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Finite free resolutions

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These are preliminary lecture notes, intended only for distribution to participants

FINITE FREE RESOLUTIONS

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INTRODUCTION

With these lectures we aim to give a survey on the theory of finite free resolutions. We will treat the Buchsbaum-Eisenbud acyclicity criterion, discuss upper and lower bounds for Betti-numbers, including the Evans-Griffith syzygy theorem, and compare the graded Betti-numbers of an ideal and with those of its generic initial ideal.

1. LECTURE: BASIC CONCEPTS; ACYCLICITY CRITERIA

Throughout these lectures $(R,\mathfrak{m},k)$ denotes either a Noetherian local ring or a standard graded k-algebra with graded maximal ideal $\mathfrak{m}$. All modules considered in these lectures will be finitely generated, and will be graded if R is graded.

Let M be an R-module, $m_1,\dots,m_r$ a minimal system of (homogeneous) generators of M. Let F_0 be a free R-module with basis $e_1,\dots,e_r$, and let $\varepsilon\colon F_0\to M$ be surjective R-module homomorphismus defined by $\varepsilon(e_i)=m_i$ for $i=1,\dots,r$. Nakayama's lemma implies that $\operatorname{Ker}(\varepsilon)\subset\mathfrak{m}F_0$. Since R is Noetherian, $\operatorname{Ker}(\varepsilon)$ is finitely generated, and there is again a free R-module F_1 and an epimorphism $F_1\to\operatorname{Ker}(\varepsilon)$, whose kernel is a submodule of $\mathfrak{m}F_1$. Composing $F_1\to\operatorname{Ker}(\varepsilon)$ with the inclusion map $\operatorname{Ker}(\varepsilon)\subset F_0$, we get a homomorphism $\varphi_1\colon F_1\to F_0$ such that

$$F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

is exact and $\operatorname{Im}(\varphi_1)\subset\mathfrak{m}F_o$. Proceeding this way one constructs an exact sequence

$$\cdots \longrightarrow F_p \xrightarrow{\varphi_p} \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Definition 1.1. Let M be an R-module. A complex

$$\mathbb{F}:\cdots \longrightarrow F_p \xrightarrow{\varphi_p} \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

of finitely generated free R-modules is called a *minimal free R-resolution of M*, if

(i) $\varphi(\mathbb{F})\subset\mathfrak{m}\mathbb{F}$;
(ii) $H_0(\mathbb{F})\cong M$ and $H_i(\mathbb{F})=0$ for $i>0$.

A minimal free resolution always exist as we have just seen. It is called minimal since for each i the basis elements of F_i are mapped to a minimal set of generators of $\operatorname{Ker}\varphi_{i-1}$.

Any two minimal free resolutions of M are isomorphic, that is, if $\mathbb{F}$ and $\mathbb{G}$ are minimal free resolutions of M, then there is an isomorphism of complexes $\mathbb{F}\cong\mathbb{G}$.

If $(\mathbb{F},\varphi)$ is a minimal free resolution of M, then

$$\operatorname{syz}_i(M)=\operatorname{Im}(\varphi_i)$$

is called the ith syzygy module of M.

In the graded case, by choosing in each step of the construction of the minimal free resolution a minimal system of *homogeneous* generators of $\operatorname{Ker}\varphi_i$, one obtains a graded minimal free resolutions, that is, a minimal free resolution $\mathbb{F}$ such that

(iii) $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$ for all i;
(iv) $\varphi_i \colon F_i \to F_{i-1}$ is homogeneous of degree 0.

Definition 1.2. Let $\mathbb{F}$ be a minimal free resolution of M. Then $\beta_i = \operatorname{rank} F_i$ is called the ith *Betti-number of* M.

Remark 1.3. (a) In the graded case, $\beta_i = \sum_j \beta_{ij}$ for all i. The numbers β_{ij} are called the *graded* Betti-numbers of M;

(b) Let $\mathbb{F}$ be a minimal free resolution of M. Since

$$\operatorname{Tor}_i^R(M,k) = H_i(\mathbb{F}\otimes k) = F_i \otimes k = F_i/\mathfrak{m}F_i,$$

it follows that $\beta_i = \dim_k \operatorname{Tor}_i(M,k)$.

In the graded case, $\operatorname{Tor}_i^R(M,k)$ is a graded k-vector space, and $\beta_{ij} = \dim_k \operatorname{Tor}_i^R(M,k)_j$.

Let $\mathbb{F}$ be the minimal free resolution of M. We say that M has a *finite free resolution*, if there exists an integer i such that $F_i = 0$.

Note that M has finite free resolution if one of the equivalent conditions are satisfied: there exists an integer i such that

(a) $F_j = 0$ for all $j \geq i$;
(b) $\operatorname{Tor}_i(M,k) = 0$;
(c) $\operatorname{Tor}_j(M,k) = 0$ for all $j \geq i$.

Suppose that M has a finite free resolution. The maximal number i with $\operatorname{Tor}_i(M,k) \neq 0$ is called the *projective dimension of* M, and denoted $\operatorname{proj}\dim M$.

If R is regular, then *all* modules have a finite free resolution. Indeed, let $\mathbf{x} = x_1,\dots,x_n$ be a regular system of parameters of R. In the graded case, R is the polynomial ring, and for $\mathbf{x}$ we may choose the variables.

Let $\mathbb{K}$ be the Koszul complex attached to $\mathbf{x}$. Then $\mathbb{K}$ is exact, since $\mathbf{x}$ is a regular sequence. Thus $\mathbb{K}$ is a minimal free resolution of k, and hence

$$\operatorname{Tor}_i(M,k) = H_i(M\otimes\mathbb{K}) \quad \text{for all} \quad i.$$

Since $\mathbb{K}_{n+1} = 0$, we see that $\operatorname{Tor}_{n+1}(M,k) = 0$. Hence we conclude that

$$\operatorname{proj}\dim M \leq n = \dim R$$

for all R-modules M.

In these lectures we are mostly interested in finite free resolutions. The following natural question arises:

What can be said about the Betti-numbers?

To be more specific we ask:

(1) What can be said about the projective dimension?
(2) Given a finite complex of free R-modules. When is it exact?
(3) Fix certain data like the projective dimension or, in the graded case, the Hilbert-function. Are there lower or upper bounds for the Betti-numbers for such modules?

(4) Suppose R is a polynomial ring and $I \subset R$ is a graded ideal. Given a term order. How are the Betti numbers of I and its initial ideal $\operatorname{in}(I)$ related to each other?
(5) What can be said about the graded Betti-numbers of a monomial ideal? In the context of (4) this question is of interest.

The answer to question (1) is classical

Theorem 1.4 (Auslander-Buchsbaum)**.** *Suppose M has a finite free resolution. Then*

$$\operatorname{proj}\dim M + \operatorname{depth} M = \operatorname{depth} R.$$

In particular, $\operatorname{proj}\dim M \leq \operatorname{depth} R$*.*

Proof. We proceed by induction on $c := \operatorname{depth} R - \operatorname{depth} M$. Suppose $c \leq 0$ and let $t = \operatorname{depth} R$. Then there exists a sequence $\mathbf{x} = x_1, \ldots, x_t$ which is regular on R and M.

Suppose that $\operatorname{proj}\dim M = p > 0$, and let $\mathbb{F}$ be the minimal free resolution of M. Then $\bar{\mathbb{F}} = \mathbb{F}/(\mathbf{x})\mathbb{F}$ is a minimal free resolution of $M/(\mathbf{x})M$, and hence $\bar{\varphi}_p : \bar{F}_p \to \bar{F}_{p-1}$ is injective. However, since $\operatorname{depth} \bar{F}_p = 0$, there exists $a \in \bar{F}_p$, $a \neq 0$ with $\mathfrak{m}a = 0$. Since $\bar{\varphi}_p(\bar{F}_p) \subset \mathfrak{m}\bar{F}_{p-1}$, it follows that $\bar{\varphi}_p(a) = 0$, contradiction.

Suppose now that $c > 0$. Then $\operatorname{depth} \operatorname{syz}_1(M) = \operatorname{depth} M + 1$, so that

$$\operatorname{depth} R - \operatorname{depth} \operatorname{syz}_1(M) = c - 1.$$

By induction hypothesis, we have $\operatorname{proj}\dim \operatorname{syz}_1(M) + \operatorname{depth} \operatorname{syz}_1(M) = \operatorname{depth} R$. Hence, since $\operatorname{proj}\dim M = \operatorname{proj}\dim \operatorname{syz}_1(M) - 1$, the assertion follows.

□

Now we will deal with the second question. Suppose $\mathbb{F}$ is a finite complex of free R-modules, and suppose we want to prove it is acyclic, i.e. $H_i(\mathbb{F}) = 0$ for $i > 0$. Assuming it is not acyclic, we could localize at a suitable prime ideal P such that after localization, $H_i(\mathbb{F})$ is finite length module for all $i > 0$. In this situation we can apply

Theorem 1.5 (Lemme d'acyclicité, Peskine-Szpiro [28])**.** *Let*

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be finite complex of free R-modules with $p \leq \operatorname{depth} R$, and suppose that $\operatorname{depth} H_i(\mathbb{F}) = 0$ *for all $i > 0$. Then $\mathbb{F}$ is acyclic.*

Proof. We may assume that $p > 0$, and prove by induction on i that $H_{p-i}(\mathbb{F}) = 0$.

For $i = 0$, $H_p(\mathbb{F})$ is submodule of F_p of depth 0. Since $\operatorname{depth} F_p > 0$, this submodule must be zero.

Now given i with $0 < i < p$. By induction hypothesis we have that $H_p(\mathbb{F}) = H_{p-1}(\mathbb{F}) = H_{p-i+1}(\mathbb{F}) = 0$. Hence

$$0 \to F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_{p-i+1} \longrightarrow \operatorname{Im}(\varphi_{p-i+1}) \longrightarrow 0$$

is exact. It follows that $\operatorname{depth} \operatorname{Im}(\varphi_{p-i+1}) = \operatorname{depth} R - (i-1) \geq p - i + 1 > 1$.

Suppose that $H_{p-i}(\mathbb{F}) \neq 0$. Then $\operatorname{depth} H_{p-i}(\mathbb{F}) = 0$, and since $\operatorname{depth} \operatorname{Ker}(\varphi_{p-i}) > 0$, the exact sequence

$$0 \longrightarrow \operatorname{Im}(\varphi_{p-i+1}) \longrightarrow \operatorname{Ker}(\varphi_{p-i}) \longrightarrow H_{p-i}(\mathbb{F}) \longrightarrow 0$$

implies that $\operatorname{depth} \operatorname{Im}(\varphi_{p-i+1}) = 1$, a contradiction. □

In the proof of Theorem 1.5 it was not important that the modules F_i are free, and one could have replaced them by any other modules satisfying $\operatorname{depth} F_i \geq i$, and would have obtained the same conclusion.

Let Q be the ring of fractions of R. An R-module M has *rank* r if $M \otimes Q$ is free of rank r. It is easy to see that M has rank r, if and only if $M_{\mathfrak{p}}$ is free of rank r for all $\mathfrak{p} \in \operatorname{Ass}(R)$.

The rank is additive on short exact sequences: suppose $0 \to U \to M \to N \to 0$ is an exact sequence of R-modules. If two of the modules U, M or N have a rank, then the third does, and $\operatorname{rank} M = \operatorname{rank} U + \operatorname{rank} N$.

