Around the Eakin-Sathaye Theorem

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Abstract

- A generalisation of the Eakin-Sathaye Theorem on reductions to the case of complete, and so of joint, reductions (in the sense of Rees)
- Applications of these results
- Generalisations of related work of Lipman, Eakin-Sathaye and Sally-Vasconcelos

1. *Introduction*

R a commutative Noetherian ring with identity and I an ideal of R.

A *reduction* of I is an ideal $J \subseteq I$ such that $I^{n+1} = JI^n$ for some non-negative integer n.

Reductions introduced by Northcott and Rees in the mid 1950s and play a major rôle in:

- multiplicity theory and the study of Hilbert-Samuel polynomials
- the theory of blow-up rings and their normalisations
- the study of invariants measuring the complexity of ideals and associated blow-up rings

In particular, we have the famous Eakin-Sathaye Theorem:

[EaSa] *Let R be a local ring with infinite residue field and let I be an ideal in R such that for some integers n and r* with $n \geq 1$ and $r \geq 0$, *I^n can be generated by fewer than $\binom{n+r}{r}$ elements. Then for 'general' elements $y_1,...,y_r$ in I, $(y_1,...,y_r)I^{n-1} = I^n$.*

- In the original paper, the theorem proved by adding on indeterminates to R to form a 'Nagata extension' and acting on this extension by a permutation group.
- In [O'Ca, 1989], a purely 'internal' proof given that used elements parameterised by points in Zariski-open sets.
- Recently, [Ho-Tr, 2003] gave a combinatorial proof using the theory of generic initial ideals and Borel-fixed ideals.

- Eakin and Sathaye (1976) apply their result to complement Lipman's work [Lip, 1971] on 'stable' and 'prestable' ideals. Sally and Vasconcelos [SaVa, 1974] used Lipman's work to give a theory of 'stable rings'.

Rees (1984) introduced the 'complete' and 'joint' reductions (in restrictive context later made general in [O'Ca]) to give an algebraic theory of mixed multiplicities:

Let $I_1,\dots,I_s$ (not necessarily distinct) ideals in $(R,\mathfrak{m},k)$ with k infinite and let $I = I_1\cdots I_s$. Elements $x_{ij}\in I_i$ $(i=1,\dots,s,\ j=1,\dots,r)$ form a *complete reduction of* $I_1,\dots,I_s$ whenever $(y_1,\dots,y_r)$ is a reduction of the ideal I, with $y_j = x_{1j}\cdots x_{sj},\ j=1,\dots,r$.

[Rees (*loc. cit.*) established the existence of complete reductions when $r=\dim R$; [O'Ca] improved this to the best possible case

$$r = l(I) := \dim \oplus_{t\geq 0} I^t/\mathfrak{m}I^t (\leq \dim R),$$

and so to all cases $r \geq l(I)$.]

Elements $x_i \in I_i$, $i=1,\dots,s$, are said to be a *joint reduction of* $I_1,\dots,I_s$ whenever $x_1I_2\cdots I_s + \dots + x_sI_2\cdots I_{s-1}$ is a reduction of the ideal I.

It is immediate that joint reductions exist whenever $s\geq l(I)$ (as then a complete reduction consisting of s elements exists and immediately yields a joint reduction).

Note the particular case $s=\dim R$, especially the case each I_i $\mathfrak{m}$-primary (so that $l(I)=\dim R$).

[Of course the case $s\geq l(I)$ can be achieved by padding out the I_i with additional copies of R, if necessary.]

Analogues of 'classical' results for reductions and multiplicities have been developed for joint reductions and mixed multiplicities. Here we supply an analogue of the Eakin-Sathaye Theorem in the case of a complete reduction, using an adaptation of the ideas of [O'Ca]; the analogue for joint reductions then follows immediately.

We note applications to Lipman's work on stable elements and show that related earlier results, in which a certain parameter (dimension, resp. the Eakin-Sathaye number r) is held at the value 1, can be recast quite generally.

2. *The generalised Eakin-Sathaye Theorems*

We have the following analogues of the Eakin-Sathaye Theorem for the case of complete and joint reductions.

