



the
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international centre for theoretical physics

ICTP 40th Anniversary

SMR1563/1

School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics

(24 May - 11 June 2004)

Toric rings and discrete convex geometry

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These are preliminary lecture notes, intended only for distribution to participants

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Lectures for the School on
Commutative Algebra and Interactions
with Algebraic Geometry and Combinatorics
Trieste, May/June 2004

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Preface

This text contains the computer presentation of 4 lectures:

1. Affine monoids and their algebras
2. Homological properties and combinatorial applications
3. Unimodular covers and triangulations
4. From vector spaces to polytopal algebras

Lecture 1 introduces the affine monoids and relates them to the geometry of rational convex cones. Lecture 2 contains the homological theory of normal affine semigroup rings and their applications to enumerative combinatorics developed by Hochster and Stanley.

Lectures 3 and 4 are devoted to lines of research that have been pursued in joint work with Joseph Gubeladze (Tbilisi/San Francisco).

A rather complete expository treatment of Lectures 1 and 2 is contained in

W. Bruns. *Commutative algebra arising from the Anand-Dumir-Gupta conjectures*. Preprint.

Most of Lecture 3 and much more – in particular basic notions and results of polyhedral convex geometry – is to be found in

W. Bruns and J. Gubeladze. *K -theory, rings, and polytopes*. Draft version of Part 1 of a book in progress.

For Lecture 4 there exists no coherent expository treatment so far, but a brief overview is given in

W. Bruns and J. Gubeladze. *Polytopes and K -theory*. Preprint.

A previous exposition, covering various aspects of these lectures is to be found in

W. Bruns and J. Gubeladze. *Semigroup algebras and discrete geometry*. In L. Bonavero and M. Brion (eds.), *Toric geometry. Séminaires et Congrès 6* (2002), 43–127

All these texts can be downloaded from (or via)

<http://www.math.uos.de/staff/phpages/brunsw/course.htm>

They contain extensive lists of references.

Osnabrück, May 2004

Winfried Bruns

Lecture 1

Affine monoids and their algebras

Affine monoids and their algebras

An **affine monoid** M is (isomorphic to) a finitely generated submonoid of \mathbb{Z}^d for some $d \geq 0$, i. e.

- $M + M \subset M$ (M is a **semigroup**);
- $0 \in M$ (now M is a **monoid**);
- there exist $x_1, \dots, x_n \in M$ such that
$$M = \mathbb{Z}_+x_1 + \dots + \mathbb{Z}_+x_n.$$

Often affine monoids are called **affine semigroups**.

$\text{gp}(M) = \mathbb{Z}M$ is the group generated by M .

$\text{gp}(M) \cong \mathbb{Z}^r$ for $r = \text{rank } M = \text{rank gp}(M)$.

Let K be a field (or a commutative ring). Then we can form the monoid algebra

$$K[M] = \bigoplus_{a \in M} KX^a, \quad X^a X^b = X^{a+b}$$

X^a = the basis element representing $a \in M$.

$M \subset \mathbb{Z}^d \Rightarrow K[M] \subset K[\mathbb{Z}^d] = K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ is a monomial subalgebra.

Proposition 1.1. *Let M be a monoid.*

- (a) M is finitely generated $\iff K[M]$ is a finitely generated K -algebra.
- (b) M is an affine monoid $\iff K[M]$ is an affine domain.

Proposition 1.2. *The Krull dimension of $K[M]$ is given by*

$$\dim K[M] = \text{rank } M.$$

Proof. $K[M]$ is an affine domain over K . Therefore

$$\begin{aligned} \dim K[M] &= \text{trdeg QF}(K[M]) \\ &= \text{trdeg QF}(K[\text{gp}(M)]) \\ &= \text{trdeg QF}(K[\mathbb{Z}^r]) \\ &= r \end{aligned}$$

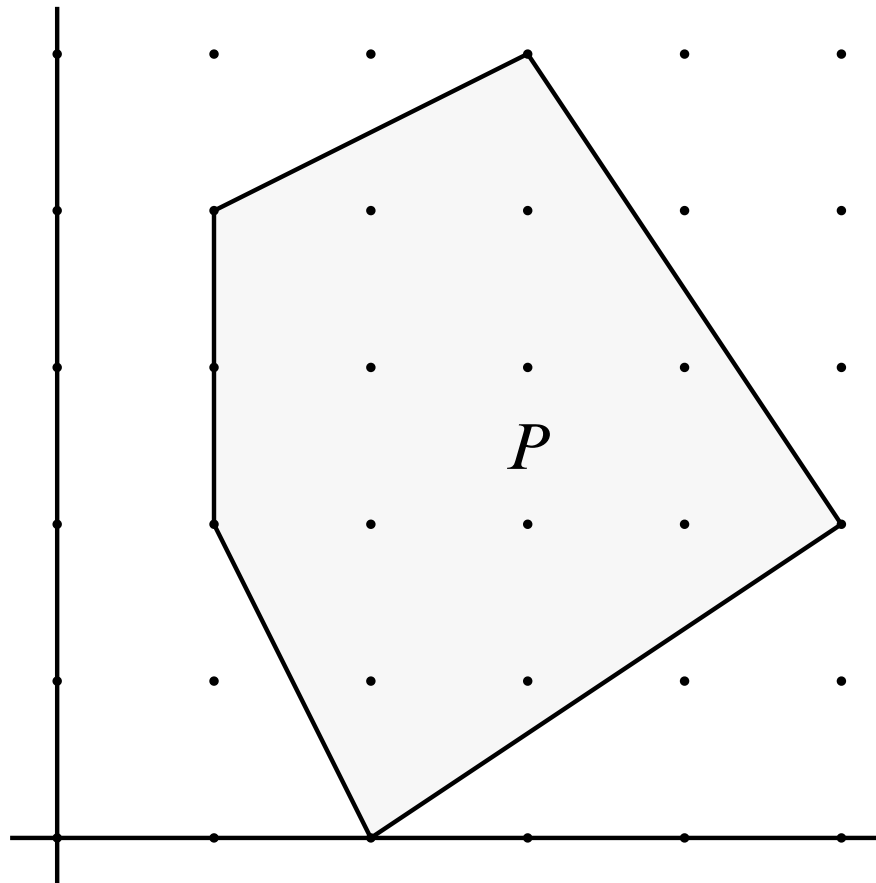
where $r = \text{rank } M$.

Sources for affine monoids (and their algebras) are

- monoid theory,
- ring theory,
- initial algebras with respect to monomial orders,
- invariant theory of torus actions,
- enumerative theory of linear diophantine systems,
- lattice polytopes and rational polyhedral cones,
- coordinate rings of toric varieties.

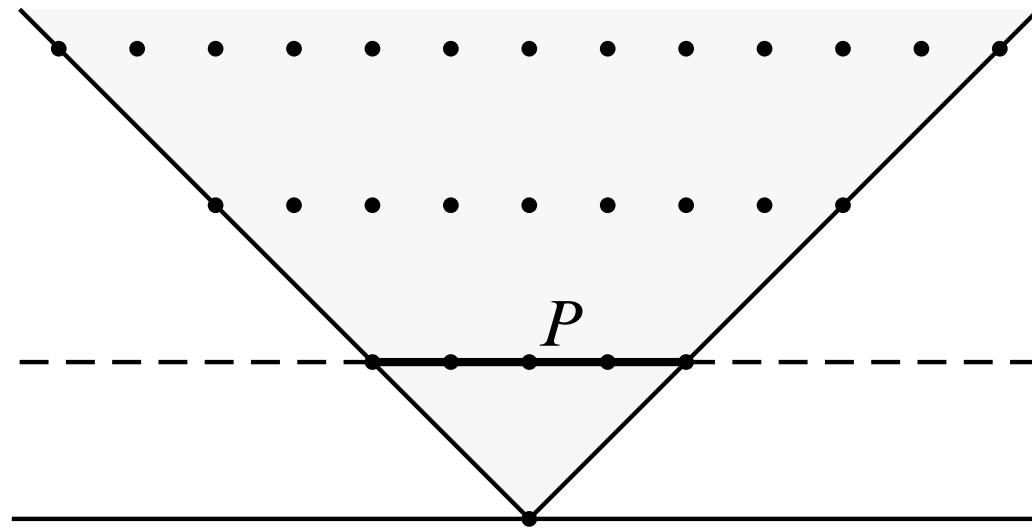
Polytopal monoids

Definition 1.3. The convex hull $\text{conv}(x_1, \dots, x_m)$ of points $x_i \in \mathbb{Z}^n$ is called a **lattice polytope**.



With a lattice polytope $P \subset \mathbb{R}^n$ we associate the **polytopal monoid**

$$M_P \subset \mathbb{Z}^{n+1} \text{ generated by } (x, 1), x \in P \cap \mathbb{Z}^n.$$



Vertical cross-section of a polytopal monoid

Such monoids will play an important role in Lectures 3 and 4.

Presentation of an affine monoid algebra

Let $R = K[x_1, \dots, x_n]$. Then we have a presentation

$$\pi : K[X] = K[X_1, \dots, X_n] \rightarrow K[x_1, \dots, x_n], \quad X_i \mapsto x_i.$$

Let $I = \text{Ker } \pi$ and $M = \{\pi(X^a) : a \in \mathbb{Z}_+^n\}$

Theorem 1.4. *The following are equivalent:*

- (a) M is an affine monoid and $R = K[M]$;
- (b) I is prime, generated by binomials $X^a - X^b$, $a, b \in \mathbb{Z}_+^n$;
- (c) $I = K[X] \cap IK[X^{\pm 1}]$, I is generated by binomials $X^a - X^b$, and $U = \{a - b : X^a - X^b \in I\}$ is a direct summand of \mathbb{Z}^n .

Proof. (a) \Rightarrow (b) Since R is a domain, I is prime. Let

$$f = c_1 X^{a_1} + \cdots + c_m X^{a_m} \in I, c_i \in K, c_i \neq 0, a_1 >_{\text{lex}} \cdots >_{\text{lex}} a_m.$$

There exists $j > 1$ with $\pi(X^{a_1}) = \pi(X^{a_j})$, and so $X^{a_1} - X^{a_j} \in I$.

Apply **lexicographic induction** to $f - c_1(X^{a_1} - X^{a_j})$.

(b) \Rightarrow (c) Since I is an ideal, U is a subgroup. Since I is prime and $X_i \notin I$ for all i , $I = K[X] \cap IK[X^{\pm 1}]$. Let $u \in \mathbb{Z}^n$, $m > 0$ such that $mu \in U$, $u = v - w$ with $v, w \in \mathbb{Z}_+^n$. Clearly $X^{um} - X^{vm} \in I$. We can assume $\text{char } K \nmid m$. Then

$$X^{um} - X^{vm} = (X^u - X^v)(X^{u(m-1)} + X^{u(m-2)v} + \cdots + X^{(m-1)u}),$$

and the second term is not in $(X_1 - 1, \dots, X_n - 1) \supset I$.

(c) \Rightarrow (a) Consider $K[X] \rightarrow K[X^{\pm 1}] = K[\mathbb{Z}^n] \rightarrow K[\mathbb{Z}^n / U]$.

Cones

An affine monoid M generates the cone

$$\mathbb{R}_+ M = \left\{ \sum a_i x_i : x_i \in M, a_i \in \mathbb{R}_+ \right\}$$

Since $M = \sum_{i=1}^n \mathbb{Z}_+ x_i$ is finitely generated, $\mathbb{R}_+ M$ is finitely generated:

$$\mathbb{R}_+ M = \left\{ \sum_{i=1}^n a_i x_i : a_1, \dots, a_n \in \mathbb{R}_+ \right\}.$$

The structures of M and $\mathbb{R}_+ M$ are connected in many ways. It is necessary to understand the **geometric structure** of $\mathbb{R}_+ M$.

Finite generation \iff intersection of finitely many halfspaces:

Theorem 1.5. *Let $C \neq \emptyset$ be a subset of \mathbb{R}^m . Then the following are equivalent:*

- *there exist finitely many elements $y_1, \dots, y_n \in \mathbb{R}^m$ such that $C = \mathbb{R}_+ y_1 + \dots + \mathbb{R}_+ y_n$;*
- *there exist finitely many linear forms $\lambda_1, \dots, \lambda_s$ such that C is the intersection of the half-spaces $H_i^+ = \{x : \lambda_i(x) \geq 0\}$.*

For full-dimensional cones the **(essential) support hyperplanes** $H_i = \{x : \lambda_i(x) = 0\}$ are unique:

Proposition 1.6. *If C generates \mathbb{R}^m as a vector space and the representation $C = H_1^+ \cap \cdots \cap H_s^+$ is **irredundant**, then the hyperplanes H_i are **uniquely determined** (up to enumeration). Equivalently, the linear forms λ_i are unique up to positive scalar factors.*

rational generators \iff rationality of the support hyperplanes:

Proposition 1.7. *The generating elements y_1, \dots, y_n can be chosen in \mathbb{Q}^m (or \mathbb{Z}^m) if and only if the λ_i can be chosen as linear forms with rational (or integral) coefficients.*

Such cones are called **rational**.