The additivity of rank implies

Proposition 1.6. *Suppose M has a finite free resolution $\mathbb{F}$. Then*

$$\operatorname{rank} M = \sum_i (-1)^i \beta_i.$$

Corollary 1.7. *Let $I \neq 0$ be an ideal with finite free resolution. Then I contains a non-zerodivisor.*

Proof. By Proposition 1.6, I has a rank, and $\operatorname{rank} I + \operatorname{rank} R/I = \operatorname{rank} R = 1$. Since $I \neq 0$ and $I \otimes Q \to R \otimes Q = Q$ is injective, it follows that $\operatorname{rank} I = 1$. Therefore $\operatorname{rank} R/I = 0$, and so R/I is annihilated by a non-zerodivisor. $\square$

Let $\varphi \colon M \to N$ be an R-module homomorphism. We say that φ has rank r, if $\operatorname{Im} \varphi$ has rank r.

An R-module homomorphism $\varphi \colon F \to G$ of finite free R-modules is given by a matrix A with respect to bases of F and G. We denote by $I_t(\varphi)$ the ideal generated by all t-minors of φ, and set $I(\varphi) = I_r(\varphi)$, if r is the rank of φ. We also set $I_t(\varphi) = R$ if $t \leq 0$ and $I_t(\varphi) = 0$ if $t > \min\{\operatorname{rank} F, \operatorname{rank} G\}$. The definitions do not depend on the chosen bases of F and G.

For the next theorem we shall need the following facts:

Proposition 1.8. *Let $\varphi : F \to G$ be homomorphism of finite free R-modules, and $\mathfrak{p}$ a prime ideal. Then*

(a) $I_t(\varphi) \not\subset \mathfrak{p} \iff (\operatorname{Im} \varphi)_{\mathfrak{p}}$ *contains a free direct summand of $G_{\mathfrak{p}}$ of rank t;*
(b) $I_t(\varphi) \not\subset \mathfrak{p}$ *and* $I_{t+1}(\varphi)_{\mathfrak{p}} = 0 \iff (\operatorname{Im} \varphi)_{\mathfrak{p}}$ *is a free direct summand of $G_{\mathfrak{p}}$ of rank t;*
(c) $\operatorname{rank} \varphi = r \iff \operatorname{grade} I_r(\varphi) \geq 1$ *and* $I_{r+1}(\varphi) = 0$.

Theorem 1.9 (Buchsbaum-Eisenbud [13])**.** *Let*

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a finite complex of free R-modules, and let $r_i = \sum_{j \geq i} (-1)^{j-i} \operatorname{rank} F_j$. Then the following conditions are equivalent:

(a) $\mathbb{F}$ *is acyclic;*
(b) $\operatorname{grade} I_{r_i}(\varphi_i) \geq i$ *for* $i = 1, \ldots, p$*;*
(c) (i) $\operatorname{rank} F_i = \operatorname{rank} \varphi_i + \operatorname{rank} \varphi_{i+1}$ *for* $i = 1, \ldots p$*;*
  (ii) $\operatorname{grade} I(\varphi_i) \geq i$ *for* $i = 1, \ldots, p$.

Proof. (a) ⇒ (b): The acyclicity of $\mathbb{F}$ and the additivity of rank imply that $r_i = \operatorname{rank}\varphi_i$. Therefore, Proposition 1.8 implies that $\operatorname{grade} I_{r_i}(\varphi_i) \geq 1$ for $i = 1,\dots,p$. Hence there exists a non-zerodivisor x which is contained in all the ideals $I_{r_i}(\varphi_i)$. If x is a unit, then $I_{r_i}(\varphi_i) = R$ for all i and we are done. Otherwise $x \in \mathfrak{m}$, and x is non-zerodivisor on all F_i and on $\operatorname{Im}(\varphi_1)$. Let $\bar{}$ denote residue classes modulo x. Then $0 \to \bar{F}_p \to \bar{F}_{p-1} \to \dots \to \bar{F}_2 \to \bar{F}_1 \to 0$ is acyclic. By induction we have $\operatorname{grade} I_{r_i}(\bar{\varphi}_i) \geq i-1$. Hence, since $\overline{I_{r_i}(\varphi_i)} = I_{r_i}(\bar{\varphi}_i)$, we conclude that $\operatorname{grade} I_{r_i}(\varphi_i) \geq i$ for $i = 2,\dots,p$.

(b) ⇒ (a): By induction on p, may assume that $0 \to F_p \to \cdots \to F_1 \to 0$ is acyclic, and have to show that $H_1(\mathbb{F}) = 0$.

Set $M_i = \operatorname{Coker}(\varphi_{i+1})$ for $i = 1,\dots,p$. We first show by descending induction that $\operatorname{depth}(M_i)_\mathfrak{p} \geq \min\{i, \operatorname{depth} R_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ and $i = 1,\dots,p$.

The assertion is trivial for $i = p$, since $M_p = F_p$. Now let $i < p$ and consider the exact sequence $0 \to M_{i+1} \to F_i \to M_i \to 0$.

If $\operatorname{depth} R_\mathfrak{p} \geq i+1$, then our induction hypothesis implies that $\operatorname{depth}(M_{i+1})_\mathfrak{p} \geq i+1$, and hence $\operatorname{depth}(M_i)_\mathfrak{p} \geq i$.

If $\operatorname{depth} R_\mathfrak{p} \leq i$, then (b) implies that $I_{r_{i+1}}(\varphi_{i+1}) \not\subset \mathfrak{p}$, and since $\operatorname{rank} M_{i+1} = r_{i+1}$ we have $I_t(\varphi_{i+1}) = 0$ for $t > r_{i+1}$. Thus Proposition 1.8 implies that $(M_i)_\mathfrak{p}$ is free, and hence $\operatorname{depth}(M_i)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$.

Now assume that $H_1(\mathbb{F}) \neq 0$, and let $\mathfrak{p} \in \operatorname{Ass} H_1(\mathbb{F})$. If $\operatorname{depth} R_\mathfrak{p} \geq 1$, then $\operatorname{depth}(M_1)_\mathfrak{p} \geq 1$, and hence $\operatorname{depth} H_1(\mathbb{F})_\mathfrak{p} \geq 1$, since $H_1(\mathbb{F}) = \operatorname{Ker}(M_1 \to F_0)$. This is a contradiction.

On the other hand, if $\operatorname{depth} R_\mathfrak{p} = 0$, then $I_{r_1}(\varphi_1) \not\subset \mathfrak{p}$ and

$$U := \operatorname{Im}((\varphi_1)_\mathfrak{p}) = \operatorname{Im}((M_1)_\mathfrak{p} \to (F_0)_\mathfrak{p})$$

contains a free direct summand of $(F_0)_\mathfrak{p}$ of rank r_1, see 1.8. However since $(M_1)_\mathfrak{p}$ is a free module of rank r_1, the surjective map $(M_1)_\mathfrak{p} \to U$ must be an isomorphism, i.e. $H_1(\mathbb{F})_\mathfrak{p} = 0$. This is again a contradiction.

(a), (b) ⇒ (c): Since $\mathbb{F}$ is acyclic, the sequences $0 \to \operatorname{Im}\varphi_{i+1} \to F_i \to \operatorname{Im}\varphi_i \to 0$ are exact. Thus the additivity of rank implies condition (c)(i).

As noticed in (a) ⇒ (b), we have $r_i = \operatorname{rank}\varphi_i$ for $i = 1,\dots,p$. Hence (b) implies (c)(ii).

(c) ⇒ (b): It follows from (c)(i) that $r_i = \operatorname{rank}\varphi_i$. Hence (ii) implies (b). □

As an application we prove

Theorem 1.10 (Hilbert-Burch)**.** *Let I be an ideal with free resolution*

$$0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists $a \in R$ such that $I = aI_n(\varphi)$. Moreover, if $\operatorname{grade} I \geq 2$, *then $I = I_n(\varphi)$ and* $\operatorname{grade} I = 2$.

Proof. Let φ be given by the $(n+1) \times n$ matrix A with respect to the canonical bases of R^n and R^{n+1}, and let $\pi\colon R^{n+1} \to R$ the homomorphism which sends the canonical basis element e_i to $(-1)^i\delta_i$, where δ_i denotes the minor of A with the ith row deleted. Let B be the $(n+1) \times (n+1)$-matrix which is obtained from A by adding the j the column of A to A as an $(n+1)$th column. Then B has two equal columns, and hence $\det B = 0$. Expanding $\det B$ with respect to the $(n+1)$th column we therefore get

$$0 = \sum_i a_{ij}(-1)^j \delta_i.$$

This shows that

$$0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \xrightarrow{\pi} R \longrightarrow 0 \tag{1}$$

is a complex.

Since we assume that $0 \to R^n \to R^{n+1} \to I \to 0$ is exact, Theorem 1.9 implies that $\operatorname{grade} I_n(\varphi) \geq 2$. Therefore, since $I_1(\pi) = I_n(\varphi)$, it follows from Theorem 1.9 that complex (1) is exact. Hence $I_n(\varphi) \cong \operatorname{Coker}\varphi \cong I$. Composing this isomorphism with the inclusion map $I \subset R$ we obtain a monomorphism $I_n(\varphi) \to R$. However since $\operatorname{grade} I_n(\varphi) \geq 2$, we have $\operatorname{Hom}_R(I_n(\varphi), R) = R$. Thus the monomorphism $I_n(\varphi) \to R$ is multiplication by an element $a \in R$. It follows that $I = aI_n(\varphi)$. By Corollary 1.7 the element a must be a non-zerodivisor.

Suppose now that $\operatorname{grade} I \geq 2$. Then, since $I = aI_n(\varphi) \subset (a)$, it follows that a is unit, and $I = I_n(\varphi)$. Finally, since $\operatorname{grade} I \leq \operatorname{proj\,dim} R/I = 2$, we get $\operatorname{grade} I = 2$. □

2. SECOND LECTURE: LOWER BOUNDS

In our discussion on the question which are the possible Betti-numbers of a module of finite projective dimension, we will concentrate in this section on lower bounds.

The following simple result gives us a hint what kind of bounds could be expected.

Proposition 2.1. *Suppose R is regular, and let $I \subset R$ be a radical ideal of grade g. Then $\beta_i(R/I) \geq \binom{g}{i}$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of I. Then $R_\mathfrak{p}$ is a regular local ring of dimension $\geq g$ with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Since I is a radical ideal it follows that $IR_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$.

Let $\mathbb{F}$ be a minimal free resolution of R/I. Since localization is an exact functor, $\mathbb{F}_\mathfrak{p}$ is a resolution of $(R/I)_\mathfrak{p}$. This resolution may not be minimal. Nevertheless we conclude that $\beta_i(R/I) \geq \beta_i(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \geq \binom{g}{i}$. The last inequality follows since $\mathfrak{p}R_\mathfrak{p}$ is generated by a regular sequence of length $\geq g$. □

Corollary 2.2. *Let R be the polynomial ring and $I \subset R$ a monomial ideal of grade g. Then $\beta_i(R/I) \geq \binom{g}{i}$.*

Proof. Let $u_1, \ldots, u_m$ be the minimal set of monomial generators of I, say $u_i = \prod_j^n x_j^{a_{ij}}$, and let S be the polynomial ring in the new set of variables x_{ij}.

The monomial $v_i = \prod_{j=1}^n \prod_{k=1}^{a_{ij}} x_{jk}$ is called the polarization of u_i, and the ideal $I^p = (v_1, \ldots v_m) \subset S$ the polarization of I.

