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ICTP 40th Anniversary

SMR1563/12

School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics

(24 May - 11 June 2004)

Commutative algebra and duality in the cohomology of groups

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COMMUTATIVE ALGEBRA AND DUALITY IN THE COHOMOLOGY OF GROUPS

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ABSTRACT. I shall use duality in the cohomology rings of finite groups as a motivation for a discussion of what the word "Gorenstein" should really mean in a more general context than usual.

Contents.

LECTURE 1: WHAT IS GROUP COHOMOLOGY?

Let G be a finite group and k a commutative ring of coefficients. Definition of cohomology of groups via algebra: if M is a kG-module, then

$$H^*(G, M) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M) \cong \mathrm{Ext}^*_{kG}(k, M).$$

Definition via topology: by a theorem of Milnor, there exists a contractible space EG and a free action of G on EG. We define $BG = EG/G$ and $H^*(G, k) = H^*(BG; k)$.

The cohomology $H^*(G, k)$ has a multiplication, which can be described either using Yoneda multiplication in Ext, or cup product in cohomology of BG. If k is Noetherian, this makes it a finitely generated graded commutative ring. Here *graded commutative* means that

$$ba = (-1)^{|a||b|}ab.$$

For example, let us compute $H^*(\mathbb{Z}/2, k)$ both algebraically and topologically, where k is a field of characteristic two. The group algebra $k\mathbb{Z}/2$ is isomorphic to $k[t]/(t^2)$, where $\mathbb{Z}/2 = \langle g \mid g^2 = 1\rangle$ and $t = 1 + g \in kG$. We can compute $H^*(G, k) = \mathrm{Ext}^*_{k\mathbb{Z}/2}(k, k)$ by forming a free resolution of k as follows:

$$\cdots \xrightarrow{t} k\mathbb{Z}/2 \xrightarrow{t} k\mathbb{Z}/2 \xrightarrow{t} k\mathbb{Z}/2 \to 0$$

The author is partly supported by NSF grant DMS-0242909.

Taking homomorphisms into k and computing cohomology of the resulting complex, we find that each $H^i(G,k)$ is one dimensional over k, and the Yoneda class of the exact sequence

$$0 \to k \to k\mathbb{Z}/2 \xrightarrow{t} \cdots \xrightarrow{t} k\mathbb{Z}/2 \to k \to 0$$

can be chosen as a basis element. Multiplication is given by Yoneda splice, so writing x for the generator in degree one, we find that the above sequence represents the cohomology element x^i. So we have

$$H^*(\mathbb{Z}/2,k) \cong k[x]$$

with $|x| = 1$.

On the other hand, if we think about the topological definition, let S^∞ be the union of the finite dimensional spheres S^n, with each sphere embedded as the equator in the next one. Any map from S^n into S^∞ lands in some S^m, and is then null homotopic, by sliding it off to the north pole in S^{m+1}; so S^∞ is contractible. The group $\mathbb{Z}/2$ acts freely on S^∞ by sending each point on a sphere to the antipodal point. It follows that S^∞ is an $E\mathbb{Z}/2$. The quotient of S^∞ by the antipodal map is infinite projective space $\mathbb{R}P^\infty$, so this is our model for $B\mathbb{Z}/2$. It has one hemispherical cell in each dimension, and its cohomology is $k[x]$, with $|x| = 1$.

The correspondence between these two views is that the cellular chains on S^∞ gives the free resolution we discussed above. More generally, for any group G, chains on EG form a free resolution of the coefficient ring k as a kG-module. We have

$$C^*(BG;k) \cong \mathrm{Hom}_k(C_*(BG;k),k) \cong \mathrm{Hom}_{kG}(C_*(EG;k),k)$$

and so

$$H^*(BG;k) \cong \mathrm{Ext}^*_{kG}(k,k).$$

As our next example, we can compute $H^*(\mathbb{Z}/2 \times \mathbb{Z}/2,k)$ using the Künneth theorem. Namely, we can use $E\mathbb{Z}/2 \times E\mathbb{Z}/2$ as our EG, which gives $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$ for our BG. So $H^*(G,k) \cong k[x,y]$ with $|x| = |y| = 1$. The same method works for any elementary abelian 2-group, and shows that

$$H^*((\mathbb{Z}/2)^r,k) \cong k[x_1,\dots,x_r],$$

a polynomial ring on r generators in degree one.

Next, we look at the quaternion group Q_8 of order eight. This can be regarded as a subgroup of the multiplicative group of the four dimensional quaternion division algebra $\mathbb{H}$. The group of quaternions of norm one is isomorphic to SU(2), which is topologically just the unit sphere S^3 in $\mathbb{R}^4$. So by left multiplication, Q_8 acts freely on S^3. Although S^3 isn't contractible, we can use this action to build a resolution, and then obtain cohomological information. Namely, we can choose a cellular decomposition of S^3 so that the action of Q^8 is cellular, and then the cellular chains give us a sequence of free kQ_8-modules

$$0 \to C_3(S^3;k) \to C_2(S^3;k) \to C_1(S^3;k) \to C_0(S^3;k) \to 0$$

whose homology is k in degrees zero and three, and is exact elsewhere. Splicing this sequence to itself infinitely often gives us a free resolution of k as a kQ_8-module

$$\cdots \to C_0(S^3;k) \to C_3(S^3;k) \to \cdots \to C_0(S^3;k) \to 0.$$

Computing cohomology using this free resolution, we see that there is an element $z \in H^4(Q_8,k)$ inducing the periodicity, and

$$H^*(Q_8,k)/(z) \cong H^*(S^3/Q_8;k).$$

Since S^3/Q_8 is an orientable manifold, its cohomology satisfies Poincaré duality. The way to say this in terms of commutative algebra is that $H^*(Q_8,k)$ is a *one-dimensional Gorenstein ring.*

