIES !

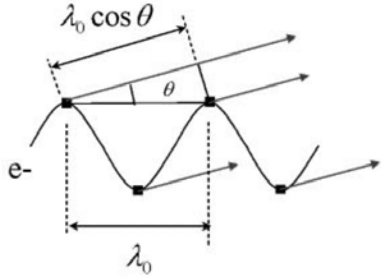
Medium energy machines can only get there by:

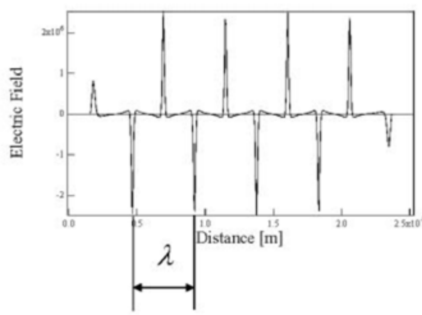
- SMALL PERIOD (LOW GAP) UNDULATORS

- THE USE OF HIGHER HARMONICS

- SOLVE LIFETIME PROBLEM + STABILITY WITH TOP UP INJECTION

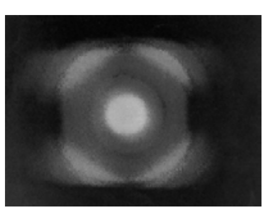
Radiation Field from a Planar Undulator in time Domain

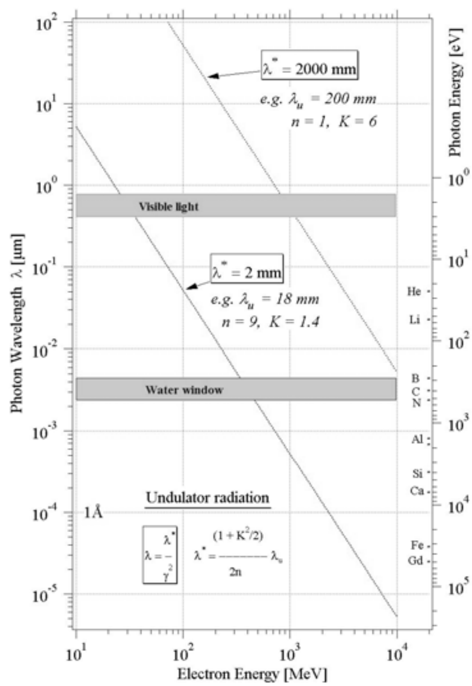


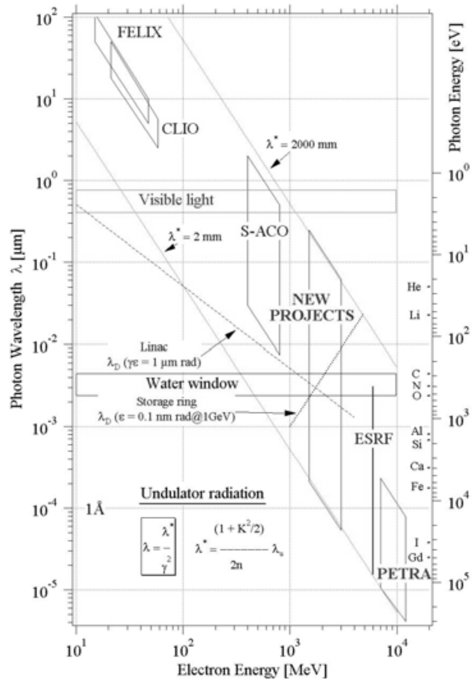


$$\lambda = c\left(\frac{\lambda_0}{v_s} - \frac{\lambda_0}{c}\cos\theta\right) \cong \lambda_0\left(1 - \frac{v_s}{c} + \frac{\theta^2}{2}\right) \cong \frac{\lambda_0}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

I, 12/38 , P. Elleaume, CAS, Brunnen July 2-9, 2003.







Undulator based sources

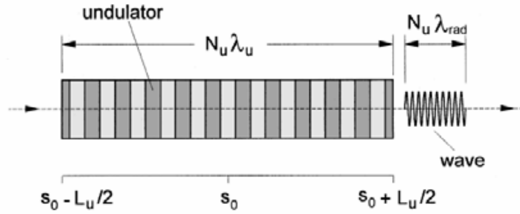
Brightness $$B = \frac{N_{ph}}{\Delta t} \cdot \frac{1}{\Delta S \cdot \Delta \Omega} \cdot \frac{1}{\Delta\lambda/\lambda}$$

Flux $N_{ph} \propto N_u$ **(periods)**

The line width $\frac{\Delta\lambda}{\lambda} \sim \frac{1}{N_u}$ **if** $\frac{1}{N_u} > 2\pi \cdot \frac{\sigma_E}{E}$

If energy spread is small enough $B \sim N_u^2$

Undulator line width



Undulator of infinite length $N_u = \infty \quad \Rightarrow \quad \frac{\Delta\lambda}{\lambda} = 0$

Finite length undulator

- radiation pulse has as many periods as the undulator
- the line width is $\frac{\Delta\lambda}{\lambda} \sim \frac{1}{N_u}$

Due to the electron energy spread $\frac{\Delta\lambda}{\lambda} = 2\frac{\sigma_E}{E}$

Undulator line width

$$I(\Delta\omega) \propto \left[\frac{\sin\left(\pi N_u \frac{\Delta\omega}{\omega_w}\right)}{\pi N_u \frac{\Delta\omega}{\omega_w}} \right]^2$$

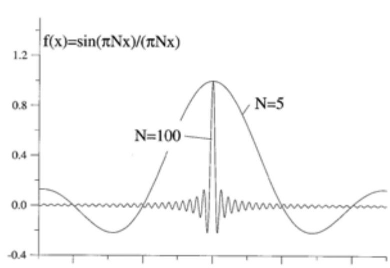


Fig. 10.7. $\frac{\sin \pi N_p x}{\pi N_p x}$ distribution for $N_p = 5$ and $N_p = 100$

Electron Dynamics
with radiation

L. Rivkin

Swiss Light Source

Radiation effects in electron storage rings

Average radiated power restored by RF

$U_0 \cong 10^{-3}$ of E_0

- **Electron loses energy each turn**
- **RF cavities provide voltage to accelerate electrons back to the nominal energy**

$V_{RF} > U_0$

Radiation damping

- **Average rate of energy loss produces DAMPING of electron oscillations in all three degrees of freedom (if properly arranged!)**

Quantum fluctuations

- **Statistical fluctuations in energy loss (from quantised emission of radiation) produce RANDOM EXCITATION of these oscillations**

Equilibrium distributions

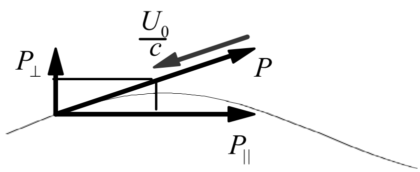
- **The balance between the damping and the excitation of the electron oscillations determines the equilibrium distribution of particles in the beam**

Average energy loss per turn

- Every turn electron radiates small amount of energy

$$E_1 = E_0 - U_0 = E_0\left(1 - \frac{U_0}{E_0}\right)$$

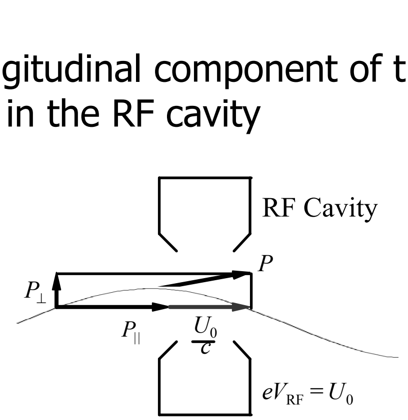
- Since the radiation is emitted along the tangent to the trajectory, only the amplitude of the momentum changes



$$P_1 = P_0 - \frac{U_0}{c} = P_0\left(1 - \frac{U_0}{E_0}\right)$$

Energy gain in the RF cavities

- Only the longitudinal component of the momentum is increased in the RF cavity



- The transverse momentum, or the amplitude of the betatron oscillation remains small

Energy of betatron oscillation

- Transverse momentum corresponds to the energy of the betatron oscillation

$$E_\beta \propto A^2$$

$$A_1^2 = A_0^2\left(1 - \frac{U_0}{E_0}\right) \quad \text{or} \quad A_1 \cong A_0\left(1 - \frac{U_0}{2E_0}\right)$$

- The relative change in the betatron oscillation amplitude that occurs in one turn (time T_0)

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

Exponential damping

- But this is just the exponential decay law!