It is a basic fact [12, Lemma 4.2.16] that the sequence of linear forms $x_{j1} - x_{jk}$ with $j = 1, \ldots, n$ and $k = 2, 3, \ldots$ form a regular sequence ℓ on S/I^p, and that $(S/I^p)/\ell(S/I^p) \cong R/I$. In particular, $\beta_i(R/I) = \beta_i(S/I^p)$. Since $\operatorname{grade} I = \operatorname{grade} I^p$, and since I^p is a radical ideal, the conclusion follows. □

These results indicate that the following may be true

Conjecture 2.3. *Let M be an R-module of grade g with finite free resolution. Then $\beta_i(M) \geq \binom{g}{i}$.*

In case R is regular and M is a module of finite length, this conjecture is known as the Buchsbaum-Eisenbud and Horrocks conjecture.

The conjecture is widely open. There are few cases in which the conjectured lower bound for the Betti numbers is known:

(a) (Buchsbaum-Eisenbud, [14]) R is regular, $M = R/I$ has finite length, and the free resolution $\mathbb{F}$ of R/I has an algebra structure;
(b) (Huneke-Ulrich, [25]) R is regular, $M = R/I$, and I is in the linkage class of a complete intersection;
(c) (Herzog-Hibi-Kühl, [22] and [21]) R is regular and M is componentwise linear.

The argument of Buchsbaum-Eisenbud is as follows: choose a regular sequence $\mathbf{x} = x_1,\ldots,x_n$ with all $x_i \in I$. We may assume that $n > 1$. Let $\mathbb{K}$ be the Koszul complex of $\mathbf{x}$ with $K_1 = \bigoplus_i^n Re_i$ and $\partial(e_i) = x_i$. Then the inclusion $(\mathbf{x}) \subset I$ can be lifted to a linear map $\alpha_1\colon K_1 \to F_1$ such that $\varphi_1(\alpha_1(e_i)) = x_i$ for all i. Now for each integer $k \geq 1$ let $\alpha_k\colon K_k \to F_k$ be defined by

$$\alpha_j(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \alpha_1(e_{i_1}) \cdot \ldots \cdot \alpha_1(e_{i_k}).$$

Then $\alpha\colon \mathbb{K} \to \mathbb{F}$ is an algebra and complex homomorphism, whose kernel is a graded ideal $\mathfrak{a}$ in $\mathbb{K}$.

Suppose $\mathfrak{a} \neq 0$. Let $a \in \mathfrak{a}$ be a non-zero element of degree j. Then there exist $b \in K_{n-j}$ with $b \wedge a \neq 0$. Since $b \wedge a \in \mathfrak{a}$ it follows that $\mathfrak{a}_n \neq 0$. We identify K_n with R. Then the image of α_n is a cyclic submodule of $\mathbb{F}_n$ which is isomorphic to $R/\mathfrak{a}_n$. Since R is a domain the annihilator of a non-zero submodule $\mathbb{F}_n$ is zero. It follows that $R/\mathfrak{a}_n = 0$, so that $\alpha_n = 0$. The canonical epimorphism $R/(\mathbf{x}) \to R/I$ with kernel, say C, induces the exact sequence

$$\mathrm{Ext}_R^{n-1}(C,R) \longrightarrow \mathrm{Ext}_R^n(R/I,R) \xrightarrow{\psi} \mathrm{Ext}_R^n(R/(\mathbf{x}),R).$$

Here the homomorphism ψ is induced by α_n, and hence is the zero map, and $\mathrm{Ext}^{n-1}(C,R) = 0$, since C is of dimension zero. It follows that $\mathrm{Ext}^n(R/I,R) = 0$, a contradiction.

Thus we conclude that $\mathfrak{a} = 0$. Therefore α is injective, and it follows that

$$\beta_i(R/I) = \operatorname{rank} F_i \geq \operatorname{rank} K_i = \binom{n}{i} \quad \text{for all} \quad i.$$

Unfortunately, not all finite minimal free resolutions admit an algebra structure. In [4] Avramov discovered obstructions to the existence of such structures, and later Srinivasan [31] showed that despite the vanishing of the obstructions defined by Avramov, a finite minimal free resolution still may not admit an algebra structure.

Discussion of (c): In [21] componentwise linear modules are introduced: a graded R-module is called *componentwise linear* if for all j the submodule $M_{\langle j\rangle}$ generated by the jth component M_j of M has a linear resolution.

By assumption, $R = k[x_1,\ldots,x_n]$ is the polynomial ring. We may assume that k is infinite. Then for a generic choice of linear forms $y_1,\ldots,y_n$ one has for $i = 1,\ldots,n$ that

$$A_i = \mathrm{Ker}(M/(y_1,\ldots,y_{i-1})M \xrightarrow{y_i} M/(y_1,\ldots,y_{i-1})M)$$

is a module of finite length. We set

$$\alpha_i = \ell(A_i),$$

and call $\alpha_1,\ldots,\alpha_n$ the *generic annihilation numbers of M*.

It will be shown in the next section (see Corollary 3.2 and Theorem 3.5) that

$$\beta_i \leq \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j,$$

with equality if and only if M is componentwise linear.

For the proof of (c) we need the following

Lemma 2.4. *With the notation and assumptions introduced suppose that* $\operatorname{depth} M = t$*, and let* $\alpha_1, \ldots, \alpha_n$ *be the generic annihilation numbers of* M*. Then* $\alpha_i = 0$ *for* $i \leq t$*, and* $\alpha_i \neq 0$ *for* $i > t$*.*

Proof. Suppose $\operatorname{depth} M > 0$. Then a generic linear form y is a non-zerodivisor. This shows that $\alpha_i = 0$ for $i \leq t$.

In order to prove that $\alpha_i \neq 0$ for $i > t$, it suffices to show: if $\operatorname{depth} M = 0$, and y is a generic linear form, then (i) $(0 :_M y) \neq 0$, and (ii) $\operatorname{depth} M/yM = 0$.

Statement (i) is obvious. For the proof of (ii) we consider for all i the map

$$y^{i-1}M/y^iM \to y^iM/y^{i+1}M$$

induced by multiplication by y.

Let C_i be the kernel of this map, and let $c + y^iM \in C_i$. Then $c = y^{i-1}a$ with $a \in M$ and there exists $b \in M$ such that $yc = y^ia = y^{i+1}b$. Hence $y(c - y^ib) = 0$, and so $\mathfrak{m}^n c \in y^iM$ for some n, since y is a generic linear form. This shows that C_i is a finite length module for all i.

Suppose now that $\operatorname{depth} M/yM > 0$. We show by induction on i, that $y^{i-1}M/y^iM \to y^iM/y^{i+1}M$ is an isomorphism. In fact, for each i we have the exact sequence

$$0 \longrightarrow C_i \longrightarrow y^{i-1}M/y^iM \longrightarrow y^iM/y^{i+1}M \longrightarrow 0.$$

For $i = 1$, $\operatorname{depth} M/yM > 0$ and $\ell C_1 < \infty$. This implies that $C_1 = 0$. Therefore $M/yM \to yM/y^2M$ is an isomorphism.

Now let $i > 0$. By induction we may assume that $y^{j-1}M/y^jM \cong y^jM/y^{j+1}M$ for all $j < i$. In particular, it follows that $M/yM \cong y^{i-1}M/y^iM$, so that $\operatorname{depth} y^{i-1}M/y^iM > 0$. However since $\ell C_i < \infty$, the above exact sequence shows again that $C_i = 0$ and that $y^{i-1}M/y^iM \to y^iM/y^{i+1}M$ is an isomorphism.

On the other hand, since $\operatorname{depth} M = 0$, there exists $c \in M$, $c \neq 0$, such that $yc = 0$. Let i be such that $c \in y^{i-1}M \setminus y^iM$. Then $c + y^iM \neq 0$ but $y(c + y^iM) = 0$. This is a contradiction since $C_i = 0$. □

Now statement (c) will be a consequence of the following stronger result.

Theorem 2.5. *Let* R *be the polynomial ring, and* M *a componentwise linear* R*-module with* $\operatorname{proj\,dim} M = p$*. Then* $\beta_i(M) \geq \binom{p}{i}$*.*

Proof. Let $t = \operatorname{depth} M$. Then $t = n - p$, by the Auslander-Buchsbaum formula. Therefore, $\alpha_i > 0$ for $i = n - p + 1, \ldots, n$, by Lemma 2.4. Thus, since M is componentwise linear,

$$\begin{aligned}\beta_i(M) &= \sum_{j=n-p+1}^{n-i+1} \binom{n-j}{i-1}\alpha_j \geq \sum_{j=n-p+1}^{n-i+1} \binom{n-j}{i-1} \\ &= \sum_{j=i-1}^{p-1} \binom{j}{i-1} = \binom{p}{i}.\end{aligned}$$

□

In view of this result one may hope that for any R-module M of projective dimension p one has $\beta_i(M) \geq \binom{p}{i}$. However by a theorem of Bruns [10, Satz 3], if N is an ith syzygy module of a module of finite projective dimension, then N is also the ith syzygy module of an ideal generated by 3 elements. In particular, if N is the second syzygy module of a module of projective dimension p, then there exists an ideal I generated by 3 elements whose second syzygy module is N, one has $\beta_2(R/I) = 3 < \binom{p}{2}$, if $p > 3$.

The following concrete very simple example was communicated to us by Conca: let $I = (-x_1x_2 + x_3x_4, x_2^2, x_3^2) \subset R = k[x_1, x_2, x_3, x_4]$. Then R/I has the resolution

$$0 \longrightarrow R \longrightarrow R^4 \longrightarrow R^5 \longrightarrow R^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

The theorem of Bruns also tells us that the resolution of an ideal generated by 3 elements can have arbitrary high projective dimension. On the other hand, it is conjectured by Stillman that if we fix a sequence of numbers $d_1, \ldots, d_r$, then there is a number p such that any ideal in a polynomial (over a field K) which is generated by forms of degree $d_1, \ldots, d_r$ has projective dimension $\leq p$. This conjecture is known to be true only in a few special cases.

For monomial ideals the strong lower bound for the Betti-numbers holds. More generally one has the following result ([9, Theorem 1.1])

Theorem 2.6 (Brun, Römer)**.** *Let M be a $\mathbb{Z}^n$-graded module with* $\operatorname{proj\,dim} M = p$. *Then* $\beta_i(M) \geq \binom{p}{i}$.

There is a strengthening of Conjecture 2.3 in a different direction

Conjecture 2.7. *Let M be an R-module of grade g with finite projective dimension. Then* $\operatorname{rank} \operatorname{syz}_i(M) \geq \binom{g-1}{i-1}$.

Of course the additivity of rank yields that Conjecture 2.7 implies Conjecture 2.3.

The best known general result concerning lower bounds for the syzygy modules is the famous

Theorem 2.8 (Evans-Griffith [19])**.** *Suppose that R contains a field. Let M be an R-module with* $\operatorname{proj\,dim} M = p$. *Then* $\operatorname{rank} \operatorname{syz}_i(M) \geq i$ *for* $i = 1, \ldots, p-1$.

Of course we must exclude $i = p$ in the statement of the theorem, since for example the pth syzygy module of a regular sequence of length p is only of rank 1.

Since the rank is additive we immediately obtain

Corollary 2.9. *With the assumptions of 2.8 one has*

$$\beta_i(M) \geq \begin{cases} 2i+1, & \text{for } i = 0,\dots,p-, \\ p, & \text{for } i = p-1, \\ 1, & \text{for } i = p. \end{cases}$$

For the proof of Theorem 2.8 we follow the presentation given in [12] and in the paper [11] of Bruns. This requires some preparations: let M be an R-module, and $x \in M$. Then

$$\mathscr{O}(x) = \{\varphi(x) \colon \varphi \in \operatorname{Hom}_R(M,R)\},$$

is an ideal, the so-called *order ideal of x*.