There are several things we can take from this example. First, if a finite group G acts freely on a sphere, of any dimension, then the cohomology is periodic. So for example, our computation of $H^*(\mathbb{Z}/2 \times \mathbb{Z}/2,k)$ shows that $\mathbb{Z}/2 \times \mathbb{Z}/2$ cannot act freely on a sphere of any dimension. A similar calculation for p odd shows that $\mathbb{Z}/p \times \mathbb{Z}/p$ also cannot act freely on a sphere of any dimension. So if G acts freely on a sphere, then the p-rank of G (i.e., the maximal rank r of an elementary abelian p-subgroup $(\mathbb{Z}/p)^r$ of G) is at most one.

The second thing that we can take from this example is the general statement that if G acts freely on a sphere then $H^*(G,k)$ is (trivial or) a one dimensional Gorenstein ring. The following is a definition of Gorenstein suitable for our situation.

Definition 1.1. Let R be a finitely generated graded commutative ring with $R^0 = k$ and $R^i = 0$ for $i < 0$. We say that R is *Cohen–Macaulay* if there are homogeneous elements $\zeta_1, \dots, \zeta_r$ in R generating a polynomial subring $k[\zeta_1, \dots, \zeta_r]$ over which R is a finitely generated free module. This is equivalent to the condition that $\zeta_1, \dots, \zeta_r$ is a *regular sequence.*

We say that R is *Gorenstein* if, in addition, the quotient $\bar{R} = R/(\zeta_1, \dots, \zeta_r)$ satisfies Poincaré duality. This means that there is a "top" degree d such that the dimensions of $\bar{R}^{d-i}$ and $\bar{R}^i$ are equal for all values of i, and

$$\bar{R}^i \times \bar{R}^{d-i} \to \bar{R}^d$$

is a perfect pairing.

If we want to be more precise, we say that R is *Gorenstein with shift* a, where

$$a = -d + \sum_{i=1}^{r}(|\zeta_i| - 1)$$

is independent of the choice of parameters.

For example, if R is the cohomology of an orientable manifold of dimension d then R is Gorenstein with shift $a = -d$.

Another equivalent way of expressing the Gorenstein condition that works more generally is that if R has Krull dimension r, then it is Gorenstein with shift a if

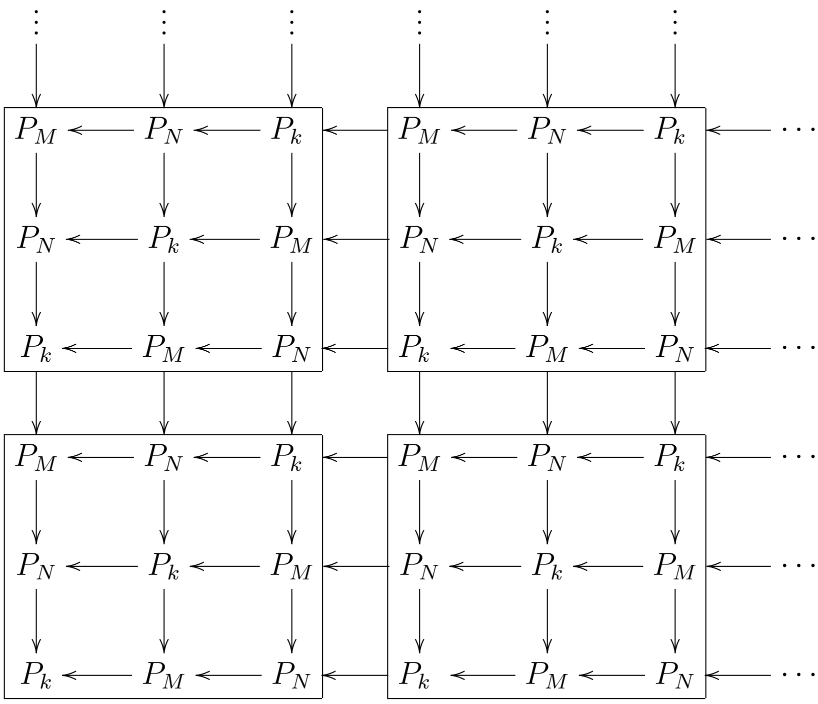
$$\mathrm{Ext}_R^{i,j}(k, R) = \begin{cases} k & i = r \text{ and } j = -a - r \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Here, Ext is bigraded, with the Ext degree as the first index and the internal degree coming from the grading on R as the second.

Theorem 1.3 (Benson, Carlson [4]). *If G is a finite group and $H^*(G, k)$ is Cohen–Macaulay, then it is Gorenstein shift $a = 0$.*

Corollary 1.4. *If $H^*(G, k)$ is a polynomial ring then $p = 2$, the generators are in degree one, and (modulo an odd order normal subgroup invisible to mod 2 cohomology) $G \cong (\mathbb{Z}/2)^r$.*

Explicitly compute a resolution for $GL(3, \mathbb{F}_2)$ in characteristic two, and display the duality. Total complex of the double complex



2. Lecture 2: Local cohomology of group cohomology

In this lecture we discuss local cohomology. This can be computed using the stable Koszul complex, as follows. Let R be a Noetherian graded commutative local ring with maximal ideal $\mathfrak{m}$, and let M be a graded R-module. Choose a homogeneous set of parameters $\zeta_1, \dots, \zeta_r$ in $\mathfrak{m}$ for M, and form the complex

$$0 \to M \to \bigoplus_i M[\zeta_i^{-1}] \to \bigoplus_{i<j} M[(\zeta_i\zeta_j)^{-1}] \to \cdots \to M[(\zeta_1 \dots \zeta_r)^{-1}] \to 0.$$

The cohomology of this complex is the doubly graded local cohomology

$$H^{**}_{\mathfrak{m}} M.$$

The first grading is the local cohomological degree, and the second grading comes from the internal grading on R and M. For further details, see Bruns and Herzog [6]. A theorem of Grothendieck states that $H^{s,t}_{\mathfrak{m}} M \cong (R^s\Gamma_{\mathfrak{m}} M)^t$, the right derived functors of the functor $\Gamma_{\mathfrak{m}}$ on graded R-modules defined by

$$\Gamma_{\mathfrak{m}} M = \{x \in M \mid \exists\, n \quad \mathfrak{m}^n.x = 0\}.$$

In general, we have $H^{s,t}_{\mathfrak{m}} M = 0$ except when

$$\mathrm{depth}(M) \le s \le \dim(M);$$

furthermore, if $s = \mathrm{depth}(M)$ or $s = \dim(M)$ then for some value of t we have $H^{s,t}_{\mathfrak{m}} M \neq 0$.

For example, if R is Cohen–Macaulay of Krull dimension r, then $H^{s,t}_{\mathfrak{m}} R = 0$ except when $s = r$. The Matlis dual of $H^{r,*}_{\mathfrak{m}} R$ is the *canonical module* Ω_R. The condition that R is Gorenstein with shift a is the condition that $\Omega_R \cong R[a + r]$. Note that if $R^0 = k$ and $R^n = 0$ for $n < 0$ then the Matlis dual of an R-module M is the same as the graded dual $\mathrm{Hom}_k(M, k)$, so that in this case, Gorenstein with shift a is the statement that

$$H^{r,*}_{\mathfrak{m}} R \cong \mathrm{Hom}_k(R, k)[a + r]. \tag{2.1}$$

Do the computations for the semidihedral 2-groups

$$SD_{2^n} = \langle g, h \mid g^{2^{n-1}} = 1, h^2 = 1, hgh^{-1} = g^{2^{n-2}-1} \rangle$$

($n \ge 4$). The cohomology ring of this group is

$$H^*(SD_{2^n}, \mathbb{F}_2) = \mathbb{F}_2[x, y, z, w]/(xy, y^3, yz, z^2 + x^2w)$$

with $|x| = |y| = 1$, $|z| = 3$ and $|w| = 4$. This example has Krull dimension two and depth one.

Greenlees developed a spectral sequence [9]

$$H^{s,t}_{\mathfrak{m}}\, H^*(G, k) \Rightarrow H_{-s-t}(G, k) \tag{2.2}$$

which generalizes Theorem 1.3. Interpret $H_*(G,k)$ as the injective hull of k as a graded $H^*(G,k)$-module, namely the graded dual of $H^*(G,k)$.

The Greenlees spectral sequence is an algebraic shadow of the statement that $H^*(G,k)$ is "Gorenstein" in a suitable derived sense to be discussed in the next two lectures. Of course, if $H^*(G,k)$ were Cohen–Macaulay as a ring, then this spectral sequence would collapse and say that it is Gorenstein. The semidihedral example shows that this is not always the case.

Examine the example of $\Gamma_7 a_2$ in detail. Set up the spectral sequence using the stable Koszul construction from [2].

Since $H_j(G,k) = 0$ for j negative, the E_∞ page of the Greenlees spectral sequence vanishes for $s+t>0$. The regularity conjecture from [3] says that already the E_2 page has this property. In other words, it says that $H^{s,t}_{\mathfrak{m}} H^*(G,k) = 0$ if $s+t>0$. This conjecture has been proved as long as the Krull dimension is at most two more than the depth.

3. Lecture 3: What is Gorenstein, really?

What should "Gorenstein" really mean for a cohomology ring? The problem with this question is that the ring structure alone is not the only structure carried by the cohomology ring. There are also Massey products and Steenrod operations. The Massey product are shadows of "higher associativities" and the Steenrod operations are shadows of "higher commutativities." All of these operations are obtained from the differential graded algebra (DGA) of cochains, but is lost when we pass to the cohomology as a ring. It turns out that we need to formulate the Gorenstein property at the level of the DGA of cochains, rather than at the level of the cohomology ring. Later, we shall discuss how to put enough extra structure on the cohomology to formulate things there.

Denote by $P(k)$ a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to$$

of k as a kG-module, and write $\mathcal{R}$ for the differential graded algebra $\mathrm{End}_{kG}(P(k))$ of endomorphisms of the complex $P(k)$. So

$$\mathcal{R}^n = \prod_i \mathrm{Hom}_{kG}(P_{n+i}, P_i)$$

with multiplication defined by composition of maps and differential

$$d\colon \mathcal{R}^n \to \mathcal{R}^{n+1}$$

defined by

$$d(f)\,(x) = d(f(x)) - (-1)^{|f|} f(d(x)).$$

We have $H^n(\mathcal{R}) \cong H^n(G,k)$.

For any DGA R over k, we can form the homotopy category of DG R-modules. A map in this category is called a *quasiisomorphism* if it induces an isomorphism in homology. The category of fractions obtained by inverting the quasiisomorphisms is the *derived category* $\mathbf{D}(R)$. If X and Y are objects in $\mathbf{D}(R)$, then we can form an object $\mathbb{R}\mathrm{Hom}_R(X, Y)$ in $\mathbf{D}(k)$ whose cohomology groups are the maps in $\mathbf{D}(R)$ from X to shifts of Y. These work in the way one would expect: if X is an R-S-bimodule and Y is a left R-module, for example, then $\mathbb{R}\mathrm{Hom}_R(X, Y)$ is a left S-module. Similarly, if X is a *right* R-module and Y is a left R-module, we can form the derived tensor object $X \overset{\mathbb{L}}{\otimes}_R Y$.

Dwyer, Greenlees and Iyengar [7] define an augmented DGA $R \to k$ over a field k to be *Gorenstein* with shift a if

$$\mathbb{R}\mathrm{Hom}_R(k, R) \simeq k[-a] \tag{3.1}$$

as objects in $\mathbf{D}(k)$. This should be compared with (1.2). They also impose another technical condition (see Definition 6.1 of [7]) which is designed to ensure that Matlis duality behaves well, and which we shall not state here. Also, their context is more general than DGAs; they work with ring spectra in the sense of algebraic topology. They show that the DGA $\mathcal{R}$ described above is Gorenstein in their sense, with shift zero. They show that under suitable circumstances, in the presence of the "higher commutativities" mentioned above, the Gorenstein condition gives a spectral sequence

$$H^{s,t}_{\mathfrak{m}} H^* R \Rightarrow H^{s+t+a}\mathrm{Hom}_k(R, k). \tag{3.2}$$

In the case of the DGA $\mathcal{R}$, this is the Greenlees spectral sequence (2.2).

Remarks. Much of what we are describing in this section and the next comes from Dwyer, Greenlees and Iyengar [7]. They use the language of $\mathbb{S}$-algebras and their modules. An $\mathbb{S}$-algebra is a ring spectrum, using a category of spectra in which the smash product is strictly commutative and associative, not just up to all higher homotopies. Such a category has been constructed by Elmendorf, Kříž, Mandell and May [8] among others. This is necessary in order to perform some of the operations of commutative algebra. So for example if R is a commutative $\mathbb{S}$-algebra then a localization of R at a prime ideal in its homotopy is again a commutative $\mathbb{S}$-algebra; and the tensor product of two R-modules is again an R-module. A DGA is an example of an $\mathbb{S}$-algebra, but be warned that commutativity as an $\mathbb{S}$-algebra only means that it is commutative "up to all higher homotopies" (in the E_∞ sense) as a DGA. So for example, $C^*(BG; k)$ is a commutative $\mathbb{S}$-algebra, but not a commutative DGA.

If R is a DGA, the its homotopy as an $\mathbb{S}$-algebra is the same as its homology as a DGA. However, for this really to be true, it is necessary to grade DGAs homologically rather than cohomologically. We are numbering ours cohomologically, but it is easy to reindex by negating the grading, $X_n = X^{-n}$.

If H^*R were Gorenstein with shift a as a ring, the spectral sequence (3.2) would degenerate to

$$H^{r,t}_{\mathfrak{m}} H^*R \cong \mathrm{Hom}_k(H^*R, k)^{r+t+a}$$

where r is the Krull dimension of R. Comparing this with (2.1), we see that under the above circumstances, if R is Gorenstein and H^*R is Cohen–Macaulay, then H^*R is Gorenstein. This can be viewed as a generalization of Theorem 1.3.

So what extra information is there in $\mathcal{R}$ that is not reflected in the ring structure of $H^*\mathcal{R} = H^*(G,k)$, and that gives rise to the difference between the statement that $\mathcal{R}$ is Gorenstein and the statement that $H^*\mathcal{R}$ is Gorenstein? As we indicated earlier, there are two kinds of extra information available: higher associativities and higher commutativities. It turns out that for the definition of Gorenstein, only the higher associativities are important. This is captured in the notion of an A_∞ algebra, as we now explain. If higher commutativities are also incorporated, the resulting structure is called an E_∞ algebra; this is important when we want to perform constructions from commutative algebra such as localization, and also if we want the tensor product of two modules to be another module. A very good survey of A_∞ algebras is Keller [11], to which we refer for all details not supplied here.

Let us begin by recalling the definition of the Massey triple product in cohomology. Suppose that $[x]$, $[y]$ and $[z]$ are elements of $H^*(G,k)$ with the property that $[x][y] = 0$ and $[y][z] = 0$. Choose cocycles x, y and z in $\mathcal{R}$ representing them. Then there are cochains u and v such that $xy = du$ and $yz = dv$. Look at the element

$$uz - (-1)^{|x|}xv. \tag{3.3}$$

If we apply d to this, then using the fact that $d(z) = 0$ and $d(x) = 0$, we get

$$d(u)z - xd(v) = xyz - xyz = 0.$$

So (3.3) is a cocycle, and we define the *Massey triple product* to be its cohomology class

$$\langle x, y, z\rangle = [uz - (-1)^{|x|}xv].$$

Replacing x, y and z by cohomologous cocycles will alter the Massey triple product, and it is easy to check that it is well defined modulo $xH^*(G,k) + H^*(G,k)z$.

Higher Massey products are defined similarly. For example, if w, x, y and z are elements such that $wx = 0$, $xy = 0$, $yz = 0$, $\langle w, x, y\rangle = 0$ and $\langle x, y, z\rangle = 0$, then the Massey product $\langle w, x, y, z\rangle$ is defined. Unfortunately, the indeterminacies become more and more complicated. Matric Massey products are a variation where use is made of relations that are not monomial. So for example if $wx = 0$, $wy = 0$ and $xz + yt = 0$, then we can form the matric Massey product

$$\langle w, \begin{pmatrix} x & y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix} \rangle$$

by the same sort of process as we used above.

The concept of an A_∞ algebra somehow rigidifies the Massey products, and makes them universally defined, not just under some vanishing conditions, but at the expense of having to make some choices. The beginning of the process is as follows. Let R be a DGA over a field k, and look at H^*R. Choosing representative cocycles in a linear way can be codified by the choice of a k-linear map $f_1 \colon H^*R \to R$ which induces the identity map on cohomology; here, we are thinking of H^*R as a DGA with zero differential. Now, in general, f_1 cannot be chosen to be compatible with the multiplication. However, the difference can be written as a coboundary. In other words, we can choose a linear map of degree -1

$$f_2 \colon H^*R \otimes H^*R \to R$$

such that

$$f_1(xy) - f_1(x)f_1(y) = df_2(x, y).$$

Next, if $x, y, z \in H^*R$, we look at the element

$$(-1)^{|x|} f_1(x)f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y)f_1(z) \tag{3.4}$$

and we see that applying d gives zero. So we can define a map of degree -1

$$m_3 \colon H^*R^{\otimes 3} \to H^*R$$

via

$$m_3(x, y, z) = [(-1)^{|x|} f_1(x)f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y)f_1(z)].$$

Furthermore, $f_1 m_3(x, y, z)$ differs from (3.4) by a coboundary. So we can choose a linear map of degree -2

$$f_3 \colon H^*R^{\otimes 3} \to R$$

so that

$$(-1)^{|x|} f_1(x)f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y)f_1(z) = f_1 m_3(x, y, z) - df_3(x, y, z).$$

Continuing this way, we obtain a sequence of maps

$$m_n \colon H^*R^{\otimes n} \to H^*R$$

($n \geq 1$) of degree $2 - n$ (with $m_1 = 0$ and $m_2(x, y) = xy$) and

$$f_n \colon H^*R^{\otimes n} \to R$$

($n \geq 1$) of degree $1-n$ satisfying the identities

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \tag{3.5}$$

and

$$\sum_{n=r+s+t} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{n=i_1+\dots+i_r} (-1)^{\sigma} m_r(f_{i_1} \otimes \dots \otimes f_{i_r}) \tag{3.6}$$

where

$$\sigma = (r-1)(i_1-1) + (r-2)(i_2-1) + \dots + 2(i_{r-2}-1) + (i_{r-1}-1).$$

Definition 3.7 (Stasheff)**.** An A_∞ *algebra* is a graded vector space A together with linear maps $m_n \colon A^{\otimes n} \to A$ ($n \geq 1$) of degree $2-n$ satisfying relations (3.5).

A *morphism* of A_∞ algebras $A \to B$ is a sequence of maps $f_n \colon A^{\otimes n} \to B$ ($n \geq 1$) of degree $1-n$ satisfying relations (3.6).

A *quasiisomorphism* of A_∞ algebras is a morphism such that f_1 induces an isomorphism on homology with respect to m_1.

Remarks. The case $n = 1$ of the relations (3.5) states that m_1 is a differential, while the case $n = 2$ states that m_1 is a derivation with respect to the multiplication m_2. However, the multiplication m_2 does not need to be associative; the remaining m_i ($i \geq 3$) are "higher associativities" filling in this discrepancy. A DGA is a special example of an A_∞ algebra in which m_2 is strictly associative and $m_i = 0$ for $i \geq 3$. However, in general an A_∞ algebra for which m_2 is strictly associative still need not have $m_i = 0$ for $i \geq 3$.

For a DGA R, the A_∞ structure on H^*R rigidifies the Massey product structure. For example, if x, y and z are elements of H^*R then $m_3(x, y, z)$ is a particular, well defined representative of the Massey triple product $\langle x, y, z \rangle$. The advantages of m_3 over the Massey triple product are that it is defined for *all* triples, not just ones where $xy = 0$ and $yz = 0$, and there is no indeterminacy. The disadvantage is that there were some choices made when constructing the A_∞ structure on H^*R. The resulting A_∞ structure is in fact only well defined up to quasiisomorphism.

The argument presented above for constructing m_3 and f_3 for a DGA R is the beginning of the proof of the following theorem.

Theorem 3.8 (Kadeishvili [10])**.** *If A is an A_∞ algebra (for example a DGA) then there is an A_∞ algebra structure on H^*A and a quasiisomorphism of A_∞ algebras $H^*A \to A$.*

The theory of A_∞ modules is defined similarly. If A is an A_∞ algebra, then an A_∞ *module* M comes with maps

$$m_n \colon M \otimes A^{\otimes(n-1)} \to M$$

($n \geq 1$) of degree $2-n$. These should satisfy exactly the same identities (3.5) as for an A_∞ algebra, but some of the m_i in the formula are the ones for the module and some are for the algebra. Similarly, a *morphism* of A_∞ modules $M' \to M$ is given by maps

$$f_n \colon M' \otimes A^{\otimes(n-1)} \to M$$

($n \geq 1$) of degree $1-n$, satisfying the analog of (3.6). But this time, most of the right hand side disappears because we're using the identity morphism on A. So it becomes

$$\sum_{n=r+s+t} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{n=r+s} (-1)^{(r+1)s} m_{1+s}(f_r \otimes 1^{\otimes s}).$$

A *quasiisomorphism* is a morphism such that f_1 induces an isomorphism on the cohomology with respect to m_1, as before. Composition of morphisms $f \colon M' \to M$ and $g \colon M'' \to M'$ is given by

$$(fg)_n = \sum_{n=r+s} (-1)^{(r+1)s} f_{1+s}(g_r \otimes 1^{\otimes s}).$$

If $f, g \colon M' \to M$ are morphisms, a *homotopy* from f to g is a collection of maps

$$h_n \colon M' \otimes A^{\otimes(n-1)} \to M$$

($n \geq 1$) of degree $-n$ satisfying

$$f_n - g_n = \sum_{n=r+s} (-1)^{rs} m_{1+s}(h_r \otimes 1^{\otimes s}) + \sum_{n=r+s+t} (-1)^{r+st} h_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}).$$

We form the derived category $\mathbf{D}(A)$ of an A_∞ algebra A just as we did for a DGA, except that it's slightly easier. We begin with the *homotopy category* whose objects are A-modules and whose arrows are the homotopy classes of maps. Working by analogy, we expect to have to invert the quasiisomorphisms. However, it turns out that all quasiisomorphisms in the homotopy category are already invertible. So the homotopy category *is* the derived category.

There are also Hom and tensor objects $\mathbb{R}\mathrm{Hom}_R(X, Y)$ and $X \overset{\mathbb{L}}{\otimes}_R Y$ which work in the same way as for a DGA. These are described in §6.3 of Keller [11] (with different notation).

Theorem 3.9. *A quasiisomorphism of A_∞ algebras $A \to B$ induces an equivalence of derived categories $\mathbf{D}(B) \to \mathbf{D}(A)$.*

So we can now make the same definition of Gorenstein for an A_∞ algebra as we did for a DGA. Namely, that condition (3.1) holds (plus the same technical condition as before).

Theorem 3.10. *Let G be a finite group and k a field of characteristic p. Regarding $H^*(G,k)$ as an A_∞ algebra as described above, it is Gorenstein with shift equal to zero.*

Remark 3.11. More generally, if G is a compact Lie group and k is a field then regarding $H^*(BG;k)$ as an A_∞ algebra, it is Gorenstein with shift equal to the dimension of G as a manifold, *provided* that the conjugation action of G on $\mathsf{Lie}(G)$ preserves orientation. This condition is satisfies provided G is finite or connected, but for example it is not satisfied for $G = T \rtimes H$ with T a torus and H a finite group of automorphisms which does not preserve orientation. The reader may find it instructive to compute $H^*(B(T^3 \rtimes \mathbb{Z}/2); \mathbb{F}_2)$, where the action of $\mathbb{Z}/2$ is given by negation. This ring is Cohen–Macaulay, but not Gorenstein.

Finally, it is worth remarking that as long as we regard $H^*(G,k)$ as an A_∞ algebra, we can recover the module category $\mathsf{Mod}(kG)$ from it. Actually, this is only the case when G is a p-group, because then k is the only simple module. But for a more general finite group, or even a general finite dimensional algebra Λ, $\mathsf{Mod}(\Lambda)$ can be recovered from the A_∞ algebra $\mathrm{Ext}^*_\Lambda(S,S)$, where S is a direct sum of one representative of each isomorphism class of simple Λ-modules; see §§5-6 of Keller [11] for further details. This can be regarded as a vindication of Alperin's dictum [1] that

Cohomology is representation theory.

4. Lecture 4: Duality in algebra and topology

In this lecture, following Dwyer, Greenlees and Iyengar [7], we discuss how to do local cohomology at the level of DGAs, and what this has to do with the Gorenstein condition. The discussion applies verbatim to A_∞ algebras.

Our discussion begins with local cohomology for a Noetherian local ring R with maximal ideal $\mathfrak{m}$. Recall that $H^i_\mathfrak{m}R$ are the right derived functors of the functor $\Gamma_\mathfrak{m}$. The functor $\Gamma_\mathfrak{m}$, applied to an R-module M, picks out the largest submodule that can be *built* out of copies of $k = R/\mathfrak{m}$ using the operations of direct sums, extensions, and cokernels. This is because setting

$$\Gamma^n_\mathfrak{m}M = \{x \in M \mid \mathfrak{m}^n.x = 0\},$$

the quotient $\Gamma^n_\mathfrak{m}M/\Gamma^{n-1}_\mathfrak{m}M$ is a direct sum of copies of k, so $\Gamma^n_\mathfrak{m}M$ can be built from k using direct sums and extensions. It follows that

$$\Gamma_\mathfrak{m}M = \bigcup_{n=1}^{\infty} \Gamma^n_\mathfrak{m}M$$

sits in a short exact sequence

$$0 \to \bigoplus_{n=1}^{\infty} \Gamma^n_{\mathfrak{m}} M \to \bigoplus_{n=1}^{\infty} \Gamma^n_{\mathfrak{m}} M \to \Gamma_{\mathfrak{m}} M \to 0,$$

where the first nonzero map sends an element in $\Gamma^n_{\mathfrak{m}} M$ to itself minus its image in $\Gamma^{n+1}_{\mathfrak{m}} M$.

Now consider the case of an injective module. The classification of injective modules over a Noetherian commutative ring states that every injective is a direct sum of indecomposable injectives, and that the indecomposable injectives are precisely the injective hulls $I_{\mathfrak{p}}$ of the modules $R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in R. Applying $\Gamma_{\mathfrak{m}}$ picks out the copies of $I_{\mathfrak{m}}$ and throws away the summands $I_{\mathfrak{p}}$ with $\mathfrak{p} \neq \mathfrak{m}$.

If M is an R-module, considered as an object in $\mathbf{D}(R)$ concentrated in degree zero, then M is equivalent to an injective resolution

$$0 \to I^0 \to I^1 \to I^2 \to \cdots$$

of M. So the complex $\mathbb{R}\Gamma_{\mathfrak{m}} M$, namely

$$0 \to \Gamma_{\mathfrak{m}} I^0 \to \Gamma_{\mathfrak{m}} I^1 \to \Gamma_{\mathfrak{m}} I^2 \to \cdots$$

comes with a map $\mathbb{R}\Gamma_{\mathfrak{m}} M \to M$ in $\mathbf{D}(R)$ having the following properties:

(i) $\mathbb{R}\Gamma_{\mathfrak{m}} M$ is built from k, and

(ii) If X is an object in $\mathbf{D}(R)$ built from k and $X \to M$ is a map in $\mathbf{D}(R)$ then there is a unique factorization through $\mathbb{R}\Gamma_{\mathfrak{m}} M$:

$$\begin{array}{ccc} & & \mathbb{R}\Gamma_{\mathfrak{m}} M \\ & \nearrow & \downarrow \\ X & \longrightarrow & M. \end{array}$$

We use this as the basis for the following definitions. Let R be a DGA (or an A_∞ algebra or an $\mathbb{S}$-algebra) and let $\mathbf{D}(R)$ be its derived category.

Definition 4.1. If X is an object in $\mathbf{D}(R)$, we say that Y is *built* from X if it is in the smallest triangulated subcategory of $\mathbf{D}(R)$ containing X and closed under isomorphism, direct sums and direct summands. If only *finite* direct sums are used, then Y is *finitely built* from X.

A map $U \to V$ is an X-*equivalence* if $\mathbb{R}\mathrm{Hom}_R(X, U) \to \mathbb{R}\mathrm{Hom}_R(X, V)$ is an isomorphism. Y is X-*cellular* if every X-equivalence $U \to V$ induces an isomorphism $\mathbb{R}\mathrm{Hom}_R(Y, U) \to \mathbb{R}\mathrm{Hom}_R(Y, V)$.

Theorem 4.2. *An object Y is X-cellular if and only if it is built from X.*

For any X and Y, there exists an X-cellular object $\mathsf{Cell}_X(Y)$ together with an X-equivalence $\mathsf{Cell}_X(Y) \to Y$. This is unique up to canonical isomorphism in $\mathbf{D}(R)$, and is called the *X-cellular approximation* to Y.

In terms of these definitions, if R is a Noetherian local ring, regarded as a DGA concentrated in degree zero, then

$$\mathbb{R}\Gamma_{\mathfrak{m}}M \cong \mathsf{Cell}_k M.$$

So in general, if $R \to k$ is an augmented DGA, we can regard $\mathsf{Cell}_k M$ as a "local cohomology object" for M in the derived category.

Definition 4.3. Let $R \to k$ be an augmented DGA. We say that k is *proxy-small* if there is an object K in $\mathbf{D}(R)$, called a *Koszul complex*, such that K is finitely built from R, K is finitely built from k, and k is built from K.

Example 4.4. If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, and $k = R/\mathfrak{m}$, then k is proxy-small, and the usual Koszul complex for a system of parameters in $\mathfrak{m}$ can be used as the Koszul complex K in the definition above.

The following is Proposition 2.10 of [7], which gives a model for $\mathsf{Cell}_k(X)$ in case k is proxy-small.

Proposition 4.5. *Suppose that k is proxy-small, and set $\mathcal{E} = \mathbb{R}\mathrm{Hom}_R(k,k)$. Then for any X in $\mathbf{D}(R)$, the natural map*

$$\mathbb{R}\mathrm{Hom}_R(k,X) \overset{\mathbb{L}}{\otimes}_{\mathcal{E}} k \to X$$

is a k-cellular approximation to X.

We can regard the map described in the Proposition above as the counit of an adjunction

$$\mathbf{D}(R) \underset{-\overset{\mathbb{L}}{\otimes}_{\mathcal{E}} k}{\overset{\mathbb{R}\mathrm{Hom}_R(k,-)}{\rightleftarrows}} \mathbf{D}(\mathcal{E}^{\mathrm{op}}) .$$

This adjunction gives a close relationship between the two categories $\mathbf{D}(R)$ and $\mathbf{D}(\mathcal{E})$.

Example 4.6. Let $R = kG$. Then $\mathcal{E}$ is the differential graded algebra $\mathcal{R} = \mathrm{End}_{kG}(P(k))$ discussed in the previous lecture. This DGA is quasiisomorphic to $C^*(BG;k)$, and so we are relating $\mathbf{D}(kG)$ with $\mathbf{D}(C^*(BG;k))$.

If G is a finite p-group and k has characteristic p, then every object in $\mathbf{D}(kG)$ is built from k. It follows that the adjunction above gives an embedding of $\mathbf{D}(kG)$ in $\mathbf{D}(C^*(BG;k))$.

Double Centralizers. If $R \to k$ is an augmented DGA, and $\mathcal{E} = \mathbb{R}\mathrm{Hom}_R(k,k)$, we define the *double centralizer* of R to be $\hat{R} = \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(k,k)$. There is an obvious homomorphism $R \to \hat{R}$ given by left multiplication. We say that R is *dc-complete* if $R \to \hat{R}$ is a quasiisomorphism.

Example 4.7 (Proposition 9.18 of [7])**.** If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$ then $\hat{R}$ is the $\mathfrak{m}$-adic completion $R^\wedge_\mathfrak{m} = \varprojlim_n R/\mathfrak{m}^n$.

Example 4.8 (§3 of [7])**.** If G is a finite p-group then kG is dc-complete. If G is any finite group (or more generally, a compact Lie group) then $C^*(BG;k)$ is dc-complete. But the double centralizer of kG is $C_*(\Omega(BG^\wedge_p);k)$, which is only equivalent to kG if G is a p-group (in the case of a compact Lie group, it's equivalent to $C_*(G;k)$ if $\pi_0(G)$ is a finite p-group).

We can now "explain" why $C^*(BG;k)$ is Gorenstein.

Proposition 4.9 (Proposition 6.5 of [7])**.** *Suppose that $R \to k$ is an augmented DGA, and set $\mathcal{E} = \mathbb{R}\mathrm{Hom}_R(k,k)$. If R is dc-complete and k is proxy-regular, then R is Gorenstein with shift a if and only if $\mathcal{E}$ is Gorenstein with shift a.*

The proof is based on the identities

$$\mathbb{R}\mathrm{Hom}_R(k,R) \cong \mathbb{R}\mathrm{Hom}_R(k,\mathbb{R}\mathrm{Hom}_{\mathcal{E}}(k,k)) \cong \mathbb{R}\mathrm{Hom}_{R\otimes_k\mathcal{E}}(k \overset{\mathbb{L}}{\otimes}_k k, k)$$

$$\mathbb{R}\mathrm{Hom}_{\mathcal{E}}(k,\mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(k,\mathbb{R}\mathrm{Hom}_R(k,k)) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{E}\otimes_k R}(k \overset{\mathbb{L}}{\otimes}_k k, k).$$

If G is a finite p-group, then the conditions of Proposition 4.9 are satisfied. The group algebra kG is self-injective, and so it is Gorenstein with shift zero. So by the Proposition, $C^*(BG;k)$ is also Gorenstein with shift zero.

If G is a finite group but not a p-group, the argument is more subtle, because kG is not dc-complete. One must work via an embedding of G into $SU(n)$. Then $C^*(BSU(n);k)$ is dc-complete and Gorenstein with shift n^2-1, and $C^*(SU(n)/G;k)$ is Gorenstein with shift $-(n^2-1)$. One then uses the fibration

$$SU(n)/G \to BG \to BSU(n)$$

to show that $C^*(BG;k)$ is Gorenstein with shift zero.

There is another model for $\mathsf{Cell}_k(X)$, which explains the spectral sequence (3.2). This is analogous to the theorem of Grothendieck mentioned in Lecture 2, stating that $H^{s,t}_\mathfrak{m} M \cong (R^s\Gamma_\mathfrak{m} M)^t$. Let R be an E_∞ DGA (plus suitable technical conditions that can be found in [7]). If x is an element of $H^i(R)$, we can represent x by a cocycle in R and look at right multiplication by x as a morphism $R \xrightarrow{x} \Sigma^i R$. We complete to a triangle in $\mathbf{D}(R)$:

$$K(x) \to R \xrightarrow{x} \Sigma^i R.$$

Similarly, if we use x^m, we write

$$K_m(x) \to R \xrightarrow{x^m} \Sigma^{im} R,$$

and

$$K_\infty(x) = \underset{\substack{\longrightarrow \\ m}}{\mathrm{Hocolim}}\, K_m(x).$$

The diagram

$$\begin{array}{ccccc}
K(x) & \longrightarrow & R & \xrightarrow{x} & \Sigma^i R \\
\downarrow & & \| & & \downarrow{\scriptstyle x} \\
K_2(x) & \longrightarrow & R & \xrightarrow{x^2} & \Sigma^{2i} R \\
\downarrow & & \| & & \downarrow{\scriptstyle x} \\
K_3(x) & \longrightarrow & R & \xrightarrow{x^3} & \Sigma^{3i} R \\
\downarrow & & \| & & \downarrow{\scriptstyle x} \\
\vdots & & \vdots & & \vdots
\end{array}$$

and the exactness of Hocolim shows that there is a triangle

$$K_\infty(x) \to R \to R[x^{-1}].$$

If $x_1, \ldots, x_r$ is a homogeneous sequence of parameters for M in H^*R then

$$K_\infty(x_1) \overset{\mathbb{L}}{\otimes}_R \cdots \overset{\mathbb{L}}{\otimes}_R K_\infty(x_r) \overset{\mathbb{L}}{\otimes}_R M$$

is a model for $\mathsf{Cell}_k(M)$. Filtering this stable Koszul complex in a suitable sense is what gives rise to the local cohomology spectral sequence (3.2).

We end by remarking that the relationships between $\mathbf{D}(kG)$ and $\mathbf{D}(C^*(BG;k))$ described above have recently been used in a paper of Benson and Greenlees [5] to prove a conjecture in modular representation theory via algebraic topology. The theorem is too technical to state here.

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