Proposition 1.8. *If $Y = \{y_1, \dots, y_n\} \subset \mathbb{Q}^m$, then $\mathbb{Q}^m \cap \mathbb{R}_+ Y = \mathbb{Q}_+ Y$.*

Gordan's lemma and normality

As seen above, affine monoids define rational cones. The converse is also true.

Lemma 1.9 (Gordan's lemma). *Let $U \subset \mathbb{Z}^d$ be a subgroup and $C \subset \mathbb{R}^d$ a rational cone. Then $U \cap C$ is an **affine monoid**.*

Proof. Let $V = \mathbb{R}U \subset \mathbb{R}^d$. Then:

- $V \cap \mathbb{Q}^d = \mathbb{Q}U$;
- $C \cap V$ is a rational cone in V
- \Rightarrow We may assume that $U = \mathbb{Z}^d$.

C is generated by elements $y_1, \dots, y_n \in M = C \cap \mathbb{Z}^d$.

$$x \in C \Rightarrow x = a_1 y_1 + \dots + a_n y_n \quad a_i \in \mathbb{R}_+.$$

$$x = x' + x'', \quad x' = \lfloor a_1 \rfloor y_1 + \cdots + \lfloor a_n \rfloor y_n.$$

Clearly $x' \in M$. But

$$x \in M \Rightarrow x'' \in \text{gp}(M) \cap C \Rightarrow x'' \in M.$$

x'' lies in a bounded set $B \Rightarrow$

M generated by y_1, \dots, y_n and the finite set $M \cap B$.

The monoid $M = U \cap C$ has a special property:

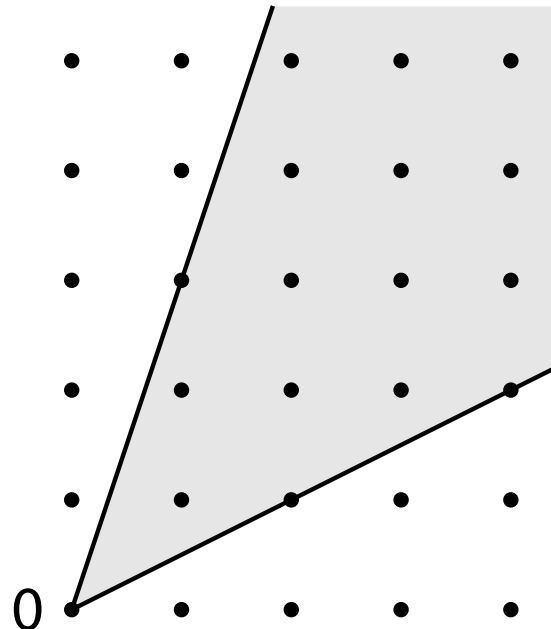
Definition 1.10. A monoid M is **normal** \iff

$$x \in \text{gp}(M), \quad kx \in M \text{ for some } k \in \mathbb{Z}, k > 0 \quad \Rightarrow \quad x \in M.$$

Proposition 1.11.

- $M \subset \mathbb{Z}^d$ *normal affine monoid* \iff there exists a rational cone C such that $M = \text{gp}(M) \cap C$;
- $M \subset \mathbb{Z}^d$ *affine monoid* \implies the *normalization* $\bar{M} = \text{gp}(M) \cap \mathbb{R}_+ M$ is affine.

Briefly: **Normal affine monoids are discrete cones.**



Positivity, gradings and purity

Definition 1.12. A monoid M is **positive** if $x, -x \in M \Rightarrow x = 0$.

Definition 1.13. A **grading** on M is a homomorphism $\deg : M \rightarrow \mathbb{Z}$. It is **positive** if $\deg x > 0$ for $x \neq 0$.

Proposition 1.14. For M affine the following are equivalent:

- (a) M is **positive**;
- (b) $\mathbb{R}_+ M$ is **pointed** (i. e. contains no full line);
- (c) M is isomorphic to a **submonoid of \mathbb{Z}_+^s** for some s ;
- (d) M has a **positive grading**.

Proof. (c) \Rightarrow (d) \Rightarrow (a) trivial.

(a) \Rightarrow (b) Set $C = \mathbb{R}_+ M$. One shows:

$$\{x \in C : -x \in C\} = \mathbb{R}\{x \in M : -x \in M\}.$$

Therefore: M positive $\Rightarrow C$ pointed.

(b) \Rightarrow (c) Let C be positive. For each facet F of C there exists a unique linear form $\sigma_F : \mathbb{R}^d \rightarrow \mathbb{R}$ with the following properties:

- $F = \{x \in C : \sigma_F(x) = 0\}$, $\sigma_F(x) \geq 0$ for all $x \in C$;
- σ has integral coefficients, $\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

These linear forms are called the **support forms** of C . Let $s = \#\text{facets}(C)$ and define

$$\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^s, \quad \sigma(x) = (\sigma_F(x) : F \text{ facet}).$$

Then $\sigma(M) \subset \sigma(\bar{M}) \subset \mathbb{Z}_+^s$. Since C is positive, σ is injective!

We call σ the **standard embedding**.

For M normal the standard embedding has an important property:

Proposition 1.15. M positive affine monoid. Then the following are equivalent:

(a) M is *normal*;

(b) σ maps M isomorphically onto $\mathbb{Z}_+^s \cap \sigma(\text{gp}(M))$.

M **pure submonoid** of $N \iff M = N \cap \text{gp}(M)$.

Corollary 1.16. M affine, positive, normal $\iff M$ isomorphic to a pure submonoid of \mathbb{Z}_+^s for some s .

Normality and purity of $K[M]$

An integral domain R is **normal** if R integrally closed in $\text{QF}(R)$.

R **pure subring** of $S \iff S = R \oplus T$ as an R -module.

Theorem 1.17. M positive affine monoid.

(a) M normal $\iff K[M]$ normal

(b) $M \subset \mathbb{Z}_+^s$ pure submonoid $\iff K[M]$ pure subalgebra of
 $K[\mathbb{Z}_+^s] = K[Y_1, \dots, Y_s]$

Proof. (a) $x \in \text{gp}(M)$, $mx \in M$ for some $m > 0$

$$\Rightarrow X^x \in K[\text{gp}(M)] \subset \text{QF}(K[M]), \quad (X^x)^m \in K[M].$$

Thus: $K[M]$ normal $\Rightarrow X^x \in K[M] \Rightarrow x \in M$.

Conversely, let M be normal, $\text{gp}(M) = \mathbb{Z}^r$, $C = \mathbb{R}_+ M$

$$\Rightarrow M = \mathbb{Z}^r \cap C$$

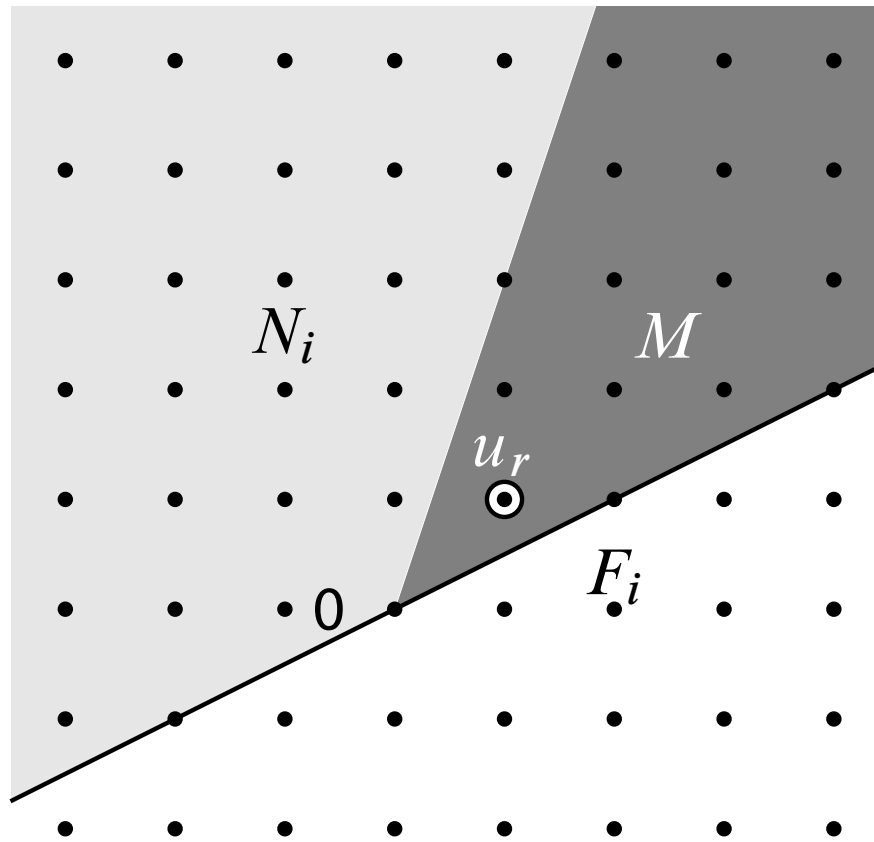
$C = H_1^+ \cap \dots \cap H_s^+$, H_i^+ rational closed halfspace \Rightarrow

$$M = N_1 \cap \dots \cap N_s, \quad N_i = \mathbb{Z}^r \cap H_i^+$$

$$\Rightarrow K[M] = K[N_1] \cap \dots \cap K[N_s]$$

N_i **discrete halfspace**

Consider the hyperplane H_i bounding H_i^+ . Then $H_i \cap \mathbb{Z}^r$ direct summand of \mathbb{Z}^r .



$\Rightarrow \mathbb{Z}^r$ has basis u_1, \dots, u_r with $u_1, \dots, u_{r-1} \in H_i, u_r \in H_i^+$

$\Rightarrow K[N_i] \cong K[\mathbb{Z}^{r-1} \oplus \mathbb{Z}_+] \cong K[Y_1^{\pm 1}, \dots, Y_{r-1}^{\pm 1}, Z]$

Thus $K[M]$ intersection of factorial (hence normal) domains $\Rightarrow K[M]$ normal

(b) $T = K\{X^x : x \in \mathbb{Z}^s \setminus M\} \Rightarrow K[Y_1, \dots, Y_s] = K[M] \oplus T$ as K -vector space

M pure submonoid $\Rightarrow T$ is $K[M]$ -submodule

Converse not difficult.

A grading on M induces a grading on $K[M]$:

Proposition 1.18. *Let M be an affine monoid with a grading deg .*

Then

$$K[M] = \bigoplus_{k \in \mathbb{Z}} K\{X^x : \text{deg } x = k\}$$

is a grading on $K[M]$.

If deg is positive, then $K[M]$ is positively graded.

The class group

R normal Noetherian domain (or a Krull domain).

$I \subset \text{QF}(R)$ **fractional ideal** \iff there exists $x \in R$ such that xI is a non-zero ideal

I is **divisorial** $\iff (I^{-1})^{-1} = I$ where

$$I^{-1} = \{x \in \text{QF}(R) : xI \subset R\}.$$

$(I, J) \mapsto ((IJ)^{-1})^{-1}$ defines a group structure on

$$\text{Div}(R) = \{\text{div. ideals}\}$$

Fact: $\text{Div}(R)$ free abelian group with basis $\mathbb{Z} \text{div}(\mathfrak{p})$, \mathfrak{p} height 1 prime ideal
($\text{div}(I)$ denotes I as an element of $\text{Div}(R)$)

Princ(R) = $\{xR : x \in \text{QF}(R)\}$ is a subgroup

$$\text{Cl}(R) = \frac{\text{Div}(R)}{\text{Princ}(R)}$$

is called the (divisor) class group.

It parametrizes the isomorphism classes of divisorial ideals.

R is factorial $\iff \text{Cl}(R) = 0$.

Let M be a positive normal affine monoid, $R = K[M]$. Choose

$$x \in \text{int}(M) = \{y \in M : \sigma_F(y) > 0 \text{ for all facets } F\}.$$

$$\Rightarrow M[-x] = \text{gp}(M)$$

$$\Rightarrow R[(X^x)^{-1}] = K[\text{gp}(M)] = L = \text{Laurent polynomial ring}$$

Nagata's theorem \Rightarrow exact sequence

$$0 \rightarrow U \rightarrow \text{Cl}(R) \rightarrow \text{Cl}(L) \rightarrow 0,$$

U generated by classes $[\mathfrak{p}]$ of minimal prime overideals of X^x

L factorial $\Rightarrow \text{Cl}(L) = 0 \Rightarrow \text{Cl}(R) = U$.