Suppose for example that $M = F$ is free with basis $e_1,\dots,e_n$, and that $x \in F$. Then $x = \sum_{i=1} a_i e_i$ for some $a_i \in R$. Since the linear forms $\varphi_i \colon F \to R$ with $\varphi_i(e_j) = \delta_{ij}$ generate $\operatorname{Hom}_R(F,R)$, and since $\varphi_i(x) = a_i$ for $i = 1,\dots,n$, it follows that in this case

$$\mathscr{O}(x) = (a_1,\dots,a_n).$$

We have

Lemma 2.10. *Let M be an R-module, $x \in M$ and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $x \in M$ generates a free direct summand of $M_\mathfrak{p}$ if and only if $\mathscr{O}(x) \not\subset \mathfrak{p}$.*

Proof. The order ideal $\mathscr{O}(x)$ localizes since $\operatorname{Hom}_R(M,R)_\mathfrak{p}$ is naturally isomorphic to $\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},R_\mathfrak{p})$. Thus we assume that $\mathfrak{p} = \mathfrak{m}$, and hence $\mathscr{O}(x) \not\subset \mathfrak{m}$ if and only if $\mathscr{O}(x) = R$. This is equivalent to say that there exists $\varphi \colon M \to R$ with $\varphi(x) = 1$.

Suppose $M = Rx \oplus N$, then the projection to the first summand composed with the isomorphism $Rx \to R$, $x \mapsto 1$, yields $\varphi \colon M \to R$ with $\varphi(x) = 1$. Conversely, given such φ we have $M = Rx \oplus \operatorname{Ker}\varphi$. □

The next result is one important step in the proof of Theorem 2.8

Theorem 2.11. *Suppose R contains a field. Let*

$$\mathbb{F} \colon 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a complex of finitely free R-modules such that $\operatorname{Im}(\varphi_i) \subset \mathfrak{m}F_{i-1}$ for all i. Let $t \geq 0$ be an integer and set $r_i = \sum_{j=i}^{p}(-1)^{j-i}\operatorname{rank} F_j$. Suppose that $\operatorname{codim} I_{r_i}(\varphi_i) \geq i+t$ for all i. Then, for $j = 1,\dots,p$ and every $e \in F_j \setminus \mathfrak{m}F_j$, one has $\operatorname{codim}\mathscr{O}(\varphi_j(e)) \geq j+t$.

Proof. Let $J = \mathscr{O}(\varphi_j(e))$. We may assume that $J \subset \mathfrak{m}$. We set $\bar{R} = R/J$ and $\bar{\mathbb{F}} = \mathbb{F} \otimes \bar{R}$. Then $\bar{\varphi}_j(\bar{e}) = 0$, and $I_{r_i}(\bar{\varphi}_i) = (I_{r_i}(\varphi_i)+J)/J$.

Assume that $\operatorname{codim} J \leq j+t-1$. Then we obtain

$$\dim(\bar{R}/I_{r_i}(\bar{\varphi}_i)) \leq \dim(R/I_{r_i}(\varphi_i)) \leq \dim R - i - t \leq \dim \bar{R} - i + j - 1,$$

which implies that $\operatorname{codim} I_{r_i}(\bar{\varphi}_i) \geq i - j + 1$ for all $i \geq j$.

Let

$$\mathbb{G} \colon 0 \longrightarrow G_{p-j+1} \xrightarrow{\psi_{p-j+1}} G_{p-j} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\psi_1} G_0 \longrightarrow 0,$$

with $G_i = \bar{F}_{i+j-1}$ and $\psi_i = \bar{\varphi}_{i+j-1}$. Then $\mathbb{G}$ is a complex with $\operatorname{codim} I_{r_i}(\psi_i) \geq i$ for $i = 1,\dots p-j+1$. If we would have $\operatorname{grade} I_{r_i}(\psi_i) \geq i$ for all i, then the Eisenbud-Buchsbaum acyclicity criterion would imply that $\mathbb{G}$ is acyclic. In order to remedy this defect, we choose a balanced big Cohen-Macaulay module for $\bar{R}$ and consider the complex $\mathbb{G}\otimes M$. This is precisely the step in the proof where we need that $\bar{R}$ contains a field, because in this case it is known that there exists a balanced big Cohen-Macaulay $\bar{R}$-module, that is, a Cohen-Macaulay $\bar{R}$-module (not necessarily finitely generated) such that every system of parameters of $\bar{R}$ is an M-regular sequence, see [12, Corollary 8.5.3]. It follows that $\operatorname{grade}(I_{r_i}(\psi_i),M)\geq i$ for all i. An obvious modification of the Eisenbud-Buchsbaum acyclicity criterion then implies that $\mathbb{G}\otimes M$ is acyclic.

Since $\psi_1(\bar{e})=0$, it follows that $(\psi_1\otimes M)(\bar{e}\otimes M)=0$. However, since $\mathbb{G}\otimes M$ is acyclic it follows that $\operatorname{Ker}(\psi_1\otimes M)=\operatorname{Im}(\psi_2\otimes M)$. Therefore $\bar{e}\otimes M\subset\operatorname{Im}(\psi_2\otimes M)\subset\mathfrak{m}(G_1\otimes M)$.

On the other hand, since $\bar{e}\notin\mathfrak{m}G_1$, it follows that the image of $\bar{e}\otimes M$ under the canonical epimorphism $G_1\otimes M\to(G_1\otimes M)/\mathfrak{m}(G_1\otimes M)=G_1/\mathfrak{m}G_1\otimes M/\mathfrak{m}M$ is isomorphic to $M/\mathfrak{m}M\neq 0$. This implies that $\bar{e}\otimes M\not\subset\mathfrak{m}G_1\otimes M$, a contradiction. □

For the inductive proof of the Evans-Griffith theorem we need the following technical

Lemma 2.12. *Let M be an R-module. Then there exists a free R-module F and a homomorphism $\varphi\colon M\to F$ with the following property: If $\mathfrak{p}$ is a prime ideal and $N\subset M_{\mathfrak{p}}$ is free direct $R_{\mathfrak{p}}$-summand, then $\varphi_{\mathfrak{p}}(N)$ is a free direct summand of $F_{\mathfrak{p}}$ with* $\operatorname{rank}\varphi_{\mathfrak{p}}(N)=\operatorname{rank}N$.

Proof. Denote by W^*, the R-dual of an R-modules W. We choose a G a free R-module and an epimorphism $\pi: G\to M^*$, and let $h\colon M\to M^{**}$ be the canonical homomorphism. Then $F=G^*$ and $\varphi=\pi^*\circ h$ have the desired property. Indeed, since R is Noetherian and all modules are finitely generated, the construction of F and φ localize. Thus we may assume that $R=R_{\mathfrak{p}}$. Since N is a free direct summand of M, there exist $g_1,\dots,g_r\in N$ and $\alpha_1,\dots,\alpha_r\in M^*$ such that $\alpha_i(g_j)=\delta_{ij}$. Choose $\beta_i\in G$ with $\pi(\beta_i)=\alpha_i$ for $i=1,\dots,r$. Then

$$\varphi(g_i)(\beta_j)=h(g_i)(\pi(\beta_j))=h(g_i)(\alpha_j)=\alpha_j(g_i)=\delta_{ij}.$$

This proves that $\varphi(N)$ is a free direct summand of F with $\operatorname{rank}N=\operatorname{rank}\varphi(N)$. □

Proof of Theorem 2.8. We prove more generally the following statement $(*)$: let

$$\mathbb{F}\colon 0\longrightarrow F_p\xrightarrow{\varphi_p}F_{p-1}\longrightarrow\cdots\xrightarrow{\varphi_2}F_1\xrightarrow{\varphi_1}F_0\longrightarrow 0$$

be a complex of finitely generated free R-modules such that $\operatorname{Im}(\varphi_i)\subset\mathfrak{m}F_{i-1}$ for all i, and set $r_i=\sum_{j=i}^{p}(-1)^{j-i}\operatorname{rank}F_j$. Suppose that there exists an integer $t\geq 0$ such that $\operatorname{codim}I_{r_i}(\varphi_i)\geq i+t$ for all i. Then $r_i\geq i+t$ for $i=1,\dots,p-1$.

Let M be a balanced big Cohen-Macaulay module of R. Since $\operatorname{codim}I_{r_i}(\varphi_i)\geq i$, there exists a sequence $x_1,\dots,x_i$ in $I_{r_i}(\varphi_i)$ which is part of a system of parameters of R, and hence a regular sequence on M. This implies that $\operatorname{grade}(\varphi_i,M)\geq i$ for all i. Thus $\mathbb{F}\otimes M$ is acyclic. In particular, $r_i\geq 1$ for $i=1,\dots,p$.

We prove $(*)$ by induction on p. If $p=1$, then there is nothing to show. Suppose now that $p>1$, and consider the complex

$$\mathbb{G}\colon 0\longrightarrow G_{p-1}\xrightarrow{\psi_{p-1}}G_{p-2}\longrightarrow\cdots\longrightarrow G_1\xrightarrow{\psi_1}G_0\longrightarrow 0,$$

with $G_i=F_{i+1}$ and $\psi_i=\varphi_{i+1}$ for $k=1,\dots,p-1$. Let $s_i=\sum_{j_i}^{p-1}(1)^{j-i}\operatorname{rank}G_j$. Then $\operatorname{codim}I_{s_i}(\psi_i)\geq i+(1+t)$ for all i. Hence by induction hypothesis, we have $r_{i+1}=s_i\geq(i+1)+t$ for $i=1,\dots,p-1$.

Thus it remains to show that $r_1 \geq 1+t$. We show this by induction on t. The assertion is clear if $t=0$. Suppose now that $t>0$. We choose $e \in F_1 \setminus \mathfrak{m}F_1$, replace F_1 by $F_1' = F_1/Re$ and φ_2 by the induced map $\varphi_2' \colon F_2 \to F_1'$. Furthermore, we choose F_0' and $\operatorname{Coker}\varphi_2' \to F_0'$ as described in Lemma 2.12. This yields a map $\varphi_1' \colon F_1' \to F_0'$, so that we obtain the complex

$$\mathbb{F}' \colon 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\varphi_2'} F_1' \xrightarrow{\varphi_1'} F_0' \longrightarrow 0.$$

We show (i) $\operatorname{codim} I_{r_2'}(\varphi_2') \geq t+1$ and (ii) $\operatorname{codim} I_{r_1'}(\varphi_1') \geq t$. Then $\mathbb{F}'$ satisfies the hypotheses of $(*)$ with $t-1$ instead of t.

It may be that $\operatorname{Im}\varphi_1' \not\subseteq \mathfrak{m}F_0'$. In this case one can split off a direct summand without affecting (i) ansd (ii). Applying our induction hypothesis to this cancelled complex, we obtain $r_1' \geq t$, and hence $r_1 \geq t+1$, as desired.