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

- The amplitudes are exponentially **damped**

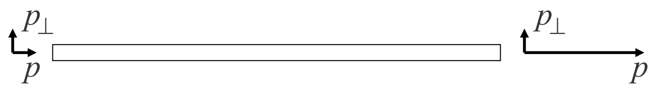
$$A = A_0 \cdot e^{-t/\tau}$$

with the damping decrement

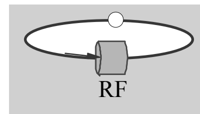
$$\frac{1}{\tau} = \frac{U_0}{2E\,T_0}$$

Adiabatic damping in linear accelerators

In a **linear accelerator**: $x' = \dfrac{p_\perp}{p}$ decreases $\propto \dfrac{1}{E}$



In a **storage ring** beam passes many times through same RF cavity



- **Clean loss of energy every turn (no change in x')**
- **Every turn is re-accelerated by RF (x' is reduced)**
- **Particle energy on average remains constant**

Damping time

- the time it would take particle to lose all of its energy

$$\tau_\varepsilon = \frac{E\,T_0}{U_0}$$

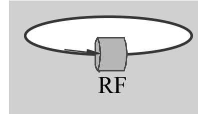
- or in terms of radiated power

$$\tau_\varepsilon = \frac{E\,T_0}{U_0} = \frac{E}{P_\gamma}$$

remember that $P_\gamma \propto E^4$

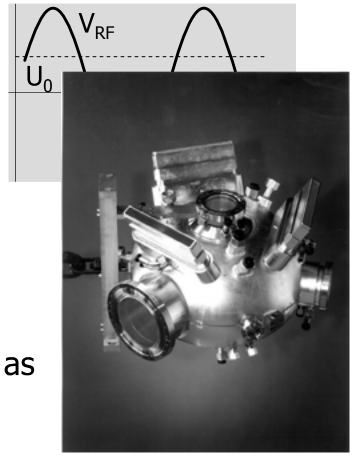
$$\tau_\varepsilon \propto \frac{1}{E^3}$$

Longitudinal motion: compensating radiation loss U_0



$$f_{RF} = h \cdot f_0$$

- RF cavity provides accelerating field with frequency
  - h – harmonic number

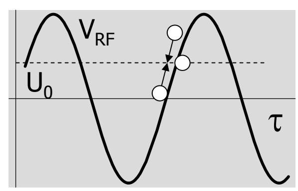


- The energy gain:

$$U_{RF} = eV_{RF}(\tau)$$

- Synchronous particle:
  - has design energy
  - gains from the RF on the average as much as it loses per turn U_0

Longitudinal motion: phase stability



- Particle ahead of synchronous one
  - gets too much energy from the RF
  - goes on a longer orbit (not enough B)
    >> takes longer to go around
  - comes back to the RF cavity closer to synchronous part.

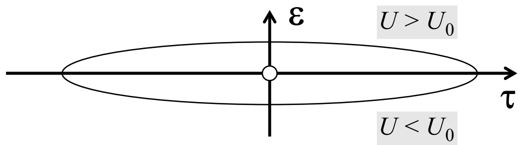
- Particle behind the synchronous one
  - gets too little energy from the RF
  - goes on a shorter orbit (too much B)
  - catches-up with the synchronous particle

Longitudinal motion: damping of synchrotron oscillations

$P_\gamma \propto \boldsymbol{E}^2 B^2$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The synchrotron motion is damped

- the phase space trajectory is spiraling towards the origin

Radiation loss

$\mathbf{P_\gamma \propto E^2 B^2}$

Displaced off the design orbit particle sees fields that are different from design values

- betatron oscillations: zero on *average*
  - linear term in B^2 - *averages to zero*
  - quadratic term - *small*
- energy deviation
  - different energy: $\mathrm{P}_\gamma \propto \mathrm{E}^2$
  - different magnetic field
    particle moves on a different orbit, defined by the *off-energy* or *dispersion* function D_x

$\Rightarrow$ both contribute to linear term in $\mathrm{P}_\gamma(\varepsilon)$

Radiation loss

$$\mathbf{P_\gamma \propto E^2 B^2}$$

To first order in ε

$$\mathbf{U_{rad} = U_0 + U' \cdot \varepsilon}$$

electron energy changes slowly, at any instant it is moving on an orbit defined by $\mathbf{D_x}$

$$\mathbf{U' \equiv \left.\frac{dU_{rad}}{dE}\right|_{E_0}}$$

after some algebra one can write

$$U' = \frac{U_0}{E_0}(2 + \mathcal{D})$$

$$\mathcal{D} \neq 0 \quad \textit{only when} \quad \frac{k}{\rho} \neq 0$$

Energy balance

Energy gain from the RF system: $U_{RF} = eV_{RF}(\tau) = U_0 + e\dot{V}_{RF} \cdot \tau$

- synchronous particle ($\tau = 0$) will get exactly the energy loss per turn
- we consider only linear oscillations

$$\dot{V}_{RF} = \left.\frac{dV_{RF}}{d\tau}\right|_{\tau=0}$$

- Each turn electron gets energy from RF and loses energy to radiation within one revolution time T_0

$$\Delta\varepsilon = (U_0 + e\dot{V}_{RF} \cdot \tau) - (U_0 + U' \cdot \varepsilon) \qquad \frac{d\varepsilon}{dt} = \frac{1}{T_0}(e\dot{V}_{RF} \cdot \tau - U' \cdot \varepsilon)$$

- An electron with an energy deviation will arrive after one turn at a different time with respect to the synchronous particle

$$\frac{d\tau}{dt} = -\alpha \frac{\varepsilon}{E_0}$$

Synchrotron oscillations: damped harmonic oscillator

Combining the two equations $$\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2 \varepsilon = 0$$

- where the oscillation frequency $\Omega^2 \equiv \dfrac{\alpha e \dot{V}_{RF}}{T_0 E_0}$
- the damping is slow: $\alpha_\varepsilon \equiv \dfrac{U'}{2T_0}$ typically $\alpha_\varepsilon << \Omega$
- the solution is then:

$$\varepsilon(t) = \hat{\varepsilon}_0 e^{-\alpha_\varepsilon t} \cos(\Omega t + \theta_\varepsilon)$$

- similarly, we can get for the time delay:

$$\tau(t) = \hat{\tau}_0 e^{-\alpha_\varepsilon t} \cos(\Omega t + \theta_\tau)$$

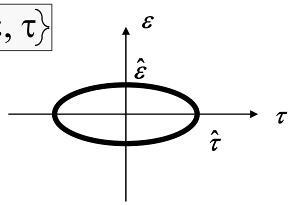
Synchrotron (time - energy) oscillations

The ratio of amplitudes at any instant $$\hat{\tau} = \frac{\alpha}{\Omega E_0} \hat{\varepsilon}$$

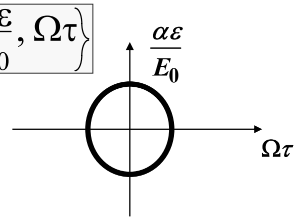
Oscillations are 90 degrees out of phase $$\theta_\varepsilon = \theta_\tau + \frac{\pi}{2}$$

The motion can be viewed in the phase space of conjugate variables

$$\{\varepsilon, \tau\}$$



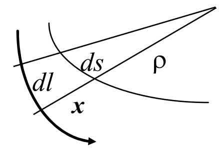
$$\left\{\frac{\alpha\varepsilon}{E_0}, \Omega\tau\right\}$$



Orbit Length

Length element depends on x

$$dl = \left(1 + \frac{x}{\rho}\right)ds$$



Horizontal displacement has two parts:

$$x = x_\beta + x_\varepsilon$$

- To first order x_β does not change L
- x_ε – has the same sign around the ring

Length of the off-energy orbit $$L_\varepsilon = \oint dl = \oint \left(1 + \frac{x_\varepsilon}{\rho}\right)ds = L_0 + \Delta L$$

$$\Delta L = \delta \cdot \oint \frac{D(s)}{\rho(s)} ds \quad \text{where} \quad \delta = \frac{\Delta p}{p} = \frac{\Delta E}{E}$$

$$\frac{\Delta L}{L} = \alpha \cdot \delta$$

Momentum compaction factor

$$\alpha \equiv \frac{1}{L} \oint \frac{D(s)}{\rho(s)} ds$$

Like the tunes Q_x, Q_y - α depends on the whole optics

- A quick estimate for separated function guide field:

$$\alpha = \frac{1}{L_0 \rho_0} \oint_{\text{mag}} D(s)\, ds = \frac{1}{L_0 \rho_0} \langle D \rangle \cdot L_{mag}$$

$\rho = \rho_0$ in dipoles
$\rho = \infty$ elsewhere

- But $L_{mag} = 2\pi\rho_0$

$$\alpha = \frac{\langle D \rangle}{R}$$

- Since dispersion is approximately

$$D \approx \frac{R}{Q^2} \quad \Rightarrow \quad \alpha \approx \frac{1}{Q^2} \quad \text{typically} \; < 1\%$$

and the orbit change for ~ 1% energy deviation

$$\frac{\Delta L}{L} = \frac{1}{Q^2} \cdot \delta \approx 10^{-4}$$

Something funny happens on the way around the ring...

Revolution time changes with energy $T_0 = \frac{L_0}{c\beta}$

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta}$$

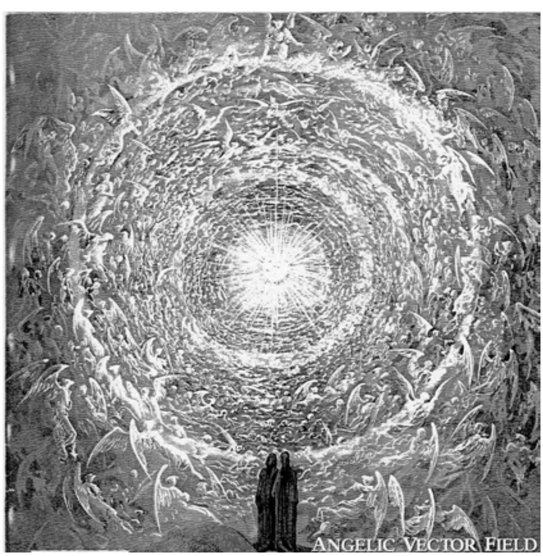
- Particle goes faster (not much!) $\frac{d\beta}{\beta} = \frac{1}{\gamma^2} \cdot \frac{dp}{p} \quad \text{(relativity)}$
- while the orbit length increases (more!) $\frac{\Delta L}{L} = \alpha \cdot \frac{dp}{p}$
- The "slip factor" $\eta \cong \alpha \quad \text{since} \quad \alpha >> \frac{1}{\gamma^2}$

$$\frac{\Delta T}{T} = \left(\alpha - \frac{1}{\gamma^2}\right) \cdot \frac{dp}{p} = \eta \cdot \frac{dp}{p}$$