For a facet F of $\mathbb{R}_+ M$ set

$$\mathfrak{p}_F = R\{x \in M : \sigma_F(x) \geq 1\}.$$

$\Rightarrow \mathfrak{p}_F$ is a prime ideal since $R/\mathfrak{p}_F \cong K[M \cap F]$.

Evidently

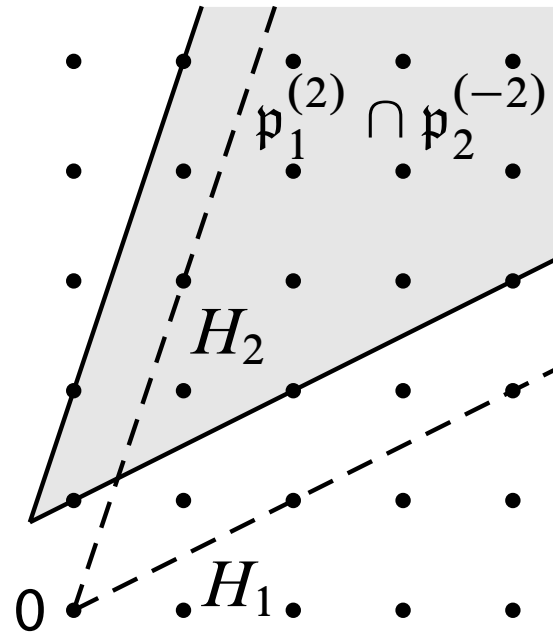
$$Rx = R\{X^y : \sigma_F(y) \geq \sigma_F(x) \text{ for all } F\} = \bigcap_F \mathfrak{p}_F^{(\sigma_F(x))}$$

$\mathfrak{p}_F^{(k)} = R\{X^y : \sigma_F(y) \geq k\}$ is the k -th symbolic power of \mathfrak{p}_F .

$$\Rightarrow \text{Cl}(R) = \sum_F \mathbb{Z}[\mathfrak{p}_F].$$

$$\left[\bigcap_F \mathfrak{p}_F^{(k_F)} \right] = \sum_F k_F [\mathfrak{p}_F]$$

\Rightarrow every divisorial ideal is isomorphic to an ideal $\bigcap_F \mathfrak{p}_F^{(k_F)}$



Monomial ideal I principal

\iff there exists a monomial X^y with $I = X^y R$

Enumerate the facets F_1, \dots, F_s , $\mathfrak{p}_i = \mathfrak{p}_{F_i}$, $\sigma_i = \sigma_{F_i}$

Theorem 1.19 (Chouinard).

$$\text{Cl}(R) \cong \frac{\bigoplus_{i=1}^s \mathbb{Z} \text{div}(\mathfrak{p}_i)}{\{\text{div}(RX^y) : y \in \text{gp}(M)\}} \cong \frac{\mathbb{Z}^s}{\sigma(\text{gp}(M))}$$

where σ is the standard embedding.

Lecture 2

Homological properties and combinatorial applications

Magic Squares

In 1966 H. Anand, V. C. Dumir, and H. Gupta investigated a combinatorial problem:

- Suppose that n distinct objects, each available in k identical copies, are distributed among n persons in such a way that each person receives exactly k objects.

What can be said about the number $H(n, k)$ of such distributions?

They formulated some conjectures:

(ADG-1) there exists a polynomial $P_n(k)$ of degree $(n-1)^2$ such that $H(n, k) = P_n(k)$ for all $k \gg 0$;

(ADG-2) $H(n, k) = P_r(n)$ for all $k > -n$; in particular $P_n(-k) = 0$, $k = 1, \dots, n-1$;

(ADG-3) $P_n(-k) = (-1)^{(n-1)^2} P_n(k-n)$ for all $k \in \mathbb{Z}$.

These conjectures were proved and extended by R. P. Stanley using methods of commutative algebra.

The weaker version (ADG-1) of (ADG-2) has been included for didactical purposes.

Reformulation:

a_{ij} = number of copies of **object** i that **person** j receives

$\Rightarrow A = (a_{ij}) \in \mathbb{Z}_+^{n \times n}$ such that

$$\sum_{k=1}^n a_{ik} = \sum_{l=1}^n a_{lj} = k, \quad i, j = 1, \dots, n.$$

$H(n, k)$ is the number of such matrices A .

The system of equations is part of the definition of **magic squares**. In combinatorics the matrices A are called magic squares, though the usually requires further properties for those.

Two famous magic squares:

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The 3×3 square can be found in ancient sources and the 4×4 appears in Albrecht Dürer's engraving **Melancholia** (1514). It has remarkable symmetries and shows the year of its creation.

A step towards algebra

Let \mathcal{M}_n be the set of all matrices $A = (a_{ij}) \in \mathbb{Z}_+^{n \times n}$ such that

$$\sum_{k=1}^n a_{ik} = \sum_{l=1}^n a_{lj}, \quad i, j = 1, \dots, n.$$

By Gordan's lemma \mathcal{M}_n is an **affine, normal monoid**, and $A \mapsto$ **magic sum** $k = \sum_{k=1}^n a_{1k}$ is a positive grading on \mathcal{M} .

It is even a pure submonoid of $\mathbb{Z}_+^{n \times n}$.

$$\text{rank } \mathcal{M}_n = (n - 1)^2 + 1$$

Theorem 2.1 (Birkhoff-von Neumann). \mathcal{M}_n is generated by the its degree 1 elements, namely the permutation matrices.

Translation into commutative algebra

We choose a field K and form the algebra

$$R = K[\mathcal{M}_n].$$

It is a normal affine monoid algebra, graded by the “magic sum”, and generated in degree 1. $\dim R = \text{rank } \mathcal{M}_n = (n - 1)^2 + 1.$

$\Rightarrow H(n, k) = \dim_K R_k = H(R, k)$ is the Hilbert function of R !

\Rightarrow (ADG-1): there exists a polynomial P_n of degree $(n - 1)^2$ such that $H(n, k) = P_n(k)$ for $k \gg 0$.

In fact, take P_n as the Hilbert polynomial of R .

A recap of Hilbert functions

Let K be a field, and $R = \bigoplus_{k=0}^{\infty} R_k$ a graded K -algebra generated by homogeneous elements x_1, \dots, x_n of degrees $g_1, \dots, g_n > 0$.

Let M be a non-zero, finitely generated graded R -module.

Then $H(M, k) = \dim_K M_k < \infty$ for all $k \in \mathbb{Z}$.

$H(M, _) : \mathbb{Z} \rightarrow \mathbb{Z}$ is the Hilbert function of M .

We form the Hilbert (or Poincaré) series

$$H_M(t) = \sum_{k \in \mathbb{Z}} H(M, k) t^k.$$

Fundamental fact:

Theorem 2.2 (Hilbert-Serre). *Then there exists a Laurent polynomial $Q \in \mathbb{Z}[t, t^{-1}]$ such that*

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^n (1 - t^{g_i})}.$$

More precisely: $H_M(t)$ is the Laurent expansion at 0 of the rational function on the right hand side.

Refinement: M is finitely generated over a **graded Noether normalization** $K[y_1, \dots, y_d]$, $d = \dim M$, of $R / \text{Ann } M$.

Special case: $g_1, \dots, g_n = 1$. Then y_1, \dots, y_d can be chosen of degree 1 (after an extension of K).

Theorem 2.3. Suppose that $g_1 = \cdots = g_n = 1$ and let $d = \dim M$.

Then

$$H_M(t) = \frac{Q(t)}{(1-t)^d}.$$

Moreover:

- There exists a **polynomial** $P_M \in \mathbb{Q}[X]$ such that

$$H(M, i) = P_M(i), \quad i > \deg H_M,$$

$$H(M, i) \neq P_M(i), \quad i = \deg H_M.$$

- $e(M) = Q(1) > 0$, and if $d \geq 1$, then

$$P_M = \frac{e(M)}{(d-1)!} X^{d-1} + \text{terms of lower degree.}$$

The general case is not much worse: P_M must be allowed to be a quasi-polynomial, i. e. a “polynomial” with periodic coefficients (instead of constant ones).

Theorem 2.4. *Suppose $R/\text{Ann } M$ has a graded Noether normalization generated by elements of degrees e_1, \dots, e_d . Then*

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^d (1 - t^{e_i})}, \quad Q(1) > 0.$$

Moreover there exists a *quasi-polynomial* P_M , whose period divides $\text{lcm}(e_1, \dots, e_d)$, such that

$$\begin{aligned} H(M, i) &= P_M(i), & i > \deg H_M, \\ H(M, i) &\neq P_M(i), & i = \deg H_M. \end{aligned}$$

Hochster's theorem

Theorem 2.5. *Let M be an affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K .*

There is no easy proof of this powerful theorem. For example, it can be derived from the Hochster-Roberts theorem, using that $K[M] \subset K[X_1, \dots, X_s]$ can be chosen as a pure embedding.

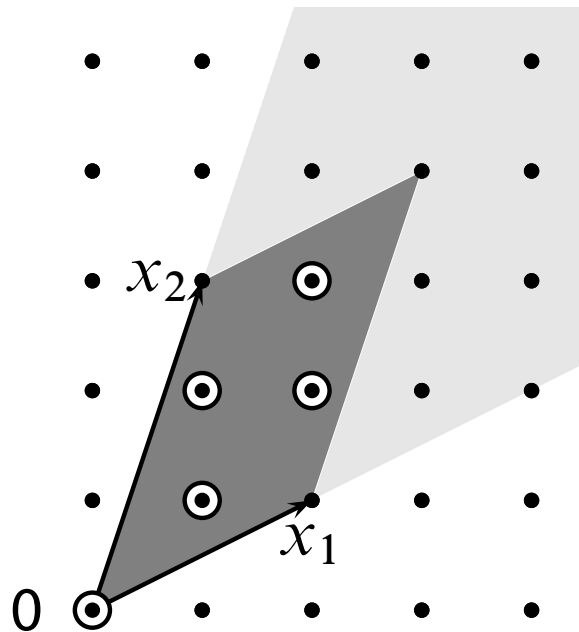
A special case is rather simple:

Definition 2.6. An affine monoid M is **simplicial** if the cone $\mathbb{R}_+ M$ is generated by rank M elements.

Proposition 2.7. *Let M be a simplicial affine normal monoid. Then $K[M]$ is Cohen-Macaulay for every field K .*

Proof. Let $d = \text{rank } M$. Choose elements $x_1, \dots, x_d \in M$ generating $\mathbb{R}_+ M$, and set

$$\text{par}(x_1, \dots, x_d) = \left\{ \sum_{i=1}^d q_i x_i : q_i \in [0, 1) \right\}.$$



$$B = \text{par}(x_1, \dots, x_d) \cap \mathbb{Z}^d$$

$$N = \mathbb{Z}_+ x_1 \cdots + \mathbb{Z}_+ x_d.$$

The arguments in the proof Gordan's lemma and the linear independence of x_1, \dots, x_d imply:

- $M = \bigcup_{z \in B} z + N$
- N is free.
- The union is disjoint.

In commutative algebra terms:

- $K[M]$ is finite over $K[N]$.
- $K[N] \cong K[X_1, \dots, X_d]$.
- $K[M]$ is a free module over $K[N]$.

$\Rightarrow K[M]$ is Cohen-Macaulay.

Combinatorial consequence of the Cohen-Macaulay property:

Theorem 2.8. Let M be a graded *Cohen-Macaulay module* over the positively graded K -algebra R and x_1, \dots, x_d a h.s.o.p. for M , $e_i = \deg x_i$. Let

$$H_M(t) = \frac{h_a t^a + \dots + h_b t^b}{\prod_{i=1}^d (1 - t^{e_i})}, \quad h_a, h_b \neq 0.$$

Then $h_i \geq 0$ for all i .

If $M = R = K[R_1]$, then $h_i > 0$ for all $i = 0, \dots, b$.

Proof. $h_a t^a + \dots + h_b t^b$ is the Hilbert series of

$$M/(x_1 M + \dots + x_d M).$$

Reciprocity

(ADG-3) compares values $P_R(k)$ of the Hilbert polynomial of $R = K[\mathcal{M}_n]$ for **all** values of k :

$$(ADG-3) \quad P_n(-k) = (-1)^{(n-1)^2} P_n(k - n) \text{ for all } k \in \mathbb{Z}.$$

According to (ADG-2) the shift $-n$ in $P_n(k - n)$ is the degree of $H_R(t)$ (not yet proved).