Proof of (i): let $\mathfrak{p}$ be a prime ideal with $\operatorname{codim}\mathfrak{p} \leq t$. Then $I_{r_i}(\varphi_i) \not\subset \mathfrak{p}$, so that $\mathbb{F} \otimes R_\mathfrak{p}$ is split acyclic. In particular, $(F_1)_\mathfrak{p} \cong (\operatorname{Im}\varphi_2)_\mathfrak{p} \oplus (\operatorname{Coker}\varphi_2)_\mathfrak{p}$ with $\operatorname{rank}(\operatorname{Im}\varphi_2)_\mathfrak{p} = r_2$ and $\operatorname{rank}(\operatorname{Coker}\varphi_2)_\mathfrak{p} = r_1$. Moreover, by Theorem 2.11 we have $\operatorname{codim}\mathscr{O}(\varphi_1(e)) \geq t+1$. Therefore Lemma 2.10 implies that $\varphi_1(e)$ generates a non-zero free summand of $(F_0)_\mathfrak{p}$. Consequently, the image $\bar{e}$ of e under the residue class map $F_1 \to \operatorname{Coker}\varphi_2$ generates a non-zero free direct summand of $(\operatorname{Coker}\varphi_2)_\mathfrak{p}$. Hence $(\operatorname{Coker}\varphi_2')_\mathfrak{p} \cong (\operatorname{Coker}\varphi_2)_\mathfrak{p}/R_\mathfrak{p}\bar{e}$ is free of rank $r_1' = r_1 - 1$, and the exact sequence

$$0 \longrightarrow (\operatorname{Im}\varphi_2')_\mathfrak{p} \longrightarrow (F_1')_\mathfrak{p} \longrightarrow (\operatorname{Coker}\varphi_2')_\mathfrak{p} \longrightarrow 0$$

splits. In particular, $(\operatorname{Im}\varphi_2')_\mathfrak{p}$ is free direct summand of $(F_1')_\mathfrak{p}$ of rank r_2. Thus Proposition 1.8 implies that $I_{r_2'}(\varphi_2') \not\subset \mathfrak{p}$, as desired.

Proof of (ii): We choose $\mathfrak{p}$ as before. We have already seen that $(\operatorname{Coker}\varphi_2')_\mathfrak{p}$ is free of rank r_1'. As φ_1' is constructed as described in Lemma 2.12, $(\operatorname{Coker}\varphi_2')_\mathfrak{p}$ is mapped isomorphically onto a free direct summand of F_0'. This implies that $I_{r_1'}(\varphi_1') \not\subset \mathfrak{p}$, and shows $\operatorname{codim} I_{r_1'}(\varphi_1') \geq 1+t$, which is even more than required. $\square$

3. Lecture: Upper bounds

In the remaining sections, unless otherwise stated, $R = k[x_1, \ldots, x_n]$ is the polynomial ring, and M is a finitely generated graded R-module. As indicated in Section 1 we want to relate the Betti-numbers $\beta_i(M)$ of M to the generic annihilator numbers $\alpha_i(M)$ of M.

Let $y = y_1, \ldots, y_n$ be generic linear forms. Then

$$A_j = ((y_1, \ldots, y_{j-1})M :_M y_j)/(y_1, \ldots, y_{j-1})M$$

is a module of finite length. We set

$$\alpha_i(M) = \ell(A_i).$$

We denote by $H_i(j;M)$ the Koszul homology $H_i(y_1, \ldots, y_j; M)$ of the partial sequence $y_1, \ldots, y_j$, and set $h_i(j;M) = \dim_K H_i(j;M)$. If there is no danger of confusion, we simply write β_i, α_i, $H_i(j)$ and $h_i(j)$ for $\beta_i(M)$, $\alpha_i(M)$, $H_i(j;M)$ and $h_i(j;M)$ respectively.

Attached with y there are long exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_i(j-1) & \xrightarrow{\varphi_{i,j-1}} & H_i(j-1) & \longrightarrow & H_i(j) & \longrightarrow & H_{i-1}(j-1) \\ \cdots & \longrightarrow & H_0(j-1) & \xrightarrow{\varphi_{0,j-1}} & H_0(j-1) & \longrightarrow & H_0(j) & \longrightarrow & 0. \end{array}$$

Here $\varphi_{i,j-1}\colon H_i(j-1) \to H_i(j-1)$ is the map given by multiplication with $\pm y_j$. Note that A_j is the Kernel of the map $\varphi_{0,j-1}$. We conclude

($*$) $h_1(j) = h_1(j-1) + \alpha_j - \dim_K \operatorname{Im} \varphi_{1,j-1}$ for $i = 1$;

($**$) $h_i(j) = h_i(j-1) + h_{i-1}(j-1) - \dim_K \operatorname{Im} \varphi_{i,j-1} - \dim_K \operatorname{Im} \varphi_{i-1,j-1}$ for $i > 1$.

With the notation introduced we now have:

Proposition 3.1. *Given integers $1 \le i \le j$ we define the set*

$$C_{i,j} = \{(a,b) \in \mathbb{N}^2 : 1 \le b \le j-1 \text{ and } \max(i-j+b,1) \le a \le i\}.$$

Then we have

(a) $h_i(j) \le \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_k$ *for all $i \ge 1$ and $j \ge 1$;*

(b) *For given $i \ge 1$ and $j \ge 1$ the following conditions are equivalent:*

(i) $h_i(j) = \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_k$*;*

(ii) $\varphi_{ab} = 0$ *for all $(a,b) \in C_{i,j}$;*

(iii) $\mathfrak{m} H_a(b) = 0$ *for all $(a,b) \in C_{i,j}$.*

Proof. By induction on j and using equations (*) and (**) one proves that

$$h_i(j) = \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_{jk} - \sum_{(a,b)\in C_{i,j}} \binom{j-b}{i-a} \dim_K \operatorname{Im} \varphi_{a,b}$$

Then (a) and the equivalence of (i) and (ii) in (b) follow immediately. For the equivalence of (ii) and (iii) we notice that a generic linear form annihilates $H_a(b)$ if and only if $\mathfrak{m} H_a(b) = 0$. $\square$

By taking $j = n$ we obtain the following upper bound

Corollary 3.2. $\beta_i \le \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$ *for all $i \ge 1$.*

When this upper bound is reached is described in the next corollary in terms of vanishing of Koszul homology

Corollary 3.3. (a) *For a given integer i the following conditions are equivalent:*

(i) $\beta_i = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$,

(ii) $\mathfrak{m} H_a(b) = 0$ *for all $(a,b) \in C_{i,n}$;*

(b) *The following conditions are equivalent:*

(i) $\beta_i = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$ *for all $i \ge 1$,*

(ii) $\mathfrak{m} H_a(b) = 0$ *for all b and for all $a \ge 1$.*

We now want to discuss when condition (b)(ii) is satisfied. We first note that it implies that $y_1, \ldots, y_n$ is a proper sequence in the sense of [23].

Definition 3.4. Let R be an arbitrary commutative ring, and M and R-module. A sequence $y_1,\dots,y_r$ of elements of R is called a *proper M-sequence*,
if $y_{j+1}H_i(j;M)=0$ for all $i\geq 1$ and $j=0,\dots,r-1$.

In [26] Kühl proved the following remarkable fact: The sequence $y_1,\dots,y_r$ is a proper M-sequence if and only if

$$y_{j+1}H_1(j;M)=0 \quad \text{for} \quad j=0,\dots,r-1.$$

Now we have

Theorem 3.5 (Conca-Herzog-Hibi)**.** *Let $I\subset R$ be a graded ideal, and let $y=y_1,\dots,y_n$ be a sequence of generic linear forms. The following conditions are equivalent:*

(a) *R/I has maximal Betti numbers, i.e.*

$$\beta_i(R/I)=\sum_{j=1}^{n-i+1}\binom{n-j}{i-1}\alpha_j(R/I) \quad \textit{for all} \quad i\geq 1;$$

(b) *y is a proper R/I-sequence;*
(c) *I is componentwise linear.*

Proof. Let z be a generic linear form. Then $zH_i(p)=0$ if and only if $\mathfrak{m}H_i(p)=0$. Thus the equivalence of (a) and (b) follows from 3.3 (b). The equivalence of (b) and (c) can be found in [16, Theorem 4.5]. □

Another important method to obtain upper bounds for resolutions is to compare the resolution of an ideal I with the resolution of its initial ideal $\text{in}(I)$ with respect to some term order $<$ on R. The basic fact is the following

Theorem 3.6. *Let $I\subset R$ be a graded ideal. Then for any term order $<$ one has*

$$\beta_{ij}(R/I)\leq\beta_{ij}(R/\text{in}_<(I)) \quad \textit{for all} \quad i,j.$$

Proof. Let $\tilde{R}$ be the $k[t]$-algebra $R[t]$, where t is an indeterminate of degree 0. By [17, Theorem 15.17] there exists a graded ideal $\tilde{I}\subset\tilde{R}$ such that the $k[t]$-algebra $\tilde{R}/\tilde{I}$ is a free $k[t]$-module (and thus flat over $k[t]$), and such that

$$(\tilde{R}/\tilde{I})/t(\tilde{R}/\tilde{I})\cong R/\text{in}_<(I), \tag{2}$$

and

$$(\tilde{R}/\tilde{I})_t\cong(R/I)\otimes_k k[t,t^{-1}], \tag{3}$$

as graded k-algebras. The ideal $\tilde{I}\subset\tilde{R}$ is constructed by means of a weight function.

Let $\mathbb{F}$ be the minimal graded free $\tilde{S}$-resolution of $\tilde{R}/\tilde{I}$. Then (2) implies that $\mathbb{F}/t\mathbb{F}$ is a graded minimal free R-resolution of R/I, so that $\beta_{ij}(\tilde{R}/\tilde{I})=\beta_{ij}(R/\text{in}_<(I))$ for all i and j, and (3) implies that the localized complex $\mathbb{F}_t$ is a graded (not necessarily minimal) free $R\otimes_K K[t,t^{-1}]$ resolution of $(R/I)\otimes_K K[t,t^{-1}]$. Thus, $\beta_{ij}(R/I)=\beta_{ij}((R/I)\otimes_K K[t,t^{-1}])\leq\beta_{ij}(\tilde{R}/\tilde{I})$, as desired. □

Let M be a finitely generated graded S-module. The *regularity* of M is defined to be the number $\text{reg}(M)=\max\{j-i\colon\ \beta_{ij}(M)\neq 0\}$. As an immediate consequence of 3.6 we have

Corollary 3.7. *Let $I \subset R$ be a graded ideal. Then for any term order $<$ one has:*

(a) $\operatorname{proj\,dim} R/I \leq \operatorname{proj\,dim} R/\operatorname{in}_<(I)$.
(b) $\operatorname{depth} R/I \geq \operatorname{depth} R/\operatorname{in}_<(I)$.
(c) *If $R/\operatorname{in}_<(I)$ is Cohen-Macaulay (Gorenstein), then so is S/I.*
(d) $\operatorname{reg} R/I \leq \operatorname{reg} R/\operatorname{in}_<(I)$.

We shall see in the next section that all inequalities of 3.7 become equalities, if $\operatorname{in}_<(I)$ is replaced by the generic initial ideal $\operatorname{Gin}(I)$ with respect to the reverse lexicographic order.

We fix a term order $<$ satisfying $x_1 > x_2 > \ldots > x_n$. Let $I \subset R$ be an ideal. The *generic initial ideal* $\operatorname{Gin}(I)$ with respect to this term order is defined as follows: let $GL(n)$ denote the general linear group with coefficients in k. Any $\varphi = (a_{ij}) \in GL(n)$ induces an automorphism of the graded k-algebra R, again denoted by φ, namely

$$\varphi(f(x_1,\ldots,x_n)) = f(\sum_{i=1}^{n} a_{i1}x_i,\ldots,\sum_{i=1}^{n} a_{in}x_i) \quad \text{for all} \quad f \in R.$$

For the proof of the following result we refer to [17, Theorem 15.18]

Theorem 3.8 (Galligo, Bayer and Stillman)**.** *Let $I \subset R$ be a graded ideal. Then there is a nonempty Zariski open set $U \subseteq GL(n)$ such that $\operatorname{in}(\varphi(I))$ does not depend on $\varphi \in U$. Moreover, U meets non trivially the Borel subgroup of $GL(n)$ consisting of all upper triangular invertible matrices.*

For $\varphi \in U$ the monomial ideal $\operatorname{in}(\varphi(I))$ is called the *generic initial ideal of I*, and will be denoted $\operatorname{Gin}(I)$.