- Ring is above "transition energy" $\alpha \equiv \frac{1}{\gamma_{tr}^2}$

isochronous ring: $\eta = 0 \ \text{ or } \ \gamma = \gamma_{tr}$

Not only accelerators work above transition



Robinson theorem
Damping partition numbers

- Transverse betatron oscillations are damped with

$$\frac{1}{\tau_x} = \frac{1}{\tau_z} = \frac{U_0}{2ET_0}$$

- Synchrotron oscillations are damped twice as fast

$$\frac{1}{\tau_\varepsilon} = \frac{U_0}{ET_0}$$

- The total amount of damping (Robinson theorem) depends only on energy and loss per turn

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_\varepsilon} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}\left(J_x + J_y + J_\varepsilon\right)$$

the sum of the partition numbers $J_x + J_z + J_\varepsilon = 4$

Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

Quantum fluctuations

- Because the radiation is emitted in quanta, radiation itself takes care of the problem!
- It is sufficient to use quasi-classical picture:
  - » *Emission time is very short*
  - » *Emission times are statistically independent (each emission - only a small change in electron energy)*

Purely stochastic (Poisson) process

Quantum nature of synchrotron radiation

Damping only

- If damping was the whole story, the beam emittance (size) would shrink to microscopic dimensions!*
- Lots of problems! (e.g. coherent radiation)

---

* How small? On the order of electron wavelength

$$E = \gamma mc^2 = h\nu = \frac{hc}{\lambda_e} \quad \Rightarrow \quad \lambda_e = \frac{1}{\gamma}\frac{h}{mc} = \frac{\lambda_C}{\gamma}$$

$\lambda_C = 2.4 \cdot 10^{-12} m$ – Compton wavelength

Diffraction limited electron emittance

$$\varepsilon \geq \frac{\lambda_C}{4\pi\gamma}\left(\times N^{1/3} - \text{fermions}\right)$$

Visible quantum effects

I have always been somewhat amazed that a purely quantum effect can have gross macroscopic effects in large machines;

and, even more,

that Planck's constant has just the right magnitude needed to make practical the construction of large electron storage rings.
A significantly larger or smaller value of $\hbar$

would have posed serious -- perhaps insurmountable -- problems for the realization of large rings.

Mathew Sands

Quantum excitation of energy oscillations

Photons are emitted with typical energy $u_{ph} \approx \hbar\omega_{typ} = \hbar c \frac{\gamma^3}{\rho}$

at the rate (photons/second) $\mathcal{N} = \frac{P_\gamma}{u_{ph}}$

Fluctuations in this rate excite oscillations

During a small interval Δt electron emits photons	$N = \mathcal{N} \cdot \Delta t$
losing energy of	$N \cdot u_{ph}$
Actually, because of fluctuations, the number is	$N \pm \sqrt{N}$
resulting in **spread in energy loss**	$\pm \sqrt{N} \cdot u_{ph}$

For large time intervals RF compensates the energy loss, providing damping towards the design energy E_0

Steady state: typical deviations from E_0
≈ typical fluctuations in energy during a damping time τ_ε

Equilibrium energy spread: rough estimate

We then expect the rms energy spread to be $\sigma_\varepsilon \approx \sqrt{N \cdot \tau_\varepsilon} \cdot u_{ph}$

and since $\tau_\varepsilon \approx \frac{E_0}{P_\gamma}$ and $P_\gamma = N \cdot u_{ph}$

$\sigma_\varepsilon \approx \sqrt{E_0 \cdot u_{ph}}$ geometric mean of the electron and photon energies!

Relative energy spread can be written then as:

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma \sqrt{\frac{\lambda\!\!\!^{-}_e}{\rho}} \qquad \lambda\!\!\!^{-}_e = \frac{\hbar}{m_e c} \simeq 4 \cdot 10^{-13} m$$

it is roughly constant for all rings

$$\frac{\sigma_\varepsilon}{E_0} \sim const \sim 10^{-3}$$

- typically $E \propto \rho^2$

Equilibrium energy spread

More detailed calculations give

- for the case of an ‘isomagnetic’ lattice $\rho(s) = \begin{cases} \rho_0 & \text{in dipoles} \\ \infty & \text{elsewhere} \end{cases}$

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon \rho_0}$$

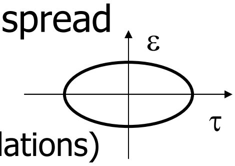
with $C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2}\right]$

It is difficult to obtain energy spread < 0.1%

- limit on undulator brightness!

Equilibrium bunch length

Bunch length is related to the energy spread



- Energy deviation and time of arrival (or position along the bunch) are conjugate variables (synchrotron oscillations)

$$\hat{\tau} = \frac{\alpha}{\Omega_s}\left(\frac{\hat{\varepsilon}}{E}\right)$$

$$\sigma_\tau = \frac{\alpha}{\Omega_s}\left(\frac{\sigma_\varepsilon}{E}\right)$$

- recall that $\Omega_s \propto \sqrt{V_{RF}}$

Two ways to obtain short bunches:

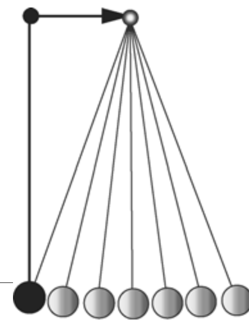
- RF voltage (power!) $\sigma_\tau \propto 1/\sqrt{V_{RF}}$
- Momentum compaction factor in the limit of α = 0 isochronous ring: particle position along the bunch is frozen $\sigma_\tau \propto \alpha$

Horizontal oscillations: equilibrium

After an electron emits a photon

- its energy decreases: $E = E_0 - u_{ph}$ $$E = E_0\left(1 - \frac{u_{ph}}{E_0}\right) = E_0(1+\delta)$$
- Neither its position nor angle change after emission
- its reference orbit has smaller radius (Dispersion)

$$x_{ref} = D \cdot \delta$$



It will start a betatron oscillation around this new reference orbit

$$x_\beta = D \cdot \delta$$

Horizontal oscillations excitation

Emission of photons is a random process

- Again we have random walk, now in **x.** How far particle will wander away is limited by the radiation damping
- The balance is achieved on the time scale of the damping time $\tau_x = 2\ \tau_\varepsilon$

$$\sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \delta = \sqrt{2} \cdot D \cdot \frac{\sigma_\varepsilon}{E}$$

- In smooth approximation for D

or, typically 10^{-3} of R,
reduced further by Q^2 focusing!
In large rings $Q^2 \sim R$, so $D \sim 1m$

$$\sigma_{x\beta} \approx \frac{\sqrt{2}R}{Q^2} \cdot \frac{\sigma_E}{E}$$

Typical horizontal beam size ~ 1 mm

Quantum effect visible to the naked eye!

Vertical size - determined by coupling

Equilibrium horizontal emittance

Detailed calculations
for isomagnetic lattice

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

where

$$\begin{aligned} \mathcal{H} &= \gamma D^2 + 2\alpha D D' + \beta D'^2 \\ &= \frac{1}{\beta}\left[D^2 + (\beta D' + \alpha D)^2\right] \end{aligned}$$

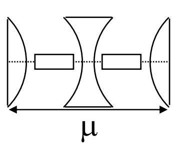
and $\langle \mathcal{H} \rangle_{mag}$ is average value in the bending magnets

$$\mathcal{H} \sim \frac{D^2}{\beta} \sim \frac{R}{Q^3}$$

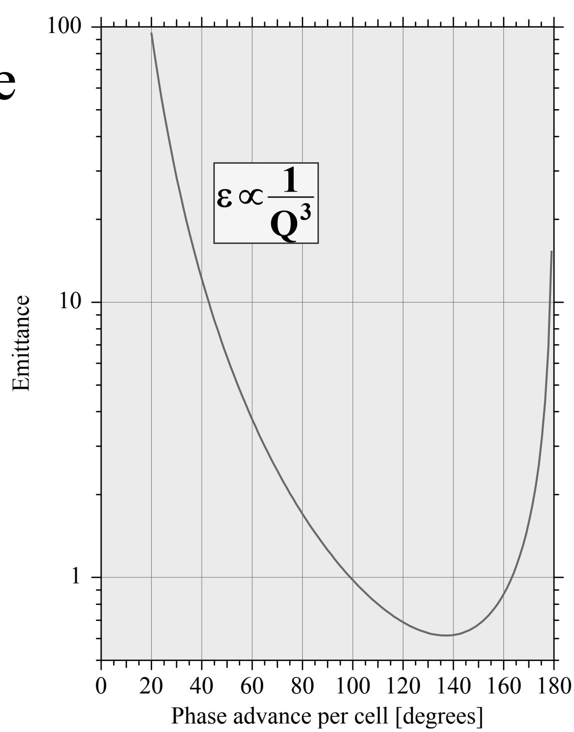
For simple lattices
(smooth approximation)

$$\varepsilon_{x0} \approx \frac{C_q E^2}{J_x} \cdot \frac{R}{\rho} \cdot \frac{1}{Q^3}$$

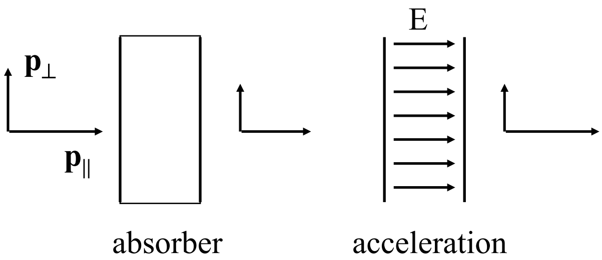
FODO Lattice emittance



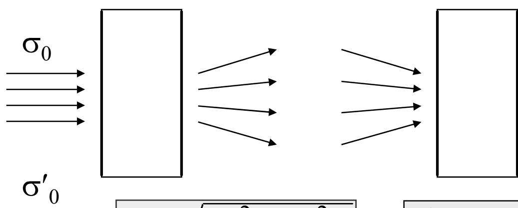
$$\varepsilon \propto \frac{\mathbf{E}^2}{\mathbf{J_x}} \theta^3 \mathrm{F_{FODO}}(\mu)$$



Ionization cooling



similar to radiation damping, but there is multiple scattering in the absorber that blows up the emittance



$$\sigma' = \sqrt{\sigma'^2_0 + \sigma'^2_{MS}}$$

$$\sigma'_0 >> \sigma'_{MS}$$

to minimize the blow up due to multiple scattering in the absorber we can **focus** the beam

Summary of radiation integrals

Momentum compaction factor

$$\alpha = \frac{I_1}{2\pi R}$$

Energy loss per turn

$$U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[\frac{\text{m}}{\text{GeV}^3}\right]$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$

$$I_2 = \oint \frac{ds}{\rho^2}$$

$$I_3 = \oint \frac{ds}{|\rho^3|}$$

$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

Summary of radiation integrals (2)

Damping parameter $\mathcal{D} = \frac{I_4}{I_2}$

Damping times, partition numbers

$$J_\varepsilon = 2 + \mathcal{D}\,, \quad J_x = 1 - \mathcal{D}\,, \quad J_y = 1$$

$$\tau_i = \frac{\tau_0}{J_i} \qquad \tau_0 = \frac{2ET_0}{U_0}$$

Equilibrium energy spread

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon} \cdot \frac{I_3}{I_2}$$

Equilibrium emittance

$$\varepsilon_{x0} = \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2}$$

$$I_1 = \oint \frac{D}{\rho}\, ds$$

$$I_2 = \oint \frac{ds}{\rho^2}$$

$$I_3 = \oint \frac{ds}{|\rho^3|}$$

$$I_4 = \oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right) ds$$

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2}\right]$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

Smooth approximation

Betatron oscillation approximated by harmonic oscillation

$$x(s) = a\sqrt{\beta(s)} \cos\left[\varphi(s) - \varphi_0\right]$$

$$\varphi(s) = \int_0^s \frac{d\dot{s}}{\beta(\dot{s})}$$

$$x \approx a\sqrt{\beta_n} \cos\left(\frac{s}{\beta_n} - \varphi_0\right) \quad \Leftarrow \quad x'' + k_{eff} \cdot x = 0, \quad k_{eff} = \frac{1}{\beta_n^2}$$

$$\beta(s) = \beta_n = \text{const}$$

- Phase advance around the ring

$$2\pi Q = \oint \frac{ds}{\beta_n} = \frac{1}{\beta_n} \cdot 2\pi R \quad \Rightarrow \quad \beta_n = \frac{R}{Q}$$

- Dispersion obeys the equation

$$D'' + k_{eff} D = \frac{1}{R} \quad \Rightarrow \quad D_n = \frac{\beta_n^2}{R} = \frac{R}{Q^2}$$

- Momentum compaction factor α

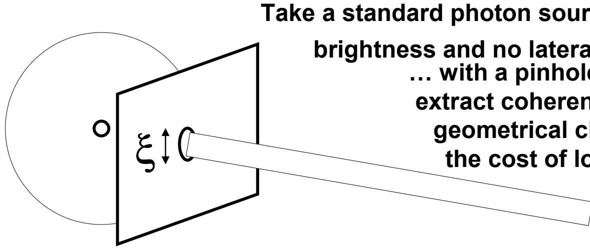
$$\alpha = \frac{\langle D \rangle}{R} = \frac{\beta_n^2}{R^2} \quad \Rightarrow \quad \alpha \approx \frac{1}{Q_x^2}$$

Synchrotron Radiation
Free Electron Lasers

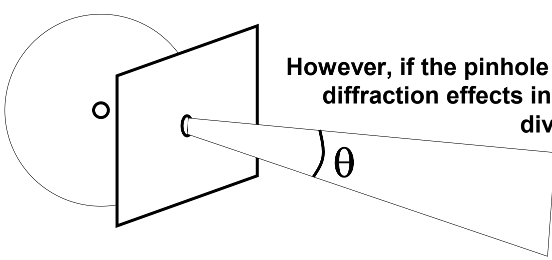
L. Rivkin

Swiss Light Source

Take a standard photon source with limited brightness and no lateral coherence …



… with a pinhole (size ξ), we can extract coherent light with good geometrical characteristics (at the cost of losing most of the emission)



However, if the pinhole size is too small diffraction effects increase the beam divergence so that:

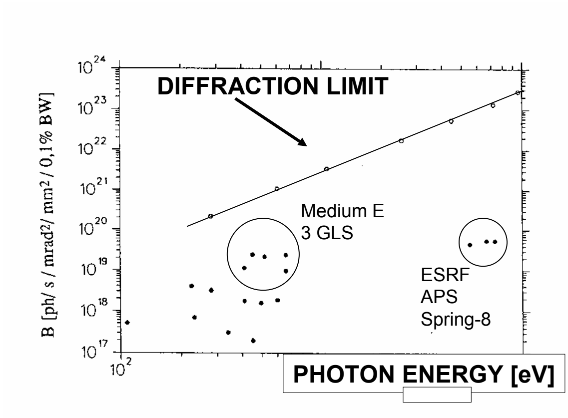
$$\xi\theta \gtrsim \lambda$$

<u>No</u> source geometry beats this <u>diffraction limit</u>

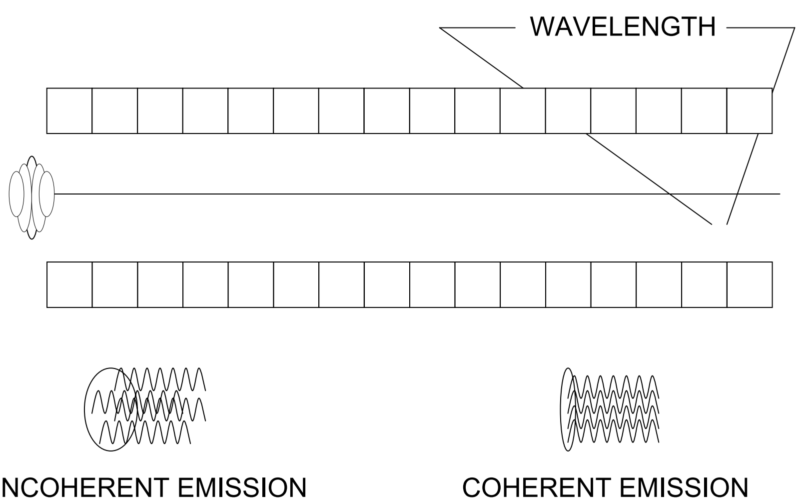
G. Margaritondo

PERFORMANCE OF 3th GENERATION LIGHT SOURCES

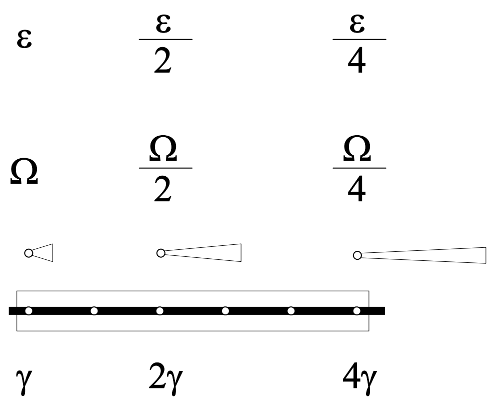
BRIGHTNESS:



MUCH HIGHER BRIGHTNESS CAN BE REACHED BY COHERENT EMISSION
OF THE ELECTRONS



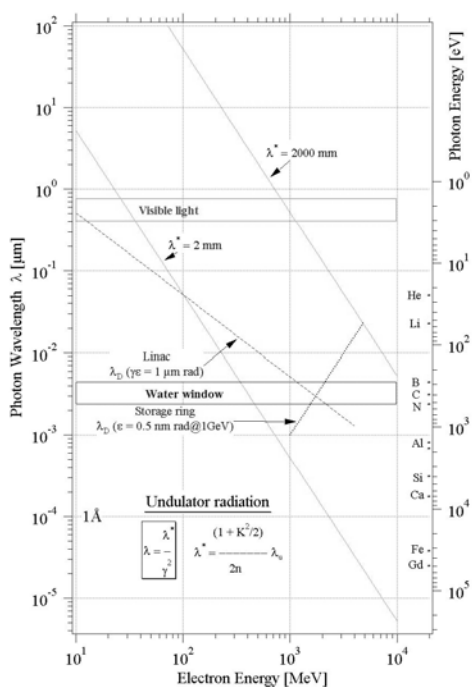
Emittance damping in linacs:



$$\varepsilon \propto \frac{1}{\gamma}$$

or

$$\gamma\varepsilon = \text{const.}$$

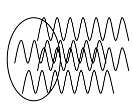


BRIGHTNESS OF SYNCHROTRON RADIATION

	electrons	*periods*	
Bending magnet	$\sim N_e$		
Wiggler	$\sim N_e$	$\sim N$	10
Undulator	$\sim N_e$	$\sim N^2$	10^4
FEL	$\sim N^2_{\mu\text{-}b}$	$\sim N^2$	10^{10}
Superradiance	$\sim N_e^2$	$\sim N^2$	10^{12}

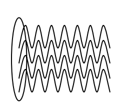
COHERENT EMISSION BY THE ELECTRONS

Intensity $\propto N$



INCOHERENT EMISSION

Intensity $\propto N^2$



COHERENT EMISSION

FIRST DEMONSTRATIONS OF COHERENT EMISSION (1989-1990)

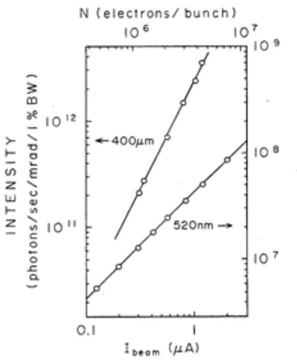


Fig. 4. Dependence of SR intensity on the beam current at $\lambda = 400$ μm and $\lambda = 520$ nm for the long pulse/short bunch beam. The ordinate is given on the left-hand side for $\lambda = 400$ μm and on the right for $\lambda = 520$ nm. The two lines show the linear and quadratic relations to the beam current. The beam current is converted to the average number of electrons in a bunch on the upper side.