- What **identity** for $H_R(t)$ is encoded in (ADG-3) ?

Lemma 2.9. Let $P : \mathbb{Z} \rightarrow \mathbb{C}$ be a quasi-polynomial. Set

$$H(t) = \sum_{k=0}^{\infty} P(k)t^k \quad \text{and} \quad G(t) = - \sum_{k=1}^{\infty} P(-k)t^k.$$

Then H and G are *rational functions*. Moreover

$$H(t) = G(t^{-1}).$$

Corollary 2.10. Let R be a positively graded, finitely generated K -algebra, $\dim R = d$, with Hilbert quasi-polynomial P . Suppose $\deg H_R(t) = g < 0$. Then the following are equivalent:

- $P(-k) = (-1)^{d-1} P(k + g)$ for all $k \in \mathbb{Z}$;
- $(-1)^d H_R(t^{-1}) = t^{-g} H_R(t)$.

Strategy:

- Find an R -module ω with $H_\omega(t) = (-1)^d H_R(t^{-1})$

This is possible for R Cohen-Macaulay: ω is the canonical module of R .

- Compute ω for $R = K[\mathcal{M}_n]$ and show that $\omega \cong R(g)$,
 $g = \deg H_R(t)$.

$R(g)$ free module of rank 1 with generator in degree $-g$. Thus
 $H_{R(g)}(t) = t^{-g} H_R(t)$.

More generally:

- Compute ω for $R = K[M]$ with M affine, normal.

The canonical module

In the following: R positively graded Cohen-Macaulay K -algebra,
 x_1, \dots, x_d h.s.o.p., $\deg x_i = g_i$.

$S = K[x_1, \dots, x_d]$ is a graded Noether normalization of R .

First $R = S = K[X_1, \dots, X_d]$:

$$(-1)^d H_S(t^{-1}) = \frac{(-1)^d}{\prod_{i=1}^d (1 - t^{-g_i})} = \frac{t^{g_1 + \dots + g_d}}{\prod_{i=1}^d (1 - t^{-g_i})} = H_\omega(t)$$

with $\omega = \omega_S = S(-(g_1 + \dots + g_d))$

The general case: R **free** over $S \cong K[X_1, \dots, X_d]$, say with homogeneous basis y_1, \dots, y_m :

$$R \cong \bigoplus_{j=1}^m S y_j \cong \bigoplus_{i=1}^u S(-i)^{h_i}, \quad h_i = \#\{j : \deg y_j = i\},$$

$$H_R(t) = (h_0 + h_1 t + \dots + h_u t^u) H_S(t) = Q(t) H_S(t)$$

Set $\omega_R = \text{Hom}_S(R, \omega_S)$. Then, with $s = g_1 + \dots + g_d$

$$\omega_R \cong \bigoplus_{i=1}^u \text{Hom}_S(S(-i)^{h_i}, S(-s)) \cong \bigoplus_{i=1}^u S(-i + s)^{h_i},$$

$$\begin{aligned} H_{\omega_R}(t) &= (h_0 t^s + \dots + h_u t^{s-u}) H_S(t) = Q(t^{-1}) t^s H_S(t) \\ &= (-1)^d Q(t^{-1}) H_S(t^{-1}) = (-1)^d H_R(t^{-1}). \end{aligned}$$

Multiplication in the first component makes ω_R an R -module:

$$a \cdot \varphi(_) = \varphi(a \cdot _).$$

- But: Is ω_R independent of S ?

Theorem 2.11. ω_R depends only on R (up to isomorphism of graded modules).

The proof requires **homological algebra**, after reduction from the graded to the local case.

Gorenstein rings

Definition 2.12. A positively graded Cohen-Macaulay K -algebra is **Gorenstein** if $\omega_R \cong R(h)$ for some $h \in \mathbb{Z}$.

Actually, there is no choice for h :

Theorem 2.13 (Stanley). *Let R be Gorenstein. Then*

- $\omega_R \cong R(g)$, $g = \deg H_R(t)$;
- $h_0 = h_{u-i}$ for $i = 0, \dots, u$: the h -vector is palindromic;
- $H_R(t^{-1}) = (-1)^d t^{-g} H_R(t)$.

Conversely, if R is a Cohen-Macaulay integral domain such that $H_R(t^{-1}) = (-1)^d t^{-h} H_R(t)$ for some $h \in \mathbb{Z}$, then R is Gorenstein.

Proof.

$$(-1)^d H_R(t) = (h_0 t^s + \cdots + h_u t^{s-u}) H_S(t)$$

$$t^{-h} H_R(t) = (h_s t^{u-h} + \cdots + h_0^{-h}) H_S(t)$$

Equality holds \iff

$$h = u - s = \deg H_R(t) \quad \text{and} \quad h_i = h_{u-i}, \quad i = 0, \dots, u$$

If R is a domain, then ω_R is torsionfree. Consider $R \mapsto \omega_R, a \mapsto ax,$
 $x \in \omega_R$ homogeneous, $\deg x = -g$.

This linear map is injective: $R(g) \hookrightarrow \omega_R$. Equality of Hilbert
functions implies **bijection**.

The canonical module of $K[M]$

In the following M affine, normal, positive monoid. We want to find the canonical module of $R = K[M]$ (Cohen-Macaulay by Hochster's theorem).

Theorem 2.14 (Danilov, Stanley). *The ideal I generated by the monomials in the interior of $\mathbb{R}_+ M$ is the canonical module of $K[M]$ (with respect to every positive grading of M).*

Note: $x \in \text{int}(\mathbb{R}_+ M) \iff \sigma_F(x) > 0$ for all facets F of $\mathbb{R}_+ M$.

\mathfrak{p}_F is generated by all monomials X^x , $x \in M$ such that $\sigma_F(x) > 0$.

$$\Rightarrow I = \bigcap_{F \text{ facet}} \mathfrak{p}_F.$$

Choose a positive grading on M and let ω be the **canonical module of R** with respect to this grading.

By definition ω is free over a Noether normalization $\Rightarrow \omega$ is a Cohen-Macaulay R -module $\Rightarrow \omega$ is (isomorphic to) a divisorial ideal \Rightarrow

- As discussed in Lecture 1, there exist $j_F \in \mathbb{Z}$ such that

$$\omega = \bigcap \mathfrak{p}_F^{(j_F)}.$$

Without further standardization we cannot conclude that $j_F = 1$ for all F .

We have to use the **natural \mathbb{Z}^r -grading, $\mathbb{Z}^r = \text{gp}(M)$** on R !

The homological property characterizing the canonical module is

$$\text{Ext}_R^j(K, \omega_R) = \begin{cases} K, & j = d, \\ 0, & j \neq d. \end{cases} \quad d = \dim R.$$

This is to be read as an isomorphism of graded modules: let \mathfrak{m} be the **irrelevant maximal ideal**; then $K = R/\mathfrak{m}$ lives in degree 0.

In our case R/\mathfrak{m} is a \mathbb{Z}^r -graded module, as is ω

$\Rightarrow \text{Ext}_R^j(K, \omega_R) \cong K$ lives in **exactly one multidegree** $v \in \mathbb{Z}^r$

$\Rightarrow \text{Ext}_R^j(K, X^{-v}\omega_R) \cong K$ in **multidegree** $0 \in \mathbb{Z}^r$

Replace ω_R by $X^{-v}\omega_R$.

Definition 2.15. Let R be a \mathbb{Z}^n -graded Cohen-Macaulay ring such that the homogeneous non-units generate a proper ideal \mathfrak{p} of R .

$\Rightarrow \mathfrak{p}$ is a prime ideal; set $d = \dim R_{\mathfrak{p}}$.

One says that ω is a \mathbb{Z}^n -graded canonical module of R if

$$\text{Ext}_R^j(R/\mathfrak{p}, \omega) = \begin{cases} R/\mathfrak{p}, & j = d, \\ 0, & j \neq d. \end{cases}$$

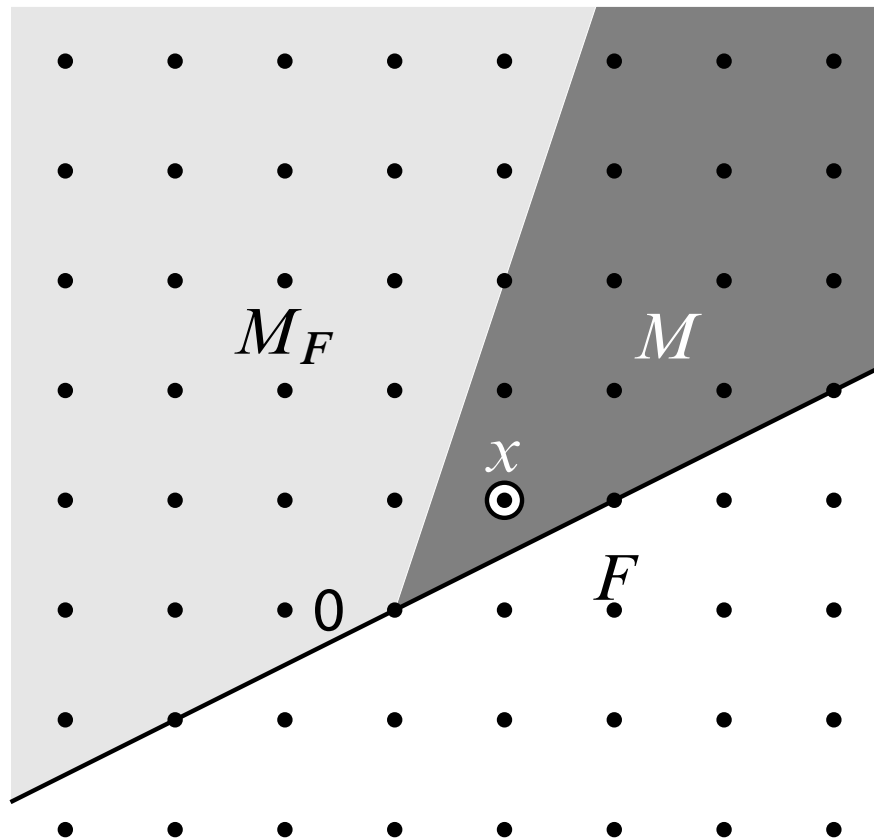
We have seen: $R = K[M]$ has a \mathbb{Z}^r -graded canonical module

$$\omega = \bigcap \mathfrak{p}_F^{(j_F)}$$

and it remains to show that $j_F = 1$ for all facets F .

Let $R_F = R[(M \cap F)^{-1}]$: we invert all the monomials in F .

$\Rightarrow R_F$ is the “discrete halfspace algebra” with respect to the support hyperplane through F .



$\Rightarrow \mathfrak{p}_F R_F$ is the \mathbb{Z}^r -graded canonical module of R_F (easy to see since $\mathfrak{p}_F R_F$ is principal generated by a monomial X^x with $\sigma_F(x) = 1$)

On the other hand: $\omega_{R_F} = (\omega_R)_F$: the \mathbb{Z}^r -graded canonical module “localizes” (a nontrivial fact)

$\Rightarrow j_F = 1$.

Back to the ADG conjectures

Recall that \mathcal{M}_n denotes the “magic” monoid. It contains the matrix $\mathbf{1}$ with all entries 1.