A monomial ideal I is called *strongly stable*, if $x_i(u/x_j) \in I$ for all monomials $u \in I$, all x_j that divides u, and all $i < j$. The ideal I is called *stable*, if $x_i(u/x_{m(u)}) \in I$ for all monomials $u \in I$, and all $i < m(u)$. Here $m(u) = \max\{i\colon x_i \text{ divides } u\}$.

For a monomial ideal I it is customary to denote the unique minimal set of monomial generators by $G(I)$. It is easy to see that I is strongly stable if $x_i(u/x_j) \in I$ for all monomials $u \in G(I)$, all x_j that divides u, and all $i < j$. A similar statement holds for stable ideals.

Stable monomial ideals were introduced by Eliahou and Kervaire [18] who also gave an explicit resolution of such ideals. Such ideals are important because of the following result [17, Theorem 15.23]

Theorem 3.9. *Suppose that $\operatorname{char} k = 0$, and let $I \subset R$ be a graded ideal. Then the generic initial ideal $\operatorname{Gin}(I)$ of I with respect to the reverse lexicographical order is strongly stable.*

Also in positive characteristic $\operatorname{Gin}(I)$ has a nice (but much more complicated) combinatorial structure.

The Koszul homology of a stable monomial ideal I can be easily computed. We let $\varepsilon\colon R \to R/I$ be the canonical epimorphism, and set $u' = u/x_{m(u)}$ for all $u \in G(I)$.

Theorem 3.10. *Let $I \subset R$ be a stable ideal. For all $j = 1,\ldots,n$ and $i > 0$, the Koszul homology $H_i(x_j,\ldots,x_n)$ is annihilated by $\mathfrak{m} = (x_1,\ldots,x_n)$. In other words, all these homology modules are k-vector spaces. A basis of $H_i(x_j,\ldots,x_n)$ is given by the homology* *classes of the cycles*

$$\varepsilon(u')e_\sigma \wedge e_{m(u)}, \quad u \in G(I), \quad |\sigma| = i-1, \quad j \le \min(\sigma), \quad \max(\sigma) < m(u).$$

Proof. We proceed by induction on $n-j$. For $j = n$, we only have to consider $H_1(x_n)$ which is obviously minimally generated by the homology classes of the elements $\varepsilon(u')e_n$ with $u \in G(I)$ such that $m(u) = n$. Since by the definition of stable ideals $x_iu' \in I$ for all i, we see that $H_1(x_n)$ is a k-vector space.

Now assume that $j < n$, and that the assertion is proved for $j+1$. Then $x_jH_i(x_{j+1},\ldots,x_n) = 0$ for all $i > 0$, so that the long exact sequence

$$\begin{aligned} \cdots \quad &\xrightarrow{x_j} H_i(x_{j+1},\ldots,x_n) \longrightarrow H_i(x_j,\ldots,x_n) \longrightarrow H_{i-1}(x_{j+1},\ldots,x_n) \\ &\xrightarrow{x_j} H_{i-1}(x_{j+1},\ldots,x_n) \longrightarrow H_{i-1}(x_j,\ldots,x_n) \longrightarrow \cdots \end{aligned}$$

splits into the exact sequences

$$(4) \qquad 0 \longrightarrow H_1(x_{j+1},\ldots,x_n) \longrightarrow H_1(x_j\ldots,x_n) \longrightarrow R_j/I_j \xrightarrow{x_j} R_j/I_j$$

and

$$(5) \qquad 0 \longrightarrow H_i(x_{j+1},\ldots,x_n) \longrightarrow H_i(x_j,\ldots,x_n) \longrightarrow H_{i-1}(x_{j+1},\ldots,x_n) \longrightarrow 0.$$

for $i > 0$. Here R_j is the polynomial ring $k[x_1,\ldots,x_j]$, I_j the ideal in R_j generated by the monomials $u \in G(I)$ which are not divisible by any x_i with $i > j$, in other words, $I_j = I \cap S_j$.

In sequence (4), $\operatorname{Ker} x_j$ is minimally generated by the residues of the monomials u' with $u \in G(I)$ and $m(u) = j$. Note that the sets $\{u \in G(I)\colon m(u) = j\}$ and $\{u \in G(I_j)\colon m(u) = j\}$ are equal, and that I_j is a stable ideal in R_j. Therefore $\operatorname{Ker} x_j$ is a k-vector space.

We now consider the short exact sequence

$$(6) \qquad 0 \longrightarrow H_1(x_{j+1},\ldots,x_n) \longrightarrow H_1(x_j,\ldots,x_n) \longrightarrow \operatorname{Ker} x_j \longrightarrow 0.$$

It is clear that the elements $\varepsilon(u')e_j$, $u' \in G(I)$, $m(u) = j$ are cycles in $K_1(x_j,\ldots,x_n)$ such that $\delta([\varepsilon(u')e_j]) = u' + I_j$. Therefore, by (6) and our induction hypothesis, it follows that the set $\mathscr{S} = \{[\varepsilon(u')e_i]\colon u \in G(I), m(u) = i \ge j\}$ generates $H_1(x_j,\ldots,x_n)$. Since I is a stable ideal we see that $x_j[\varepsilon(u')e_i] = 0$ for all $j = 1,\ldots,n$ and all $[\varepsilon(u')e_i] \in \mathscr{S}$. In other words, $H_1(x_j,\ldots,x_n)$ is a k-vector space. Finally, since the number of elements of $\mathscr{S}$ equals $\dim_k H_1(x_{j+1},\ldots,x_n) + \dim \operatorname{Ker} x_j$, we conclude that $\mathscr{S}$ is a basis of $H_1(x_j,\ldots,x_n)$.

In order to prove our assertion for $i > 1$ we consider the exact sequences (5). By induction hypothesis the homology module $H_{i-1}(x_{j+1},\ldots,x_n)$ is a k-vector space with basis

$$[\varepsilon(u')e_\sigma \wedge e_{m(u)}], \quad u \in G(I), \quad |\sigma| = i-2, \quad j+1 \le \min(\sigma), \quad \max(\sigma) < m(u).$$

Given such a homology class, consider the element $\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}$. It is clear that this element is a cycle in $K_i(x_j,\ldots,x_n)$, and that

$$\delta([\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}] = \pm[\varepsilon(u')e_\sigma \wedge e_{m(u)}].$$

Thus from the exact sequence (5) and our induction hypothesis it follows that the homology classes of the cycles described in the theorem generate $H_i(x_j,\ldots,x_n)$. Again the stability of the ideal I implies that $\mathfrak{m}$ annihilates all these homology classes, so that

$H_i(x_j,\dots,x_n)$ is a K-vector space. Finally, just as for $i=1$, a dimension argument shows that these homology classes form a basis of $H_i(x_j,\dots,x_n)$. □

Let I be a monomial ideal. We denote by $G(I)_j$ the set of monomial generators of I of degree j. The following result of Eliahou and Kervaire [18] follows immediately from 3.10.

Corollary 3.11. *Let $I\subset SR$ be a stable ideal. Then*

(a) $\beta_{ii+j}(I)=\sum_{u\in G(I)_j}\binom{m(u)-1}{i}$;
(b) $\operatorname{proj\,dim} R/I=\max\{m(u)\colon u\in G(I)\}$;
(c) $\operatorname{reg}(I)=\max\{\deg(u)\colon u\in G(I)\}$.

If we consider upper bounds for the Betti-numbers of an ideal we have to fix a class $\mathscr{C}$ of ideals and to ask if there is an upper bound for the Betti-numbers of the ideals within this class. We have already seen that the class of ideals whose residue class ring has a given sequence of annihilator numbers has such an upper bound.

Here we now consider the class $\mathscr{C}$ of ideals with given Hilbert function. Within this class there is a distinguished ideal. In fact, let $>$ be the lexicographical monomial order induced by $x_1>x_2>\dots>x_n$. Recall that a monomial ideal $I\subset R$ is called a *lexsegment ideal,* if for each monomial $u\in I$, all monomials $v>u$ belong to I as well.

Let $B\subset R_d$ be a set of monomials. Then B is called a *lexsegment* if with each $u\in B$ we have $v\in B$ for all $v>u$ in the lexicographical order. A lexsegment ideal is an ideal which is spanned in each degree by a lexsegment set of monomials.

We denote by $\operatorname{Shad}(B)$ the *shadow* of B, i.e. the set of monomials

$$\{x_1,\dots,x_n\}B=\{x_iu\colon u\in B, i=1,\dots,n\}.$$

The set B is called a *(strongly) stable* set of monomials if the ideal generated by B is (strongly) stable.

We let $m_i(B)$ the number of elements $u\in B$ with $m(u)=i$, and set $m_{\leq i}(B)=\sum_{j=1}^{i}m_j(B)$. Then we have

Lemma 3.12. *Let $B\subset M_d$ be a stable set of monomials. Then*

(a) $m_i(\operatorname{Shad}(B))=m_{\leq i}(B)$;
(b) $|\operatorname{Shad}(B)|=\sum_{i=1}^{n}m_{\leq i}(B)$.

Proof. (b) is of course a consequence of (a). For the proof of (a) we note that the map

$$\varphi\colon \{u\in B\colon m(u)\leq i\}\to\{u\in \operatorname{Shad}(B)\colon m(u)=i\},\quad u\mapsto ux_i$$

is a bijection. In fact, φ is clearly injective. To see that φ is surjective, we let $v\in\operatorname{Shad}(B)$ with $m(v)=i$. Since $v\in\operatorname{Shad}(B)$, there exists $w\in B$ with $v=x_jw$ for some $j\leq i$. It follows that $m(w)\leq i$. If $j=i$, then we are done. Otherwise, $j<i$ and $m(w)=i$. Hence, since B is stable it follows that $u=(x_j/x_i)w\in B$. The assertion follows, since $v=ux_i$. □

The following result is crucial.

Theorem 3.13 (Bayer [5]). *Let $L\subset R_d$ be a lexsegment, and $B\subset R_d$ be a stable set of monomials with $|L|\leq|B|$. Then $m_{\leq i}(L)\leq m_{\leq i}(B)$ for $i=1,\dots,n$.*

We denote by B^{lex} the unique lexsegment set of monomials with $|B^{lex}| = |B|$. Now Lemma 3.12 and Theorem 3.13 imply

Corollary 3.14. *Let $B \subset R_d$ a be stable set of monomials, then $|\operatorname{Shad}(B^{lex})| \leq |\operatorname{Shad}(B)|$.*

Using all this we now get

Theorem 3.15. *Let $I \subset R$ be a graded ideal. Then there exists a unique lexsegment ideal in R, denoted I^{lex}, such that R/I and R/I^{lex} have the same Hilbert function.*

Proof. Let $<$ be any monomial order. It is easy to see that S/I and $S/\operatorname{in}_<(I)$ have the same Hilbert function. Hence we may replace I by $\operatorname{in}_<(I)$, and thus may assume that I is a monomial ideal. Then for any field L the Hilbert function of $L[x_1,\ldots,x_n]/(G(I))$ does not depend on L. Thus we may replace k by L if necessary, and thus may as well assume that $\operatorname{char} k = 0$. Then by Theorem 3.9 the generic initial ideal $\operatorname{Gin}(I)$ of I with respect to the reverse lexicographical order is strongly stable.

For each d let I_d be spanned by the set of monomials N_d. Then N_d is a strongly stable set of monomials. Let I_d^{lex} be the subspace of R_d spanned by N_d^{lex}. We set $I^{lex} = \bigoplus_{d\geq 0} I_d^{lex}$, and show that I^{lex} is an ideal, in other words, that $\{x_1,\ldots,x_n\}I_d^{lex} \subset I_{d+1}^{lex}$ for all d.