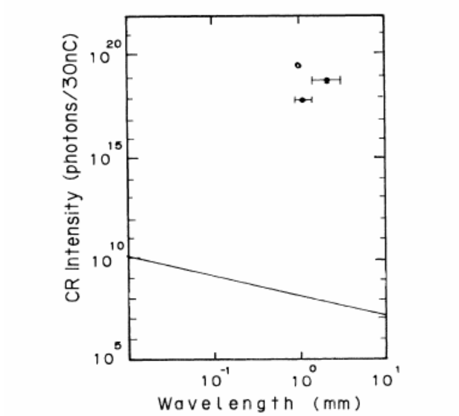


FIG. 3. The intensity of the CR measured for the bandwidths indicated with horizontal bars, the spectrum calculated according to Eq. (1) for 10% bandwidth (solid line), and the intensity expected for the complete coherence over the bunch for 10% bandwidth (open circle).

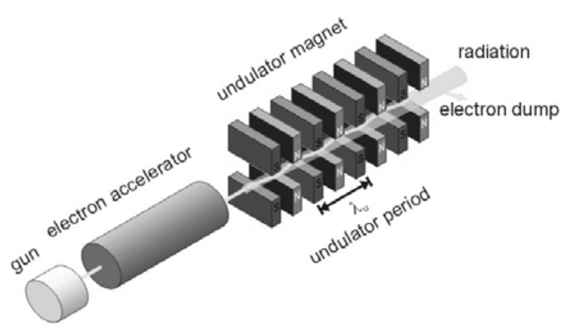
180 MeV electrons

T. Nakazato et al., Tohoku University, Japan

30 MeV electrons

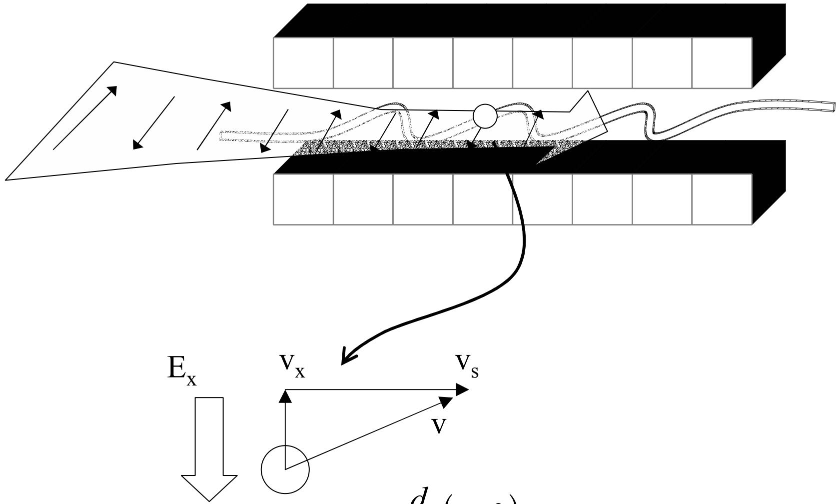
J. Ohkuma et al., Osaka University, Japan

Free Electron Laser



General layout of free-electron laser

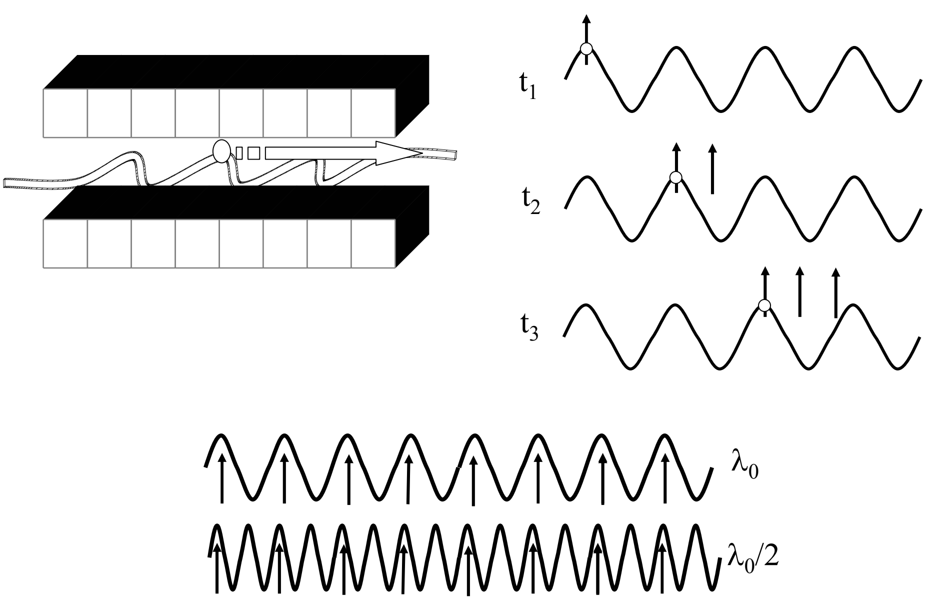
Mixing light, e- and undulator



$$\frac{d}{dt}\left(mc^2\right) = -ev_x E_x \neq 0$$

S. Werin

Undulator radiation



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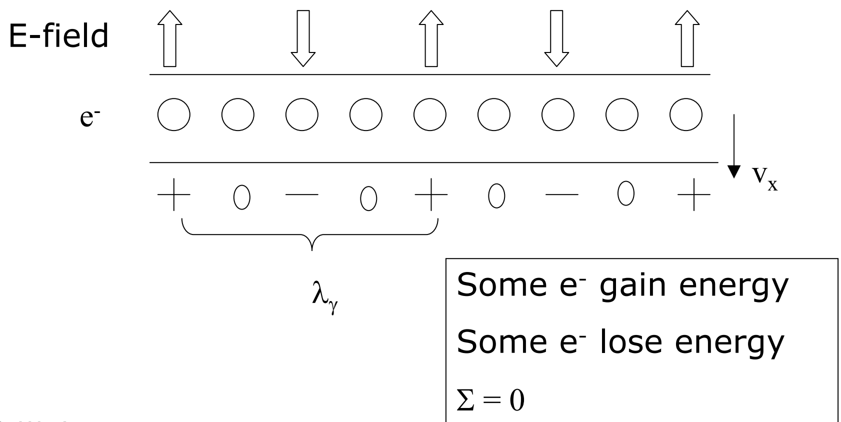
Energy exchange



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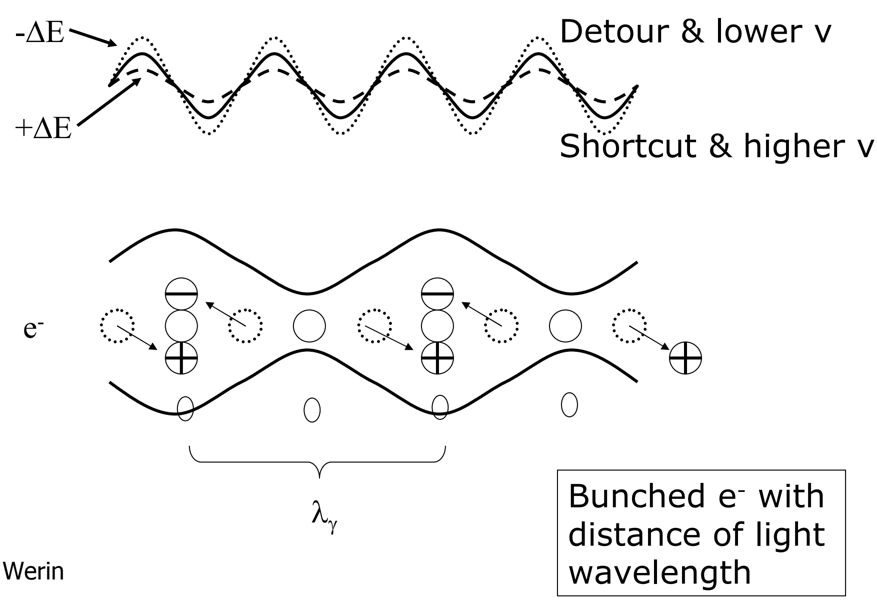
Bunching



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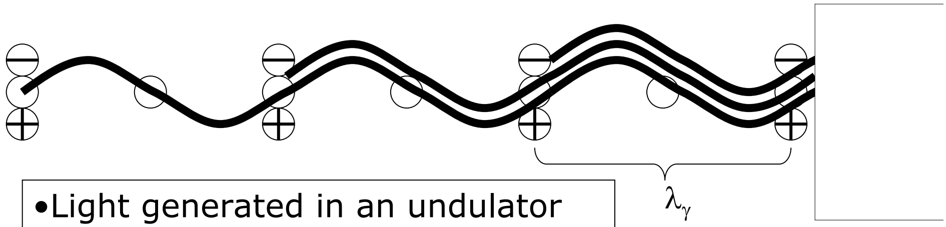
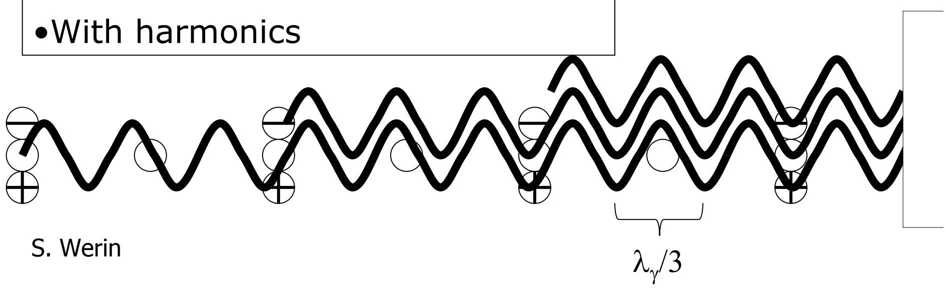
Radiator



- Light generated in an undulator
- Coherent
- With harmonics



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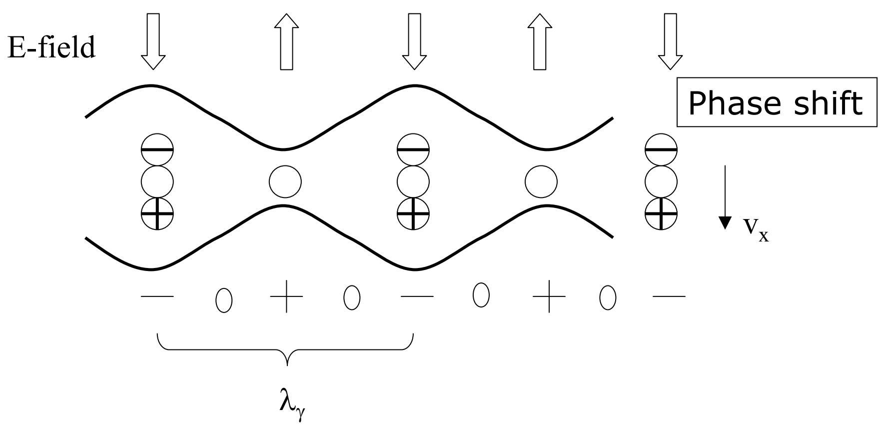
Amplification



- All e^- loose energy
- E-field gains energy
- Σ 🕒 0

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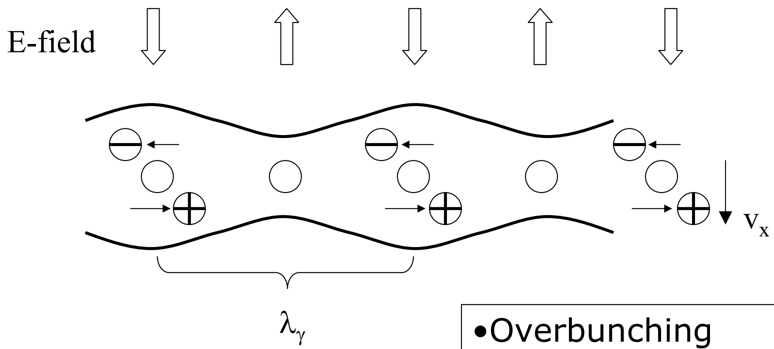
Saturation



- Overbunching
- Amplification dies off

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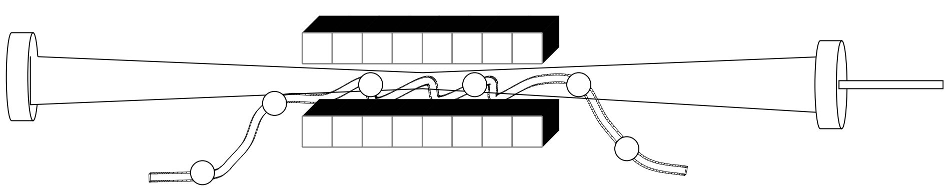
Resonator FEL



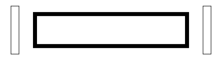
- IR 5-250 μm
- UV ¤ 200 nm
- Tunable: magnet / e- energy
- Mirrors limit

- **Storage ring**: high rep. Rate, ”stable”
- **Linac**: high peak power, ”unstable”

S. Werin

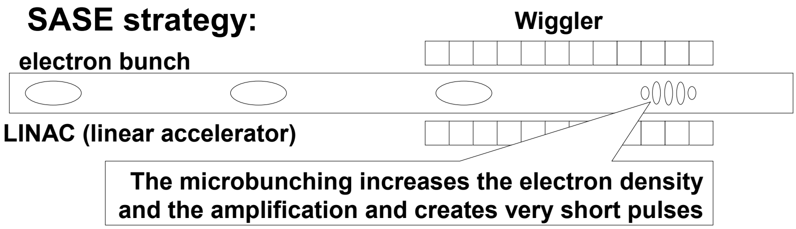
Self-amplified spontaneous emission x-ray free-electron lasers (SASE X-FEL's)

Normal (visible, IR, UV) lasers:
optical amplification in amplifying medium
plus optical cavity (two mirrors)

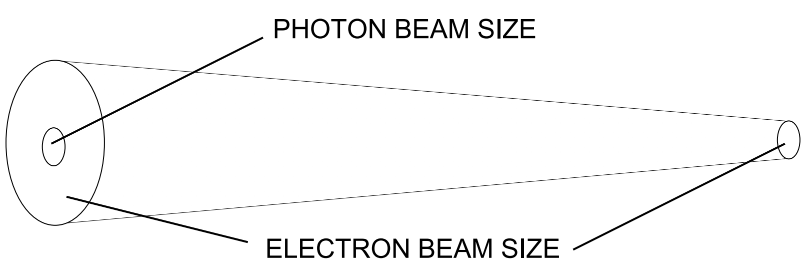


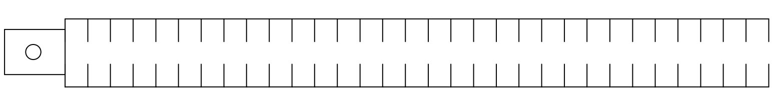
X-ray lasers: no mirrors → no optical cavity → need for one-pass high optical amplification

SASE strategy:

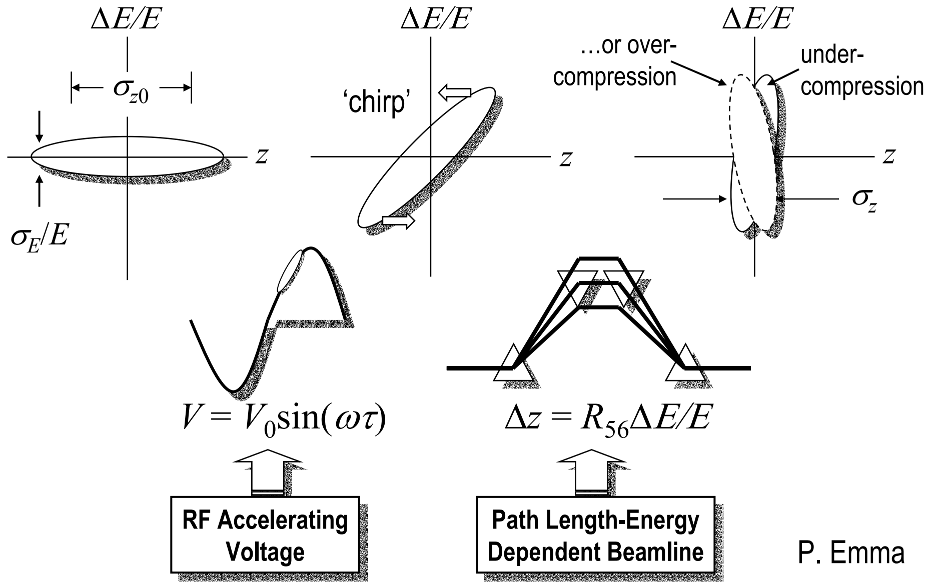


REQUIRES AN EXTREMELY SMALL ELECTRON BEAM !