Let $C = \mathbb{R}_+ \mathcal{M}_n$. Then C is cut out from $\mathbb{R} \mathcal{M}_n$ by the positive orthant

$$\Rightarrow \text{int}(C) = \{A : a_{ij} > 0 \text{ for all } i, j\}.$$

$$\Rightarrow A - \mathbf{1} \in \mathcal{M}_n \text{ for all } A \in M \cap \text{int}(C)$$

\Rightarrow interior ideal I is generated by $X^{\mathbf{1}}$; $\mathbf{1}$ has magic sum n

$\Rightarrow I \cong R(-n)$. $R = K[\mathcal{M}_n]$ is a Gorenstein ring with $\deg H_R(t) = -n$

$$\deg H_R(t) = -n \Rightarrow$$

$$\text{(ADG-2)} \quad H(n, k) = P_n(k) \text{ for all } k > -n; \text{ in particular } P_n(-k) = 0, \\ k = 1, \dots, n - 1;$$

$$\text{(ADG-2) and } R \text{ Gorenstein} \Rightarrow$$

$$\text{(ADG-3)} \quad P_n(-k) = (-1)^{(n-1)^2} P_n(k - n) \text{ for all } k \in \mathbb{Z}.$$

In terms of

$$H_R(t) = \frac{1 + h_1 t + \cdots + h_u t^u}{(1 - t)^{(n-1)^2 + 1}}, \quad h_u \neq 0,$$

we have seen that

- $u = (n - 1)^2 + 1 - n$ (ADG-2)
- $h_i > 0$ for $i = 1, \dots, u$ (R Cohen-Macaulay)
- $h_i = h_{u-i}$ for all i (ADG-3)

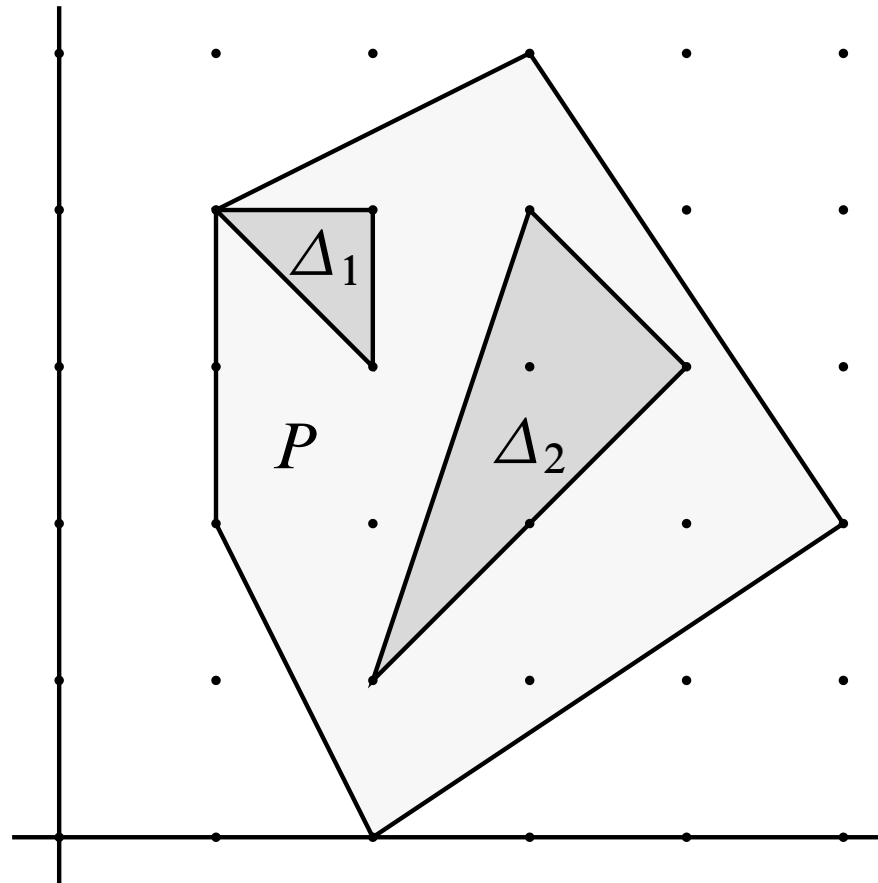
Very recent result, conjectured by Stanley and now proved by Ch. Athanasiadis:

- the sequence (h_i) is **unimodal**: $h_0 \leq h_1 \leq \cdots \leq h_{\lceil u/2 \rceil}$

Lecture 3

Unimodular covers and triangulations

Recall: $P = \text{conv}(x_1, \dots, x_n) \subset \mathbb{R}^d$, $x_i \in \mathbb{Z}^d$, is called a **lattice polytope**.



$\Delta = \text{conv}(v_0, \dots, v_d)$, v_0, \dots, v_d affinely independent, is a **simplex**.

Set $U_\Delta = \sum_{i=0}^d \mathbb{Z}(v_i - v_0)$.

$$\mu(\Delta) = [\mathbb{Z}^d : U_\Delta] = \text{multiplicity of } \Delta$$

Δ is **unimodular** if $\mu(\Delta) = 1$.

Δ is **empty** if $\text{vert}(\Delta) = \Delta \cap \mathbb{Z}^d$.


Lemma 3.1.

$$\mu(\Delta) = d! \text{vol}(\Delta) = \pm \det \begin{pmatrix} v_1 - v_0 \\ \vdots \\ v_d - v_0 \end{pmatrix}$$

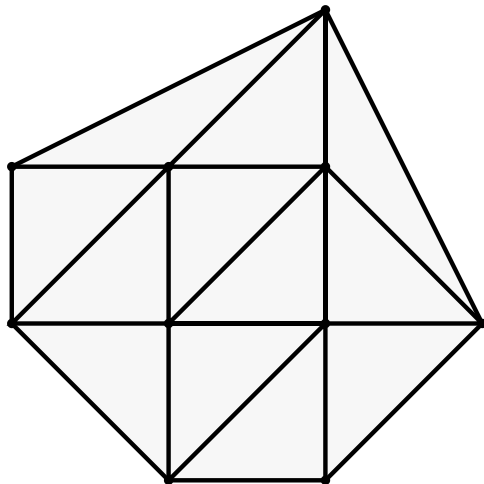
When is P covered by its unimodular subsimplices?

For short: P has **UC**.

Low Dimensions

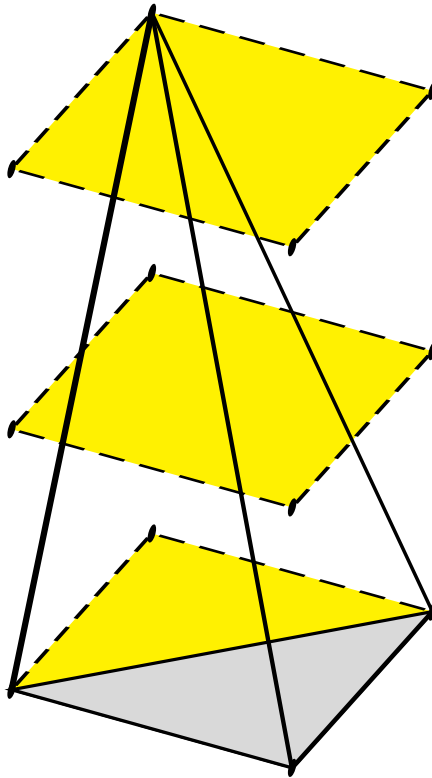
$d = 1$:  P has a unique unimodular triangulation.

$d = 2$:



Every empty lattice **triangle** is unimodular \Rightarrow every 2-polytope has a unimodular triangulation.

$d = 3$: There exist empty simplices of arbitrary multiplicity!



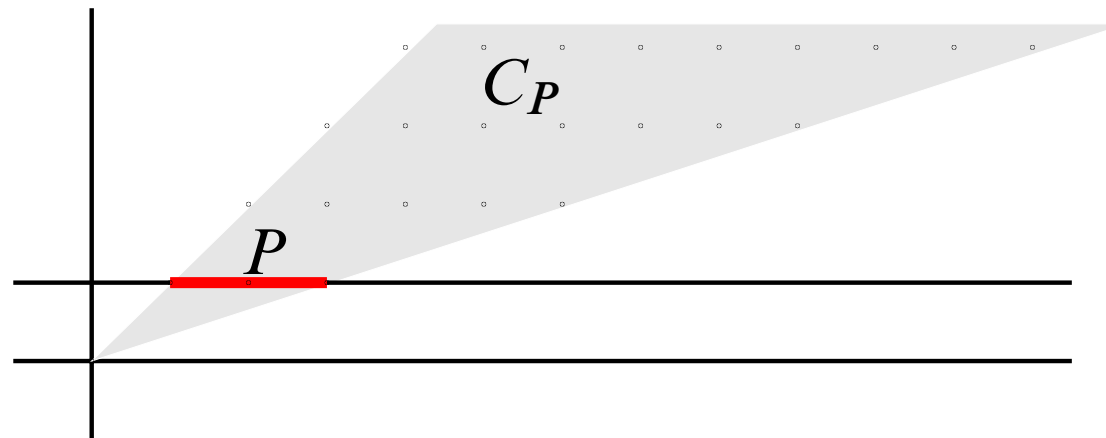
Polytopal cones and monoids

The **cone** over P is $C_P = \mathbb{R}_+ \{(x, 1) \in \mathbb{R}^{d+1} : x \in P\}$.

The **monoid** associated with P is

$$M_P = \mathbb{Z}_+ \{(x, 1) : x \in P \cap \mathbb{Z}^d\}.$$

The **integral closure** of M_P is $\widehat{M}_P = C_P \cap \mathbb{Z}^{d+1}$.



Proposition 3.2. P has UC $\Rightarrow M_P = \widehat{M}_P$ (P is *integrally closed*).

P is integrally closed \iff

(i) $\text{gp}(M_P) = \mathbb{Z}^{d+1}$ and

(ii) M_P is a **normal monoid** ($M_P = C_P \cap \text{gp}(M_P)$)

There exist **non-normal 3-dimensional polytopes**, for example

$$P = \{x \in \mathbb{R}^3 : x_i \geq 0, 6x_1 + 10x_2 + 15x_3 \leq 30\}.$$

$\implies P$ does not have UC, and this cannot be “repaired” by replacing \mathbb{Z}^3 by the smallest lattice containing $P \cap \mathbb{Z}^3$.

Monoid algebras, toric ideals and Gröbner bases

Let K be a field. The **polytopal K -algebra** $K[P]$ is the monoid algebra

$$K[P] = K[M_P] = K[X_x : x \in P \cap \mathbb{Z}^d] / I_P.$$

The **toric ideal** I_P is generated by all binomials

$$\prod_{x \in P \cap \mathbb{Z}^d} X_x^{a_x} - \prod_{x \in P \cap \mathbb{Z}^d} X_x^{b_x},$$

$$\sum a_x x = \sum b_x x, \quad \sum a_x = \sum b_x$$

expressing the **affine relations** between the lattice points in P .

Sturmfels:

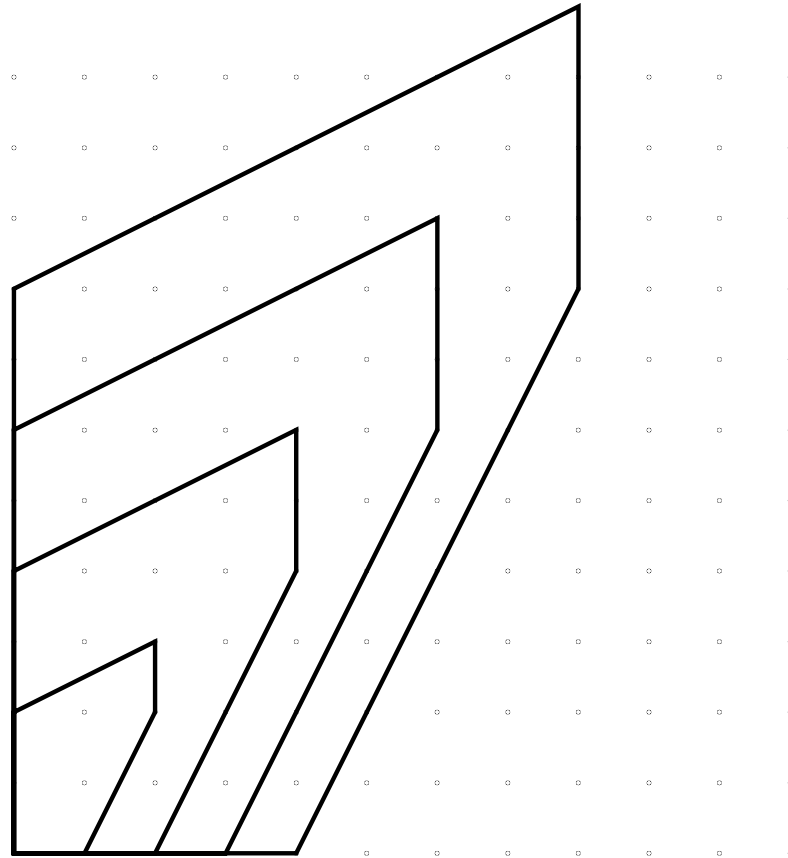
“generic” weights for $X_x \mapsto$

$$\left\{ \begin{array}{l} \text{(i) regular triangulation } \Sigma \text{ of } P, \quad \text{vert}(\Sigma) \subset P \cap \mathbb{Z}^d \\ \text{(ii) term (pre)order on } K[X_x], \quad \text{ini}(I_P) \text{ monomial ideal} \end{array} \right.$$

Theorem 3.3.

- $(\text{Stanley-Reisner ideal of } \Sigma) = \text{Rad}(\text{ini}(I_P))$
- Σ is *unimodular* \iff $\text{ini}(I_P)$ *squarefree*

Multiples of polytopes



For $c \rightarrow \infty$ ($c \in \mathbb{N}$) the lattice points $cP \cap \mathbb{Z}^d$ approximate the continuous structure of $cP \sim P$ better and better.

Algebraic results:

Theorem 3.4.