By Corollary 3.14 one has $|\operatorname{Shad}(N_d^{lex})| \leq |\operatorname{Shad}(N_d)| \leq |N_{d+1}| = |N_{d+1}^{lex}|$. On the other hand, since $\operatorname{Shad}(N_d^{lex})$ and N_{d+1}^{lex} are both lexsegments, this inequality implies $\operatorname{Shad}(N_d^{lex}) \subset N_{d+1}^{lex}$, as desired.

It is obvious from the construction that R/I and R/I^{lex} have the same Hilbert function. □

Bigatti [8] and Hulett [24] proved independently the following theorem if the base field k of R is of characteristic 0. A proof in arbitrary characteristic was later given by Pardue [27] using a suitable polarization argument.

Theorem 3.16 (Bigatti, Hulett, Pardue)**.** *Let $I \subset R$ be a graded ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{lex}) \quad \textit{for all } i \textit{ and } j.$$

In particular, among all ideals with a given Hilbert function, the unique lexsegment ideal with this given Hilbert function has the largest Betti-numbers.

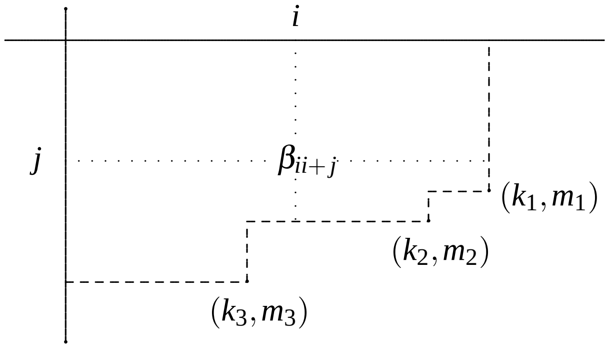
Proof. We outline the proof in case $\operatorname{char}(k) = 0$. By Theorem 3.6 we have $\beta_{ij}(I) \leq \beta_{ij}(\operatorname{Gin}(I))$ for all i and j, where $\operatorname{Gin}(I)$ denotes the generic initial ideal of I with respect to the reverse lexicographical order. Since we assume that $\operatorname{char}(k) = 0$, it follows from Theorem 3.9 that $\operatorname{Gin}(I)$ is a strongly stable ideal. We may therefore assume that I itself is a a strongly stable monomial ideal. Then $\beta_{ij}(I) = \sum_{u\in G(I)_j}\binom{m(u)-1}{i}$, by Corollary 3.11(a). A similar formula holds for I^{lex}, since I^{lex} is also strongly stable. These formulas for the Betti-numbers can be rewritten in terms of the numbers $m_{\leq i}(I)$ $m_{\leq i}(I^{lex})$. Then using Bayer's theorem 3.13 according to which $m_{\leq i}(I) \leq m_{\leq i}(I^{lex})$ for all i, one obtains the desired inequalities. □

4. Lecture: Stability

In this section we assume that $R = k[x_1,\ldots,x_n]$ is the polynomial ring over an infinite field K, and $I \subset R$ is a graded ideal. In the previous section we have seen that for any term order one has $\beta_{ij}(I) \leq \beta_{ij}(\text{in}(I))$. One may ask on what conditions on the term order or on the ideal one obtains more precise information in this comparison. A classical result in this direction is the theorem of Bayer-Stillman [7] which asserts that $\text{reg}(I) = \text{reg}(\text{Gin}(I))$. Here and throughout this section $\text{Gin}(I)$ denotes the generic initial ideal with respect to the reverse lexicographical order.

A remarkable extension of the Bayer-Stillman theorem, which we want to discuss next, was proved by Bayer, Charalambous and Popescu in [6]. Let M be a finitely generated graded S-module. A Betti number $\beta_{kk+m} \neq 0$ of M is called *extremal* if $\beta_{ii+j} = 0$ for all $(i, j) \neq (k,m)$ with $i \geq k$ and $j \geq m$.

The following picture displays the Betti diagram of a graded free resolution in the form of a MACAULAY output. The entry with coordinates (i, j) is the Betti number β_{ii+j}. In our picture the outside corners of the dashed line give the positions of the extremal Betti numbers.



Let M be a finitely generated graded R-module, and let $y = y_1,\dots,y_n$ be generic linear forms. As in Section 3 we set $H_i(j) = H_i(y_1,\dots,y_j;M)$, and

$$A_j = (y_1,\dots,y_{j-1})M :_M y_j/(y_1,\dots,y_{j-1})M.$$

All $H_i(j)$ as well as all A_j are R-modules of finite length and since M is a graded S-module, all $H_i(j)$ and all A_j are naturally graded, and there are graded isomorphisms $H_i(n)_j \cong \text{Tor}_i(k,M)_j$ for all i and j.

Let N be an Artinian graded module. We set

$$s(N) = \begin{cases} \max\{s\colon N_s \neq 0\} & \text{if} \quad N \neq 0, \\ -\infty & \text{if} \quad N = 0. \end{cases}$$

Now we introduce the following numbers attached to M and the sequence $y = y_1,\dots,y_n$. We set

$$r_j = \max\{s(H_i(j)) - i\colon i \geq 1\} \quad \text{and} \quad s_j = s(A_j) \quad \text{for} \quad j = 1,\dots,n,$$

and put $r_0 = 0$. Observe that $\text{reg}(M) = \max\{r_n, s(M/\mathfrak{m}M)\}$.

We quote the following technical result from [2].

Theorem 4.1. *With the hypotheses and notation introduced we have*

(a) $r_j = \max\{s_1,\dots,s_j\}$ *for* $j = 1,\dots,n$. *In particular,* $r_1 \leq r_2 \leq \dots \leq r_n$.

(b) *Let* $\mathscr{J} = \{j_1,\dots,j_l\}$, $1 \leq j_1 < j_2 < \dots < j_l \leq n$, *be the set of elements* $j \in [n]$ *such that* $r_j - r_{j-1} \neq 0$. *Then for all* t *with* $1 \leq t \leq l$ *and all* j *with* $j_t \leq j$ *we have*

(i) $H_i(j)_{i+s} = 0$ *for* $s > r_{j_{t-1}}$ *and* $i > j - j_t + 1$;

(ii) $H_{j-j_t+1}(j)_{j-j_t+1+r_{j_t}} \cong (A_{j_t-1})_{r_{j_t}}$.

This result yields a characterization of the extremal Betti-numbers in terms of Koszul homology

Corollary 4.2. *Let the numbers j_t be defined as in* 4.1*, and set $k_t = n - j_t + 1$ and $m_t = r_{j_t}$. Then the graded Betti number β_{ii+j} of M is extremal if and only if*

$$(i, j) \in \{(k_t, m_t) : t = 1, \ldots, l\}.$$

Moreover, $\beta_{k_t,k_t+m_t} = \dim_K (A_{j_t})_{s_{j_t}}$ for $t = 1, \ldots, l$.

Let $I \subset R$ be a graded ideal. We want to compare the graded Betti-numbers of R/I and $R/\operatorname{Gin}(I)$. Choosing generic coordinates we may assume that $\operatorname{in}(I) = \operatorname{Gin}(I)$, and that $x_n, x_{n-1}, \ldots, x_1$ is a generic sequence for R/I. For the reverse lexicographical order induced by $x_1 > x_2 > \ldots > x_n$ one has

$$\operatorname{in}((x_j, \ldots, x_n) + I) = (x_j, \ldots, x_n) + \operatorname{in}(I)$$

and

$$\operatorname{in}((x_j, \ldots, x_n) + I) : x_{j-1}) = ((x_j, \ldots, x_n) + \operatorname{in}(I)) : x_{j-1}.$$

It follows that

$$((x_j, \ldots, x_n) + I) : x_{j-1}/((x_j, \ldots, x_n) + I)$$

and

$$((x_i, \ldots, x_n) + \operatorname{in}(I)) : x_{i-1}/((x_i, \ldots, x_n) + \operatorname{in}(I))$$

have the same Hilbert function.

Let A_j be the module defined before in case that $M = R/I$. Then

$$A_j = ((x_j, \ldots, x_n) + I) : x_{j-1}/((x_j, \ldots, x_n) + I).$$

We set

$$A_j^* = ((x_i, \ldots, x_n) + \operatorname{in}(I)) : x_{i-1}/((x_i, \ldots, x_n) + \operatorname{in}(I)),$$

and $\alpha_j = \ell(A_j)$, $\alpha_j^* = \ell(A_j^*)$, $s_j = s(A_j)$ and $s_j^* = s(A_j^*)$ for $j = 1, \ldots, n$.

The preceding considerations now yield

Lemma 4.3. *The modules A_j and A_j^* have the same Hilbert functions. In particular, $\alpha_j = \alpha_j^*$ and $s_j = s_j^*$ for $j = 1, \ldots, n$.*

Combining this result with Theorem 4.1 and Corollary 4.2 we obtain

Theorem 4.4 (Bayer-Charalambous-S.Popescu)**.** *Let $I \subset R$ be a graded ideal, and let* $\operatorname{Gin}(I)$ *be the generic initial ideal of I with respect to the reverse lexicographic order. Then for any two integers $i, j \in \mathbb{N}$ one has*

(a) *the ijth Betti number of R/I is extremal if and only if the ijth Betti number of $R/\operatorname{Gin}(I)$ is extremal;*
(b) *the corresponding extremal Betti numbers of R/I and $R/\operatorname{Gin}(I)$ are equal.*

This theorem implies in particular

Corollary 4.5. *Let $I \subset R$ be a graded ideal,* $\operatorname{Gin}(I)$ *the generic initial ideal of I with respect to the reverse lexicographic order. Then*

(a) (Bayer-Stillman) $\operatorname{reg}(I)=\operatorname{reg}(\operatorname{Gin}(I))$*;*
(b) $\operatorname{proj}\dim R/I=\operatorname{proj}\dim R/\operatorname{Gin}(I)$*;*
(c) *R/I is Cohen-Macaulay, if and only if $R/\operatorname{Gin}(I)$ is Cohen-Macaulay.*

Under which circumstances do all the graded Betti-numbers of I and $\operatorname{Gin}(I)$ agree? The answer is given by

Theorem 4.6 (Aramova-Herzog-Hibi)**.** *Suppose* $\operatorname{char} k=0$*, and let $I\subset R$ be a graded ideal. The following conditions are equivalent:*

(a) *$\beta_{i,i+j}(I)=\beta_{i,i+j}(\operatorname{Gin}(I))$ for all i and j;*
(b) *I is componentwise linear.*

For the proof of this theorem we need some preparation. We write $I_{\langle j\rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to I. Moreover, we write $I_{\geq d}$ for the ideal generated by all homogeneous polynomials of I whose degree is greater than or equal to d.

The Betti-numbers of $\operatorname{Gin}(I)$ have the following properties

Proposition 4.7. *Let $I\subset R$ be a graded ideal generated in degree d. Then we have:*

(a) *if $\beta_{i,i+j}(\operatorname{Gin}(I))\neq 0$, then $\beta_{i',i'+j}(\operatorname{Gin}(I))\neq 0$ for all $i'<i$;*
(b) *if $\beta_{0,j}(\operatorname{Gin}(I))\neq 0$, then $\beta_{0,j'}(\operatorname{Gin}(I))\neq 0$ for all $d\leq j'<j$.*

Proof. Since the generic initial ideal is strongly stable, statement (a) follows from Corollary 3.11(a).