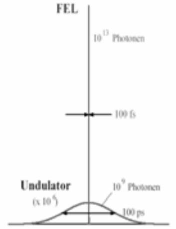
Magnetic Bunch Compression

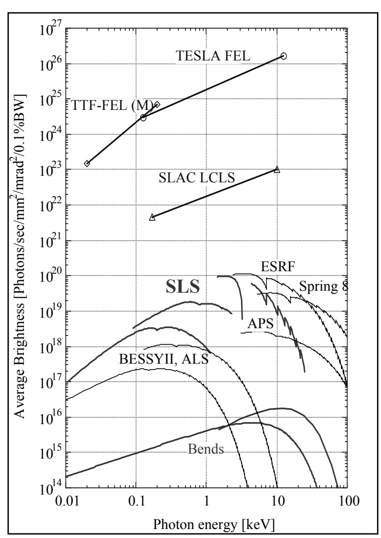


P. Emma

SASE FEL

- **UNBEATABLE BRILLIANCE**
  (10^{30} - 10^{33})
- **HIGH AVERAGE BRILLIANCE**
  (10^{22} - 10^{25})
- **SHORT PULSES**
  (1 ps – 50 fs)



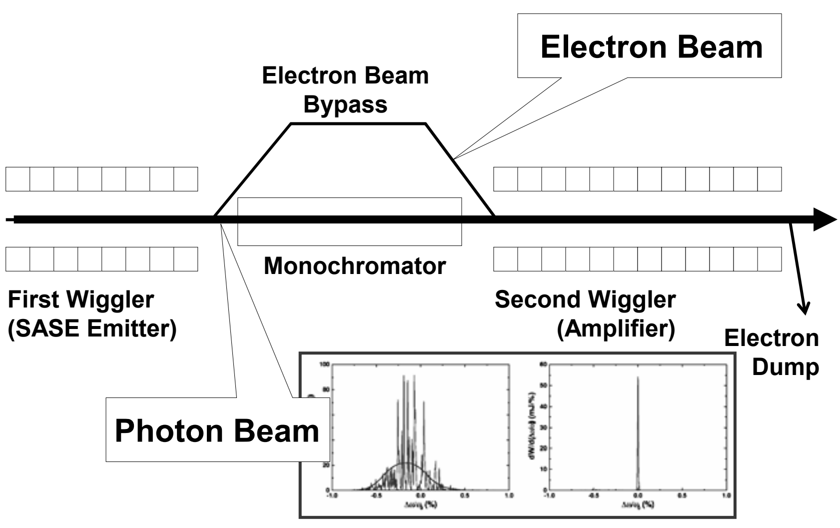


Many projects are under way …

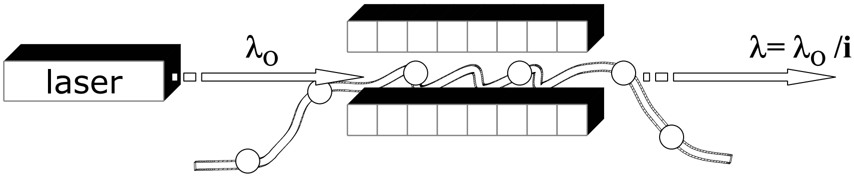
SASE FELs →

YEAR	*NAME*	*INSTITUTE*	*λ [nm]*
2000	*TTF1*	*DESY*	*90*
2000	*LEUTL*	*ARGONNE*	*530*
2004	*TTF2*	*DESY*	*24-6*
2006	*SCSS*	*SPRING-8*	*30-20*
2008	*LCLS*	*SLAC*	*0.15*
2008	*BESSY*	*BESSY*	*100-20*
2011	*X-FEL*	*DESY*	*0.1*

Seeded-Amplifier X-FELs



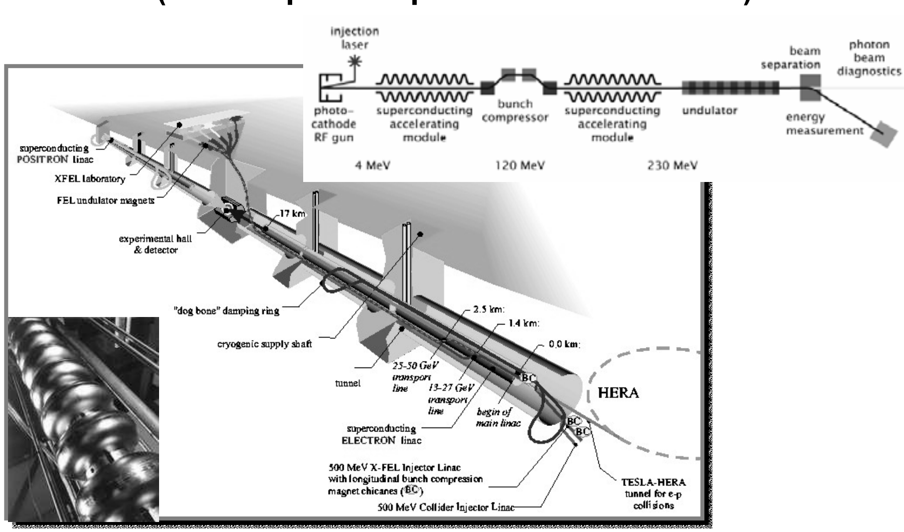
CHG - Coherent Harmonic Generation



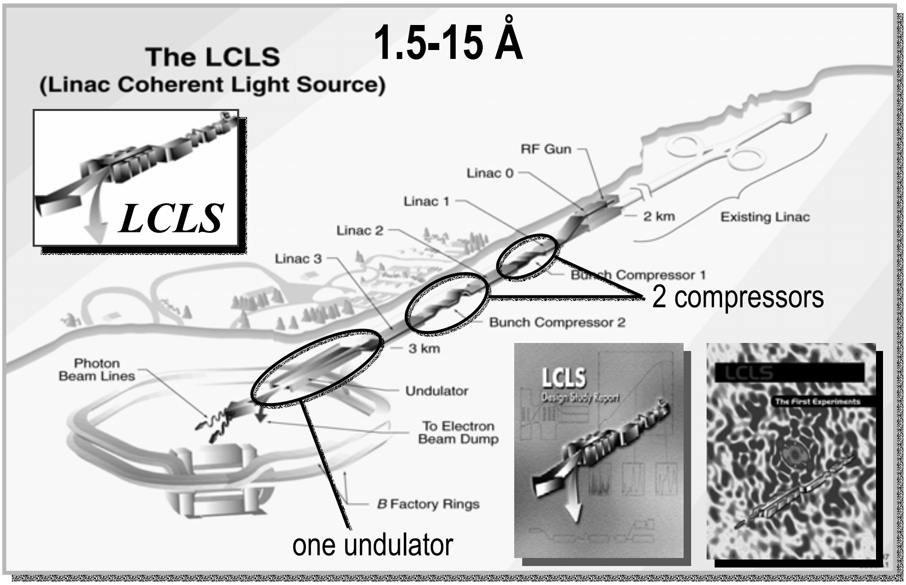
- i = 1, 3, 5, ?
- λ ~ 100 nm
- coherent

S. Werin

➔SASE FREE ELECTRON LASER (Self Amplified Spontaneous Emission)



LCLS at *SLAC*



X-FEL based on last 1-km of existing *SLAC* linac

A REDUCTION OF THE GUN EMITTANCE COULD STRONGLY REDUCE THE DIMENSION OF A FEL ➔

$$\varepsilon \le \frac{\lambda}{4\pi}$$

- FOR DIFFRACTION LIMITED BEAM

$$\varepsilon_N = \varepsilon\beta\gamma \rightarrow \gamma \ge \frac{4\pi\varepsilon_N}{\lambda}$$

- FOR REDUCED LINAC ENERGY

$$\rho^3 \approx \frac{I_{peak}\lambda}{\varepsilon\gamma^2}$$

- FOR HIGHER FEL GAIN

$$\lambda_U = \lambda\frac{2\gamma^2}{1+K^2/2}$$

- FOR SHORTER UNDULATOR LENGTHS

Smaller emittance helps!

- **Present TESLA design**

  $\varepsilon = 1$ $\quad$ $I = 5000$ A $\quad$ $L_u = 250$ m

- **TESLA + LEG**

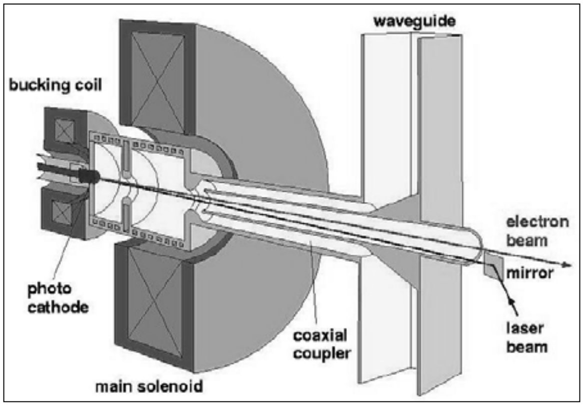
  $\varepsilon = 0.1$ $\quad$ $I = 100$ A $\quad$ $L_u = 100$ m

A POSSIBLE WAY ?

- **FIELD EMISSION**
- **NANOSTRUCTURED TIP ARRAYS**
- **UNIFORM CHARGE DISTRIBUTION WITH SPACE CHARGE COMPENSATION**
- **HIGH GRADIENT ACCELERATION**

CRITICAL ELEMENT OF A FEL = ELECTRON GUN

FOR SMALL INITIAL BEAM SIZES THE DIMENSIONS OF A FEL ARE CORRESPONDINGLY REDUCED



Ideal cathode

- Emits electrons freely, without any form of persuasion such as heating or bombardment (electrons would leak off from it into vacuum as easily as they pass from one metal to another
- Emits copiously, supplying an unlimited current density
- Lasts forever, its electron emission continuing unimpaired as long as it is needed
- Emits electrons uniformly, traveling at practically zero (transverse) velocity

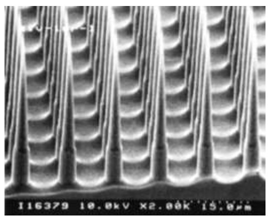
J. R. Pierce, 1946

CRITICAL ELEMENT OF A FEL = ELECTRON GUN

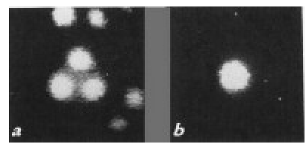
FOR SMALL INITIAL BEAM SIZES THE DIMENSIONS OF A FEL ARE CORRESPONDINGLY REDUCED

NOVEL CONCEPT OF AN ELECTRON GUN

FIELD EMISSION FROM A LARGE NUMBER OF NANOSTRUCTURED TIPS

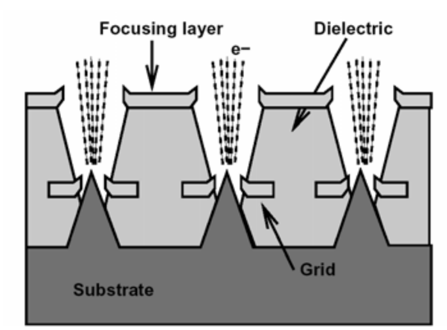


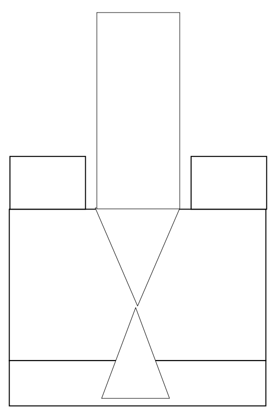
Ultimatively smallest TIP built up by 4 Tungsten atoms H.-W. Fink, UNIZH

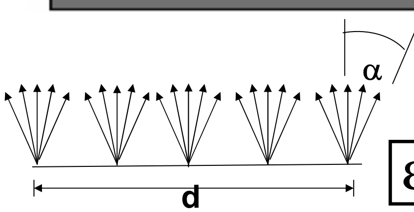


FELDEMITTER TIP ARRAY

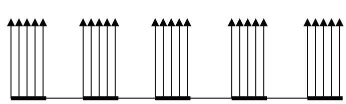
(with gate and focusing layer)





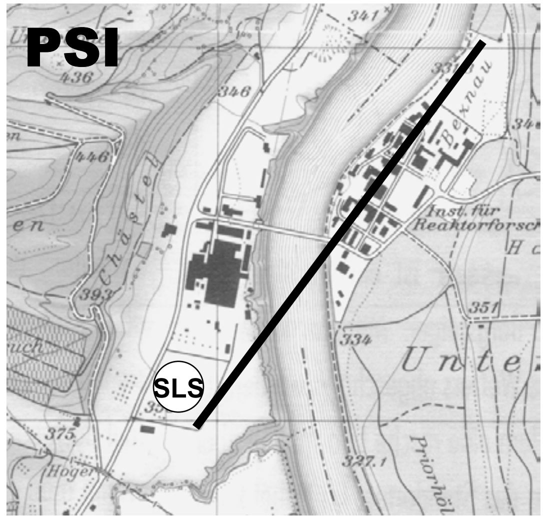


$$\varepsilon = \mathbf{d} \times \alpha$$



X-FEL FOR 1 Å

…. WITH 10 to 100 TIMES SMALLER EMITTANCE FROM THE ELECTRON GUN →

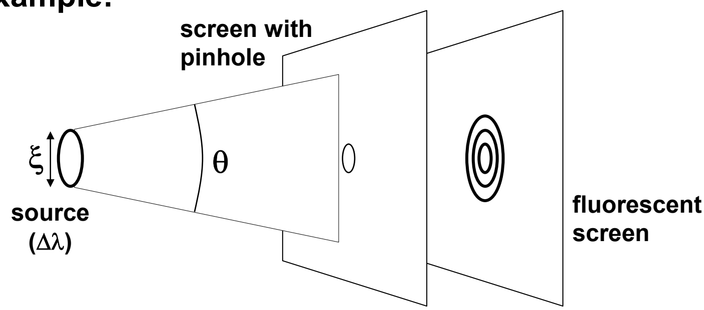


Coherence

- High brightness gives coherence
- Wave optics methods for X-rays
- Holography

Coherence: "the property that enables a wave to produce visible diffraction and interference effects"

Example:



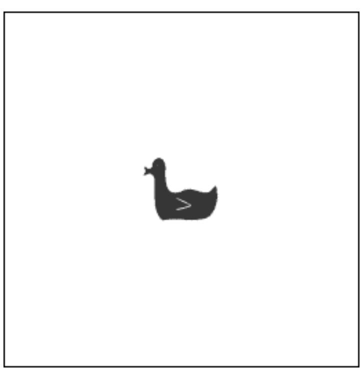
The diffraction pattern may or may not be visible on the fluorescent screen depending on the source size ξ, on its angular divergence θ and on its wavelength bandwidth Δλ

G. Margaritondo

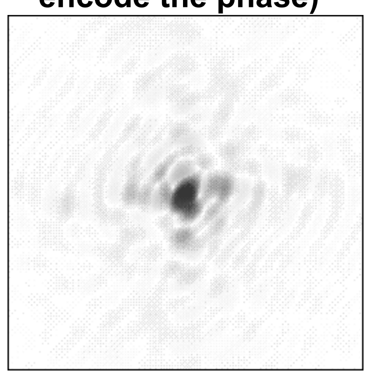
Relevance of Coherence

Diffraction Pattern of a Duck

A (two-dimensional) duck



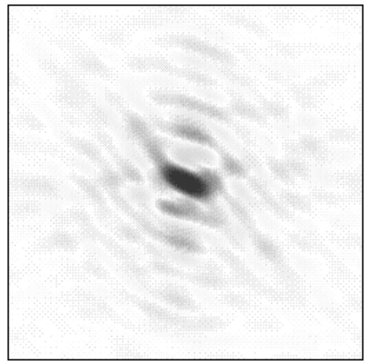
…creates this diffraction pattern (the colors encode the phase)



R. Ischebeck
Images by Kevin Cowtan, Structural Biology Laboratory, University of York

Relevance of Coherence
Diffraction Pattern of a Cat

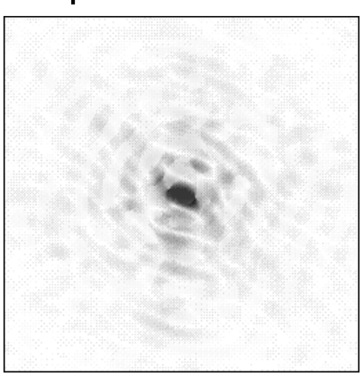
A Cat **…and its Diffraction Pattern**

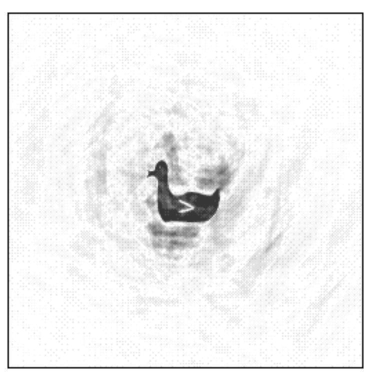


R. Ischebeck
Images by Kevin Cowtan, Structural Biology Laboratory, University of York

Relevance of Coherence
Reconstruction

Combine the amplitude of the diffraction pattern of the cat and the phase of the diffraction pattern of the duck





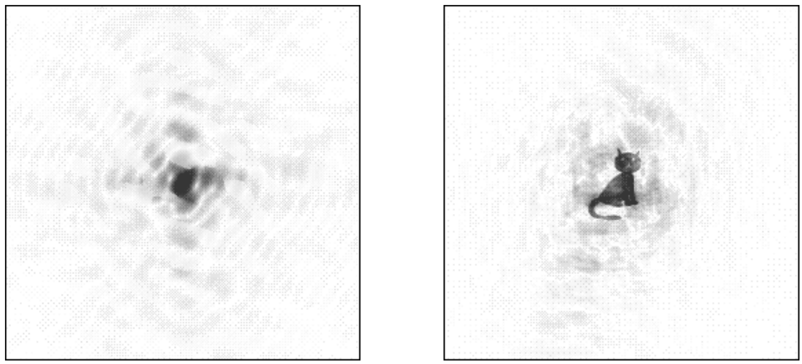
The result: a duck!

R. Ischebeck
Images by Kevin Cowtan, Structural Biology Laboratory, University of York

Relevance of Coherence

Reconstruction

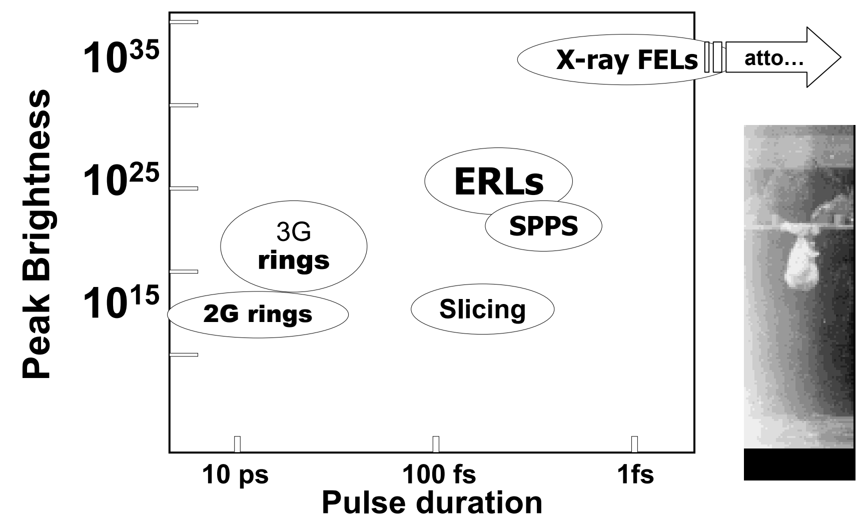
Of course, one can also do the opposite trick:
combine the amplitude of the duck and the phase of the cat



This is the famous **Phase Problem**

R. Ischebeck
Images by Kevin Cowtan, Structural Biology Laboratory, University of York

ONLY FELs CAN PROVIDE THIS EXTRAORDINARY LIGHT



H.-D. Nuhn, H. Winick