- cP integrally closed for $c \geq \dim P - 1$. Thus $K[cP]$ *normal* for $c \geq \dim P - 1$.
- I_{cP} has an initial ideal generated by degree 2 monomials for $c \geq \dim P$. Thus $K[cP]$ is *Koszul* for $c \geq \dim P$.

Proof of Koszul property uses technique of Eisenbud-Reeves-Totaro.

Questions:

- (i) Does cP have UC for $c \geq \dim P - 1$?
- (ii) Does cP have a regular unimodular triangulation of degree 2 for $c \geq \dim P$?

Positive answers: (i) $\dim P \leq 3$, (ii) $\dim P \leq 2$.

No algebraic obstructions in arbitrary dimension !

Positive rational cones and Hilbert bases

C generated by finitely many $v \in \mathbb{Z}^d$, and $x, -x \in C \Rightarrow x = 0$.

Gordan's lemma: $C \cap \mathbb{Z}^d$ is a **finitely generated monoid**.

Its irreducible elements form the **Hilbert basis** $\text{Hilb}(C)$ of C .

C is **simplicial** $\iff C$ generated by linearly independent vectors v_1, \dots, v_d .

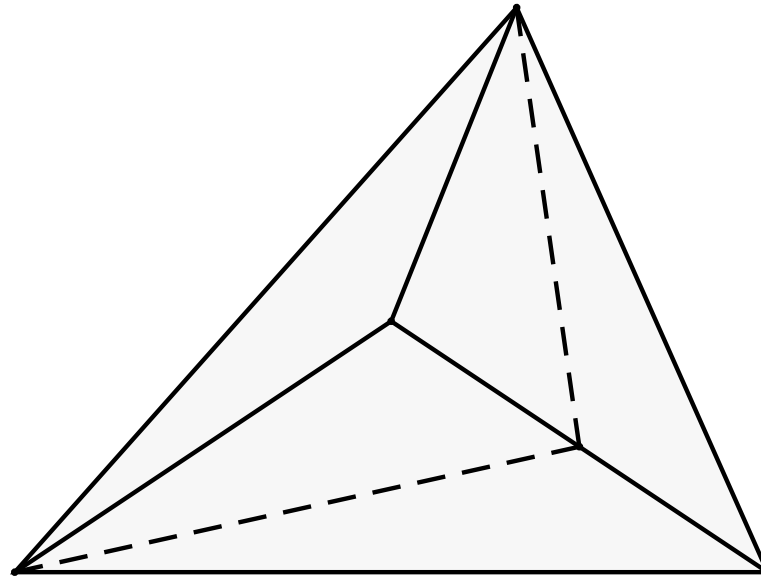
Can assume that the components of v_i are coprime. Then

$$\mu(C) = [\mathbb{Z}^d : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_d]$$

C **unimodular** $\iff C$ generated by \mathbb{Z} -basis of $\mathbb{Z}^d \iff \mu(C) = 1$

Theorem 3.5. C has a triangulation into unimodular subcones.

Proof: Start with arbitrary triangulation. Refine by **iterated stellar subdivision** to reduce multiplicities.



Here we make **no assertion on the generators** of the unimodular subcones.

But: P has a unimodular triangulation $\Rightarrow C_P$ satisfies UHT.

UHT: C has a **Unimodular Triangulation** into cones generated by subsets of $\text{Hilb}(C)$.

UHC: C is **Covered by its Unimodular subcones** generated by subsets of $\text{Hilb}(C)$.

A condition with a more algebraic flavour:

ICP: (Integral Carathéodory Property) for every $x \in C \cap \mathbb{Z}^d$ there exist $y_1, \dots, y_d \in \text{Hilb}(C)$ with $x \in \mathbb{Z}_+ y_1 + \dots + \mathbb{Z}_+ y_d$.

UHT \Rightarrow UHC \Rightarrow ICP.

UHC $\not\Rightarrow$ UHT. No example known with ICP, but without UHC.

Dimension 3

Cones of dimension 3:

Theorem 3.6 (Sebő). $\dim C = 3 \Rightarrow C$ has *UHT*

If $C = C_P$, $\dim P = 2$, this is easy since P has UT. General case is somewhat tricky.

Polytopes of dimension 3:

First triangulate P into empty simplices and then use **classification of empty simplices** (White):

$$\Delta_{pq} = \text{conv} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & 1 \end{pmatrix}, \quad 0 \leq q < p, \gcd(p, q) = 1$$

$$\mu(\Delta_{pq}) = p$$

No classification known in dimension ≥ 4 . Essential difference to dimension 3: lattice width of Δ may be > 1 .

Lagarias & Ziegler , Kantor & Sarkaria:

Proposition 3.7. cP has UC for $c \geq 2$.

Theorem 3.8.

- $2\Delta_{pq}$ has UT $\iff q = 1$ or $q = p - 1$.
- $4P$ has UT for all P .
- $c\Delta_{pq}$ has UT for $c \geq 4$.

Question: What about $3\Delta_{pq}$?

Counterexamples

$P = 2\Delta_{53}$ integrally closed 3 polytope **without UT** $\Rightarrow C_P$ has dimension 4 and **violates UHT** (first counterexample by Bouvier & Gonzalez-Sprinberg)

C_6 with Hilbert basis z_1, \dots, z_{10} , is of form C_{P_5} , $\dim P_5 = 5$, P_5 integrally closed, and **violates UHC and ICP** (B & G & Henk, Martin, Weismantel)

$$\begin{aligned} z_1 &= (0, 1, 0, 0, 0, 0), & z_6 &= (1, 0, 2, 1, 1, 2), \\ z_2 &= (0, 0, 1, 0, 0, 0), & z_7 &= (1, 2, 0, 2, 1, 1), \\ z_3 &= (0, 0, 0, 1, 0, 0), & z_8 &= (1, 1, 2, 0, 2, 1), \\ z_4 &= (0, 0, 0, 0, 1, 0), & z_9 &= (1, 1, 1, 2, 0, 2), \\ z_5 &= (0, 0, 0, 0, 0, 1), & z_{10} &= (1, 2, 1, 1, 2, 0). \end{aligned}$$

$\Rightarrow P_5$ **violates UC**

There exists a polytope of dimension 10 **with UT**, but **without a regular unimodular triangulation** (Hibi & Ohsugi)

Questions:

- Do all integrally closed polytopes P of dimensions **3 and 4** have **UC** ?
- Do all cones C of dimensions **4 and 5** have **UHC** ?
- Does there exist C **with ICP**, but **violating UHC** ?

Triangulating cP

Theorem 3.9 (Knudsen & Mumford, Toroidal embeddings). *Let P be a lattice d -polytope. Then cP has a **regular unimodular triangulation** for a **some $c \in \mathbb{Z}_+, c > 0$** .*

Not so hard: UC of d -simplices with non-overlapping interiors

Harder: UT

Most difficult: regularity

Questions: Does cP have UT for $c \gg 0$? Can we **bound c uniformly** in terms of dimension? Is $c \geq \dim P$ enough?

Covering cP

Theorem 3.10. *Let P be a d -polytope. Then there exists c_d such that cP has UC for all $c \geq c_d^{\text{pol}}$, and*

$$c_d^{\text{pol}} = O(d^{16.5}) \left(\frac{9}{4}\right)^{(\text{ld } \gamma(d))^2}, \quad \gamma(d) = (d-1) \lceil \sqrt{d-1} \rceil.$$

For the proof one needs a similar theorem about cones—cones allow induction on d .

Theorem 3.11. *Let C be a rational simplicial d -cone and Δ_C the simplex spanned by O and the extreme integral generators. Then*

- (a) (M. v. Thaden) C has a **triangulation** into unimodular simplicial cones D_i such that $\text{Hilb}(D_i) \subset c\Delta_C$ for some

$$c \leq \frac{d^2}{4} (\mu(C))^7 \left(\frac{9}{4}\right)^{(\text{ld}(\mu(C)))^2}.$$

- (b) C has a **cover** by unimodular simplicial cones D_i such that $\text{Hilb}(D_i) \subset c\Delta_C$ for some

$$c \leq \frac{d^2}{4} (d + 1) (\gamma(d))^8 \left(\frac{9}{4}\right)^{(\text{ld}(\gamma(d)))^2}.$$

Sketch of proof of Theorem 3.11:

(i) $d - 1 \rightarrow d$: we can cover the “corners” of C with unimodular subcones.

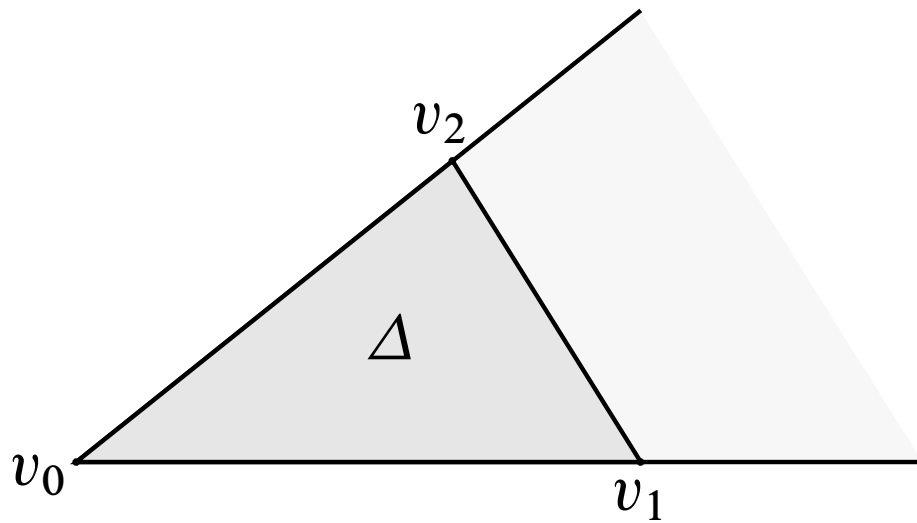
(ii) Extend the corner covers far enough into C . To have enough room, we must go “further up” in the cone. We lose unimodularity, but the multiplicity remains under control:

$$\leq \gamma(d) = \lceil \sqrt{d-1} \rceil (d-1)$$

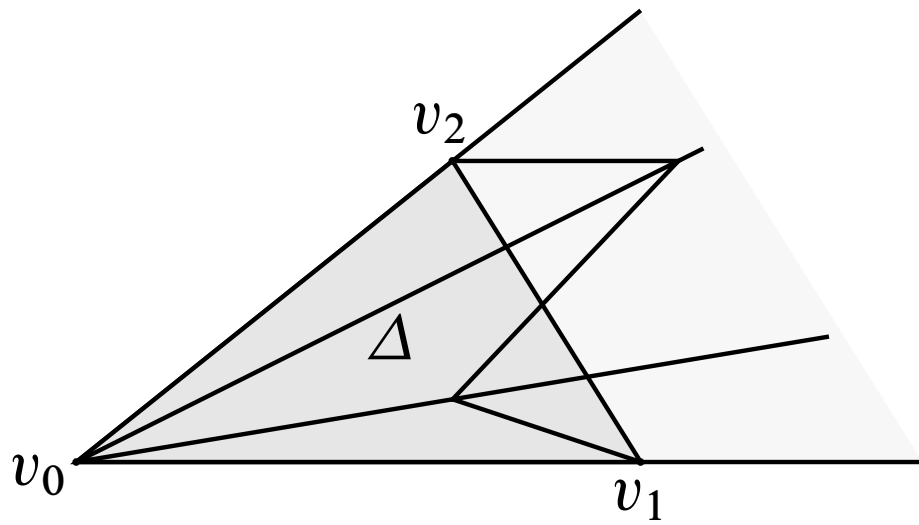
(iii) Apply part (a) of theorem to restore unimodularity.

(iv) Part (a): Control the “lengths” of the vectors in iterated stellar subdivision.