Let $g_1,\dots,g_m$ be the generators of I of degree d. Suppose that $\beta_{0,j-1}(\operatorname{Gin}(I))=0$. Then consider the ideal $I_{\geq j-2}$. Since $\operatorname{Gin}(I_{\geq j-2})=\operatorname{Gin}(I)_{\geq j-2}$, we may assume that $\beta_{0,d+1}(\operatorname{Gin}(I))=0$. We have to show that $\operatorname{Gin}(I)$ is generated in degree d. It follows from $\beta_{0,d+1}(\operatorname{Gin}(I))=0$ that all S-polynomials of degree $d+1$ reduce to zero with respect to $\{g_1,\dots,g_m\}$. Since $(\operatorname{in}(g_1),\dots,\operatorname{in}(g_m))$ is a strongly stable ideal, its first syzygy module is generated in degree $d+1$, the fact that the S-polynomials of this degree reduce to zero, implies that $\{g_1,\dots,g_m\}$ is a Gröbner basis of I. From this it follows that $\operatorname{Gin}(I)$ is generated in degree d. ☐

We shall also need

Lemma 4.8. *Let I and J be graded ideals of R generated in degree d with the same graded Betti numbers. Then $I_{\geq d+1}$ and $J_{\geq d+1}$ have the same graded Betti numbers.*

Proof. The exact sequence

$$0\longrightarrow I_{\geq d+1}\longrightarrow I\longrightarrow k(-d)^{\beta_{0,d}}\longrightarrow 0$$

induces the long exact sequence

$$\cdots\to \operatorname{Tor}_{i+1}(k,I_{\geq d+1})_{(i+1)+(j-1)}\to \operatorname{Tor}_{i+1}(k,I)_{(i+1)+(j-1)}\to \operatorname{Tor}_{i+1}(k,k)^{\beta_{0,d}}_{(i+1)+j-(d+1)}$$

$$\to \operatorname{Tor}_i(k,I_{\geq d+1})_{i+j}\to \operatorname{Tor}_i(k,I)_{i+j}\to \operatorname{Tor}_i(k,k)^{\beta_{0,d}}_{i+j-d}\to\cdots.$$

It then follows that $\beta_{i,i+j}(I_{\geq d+1}) = \beta_{i,i+j}(I)$ for all i and for all $j \neq d, d+1$. Also, $\beta_{i,i+j}(I_{\geq d+1}) = 0$ if $j \leq d$. Now, if $j = d+1$, then the above long exact sequence becomes

$$0 \to \mathrm{Tor}_{i+1}(k,I)_{i+1+d} \to \mathrm{Tor}_{i+1}(k,k)_{i+1}^{\beta_{0,d}} \to \mathrm{Tor}_i(k,I_{\geq d+1})_{i+d+1} \to \mathrm{Tor}_i(k,I)_{i+d+1} \to 0.$$

Hence, $\beta_{i,i+d+1}(I_{\geq d+1}) = \beta_{i,i+d+1}(I) + \binom{n}{i+1}\beta_{0,d}(I) - \beta_{i+1,i+1+d}(I)$.

The same formulae are valid for $\beta_{i,i+j}(J)$. This completes the proof. □

We are now in the position to give a proof of Theorem 4.6.

Proof of 4.6. First, suppose that I is componentwise linear. The following formula for the graded Betti numbers of a componentwise linear ideal I is known [21]:

$$\beta_{i,i+j}(I) = \beta_i(I_{\langle j\rangle}) - \beta_i(\mathfrak{m}I_{\langle j-1\rangle}).$$

Here $\mathfrak{m}$ is the irrelevant maximal ideal $(x_1,\ldots,x_n)$ of R. Since a strongly stable ideal is componentwise linear and since $\mathrm{Gin}(I)$ is strongly stable, the same formula is valid for $\mathrm{Gin}(I)$. Therefore, it suffices to prove that $\beta_i(I_{\langle j\rangle}) = \beta_i(\mathrm{Gin}(I)_{\langle j\rangle})$ and $\beta_i(\mathfrak{m}I_{\langle j-1\rangle}) = \beta_i(\mathfrak{m}\,\mathrm{Gin}(I)_{\langle j-1\rangle})$.

Since $I_{\langle j\rangle}$ has a linear resolution, it follows from the Bayer–Stillman theorem (Corollary 4.5), that $\mathrm{Gin}(I_{\langle j\rangle}) = \mathrm{Gin}(I)_{\langle j\rangle}$. Since $I_{\langle j\rangle}$ and $\mathrm{Gin}(I_{\langle j\rangle})$ have the same Hilbert function, and since the Betti numbers of a module with linear resolution are determined by its Hilbert function, the first equality follows. To prove the second one, we note that $\mathfrak{m}I_{\langle j-1\rangle}$ has again a linear resolution and that, by the same reason as before, $\mathfrak{m}\,\mathrm{Gin}(I)_{\langle j-1\rangle} = \mathrm{Gin}(\mathfrak{m}I_{\langle j-1\rangle})$.

Second, suppose that I and $\mathrm{Gin}(I)$ have the same graded Betti numbers. Let $\max(I)$ (resp. $\min(I)$) denote the maximal (resp. minimal) degree of a homogeneous generator of I. To show that I is componentwise linear, we work with induction on $r = \max(I) - \min(I)$. Set $d = \min(I)$.

Let $r = 0$. Since I and $\mathrm{Gin}(I)$ have the same graded Betti numbers, it follows that $\mathrm{Gin}(I)$ is generated in degree d. Since $\mathrm{Gin}(I)$ is a strongly stable ideal, we have that $\mathrm{Gin}(I)$ has a linear resolution, hence I has a linear resolution.

Now, suppose that $r > 0$. Since $\mathrm{Gin}(I_{\geq d+1}) = \mathrm{Gin}(I)_{\geq d+1}$, our induction hypothesis and Lemma 4.8 imply that $I_{\geq d+1}$ is componentwise linear. Thus, it suffices to prove that $I_{\langle d\rangle}$ has a linear resolution. Suppose this is not the case. Then, by the Bayer–Stillman theorem, $\mathrm{Gin}(I_{\langle d\rangle})$ has regularity $> d$. Moreover, since $\mathrm{Gin}(I_{\langle d\rangle})$ is strongly stable, its regularity equals $\max(\mathrm{Gin}(I_{\langle d\rangle}))$. It follows from Theorem 4.7 that $\mathrm{Gin}(I_{\langle d\rangle})$ has a generator of degree $d+1$. Now,

$$\begin{aligned}\beta_{0,d+1}(I) &= \dim I_{d+1} - \dim(\mathfrak{m}I_{\langle d\rangle})_{d+1}\\ &= \dim I_{d+1} - \dim(I_{\langle d\rangle})_{d+1},\end{aligned}$$

and

$$\begin{aligned}\beta_{0,d+1}(\mathrm{Gin}(I)) &= \dim \mathrm{Gin}(I)_{d+1} - \dim(\mathfrak{m}\,\mathrm{Gin}(I)_{\langle d\rangle})_{d+1}\\ &= \dim \mathrm{Gin}(I)_{d+1} - \dim(\mathfrak{m}\,\mathrm{Gin}(I_{\langle d\rangle}))_{d+1}\\ &> \dim \mathrm{Gin}(I)_{d+1} - \dim \mathrm{Gin}(I_{\langle d\rangle})_{d+1},\end{aligned}$$

because $(\mathfrak{m}\,\mathrm{Gin}(I_{\langle d\rangle}))_{d+1}$ is properly contained in $\mathrm{Gin}(I_{\langle d\rangle})_{d+1}$. Hence

$$\beta_{0,d+1}(\mathrm{Gin}(I)) > \beta_{0,d+1}(I),$$

a contradiction. This completes our proof. $\square$

We note that Theorem 4.6 and Theorem 4.7 (b) are not valid in positive characteristic. Indeed, if characteristic $p>0$, then $I=(x^p,y^p)$ provides a counterexample.

In the proof of Theorem 4.6 (a) $\Rightarrow$ (b) we only used that $\beta_{0j}(I)=\beta_{0j}(\mathrm{Gin}(I))$ for all j. Thus this condition implies that $\beta_i(I)=\beta_i(\mathrm{Gin}(I))$ for all i . This was first noted by Conca in [16, Theorem 1.2]. We now generalize this observation and first show

Theorem 4.9 (Conca-Herzog-Hibi)**.** *Let M be a graded R-module. Suppose $\beta_i(M)=\sum_{j=1}^{n-i+1}\binom{n-j}{i-1}\alpha_j(M)$ for some i. Then*

$$\beta_k(M)=\sum_{j=1}^{n-k+1}\binom{n-j}{k-1}\alpha_j(M) \quad \textit{for all} \quad k\geq i$$

Proof. It is enough to prove the statement for $k=i+1$. Let $y=y_1,\dots,y_n$ be a sequence of generic linear forms and denote by $H_a(b)$ the associated Koszul homology $H_a(b;M)$. By Proposition 3.1(b) we have to show that $\mathfrak{m}H_a(b)=0$ for all $(a,b)\in C_{i,n}$ implies that $\mathfrak{m}H_a(b)=0$ for all $(a,b)\in C_{i+1,n}$. But

$$C_{i+1,n}\setminus C_{i,n}=\{(i+1,b): b\leq n-1\}.$$

Thus it suffices to show: if $\mathfrak{m}H_i(b)=0$ for all b, then $\mathfrak{m}H_{i+1}(b)=0$ for all b.

We use the theorem of Kühl quoted before Theorem 3.5. The theorem implies: if $\mathfrak{m}H_1(b;M)=0$ for all b, then $\mathfrak{m}H_i(b;M)=0$ for all b an all $i\geq 1$.

Now assume that we have $H_i(b;M)=0$ for given $i>1$ and all b. Then $H_i(b;M)\cong H_1(b;\mathrm{syz}_{i-1}(M))$ and $H_{i+1}(b;M)\cong H_2(b;\mathrm{syz}_{i-1}(M))$. Assuming that $\mathfrak{m}H_i(b;M)=0$ for all b implies that $\mathfrak{m}H_1(b;\mathrm{syz}_{i-1}(M))=0$ for all b. Then the theorem of Kühl implies that $0=\mathfrak{m}H_2(b;\mathrm{syz}_{i-1}(M))=\mathfrak{m}H_{i+1}(b;M)$, as desired. $\square$

Corollary 4.10. *Assume* $\mathrm{char}(k)=0$*, and let $I\subset R$ be a graded ideal. Suppose that $\beta_i(I)=\beta_i(\mathrm{Gin}(I))$ for some i. Then*

$$\beta_k(I)=\beta_k(\mathrm{Gin}(I)) \quad \textit{for all} \quad k\geq i.$$

Proof. Since we assume $\mathrm{char}(k)=0$ the ideal $\mathrm{Gin}(I)$ is strongly stable and hence componentwise linear. It follows from 3.5 that

$$\beta_{i+1}(R/\mathrm{Gin}(I))=\sum_{j=1}^{n-i+2}\binom{n-j}{i}\alpha_j(R/\mathrm{Gin}(I))$$

By Lemma 4.3 and our assumption this implies that

$$\beta_{i+1}(R/I)=\sum_{j=1}^{n-i+2}\binom{n-j}{i}\alpha_j(R/I)$$

Now we apply Theorem 4.9 and again Lemma 4.3 to conclude that

$$\begin{aligned}\beta_k(I) = \beta_{k+1}(R/I) &= \sum_{j=1}^{n-k+2} \binom{n-j}{k} \alpha_j(R/I) \\ &= \sum_{j=1}^{n-k+2} \binom{n-j}{k} \alpha_j(R/\operatorname{Gin}(I)) \\ &= \beta_{k+1}(R/\operatorname{Gin}(I)) = \beta_k(\operatorname{Gin}(I))\end{aligned}$$

for $k = i, \ldots, n-1$. □

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