THEOREM. *(The Eakin-Sathaye Theorem for complete reductions)*

Consider $(R,\mathfrak{m},k)$ with k *infinite. Let* $I_1,\dots,I_s$ *be (not necessarily distinct) ideals in* R. *Set* $I=I_1\cdots I_s$. *Suppose that for some integers n and r* with $n\geq 1$ and $r\geq 0$, I^n *can be generated by fewer than* $\binom{n+r}{r}$ *elements. Then for 'general' elements* $y_1,\dots,y_r$ *with* $y_j = x_{1j}\cdots x_{sj},\ j=1,\dots,r$, *where* $x_{ij}\in I_i$, $i=1,\dots,s$, $(y_1,\dots,y_r)I^{n-1}=I^n$.

REMARK. There is no real requirement that R be Noetherian or that I be finitely generated: it suffices to replace I and $I_1,\dots,I_s$ by the corresponding ideals generated by the relevant finite sets of elements entering into the expressions for the finite generators of I^n.

Sketch Proof. We adapt a mixture of the ideas behind proofs in [O'Ca] (influenced by original proof), using geometrical properties of Segre varieties.

COROLLARY. *(The Eakin-Sathaye Theorem for joint reductions)*
Let $(R,\mathfrak{m},k)$ be a Noetherian local ring with infinite residue field k and let $I_1,\ldots,I_s$ be (not necessarily distinct) ideals in R. Set $I = I_1\cdots I_s$. Suppose that for some integers n and r with $n \geq 1$ and $r \geq 0$, *I^n can be generated by fewer than $\binom{n+r}{r}$ elements, and that $s \geq r$ (by padding with the list of the ideals I_i with copies of R, if necessary). Then there exist $x_i \in I_i$, $i = 1,\ldots,s$, such that*

$$x_1 I_1^{n-1} I_2^n \cdots I_s^n + \ldots + x_s I_1^n \cdots I_{s-1}^n I_s^{n-1} = I^n.$$

Proof. I has a complete reduction consisting of r elements and so has a complete reduction consisting of s elements (since 0 is common to all ideals). The latter reduction clearly results in a joint reduction with the required property.

REMARK. Note the particularly important case of the Corollary where $I_1,\ldots,I_s$ are $\mathfrak{m}$-primary and $s = r = \dim R$, and also the case of the Theorem where $I_1,\ldots,I_s$ are $\mathfrak{m}$-primary and $r = \dim R$. As regards the latter, we have an overlap with the situation considered by Lipman when $\dim R = 1$ and $I_1,\ldots,I_s$ are $\mathfrak{m}$-primary, taking r to have value 1.

3. *Applications of the original and generalised Eakin-Sathaye results*

1. The above provide another proof of the Rees/O'Carroll results on the existence of complete and joint reductions: the k-vector space dimension of the n^{th}-degree component of the special fibre ring $\oplus_{t\geq 0} I^t/\mathfrak{m}I^t$ of an ideal I of analytic spread $l(I) = r$ is eventually a polynomial in n of degree $r-1$, whereas the expression $\binom{n+r}{r}$ is a polynomial in n of degree r.

2. The following sample result generalises work of Eakin-Sathaye and Lipman to the case of arbitrary dimension and value of the Eakin-Sathaye number r, and brings to bear the generalised Eakin-Sathaye results above.

THEOREM. *Let I be an ideal in (R,m,k) with k infinite. Then the following are equivalent.*
(i) For all $n \in \mathbb{N}$, there exists a polynomial $p_I(n)$ in n of degree at most $r-1$, for some $r \in \mathbb{N}$, such that I^n has $p_I(n)$ generators.
(ii) I has a reduction consisting of r elements.
(iii) There exist $n, r \in \mathbb{N}$ such that I^n has fewer than $\binom{n+r}{r}$ generators.
Moreover, the analogues for complete and joint reductions apply as is relevant.

3. As in the 'classical' theory, estimates of the minimal number of generators yield estimates of reduction numbers, using the relevant theorem of Eakin-Sathaye type. For example, we have an easy generalisation of a result of Sally and Vasconcelos [SaVa].

PROPOSITION. *Let I be an open ideal of analytic spread r in a local Cohen-Macaulay ring R of dimension d and multiplicity μ. Suppose that t is the nilpotency degree of R/I. Then $I^n = JI^{n-1}$ for a minimal reduction J of I, where n is an integer such that $t^{d-1}n^{d-1}\mu + d \leq \binom{n+r}{r}$.*

Moreover, the analogues for complete and joint reductions apply as is relevant.

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