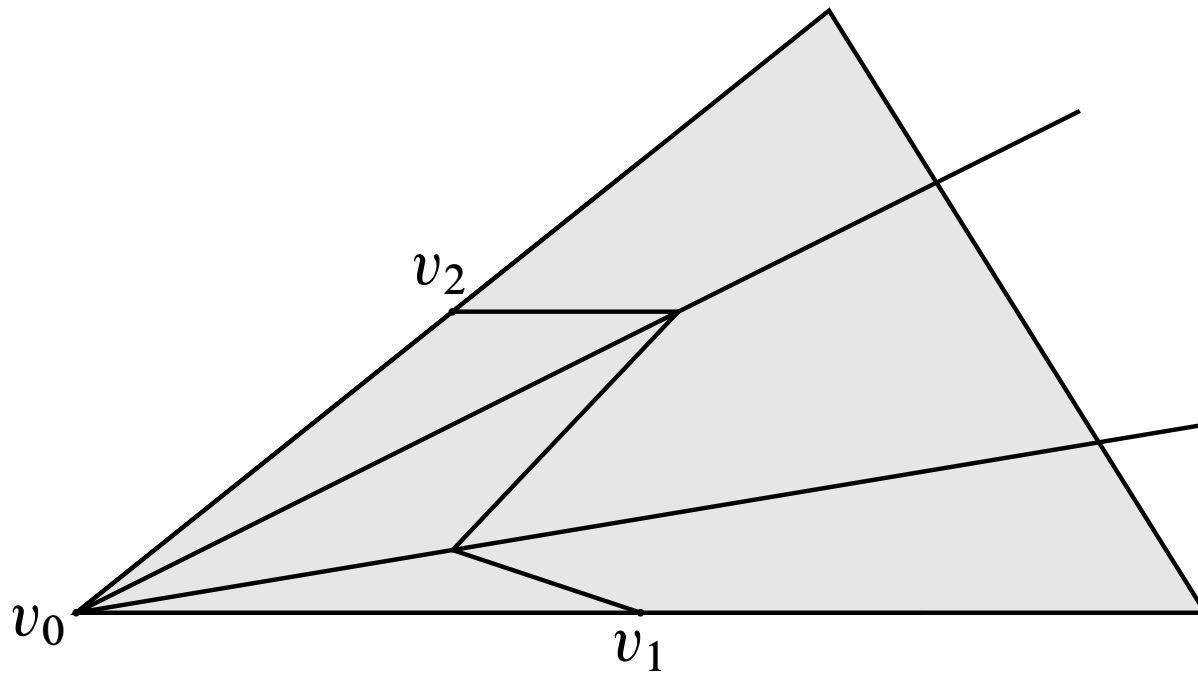
Sketch of proof of Theorem 3.10: May assume $P = \Delta$ is an (empty) simplex. Consider corner cones of Δ :



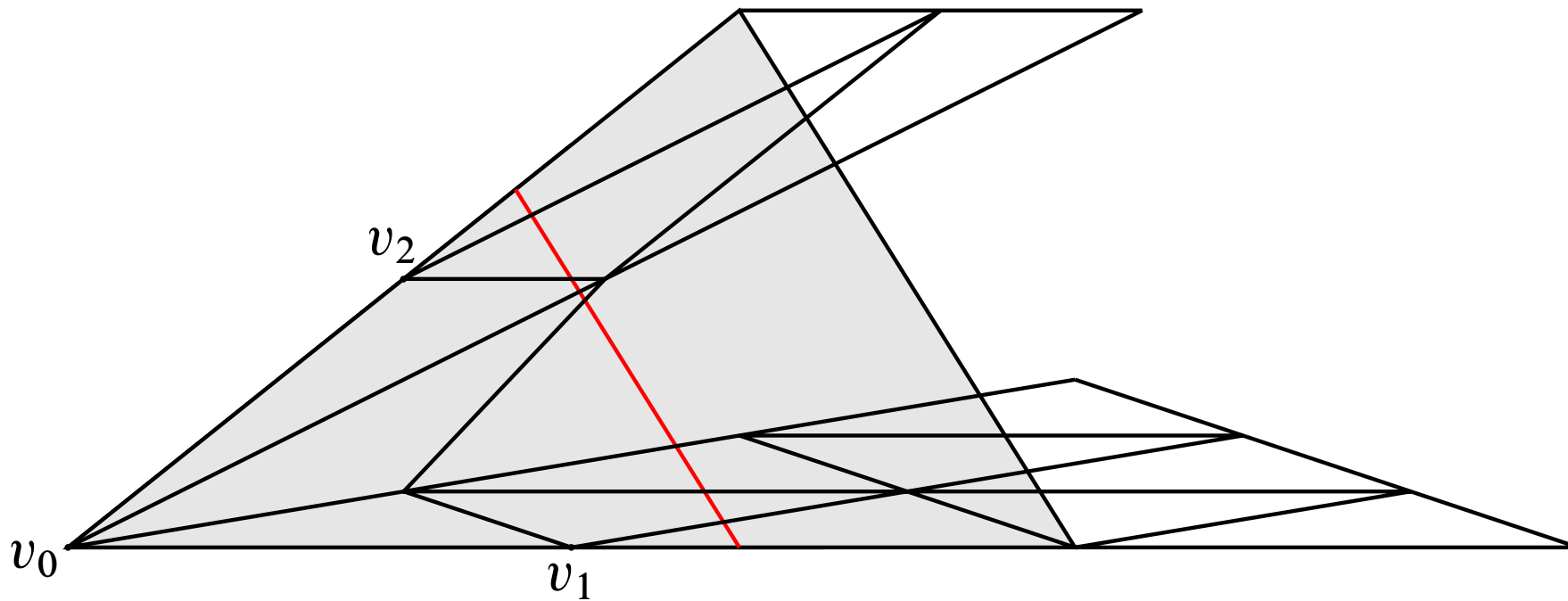
Apply Theorem 3.11 to corner cone:



Must multiply P by factor c' from Theorem 3.11 to get basic unimodular corner simplices into P



Tile corner cones:



Must multiply $c' P$ by $c'' \approx d \sqrt{d}$ to get the tiling by unimodular corner simplices close enough (= beyond red **line**) to facet opposite of v_0 .

Lecture 4

From vector spaces to polytopal algebras

Every so often you should try a damn-fool experiment —

from J. Littlewood's *A Mathematician's Miscellany*

The category $\text{Pol}(K)$

Recall from Lecture 3 that a **lattice polytope** P is the convex hull of finitely many points $x_i \in \mathbb{Z}^n$.

M_P **submonoid** of \mathbb{Z}^{n+1} generated by $(x, 1), x \in P \cap \mathbb{Z}^n$.

For a field K we let $\text{Pol}(K)$ be the category

- with objects the graded algebras $K[P] = K[M_P]$
- with morphisms the graded K -algebra homomorphisms

Main question: To what extent is $\text{Pol}(K)$ determined by combinatorial data ?

$\text{Pol}(K)$ generalizes $\text{Vect}(K)$, the category of finite-dimensional K -vector spaces:

Δ_n n -dimensional unit simplex

$$\Rightarrow K[\Delta_n] = K[X_1, \dots, X_{n+1}]$$

$$\begin{aligned} \text{Hom}_K(K^m, K^m) &\leftrightarrow \text{gr. hom}_K(S(K^m), S(K^n)) \\ &\leftrightarrow \text{gr. hom}_K(K[X_1, \dots, X_m], K[X_1, \dots, X_n]) \\ &\leftrightarrow \text{gr. hom}_K(K[\Delta_{m-1}], K[\Delta_{n-1}]) \end{aligned}$$

What properties of $\text{Vect}(K)$ can be passed on $\text{Pol}(K)$?

Note: $\text{Pol}(K)$ **not abelian**

Why not graded affine monoid algebras $K[M]$ in full generality?

Proposition 4.1. *Let P, Q be lattice polytopes. Then the K -algebra homomorphisms $K[P] \rightarrow K[Q]$ correspond bijectively to K -algebra homomorphisms $\overline{K[P]} \rightarrow \overline{K[Q]}$ of the normalizations.*

In fact, $K[P]$ equals $\overline{K[P]}$ in degree 1.

In the following the base field K is often replaced by a general commutative base ring R .

Toric automorphisms and symmetries

Elementary fact of linear algebra: $GL_n(K)$ is generated by matrices of 3 types:

- diagonal matrices
- permutation matrices
- elementary transformations

Actually, the permutation matrices are not needed. But their analogues in the general case cannot always be omitted.

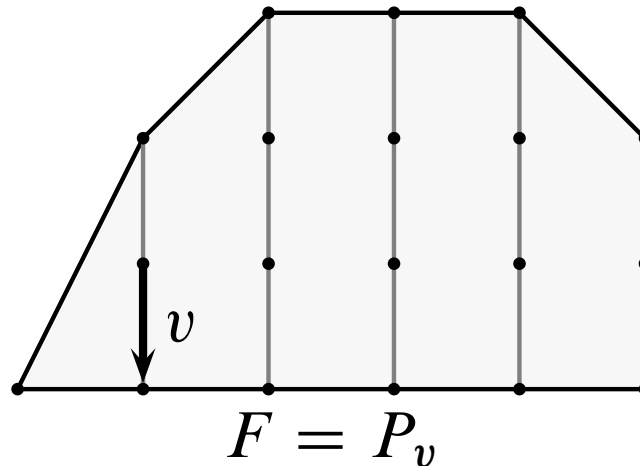
It easy to generalize diagonal matrices and permutation matrices:

- the diagonal matrices correspond to $(\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{T}_{n+1} = (K^*)^{n+1}$ acting on $K[P] \subset K[X_1^{\pm 1}, \dots, X_{n+1}^{\pm 1}]$ via the substitution $X_i \mapsto \lambda_i X_i$,
- the permutation matrices represent symmetries of Δ_{n-1} and correspond to the elements of the (affine!) symmetry group $\Sigma(P)$ of P .

How can we generalize elementary transformations?

Column structures

A **column structure** arranges the lattice points in P in columns:



More formally: $v \in \mathbb{Z}^n$ is a **column vector** if there exists a facet F , the **base facet** $P_v = F$ of v , such that

$$x + v \in P \quad \text{for all } x \in P \setminus F.$$

A column vector $v \in \mathbb{Z}^n$ is to be identified with $(v, 0) \in \mathbb{Z}^{n+1}$.

Elementary automorphisms

To each facet F of P there corresponds a facet of the cone $\mathbb{R}_+ M_P$, also denoted by F .

Recall the support form σ_F . For $F = P_v$ set $\sigma_v = \sigma_F$.

For every $\lambda \in R$ define a map from M_P to $R[\mathbb{Z}^{n+1}]$ by

$$e_v^\lambda : x \mapsto (1 + \lambda v)^{\sigma_v(x)} x.$$

σ_v \mathbb{Z} -linear and v column vector $\Rightarrow e_v^\lambda$ homomorphism from M_P into $(R[M_P], \cdot)$

$\Rightarrow e_v^\lambda$ extends to an endomorphism of $R[M_P]$

Since $e_v^{-\lambda}$ is its inverse, e_v^λ is an automorphism.

Proposition 4.2. v_1, \dots, v_s pairwise different column vectors for P with the same base facet $F = P_{v_i}$. Then

$$\varphi : (R, +)^s \rightarrow \text{gr. aut}_R(R[P]), \quad (\lambda_1, \dots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \dots \circ e_{v_s}^{\lambda_s},$$

is an *embedding of groups*.

$e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute and the inverse of $e_{v_i}^{\lambda_i}$ is $e_{v_i}^{-\lambda_i}$.

R field $\Rightarrow \varphi$ is homomorphism of algebraic groups.

\Rightarrow subgroup $\mathbb{A}(F)$ of $\text{gr. aut}_R(R[P])$ generated by e_v^λ with $F = P_v$ is an affine space over R

$\text{Col}(P)$ = set of column vectors of P (can be empty).

The polytopal linear group

Theorem 4.3. *Let K a field.*

- *Every $\gamma \in \text{gr. aut}_K(K[P])$ has a presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

$\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_{n+1}$, and $\alpha_i \in \mathbb{A}(F_i)$.

- $\mathbb{A}(F_i)$ and \mathbb{T}_{n+1} generate *conn. comp. of unity* $\text{gr. aut}_K(K[P])^0$.
- $= \{\gamma \in \text{gr. aut}_K(K[P]) \text{ inducing id on div. class group of } \overline{K[P]}\}$.
- $\dim \text{gr. aut}_K(K[P]) = \#\text{Col}(P) + n + 1$.
- \mathbb{T}_{n+1} is a *maximal torus* of $\text{gr. aut}_K(K[P])$.

The proof uses in a crucial way that **every divisorial ideal** of $K[M_P]$ is **isomorphic to a monomial ideal**.

This fact allows a **polytopal Gaussian algorithm**.

Using elementary automorphisms it corrects an arbitrary γ to an automorphism δ such that $\delta(\text{int}(K[P])) = \text{int}(K[P])$.

Lemma 4.4. $\delta(\text{int}(K[P])) = \text{int}(K[P]) \Rightarrow \delta = \tau \circ \sigma,$
 $\tau \in \mathbb{T}_{n+1}, \sigma \in \Sigma(P)$

Important fact: the divisor class group of $\overline{K[P]}$ is a **discrete object**.

To some extent one can also classify retractions of $K[P]$.

Milnor's classical K_2

Its construction is based on

- the passage to the “stable” group of elementary automorphisms
- the Steiner relations

Construction of the stable group: $E_n(R)$ subgroup generated by elementary matrices,

$$E \in E_n(R) \mapsto \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix} \in E_{n+1}(R)$$

$$\mathbb{E}(R) = \lim_{\longrightarrow} E_n(R).$$

The **Steinberg relations** for elementary matrices:

$$e_{ij}^\lambda e_{ij}^\mu = e_{ij}^{\lambda+\mu}$$

$$[e_{ij}^\lambda, e_{jk}^\mu] = e_{ik}^{\lambda\mu}, \quad i \neq k$$

$$[e_{ij}^\lambda, e_{ki}^\mu] = e_{kj}^{-\lambda\mu} \quad j \neq k$$

$$[e_{ij}^\lambda, e_{kl}^\mu] = 1 \quad i \neq l, j \neq k$$

The **stable Steinberg group** of K is defined by

- **generators** x_{ij}^λ , $i, j \in \mathbb{N}$, $i \neq j$, $\lambda \in K$ representing the elementary matrices
- the (formal) **Steinberg relations** $x_{ij}^\lambda x_{ij}^\mu = x_{ij}^{\lambda+\mu}$, $[x_{ij}^\lambda, x_{jk}^\mu] = x_{ik}^{\lambda\mu}$ etc.

Set

$$K_2(R) = \text{Ker}(\text{St}(R) \rightarrow \mathbb{E}(R)), \quad x_{ij}^\lambda \mapsto e_{ij}^\lambda.$$

Milnor's theorem:

Theorem 4.5. *The exact sequence*

$$1 \rightarrow K_2(R) \rightarrow \text{St}(R) \rightarrow \mathbb{E}(R) \rightarrow 1$$

is a *universal central extension* and $K_2(R)$ is the center of $\text{St}(R)$.

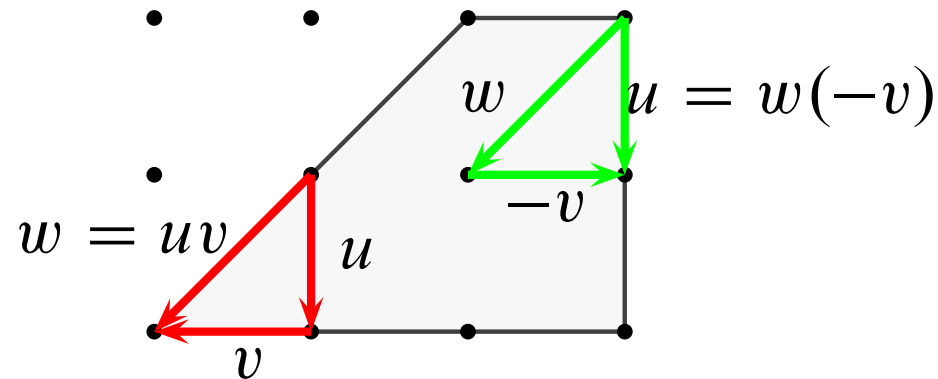
$K_2(R)$ captures the “hidden syzygies” of the elementary matrices.

Products of column vectors

Let $u, v, w \in \text{Col}(P)$. We say that

$$uv = w \iff w = u + v \quad \text{and} \quad P_w = P_u$$

Examples of products of column vectors:

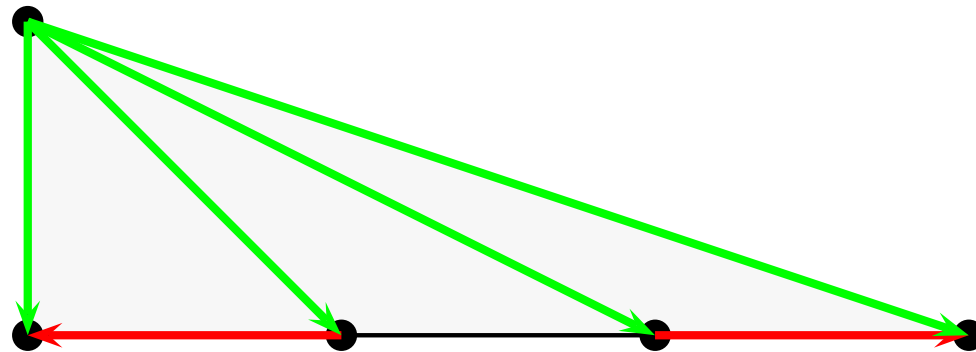


\Rightarrow **partial**, non-commutative **product structure** on $\text{Col}(P)$.

Balanced polytopes

A polytope is **balanced** if

$$\sigma_F(v) \leq 1 \quad \text{for all } v \in \text{Col}(P), \quad F = P_w.$$



A **nonbalanced** polytope

Polytopal Steinberg relations

Proposition 4.6. *P balanced, $u, v \in \text{Col}(P)$, $u + v \neq 0$, $\lambda, \mu \in R$.*

Then

$$e_v^\lambda e_v^\mu = e_v^{\lambda+\mu}$$
$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists,} \\ e_{vu}^{\mu\lambda} & \text{if } vu \text{ exists,} \\ 1 & \text{if } u + v \notin \text{Col}(P). \end{cases}$$

Note: we **know nothing** about $[e_u^\lambda, e_{-u}^\mu]$ if $u, -u \in \text{Col}(P)$!

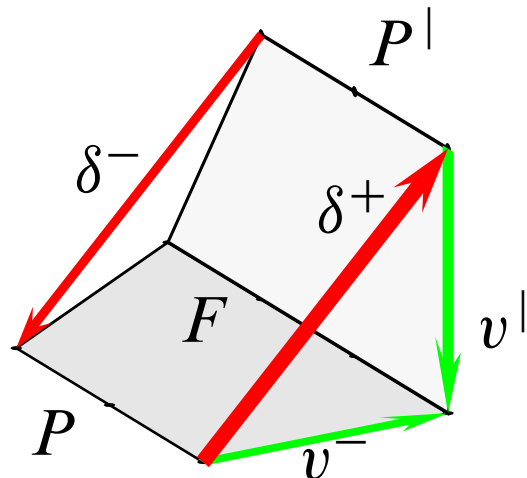
Doubling along a facet

Let $F = P_v$ be a **facet** of P and choose coordinates in \mathbb{R}^n such that \mathbb{R}^{n-1} is the affine hyperplane spanned by F .

$$P^- = \{(x', x_n, 0) : (x', x_n) \in P\}$$

$$P^\perp = \{(x', 0, x_n) : (x', x_n) \in P\}$$

$$P^{\perp F} = \text{conv}(P, P^\perp) \subset \mathbb{R}^{n+1}.$$



$$v = v^- = \delta^+ v^\perp$$

$$v^\perp = \delta^- v^-$$

Crucial facts:

$$\text{Col}(P) \hookrightarrow \text{Col}(P^{\perp F})$$

$$G \mapsto \text{conv}(G^-, G^{\perp}), \quad G \neq F$$

$$F \mapsto P^{\perp}$$

P^- = new facet

Lemma 4.7. P balanced $\Rightarrow P^{\perp F}$ balanced and

$$\text{Col}(P^{\perp F}) = \text{Col}(P)^- \cup \text{Col}(P)^{\perp} \cup \{\delta^+, \delta^-\}.$$

Doubling spectra

The chain of lattice polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset \dots)$ is called a **doubling spectrum** if

- for every $i \in \mathbb{Z}_+$ there exists a column vector $v \in \text{Col}(P_i)$ such that $P_{i+1} = P_i^{\perp v}$,
- for every $i \in \mathbb{Z}_+$ and any $v \in \text{Col}(P_i)$ there is an index $j \geq i$ such that $P_{j+1} = P_j^{\perp v}$.

Associated to \mathfrak{P} are the **'infinite polytopal' algebra**

$$R[\mathfrak{P}] = \lim_{i \rightarrow \infty} R[P_i]$$

and the filtered union

$$\text{Col}(\mathfrak{P}) = \lim_{i \rightarrow \infty} \text{Col}(P_i).$$

Now we can define a stable elementary group:

$$\mathbb{E}(R, P) = \text{subgroup of gr. aut}_R(R[\mathfrak{B}]) \text{ generated by } e_v^\lambda$$
$$v \in \text{Col}(\mathfrak{B}), \lambda \in R.$$

Note: it **depends only on P** , not on the doubling spectrum.

Theorem 4.8. $\mathbb{E}(R, P)$ is a perfect group with trivial center.

Polytopal Steinberg groups

The group $\text{St}(R, P)$ is defined by

- **generators** x_v^λ , $v \in \text{Col}(fP)$, $\lambda \in R$ representing the elementary automorphisms e_v^λ
- the (formal) **Steinberg relations** between the x_v^λ

It depends only on the partial product structure on $\text{Col}(P)$. This allows some functoriality in P .

Polytopal K_2

In analogy with Milnor's theorem we have

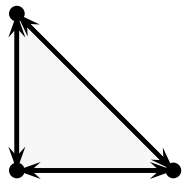
Theorem 4.9. *P balanced polytope $\Rightarrow \text{St}(R, P) \rightarrow \mathbb{E}(R, P)$ is a **universal central extension** with kernel equal to the center of $\text{St}(R, P)$.*

Definition 4.10.

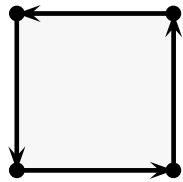
$$K_2(R, P) = \text{Ker}(\text{St}(R, P) \rightarrow \mathbb{E}(R, P)).$$

Balanced polygons

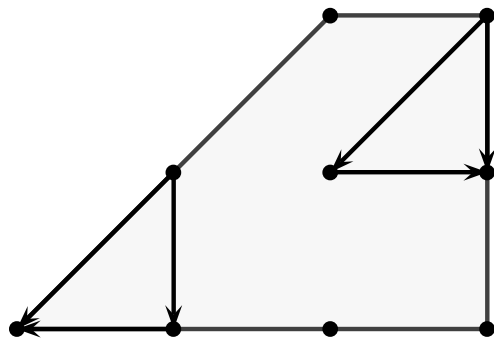
It is not difficult to classify the **balanced polygons = 2-dimensional polytopes**: ($K_2 = \text{classical } K_2$)



$$\{\pm u, \pm v, \pm w\} \quad K_2$$



$$\{\pm u, \pm v\} \quad K_2 \oplus K_2$$

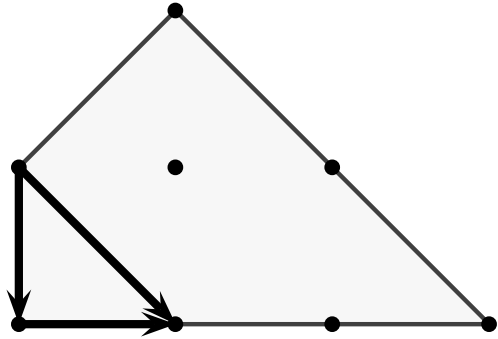


$$\{u, \pm v, w\}$$

$$w = uv$$

$$u = w(-v)$$

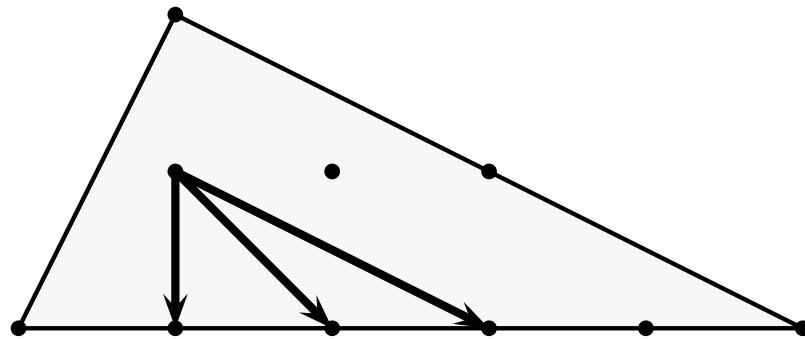
$$K_2 \oplus K_2$$



$$\{u, v, w\}$$

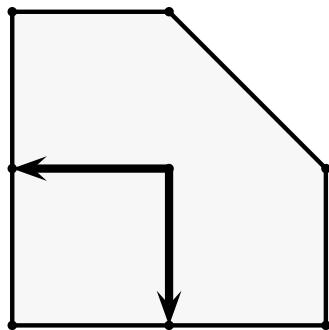
$$w = uv$$

$$K_2 \oplus K_2$$



$$\{v : P_v = F\}$$

$$K_2$$



$$\{u, v\}$$

$$K_2 \oplus K_2$$

Higher K -groups

Using [Quillen's +-construction](#) or [Volodin's construction](#) one can define higher K -groups.

For certain well-behaved polytopes both constructions yield the same result (in the classical case proved by Suslin).

Potentially difficult polytope:

