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Applications of the dimension filtration

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APPLICATIONS OF THE DIMENSION FILTRATION

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ABSTRACT. For a finitely generated A-module M we define the dimension filtration $\mathscr{M} = \{M_i\}_{0 \le i \le d}, d = \dim_A M$, where M_i denotes the largest submodule of M of dimension $\le i$. Several properties of this filtration are investigated. In particular, in case the local ring $(A, \mathfrak{m})$ possesses a dualizing complex, then this filtration occurs as the filtration of a spectral sequence related to duality. Furthermore, we call an A-module M a Cohen-Macaulay filtered provided all of the quotient modules M_i/M_{i-1} are either zero or i-dimensional Cohen-Macaulay modules. We describe a few basic properties of these kind of generalized Cohen-Macaulay modules. In the case A possesses a dualizing complex it turns out – as one of the main results – that M is a Cohen-Macaulay filtered A-module if and only if for all $0 \le i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. Furthermore basic properties of Cohen-Macaulay filtered modules with respect to localizations, completion, passing to a non-zero divisor, flat extensions are investigated.

CONTENTS

1. The Dimension Filtration

1.1. **The definitions.** Let $(A,\mathfrak{m})$ denote a local Noetherian ring. Let M be a finitely generated A-module and $d = \dim_A M$. For an integer $0 \le i < d$ let M_i denote the largest submodule of M such that $\dim_A M_i \le i$. Because of the maximal condition of a Noetherian A-module the submodules M_i of M are well-defined. Moreover it follows that $M_{i-1} \subseteq M_i$ for all $1 \le i \le d$.

Definition 1.1. The increasing filtration $\mathscr{M} = \{M_i\}_{0\le i\le d}$ of submodules of M is called the dimension filtration of M. Put $\mathscr{M}_i = M_i/M_{i-1}$ for all $1 \le i \le d$.

Primary decomposition. Note that $M_0 = H^0_{\mathfrak{m}}(M)$, where $H^0_{\mathfrak{m}}(\cdot)$ denotes the section functor with support in $\{\mathfrak{m}\}$.
Let $0 = \cap_{j=1}^n N_j$ denote a reduced primary decomposition of 0 in M. That is, $0 \neq \cap_{j=1, j\neq k}^n N_j$ for all $k = 1,\dots,n$, and N_j is a $\mathfrak{p}_j$-coprimary submodule of M such that the prime ideals $\mathfrak{p}_j$ are pairwise different and $\operatorname{Ass}_A M = \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. Hence $M_0 = \cap_{\dim A/\mathfrak{p}_j>0} N_j$.
Both of these representations of M_0 will be generalized to $M_i, 0 \le i \le d$, in the following.
Annihilation ideals. Let

$$\mathfrak{a}_i = \prod_{\mathfrak{p}\in \operatorname{Ass} M, \dim A/\mathfrak{p} \le i} \mathfrak{p}.$$

In the case that $\{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} \le i\} = \emptyset$ put $\mathfrak{a}_i = A$.

Proposition 1.2. *Let M be a finitely generated A-module. Then*

$$M_i = H^0_{\mathfrak{a}_i}(M) = \cap_{\dim A/\mathfrak{p}_j > i} N_j$$

for all $0 \le i \le d$. Here $0 = \cap_{j=1}^n N_j$ denotes a reduced primary decomposition of 0 in M.

Proof. The equality of the last two modules in the statement follows by easy arguments about the primary decomposition of the zero submodule 0 of M. Now let us prove that $M_i = H^0_{\mathfrak{a}_i}(M)$ for all $0 \le i \le d$. Clearly we have $\operatorname{Supp} H^0_{\mathfrak{a}_i}(M) = \operatorname{Supp} M \cap V(\mathfrak{a}_i)$. Therefore it follows that $M_i \subseteq H^0_{\mathfrak{a}_i}(M)$ because any element of M_i is annihilated by an ideal of dimension $\le i$. By the maximality of M_i this proves the equality. □

The previous result provides a stratification of the associated prime ideals of M in terms of those of M_i and $\mathscr{M}_i$ respectively.

Corollary 1.3. *Let $\mathscr{M} = \{M_i\}_{0\le i\le d}$ denote the dimension filtration of M. Then*

a) $\operatorname{Ass}_A M_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} \le i\}$,
b) $\operatorname{Ass}_A M/M_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} > i\}$, *and*
c) $\operatorname{Ass}_A \mathscr{M}_i = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = i\}$

for all $0 \le i \le d$.

Proof. The two first equalities are obviously true by view of 1.2. Note that

$$\mathrm{Ass}_A H^0_{\mathfrak{a}_i}(M) = \{\mathfrak{p} \in \mathrm{Ass}_A M \mid \mathfrak{p} \in V(\mathfrak{a}_i)\}.$$

The third equality is a consequence of the embedding $\mathscr{M}_i \subseteq M/M_{i-1}$ and the short exact sequence

$$0 \to M_{i-1} \to M_i \to \mathscr{M}_i \to 0.$$

Here we use the containment relation

$$\mathrm{Ass}_A M_i \subseteq \mathrm{Ass}_A M_{i-1} \cup \mathrm{Ass}_A \mathscr{M}_i$$

for the associated prime ideals of the corresponding modules. □

Remark. In a certain sense the quotients $\mathscr{M}_i, 0 \leq i \leq d$, of the dimension filtration $\mathscr{M} = \{M_i\}_{0 \leq i \leq d}$ of M are a measure for the unmixedness of M. Note that the A-module M is unmixed if

$$\dim A/\mathfrak{p} = \dim_A M \text{ for all } \mathfrak{p} \in \mathrm{Ass}_A M.$$

In this case $\mathscr{M}_i = 0$ for all $i < \dim_A M = d$ and $M_d = M$. So the filtration is discrete in the case M is unmixed.
More general let $\mathscr{M} = \{M_i\}_{0 \leq i \leq d}$ be the dimension filtration of M. Then $M_i = 0$ for all $i < \operatorname{depth}_A M$. This follows by 1.3 and the fact

$$\operatorname{depth}_A M \leq \dim A/\mathfrak{p} \text{ for all } \mathfrak{p} \in \mathrm{Ass}_A M,$$

see [M, Theorem 17.2] for this inequality.

1.2. **A Permanence Property.**

Proposition 1.4. *Let $\mathscr{M} = \{M_i\}_{0 \leq i \leq d}$ be the dimension filtration of a finitely generated A-module M. Suppose that* $\mathrm{Supp}_A M$ *is a catenary subset of* $\mathrm{Spec} A$. *Let* $\mathfrak{p} \in \mathrm{Supp} M$ *denote a prime ideal. Define*

$$M'_i = M_{i+\dim A/\mathfrak{p}} \otimes_A A_\mathfrak{p} \textit{ for all } 0 \leq i \leq \dim_{A_\mathfrak{p}} M_\mathfrak{p} = t.$$

Then $\mathscr{M}' = \{M'_i\}_{0 \leq i \leq t}$ is the dimension filtration of the $A_\mathfrak{p}$-module $M_\mathfrak{p}$.

Proof. First we mention that there is the bound

$$\dim_A M'_i \leq (i + \dim A/\mathfrak{p}) - \dim A/\mathfrak{p} = i$$

for all $i \in \mathbb{Z}$. Next we recall the following statement about associated prime ideals

$$\mathrm{Ass}_{A_\mathfrak{p}} M_\mathfrak{p} = \{\mathfrak{q}A_\mathfrak{p} \mid \mathfrak{q} \in \mathrm{Ass}_A M, \mathfrak{q} \subseteq \mathfrak{p}\},$$

see [M, Theorem 6.2]. Now let $0 = \cap_{j=1}^n N_j$ be a reduced primary decomposition of 0 in M, where N_j is $\mathfrak{q}_j$-coprimary. Suppose that $\mathfrak{q}_j \subseteq \mathfrak{p}$ for all $j = 1, \ldots, m$ and $\mathfrak{q}_j \not\subseteq \mathfrak{p}$ for all $j = m+1, \ldots, n$. Then $0 = \cap_{j=1}^m (N_j \otimes_A A_\mathfrak{p})$ is a reduced primary decomposition of 0 in $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Therefore, by view of 1.2, it yields that

$$(M_\mathfrak{p})_i = \cap_{\dim A_\mathfrak{p}/\mathfrak{q}_j A_\mathfrak{p} > i} (N_j \otimes_A A_\mathfrak{p}).$$

Moreover by the localization of $M_{i+\dim A/\mathfrak{p}}$ we get the following equality

$$M'_i = \cap_{\dim A/\mathfrak{q}_j > i + \dim A/\mathfrak{p}} (N_j \otimes_A A_\mathfrak{p}).$$

Because $\operatorname{Supp}_A M$ is supposed to be a catenary subset of $\operatorname{Spec} A$ we get that

$$\dim A/\mathfrak{q}_j = \dim A/\mathfrak{p} + \dim A_{\mathfrak{p}}/\mathfrak{q}_j A_{\mathfrak{p}}.$$

First this proves that $d = t + \dim A/\mathfrak{p}$. Because of the above statement about the associated prime ideals it shows finally that $M'_i = (M_{\mathfrak{p}})_i$ for all $0 \leq i \leq t$, as required. □

In the following we consider a variation of the notion of a system of parameters of an A-module M.

1.3. **Distinguished System of Parameters.**

Definition 1.5. Let $\underline{x} = x_1, \ldots, x_d, d = \dim_A M$, denote a system of parameters of M. Then $\underline{x} = x_1, \ldots, x_d$ is called a distinguished system of parameters of M provided $(x_{i+1}, \ldots, x_d)M_i = 0$ for all $i = 0, \ldots, d-1$.

In the next result let us prove the existence of distinguished systems of parameters of an A-module M.

Lemma 1.6. *Any finitely generated A-module M admits a distinguished system of parameters.*

Proof. First we show the existence of a parameter x_d of M such that $x_d M_i = 0$ for all $i = 0, \ldots, d-1$. To this end note that $\dim_A M_i \leq i < d$ for all $i = 0, \ldots, d-1$. Put $\mathfrak{b} = \prod_{i=0}^{d-1} \operatorname{Ann}_A M_i$. Then $\mathfrak{b} \not\subseteq \mathfrak{p}$ for any associated prime ideal $\mathfrak{p} \in \operatorname{Ass}_A M$ with $\dim A/\mathfrak{p} = d$. Therefore there is an element $x_d \in \mathfrak{b}$ and $x_d \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A M$ with $\dim A/\mathfrak{p} = d$. Whence x_d is a parameter with the desired property. Now pass to the factor module $M/x_d M$ and choose a parameter x_{d-1} of $M/x_d M$ such that $x_{d-1} M_i = 0$ for all $i = 0, \ldots, d-2$. Then an induction finishes the proof of the claim. □

It turns out that whenever $\underline{x} = x_1, \ldots, x_d$ is a distinguished system of parameters of M, the elements $x_1, \ldots, x_i$ generate an ideal of definition of $\mathscr{M}_i$. This follows since $M_i/\underline{x}M_i$ is an A-module of finite length. Therefore, whenever $\mathscr{M}_i \neq 0$, then $x_1, \ldots, x_i$ is a system of parameters of $\mathscr{M}_i$.

Lemma 1.7. *A system of parameters $\underline{x} = x_1, \ldots, x_d$ of M is a distinguished system of parameters if and only if $M_i = 0 :_M (x_{i+1}, \ldots, x_d)$ for $i = 0, \ldots, d-1$.*

Proof. Let $\underline{x} = x_1, \ldots, x_d$ denote a system of parameters of M such that

$$M_i = 0 :_M (x_{i+1}, \ldots, x_d) \text{ for all } i = 0, \ldots, d-1.$$

Then $(x_{i+1}, \ldots, x_d)M_i = 0$, i.e. $\underline{x}$ is a distinguished system of parameters. Conversely let $\underline{x}$ be a distinguished system of parameters. Then

$$M_i \subseteq 0 :_M (x_{i+1}, \ldots, x_d) \text{ for all } i = 0, \ldots, d-1$$

as follows by the definition. Moreover there is the following expression for the associated prime ideals

$$\operatorname{Ass}_A(0 :_M (x_{i+1}, \ldots, x_d)) = \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \mathfrak{p} \in V(x_{i+1}, \ldots, x_d)\}.$$

Let $\mathfrak{p}$ denote an associated prime ideal of $0 :_M (x_{i+1},\ldots,x_d)$. Then we obtain

$$\mathfrak{p} \in \operatorname{Supp}_A M/(x_{i+1},\ldots,x_d)M \text{ and } \dim A/\mathfrak{p} \le d-(d-i) = i.$$

That is, $\dim_A(0 :_M (x_{i+1},\ldots,x_d)) \le i$. Because of the maximality of M_i the equality $M_i = 0 :_M (x_{i+1},\ldots,x_d)$ follows now. □

2. A Supplement to Duality

2.1. **Dualizing complexes.** Let $(A,\mathfrak{m})$ denote a local ring possessing a dualizing complex $D^\cdot_A$. Refer to [H, Chapter V, §2] or to [S3, 1.2] for basic results about dualizing complexes. Note that the natural homomorphism of complexes

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M,D^\cdot_A),D^\cdot_A)$$

induces an isomorphism in cohomlogy for any finitely generated A-module M.

Moreover there is an integer $l \in \mathbb{Z}$ such that

$$\operatorname{Hom}_A(k,D^\cdot_A) \simeq k[l],$$

where $k = A/\mathfrak{m}$ denotes the residue field of A. Without loss of generality assume that $l = 0$. Then the dualizing complex $D^\cdot_A$ has the property

$$D_A^{-i} \simeq \oplus_{\mathfrak{p}\in \operatorname{Spec} A, \dim A/\mathfrak{p}=i} E_A(A/\mathfrak{p}),$$

where $E_A(A/\mathfrak{p})$ denotes the injective hull of $A/\mathfrak{p}$ as A-module. Therefore $D^i_A = 0$ for $i < -\dim A$ and $i > 0$.

Definition 2.1. Let M denote a finitely generated A-module and $d = \dim_A M$. For an integer $i \in \mathbb{Z}$ define

$$K^i(M) := H^{-i}(\operatorname{Hom}_A(M,D^\cdot_A)).$$

The module $K(M) := K^d(M)$ is called the canonical module of M. For $i \ne d$ the modules $K^i(M)$ are called the modules of deficiency of M. Note that $K^i(M) = 0$ for all $i < 0$ or $i > d$.

By the local duality theorem

$$H^i_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_A(K^i(M),E), i \in \mathbb{Z},$$

where $E = E_A(A/\mathfrak{m})$. Whence the modules of deficiencies of M measure the deviation of M from being a Cohen-Macaulay module. The canonical module $K(M)$ of M is a Cohen-Macaulay module provided M is a Cohen-Macaulay module.

For a finitely generated A-module M and an integer $i \in \mathbb{N}$ let

$$(\operatorname{Ass}_A M)_i = \{\mathfrak{p} \in \operatorname{Ass}_A M \mid \dim A/\mathfrak{p} = i\}.$$

For the proof of the next result see [S1, 3.1] and [S3, Lemma 1.9].

Proposition 2.2. *Let M denote a d-dimensional A-module. Then the following results are true:*

a) $\dim_A K^i(M) \le i$ *for all* $0 \le i < d$ *and* $\dim_A K(M) = d$.

b) $\mathrm{Ass}_A K(M) = (\mathrm{Ass}_A M)_d$.
c) $(\mathrm{Ass}_A K^i(M))_i = (\mathrm{Ass}_A M)_i$ *for all* $0 \le i < d$.
d) *Let M be a Cohen-Macaulay module. Then $K(M)$ is also a Cohen-Macaulay module.*

The induced homomorphisms of the cohomology of the natural map

$$M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

are isomorphisms whenever M is a finitely generated R-module.

2.2. **Dimension filtration and Duality.**

Lemma 2.3. *Let M denote a finitely generated A-module. Let $\mathfrak{p} \in \mathrm{Supp}_A M$ be a prime ideal with $t = \dim A/\mathfrak{p}$. Then there are the following isomorphisms*

$$K^i(K^j(M)) \otimes_A A_{\mathfrak{p}} \simeq K^{i-t}(K^{j-t}(M \otimes_A A_{\mathfrak{p}}))$$

for any pair $(i, j) \in \mathbb{Z}^2$.

Proof. First note that there is an isomorphism of dualizing complexes

$$D_A^{\cdot} \otimes_A A_{\mathfrak{p}} \simeq D_{A_{\mathfrak{p}}}^{\cdot}[t],$$

see e.g. [H, Chapter V, Proposition 7.1]. Now by the definition of the K^i's write

$$K^i(K^j(M)) \simeq H^{-i}(\mathrm{Hom}_A(H^{-j}(\mathrm{Hom}_A(M, D_A^{\cdot})), D_A^{\cdot})).$$

The localization functor $\cdot \otimes_A A_{\mathfrak{p}}$ is exact, i.e. it commutes with cohomology. Moreover let X denote a bounded complex of A-modules whose cohomology modules are finitely generated A-modules. Then there is the following isomorphism of complexes

$$\mathrm{Hom}_A(X, D_A^{\cdot}) \otimes_A A_{\mathfrak{p}} \simeq \mathrm{Hom}_{A_{\mathfrak{p}}}(X \otimes_A A_{\mathfrak{p}}, D_{A_{\mathfrak{p}}}^{\cdot})[t],$$

see [H, Chapter II]. Putting together all of these ingredients the statement of the proposition follows now. □

In order to compute the homology of $\mathrm{Hom}_A(\mathrm{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$ there is the following spectral sequence

$$E_1^{pq} = H^q(\mathrm{Hom}_A(\mathrm{Hom}_A(M, D_A^{\cdot}), D_A^p)),$$

see [E, Appendix 3, Part II] or [W, Section 5].
Because D_A^p is an injective A-module the corresponding E_2-term has the following form

$$E_2^{pq} = H^p(\mathrm{Hom}_A(H^{-q}(\mathrm{Hom}_A(M, D_A^{\cdot})), D_A^{\cdot})).$$

With regard to our previous notation it follows that $E_2^{pq} = K^{-p}(K^q(M))$.

Fact. Let M denote a finitely generated A-module with $d = \dim_A M$. Let us return to the above spectral sequence. Consider the stage $p + q = 0$, the only place in which non-zero cohomology occurs. Then the limit terms $E_\infty^{p,-p}, -d \le p \le 0$, are the quotients of a filtration

$$F^0 \subseteq F^{-1} \subseteq \ldots \subseteq F^{-d+1} \subseteq F^{-d} = M$$

of M. That is we have $F^p/F^{p+1} \simeq E_\infty^{p,-p}$ for all $-d \leq p \leq 0$.

Question. What is the relation of the filtration $\mathscr{F} = \{F^{-i}\}_{0\leq i\leq d}$ to the dimension filtration of M?

Theorem 2.4. *Let $\mathscr{M} = \{M_i\}_{0\leq i\leq d}$ be the dimension filtration of M. Then it follows $M_i = F^{-i}$ for all $0 \leq i \leq d$.*

It is worth to remark that in general the limit terms $E_\infty^{p,-p}$ of the spectral sequence considered above do not agree with $E_2^{p,-p} \simeq K^{-p}(K^{-p}(M))$. It would be interesting to find an explicit description of these modules.

3. Cohen-Macaulay filtered Modules

3.1. **The Definitions.** Let $(A,\mathfrak{m})$ be local Noetherian ring. Let M denote a finitely generated A-module. Let $\mathscr{M} = \{M_i\}_{0\leq i\leq d}$ denote the dimension filtration.

Definition 3.1. A finitely generated A-module M is called a Cohen-Macaulay filtered module (cf. [S2]) (or sequentially Cohen-Macaulay module), whenever $\mathscr{M}_i = M_i/M_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $0 \leq i \leq \dim_A M$.

Note that any Cohen-Macaulay module is a Cohen-Macaulay filtered module. This follows because under this assumption $M_i = 0$ for all $i < \dim_A M$. Conversely an unmixed Cohen-Macaulay filtered module is a Cohen-Macaulay module. Let M be an A-module such that $\operatorname{depth}_A M = 0$ and $M/H^0_{\mathfrak{m}}(M)$ is a Cohen-Maculay module. Then M is a Cohen-Macaulay filtered module.

Definition 3.2. Let M denote a finitely generated A-module with $d = \dim_A M$. An increasing filtration $\mathscr{C} = \{C_i\}_{0\leq i\leq d}$ of M is called a Cohen-Macaulay filtration whenever $M = C_d, d = \dim_A M$, and $\mathscr{C}_i = C_i/C_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module for all $1 \leq i \leq d$.

Proposition 3.3. *Let $\mathscr{C} = \{C_i\}_{0\leq i\leq d}$ be Cohen-Macaulay filtration of M. Then $\mathscr{C}$ coincides with the dimension filtration.*

3.2. **Approximately Cohen-Macaulay Modules.** Let $0 = \cap_{j=1}^n N_j$ denote a reduced primary decomposition. Put $u_M(0) = \cap_{\dim A/\mathfrak{p}_j = d} N_j$.

Definition 3.4. A finitely generated A-module $M, d = \dim_A M$, is called an approximately Cohen-Macaulay module whenever $M/u_M(0)$ is a Cohen-Macaulay module and $\operatorname{depth}_A M \geq d-1$.

This is the extension of the notion of an approximately Cohen-Macaulay ring introduced by S. Gôto, see [G]. Note that a Cohen-Macaulay module is always an approximately Cohen-Macaulay module. Next let us describe the relation of this notion to that of CMF modules.

Proposition 3.5. *Let M be a finitely generated A-module. Then M is approximately Cohen-Macaulay if and only if M is a Cohen-Macaulay filtered module and* $\operatorname{depth}_A M \geq \dim_A M - 1$.

Proof. First let M be an approximately Cohen-Macaulay module. Put $d = \dim_A M$. By [M, Theorem 17.2] it follows that

$$d-1 \leq \operatorname{depth}_A M \leq \dim A/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass}_A M.$$

Therefore $M_i = 0$ for $i = 0, \dots, d-2$ and $M_{d-1} = u_M(0)$, see 1.2. Now consider the short exact sequence

$$0 \to M_{d-1} \to M \to M/M_{d-1} \to 0.$$

Because M is approximately Cohen-Macaulay it follows that M/M_{d-1} is a d-dimensional Cohen-Macaulay module and $\operatorname{depth}_A M \geq d-1$. So the short exact sequence implies $\operatorname{depth}_A M_{d-1} \geq d-1$. Because of $\dim_A M_{d-1} \leq d-1$ it turns out that M_{d-1} is either zero or a $(d-1)$-dimensional Cohen-Macaulay module.
The reverse statement follows the same line of reasoning. Hence we omit the details. □

3.3. **Properties of CMF.** Now there are a few results on permanence properties of Cohen-Macaulay filtered modules. To this end $\hat{A}$ denotes the $\mathfrak{m}$-adic completion of A.

Proposition 3.6. *Let M denote a Cohen-Macaulay filtered A-module. Then the following conditions are satisfied:*

a) $\operatorname{Supp}_A M$ *is a catenary subset of* $\operatorname{Spec} A$.
b) *Let* $\mathfrak{p} \in \operatorname{Supp}_A M$. *Then*

$$\dim A/\mathfrak{p} = \dim \hat{A}/\mathfrak{q} \textit{ for all } \mathfrak{q} \in \operatorname{Ass}_{\hat{A}} \hat{A}/\mathfrak{p}\hat{A},$$

i.e. $A/\mathfrak{p}$ *is formally unmixed for all* $\mathfrak{p} \in \operatorname{Supp}_A M$.

Proof. Because M/M_{d-1} is a Cohen-Macaulay module and

$$\operatorname{Supp}_A M = \operatorname{Supp}_A M/M_{d-1}$$

both of the statements follow. For the first statement see [M, §17]. The second is a consequence of [N, (34.9)]. □

A Cohen-Macaulay filtered ring A possesses a small Cohen-Macaulay module. That is a Cohen-Macaulay module X such that $\operatorname{depth} X = \dim A$. This follows since $A/A_{d-1}, d = \dim A$, is a d-dimensional Cohen-Macaulay module. Consequently for a CMF ring A all the homological conjectures are true.

Lemma 3.7. *Let M denote a finitely generated A-module. Let $x \in \mathfrak{m}$ be an M-regular element. Then M/xM is a CMF module provided M is a CMF module.*

Proof. First note that whenever $x \in \mathfrak{m}$ is an M-regular element, then $M_0 = 0$ and x is also M/M_i-regular as well as $\mathscr{M}_i$-regular for all $i \geq 1$. Here $\mathscr{M} = \{M_i\}_{0 \leq i \leq d}$ denotes the dimension filtration and $\mathscr{M}_i = M_i/M_{i-1}$. In particular it follows that $M_i \cap xM = xM_i$ for all $1 \leq i \leq d$.

Now suppose that M is a CMF module. Let $i \geq 1$. Then

$$\mathscr{M}_i/x\mathscr{M}_i \simeq ((M_i,xM)/xM)/(M_{i-1},xM)/xM)$$

is a $(i-1)$-dimensional Cohen-Macaulay module or zero. Therefore by 3.3 it follows that M/xM is a $(d-1)$-dimensional Cohen-Macaulay filtered module since $\{(M_{i+1},xM/xM)\}_{0\leq i<d}$ is a Cohen-Macaulay filtration. □

The converse is not true as mentioned by N. T. Cuong. To this end consider a 2-dimensional local domain $(A,\mathfrak{m})$ that is not a Cohen-Macaulay ring. So it is not CMF. On the other side $A/xA, x \neq 0$, is as a 1-dimensional ring always a CMF.

In the final part of this section consider the behavior of the CMF property by passing to the completion.

Theorem 3.8. *Let M be a finitely generated A-module. Let M be a CMF A-module. Then $M \otimes_A \hat{A}$ is a CMF $\hat{A}$-module.*

Proof. Let $\mathscr{M} = \{M_i\}_{0\leq i\leq d}$ denote the Cohen-Macaulay filtration of the CMF A-module M. Then $\{M_i \otimes_A \hat{A}\}_{0\leq i\leq d}$ is clearly a Cohen-Macaulay filtration of the $\hat{A}$-module $M \otimes_A \hat{A}$. So by 3.3 $M \otimes_A \hat{A}$ is a CMF module over $\hat{A}$. □

The converse of the above statement is not true in general, see Example 5.1.

3.4. **Towards a Parametric Characterization.**

Lemma 3.9. *Let $\mathscr{M} = \{M_i\}_{0\leq i\leq d}$ denote the dimension filtration of M. Then*

$$L_A(M/\underline{x}M) \leq \sum_{i=0}^{d} L_A(\mathscr{M}_i/(x_1,\ldots,x_i)\mathscr{M}_i)$$

for any distinguished system of parameters $\underline{x} = x_1,\ldots,x_d, d = \dim_A M$, of M.

Proof. For $1 \leq i \leq d$ let us consider the following short exact sequences

$$0 \to M_{i-1} \to M_i \to \mathscr{M}_i \to 0.$$

Tensor it by $A/(x_1,\ldots,x_d)A$. Because of $x_iM_{i-1} = 0, 1 \leq i \leq d$, it induces an exact sequence

$$M_{i-1}/(x_1,\ldots,x_{i-1})M_{i-1} \to M_i/(x_1,\ldots,x_i)M_i \to \mathscr{M}_i/(x_1,\ldots,x_i)\mathscr{M}_i \to 0.$$

Because $\underline{x}$ is a distinguished system of parameters of M the elements $x_1,\ldots,x_i$ generate an ideal of definition of M_i. That is, the A-modules

$$M_i/(x_1,\ldots,x_i)M_i \text{ and } \mathscr{M}_i/(x_1,\ldots,x_i)\mathscr{M}_i, i = 0,\ldots,d,$$

are A-modules of finite length. Therefore

$$L_A(M_i/(x_1,\ldots,x_i)M_i) \leq L_A(M_{i-1}/(x_1,\ldots,x_{i-1})M_{i-1}) + \\ + L_A(\mathscr{M}_i/(x_1,\ldots,x_i)\mathscr{M}_i)$$

for all $i = 1,\ldots,d$. Because of $M = M_d$ a recurrence proves the desired inequality. □

Note that the inequality of 3.9 is also true for any system of parameters $\underline{x} = x_1, \ldots, x_d$ of M. But in this case it might happen that the modules on the right hand side are not of finite length. In this case the estimate is trivially true.

Lemma 3.10. *Let* $\mathscr{M} = \{M_i\}_{0 \leq i \leq d}$ *denote the dimension filtration of a finitely generated A-module* M *with* $d = \dim_A M$ *and* $t = \operatorname{depth}_A M$. *Let* $\underline{x} = x_1, \ldots, x_d$ *be a distinguished system of parameters.*
Suppose that M *is a CMF module. Then the following conditions are satisfied:*

a) $L_A(M/(x_1, \ldots, x_d)M) = \sum_{i=0}^{d} L_A(\mathscr{M}_i/(x_1, \ldots, x_i)\mathscr{M}_i)$.
b) $M/(x_1, \ldots, x_{d-t})M$ *is a t-dimensional Cohen-Macaulay module.*

The converse is true, i.e. the conditions a) and b) imply that M *is a CMF module, provided* $\operatorname{depth}_A M \geq d-1$.

Another partial result is the following slight generalization of [G, Lemma 2.1] for a characterization of approximately Cohen-Macaulay modules.

Proposition 3.11. *Let* M *denote a finitely generated A-module with* $d = \dim_A M$. *Let* $r \in \mathbb{N}$ *denote an integer. Suppose that there is an element* $x \in \mathfrak{m}$ *satisfying the following two conditions:*

a) $M/x^{r+1}M$ *is a* $(d-1)$*-dimensional Cohen-Macaulay module.*
b) $0 :_M x^r = 0 :_M x^{r+1}$.

Then $\operatorname{depth}_A M \geq d-1$ *and* M *is an approximately Cohen-Macaulay module with* $M_{d-1} = 0 :_M x^r$.

Proof. Put $N := 0 :_M x^r = 0 :_M x^{r+1}$. We first claim that $\operatorname{depth}_A M/x^rM \geq d-1$. Suppose the contrary, i.e. $\operatorname{depth}_A M/x^rM =: t < d-1$. Then the short exact sequence

$$0 \to M/(xM,N) \to M/x^{r+1}M \to M/x^rM \to 0$$

implies that $\operatorname{depth}_A M/(xM,N) = t+1$. Because x is an M/N-regular element it follows that

$$\operatorname{depth}_A M/N = t+2 \text{ and } \operatorname{depth}_A M/(x^sM,N) = t+1 \text{ for all } s \geq 1.$$

Therefore the short exact sequence

$$0 \to N \to M/x^sM \to M/(x^sM,N) \to 0,$$

considered for $s = r+1$, provides that $\operatorname{depth}_A N = t+2$. Then the same sequence considered for $s = r$ yields that $\operatorname{depth}_A M/x^rM \geq t+1$, a contradiction.
Therefore M/x^rM is a $(d-1)$-dimensional Cohen-Macaulay module. Now the first of the above short exact sequences proves that $M/(xM,N)$ and therefore also M/N is a Cohen-Macaulay module. Moreover the previous exact sequence considered for $s = r$ provides that N is a Cohen-Macaulay module of dimension $d-1$. By 1.2 this finishes the proof. ☐

Note. Recently Nguyen Tu Cuong (cf. [C]) has found a characterization of CMF modules in terms of multiplicities of a certain system of parameters.

4. A Cohomological Characterization

4.1. **A Preliminary Result.** Now we start the cohomological investigation of CMF modules. To this end at first we need a description of the local cohomology modules of a CMF module.

Lemma 4.1. *Let M denote a CMF module with $\mathcal{M} = \{M_i\}_{0\le i\le d}$ its dimension filtration. Let i denote an integer with $0 \le i \le d$. Then*

$$H^i_{\mathfrak{m}}(M) \simeq H^i_{\mathfrak{m}}(M_i) \simeq H^i_{\mathfrak{m}}(\mathcal{M}_i).$$

In the case A possesses a dualizing complex it follows that $K^i(M) \simeq K^i(\mathcal{M}_i)$ for all $0 \le i \le d$.

Proof. First consider the short exact sequence $0 \to M_{i-1} \to M_i \to \mathcal{M}_i \to 0$. Because of $\dim M_{i-1} \le i-1$ it induces an isomorphism $H^i_{\mathfrak{m}}(M_i) \simeq H^i_{\mathfrak{m}}(\mathcal{M}_i)$. Second for $j < i$ it yields isomorphisms $H^j_{\mathfrak{m}}(M_i) \simeq H^j_{\mathfrak{m}}(M_{i-1})$. Note that $\mathcal{M}_i$ is either zero or an i-dimensional CM module. By induction it follows that

$$H^i_{\mathfrak{m}}(M) \simeq H^i_{\mathfrak{m}}(M_d) \simeq H^i_{\mathfrak{m}}(M_{d-1}) \simeq \ldots \simeq H^i_{\mathfrak{m}}(M_{i+1}) \simeq H^i_{\mathfrak{m}}(M_i),$$

which proves the statement about the local cohomology modules. The rest of the claim for $K^i(M)$ follows by similar arguments using the dualizing complex. □

4.2. **The criterion.** Now we are prepared to prove the main result concerning a characterization of CMF modules in terms of the modules of deficiency $K^i(M), 0 \le i < d$. Moreover there is an additional information about the canonical module.
The equivalence of (i) and (ii) was announced by R. Stanley (cf. [S]). In this form the result was proved in [S2]. Another proof of the equivalence of (i) and (iii) by a different method was published by J. Herzog and E. Sbarra (cf. [HS]).

Theorem 4.2. *Let $(A, \mathfrak{m})$ denote a local ring possessing a dualizing complex $D^{\cdot}_A$. Let M be a finitely generated A-module with $d = \dim_A M$. Then the following conditions are equivalent:*

(i) *M is a CMF A-module.*
(ii) *For all $0 \le i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module.*
(iii) *For all $0 \le i \le d$ the A-modules $K^i(M)$ are either zero or i-dimensional Cohen-Macaulay modules.*

Proof. First suppose that M is a CMF module. Then the dimension filtration $\mathcal{M} = \{M_i\}_{0\le i\le d}$ has the property that for all $0 \le i \le d$ the quotient module $\mathcal{M}_i = M_i/M_{i-1}$ is either zero or an i-dimensional Cohen-Macaulay module. By view of 4.1 it follows that $K^i(M) \simeq K^i(\mathcal{M}_i)$ for all $0 \le i \le d$. Because $\mathcal{M}_i$ is either zero or an i-dimensional Cohen-Macaulay module we have

that $K^i(\mathscr{M}_i)$ is either zero or the canonical module of the i-dimensional Cohen-Macaulay module $\mathscr{M}_i$. But then the canonical module of $\mathscr{M}_i$ is also an i-dimensional Cohen-Macaulay module. So $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. This proves the implication (i) $\Rightarrow$ (ii) as well as (i) $\Rightarrow$ (iii).
In order to prove (iii) $\Rightarrow$ (i) consider the spectral sequence studied in the proof of 2.2. By view of Theorem 2.4 it will be enough to prove that all the quotients $F^p/F^{p+1} \simeq E_\infty^{p,-p}$ are either zero or $(-p)$-dimensional Cohen-Macaulay modules. We first claim that $E_\infty^{p,-p} \simeq E_2^{p,-p}$ for all $-d \leq p \leq 0$. To this end consider the subsequent stages of the spectral sequence

$$E_r^{p-r,-p+r-1} \to E_r^{p,-p} \to E_r^{p+r,-p-r+1}.$$

The term on the left hand side is zero because it is a subquotient of the modules $K^{-p+r}(K^{-p+r-1}(M)) = 0$. Recall that $\dim_A K^{-p+r-1}(M) \leq -p + r - 1$, see 2.2. The right term is a subquotient of $K^{-p-r}(K^{-p-r+1}(M))$. By our assumption we have that $K^{-p-r+1}(M)$ is either zero or an $(-p-r+1)$-dimensional Cohen-Macaulay module. But then the $(-p-r)$-th module of deficiency $K^{-p-r}(K^{-p-r+1}(M))$ is zero. That is, the modules at the right are always zero. But this implies that

$$F^p/F^{p+1} \simeq E_2^{p,-p} \simeq K^{-p}(K^{-p}(M))$$

for all $-d \leq p \leq 0$. We finish the proof by showing that $K^{-p}(K^{-p}(M))$ is either zero or a $(-p)$-dimensional Cohen-Macaulay module. By our assumption $K^{-p}(M)$ is either zero or an $(-p)$-dimensional Cohen-Macaulay module. Therefore $K^{-p}(K^{-p}(M))$ is either zero or – as the canonical module of $K^{-p}(M)$ – also a $(-p)$-dimensional Cohen-Macaulay module. By view of 3.3 this proves the claim of (i).
Finally we have to show that (ii) $\Rightarrow$ (iii). That is, we have to show that the canonical module $K(M) = K^d(M)$ is a Cohen-Macaulay module provided for all $0 \leq i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module. This will be part of another talk. □

Looking at the second part of Theorem 4.2 there is another sufficient criterion for the canonical module $K(M)$ of M being a Cohen-Macaulay module.

5. Faithful Flat Extensions and Examples

5.1. **Nagata's Example.** Let $(A, \mathfrak{m})$ denote a local Noetherian ring. Let M denote a finitely generated A-module.

Example 5.1. Let $(A, \mathfrak{m})$ denote the 2-dimensional local domain considered by M. Nagata in [N, Example 2]. Clearly it is not a Cohen-Macaulay ring. For the multiplicity $e(\mathfrak{m}, A)$ it is shown that $e(\mathfrak{m}, A) = 1$. Therefore it implies that

$$1 = e(\mathfrak{m}, A) = e(\hat{\mathfrak{m}}, \hat{A}) = e(\hat{\mathfrak{m}}, \hat{A}/u_{\hat{A}}(0)).$$

By the view of [N, (40.6)] it yields that $\hat{A}/u_{\hat{A}}(0)$ is a regular local ring, in particular a 2-dimensional Cohen-Macaulay ring. Moreover, $\operatorname{depth} A = \operatorname{depth} \hat{A} = 1$ the ideal $u_{\hat{A}}(0)$ is – considered as an $\hat{A}$-module – a 1-dimensional Cohen-Macaulay module. But this means that $\hat{A}$ is a CMF ring or equivalently an approximately Cohen-Macaulay ring. But this is not true for A. Otherwise A would be a Cohen-Macaulay ring since it is a domain.

5.2. **Passing to the Completion.** Before we shall formulate our next result let us recall the definition of a Cohen-Macaulay filtration, 3.2.
Now let $(A,\mathfrak{m}) \to (B,\mathfrak{n})$ be a faithful flat homomorphism of local rings. Let M be a finitely generated A-module with $d = \dim_A M$. Let $\mathscr{C} = \{C_i\}_{0\le i\le d}$ denote an increasing filtration of M such that $M = C_d$. Let $\mathscr{C}_B = \{(C_B)_i\}_{0\le i\le n}$ denote the induced filtration defined by $(C_B)_i = C_{i+t} \otimes_A B$, where $t = \dim B/\mathfrak{m}B$ denotes the dimension of the fibre ring.

Theorem 5.2. *Let $(A,\mathfrak{m}) \to (B,\mathfrak{n})$ be a faithful flat homomorphism of local rings. Let M be a finitely generated A-module with $d = \dim_A M$. Then the following conditions are equivalent:*

(i) *The filtration $\mathscr{C}$ is a Cohen-Macaulay filtration of M and the fibre ring $B/\mathfrak{m}B$ is a Cohen-Macaulay ring.*
(ii) *The induced filtration $\mathscr{C}_B$ is a Cohen-Macaulay filtration of the B-module $M \otimes_A B$.*

Proof. Let X denote an arbitrary finitely generated A-module. By virtue of [M, Theorem 15.1] and [M, Theorem 23.3] it follows that

$$\dim_B X \otimes_A B = \dim_A X + \dim B/\mathfrak{m}B \text{ and}$$
$$\operatorname{depth}_B X \otimes_A B = \operatorname{depth}_A X + \operatorname{depth} B/\mathfrak{m}B.$$

First of all this proves that $\dim_B X \otimes_A B = d + t$, i.e. $(C_B)_{d+t} = M \otimes_A B$. Now suppose that condition (i) is satisfied. Then the above equalities show that each of the B-modules

$$(C_B)_i/(C_B)_{i-1} \simeq (C_{i-t}/C_{i-1-t}) \otimes_A B$$

are either zero or i-dimensional Cohen-Macaulay modules. The converse follows the same line of reasoning. Hence we omit it. □

Note. The previous result 5.2 does not apply to the example considered in 5.1. In the example there does not exist a Cohen-Macaulay filtration in A, while there is one in $\hat{A}$. The Cohen-Macaulay filtration in $\hat{A}$ does not occur as the extension of a Cohen-Macaulay filtration of A.

5.3. **Examples.**

Example 5.3. a) Let M be a Cohen-Macaulay module. Then M is also a CMF module.
b) Let $(A,\mathfrak{m})$ be a local ring with $d = \dim A$. Let $N_i, i = 0,\ldots,d$, be a family of A-modules such that either $N_i = 0$ or N_i is an i-dimensional Cohen-Macaulay module. Then $M = \oplus_{i=0}^d N_i$ is a CMF module over A. This follows

easily by 3.3 since M admits a filtration $M_i = \oplus_{j=0}^{i} N_j$ such that $M_i/M_{i-1} \simeq N_i, i = 0,\dots,d$, is either zero or an i-dimensional Cohen-Macaulay module.
c) Let $(A,\mathfrak{m})$ denote a local ring. Let M be a finitely generated A-module. Then consider $A \ltimes M$, the idealization of M over A. That is, the additive group of $A \ltimes M$ coincides with the direct sum of the abelian groups A and M. The multiplication is given by

$$(a,m)\cdot(b,n) := (ab, an+bm).$$

Then $A \ltimes M$ is a d-dimensional local ring, see [N, (1.1)] or [BH, 3.3.22] for these and related facts.
Now suppose that $(A,\mathfrak{m})$ is a d-dimensional Cohen-Macaulay ring. Let M be a CMF module with $\dim M = t < d$. Then $A \ltimes M$ is a d-dimensional CMF ring. To this end let $\mathscr{M} = \{M_i\}_{0 \le i \le t}$ denote the dimension filtration of M. Now put

$$R_i = \begin{cases} A \ltimes M & \text{for} \quad i = d, \\ 0 \ltimes M & \text{for} \quad i = t+1,\dots,d-1, \text{ and} \\ 0 \ltimes M_i & \text{for} \quad i = 0,\dots,t. \end{cases}$$

Then $\{R_i\}_{0 \le i \le d}$ is a filtration of $R = A \ltimes M$ such that $R_d = A \ltimes M$ and R_i/R_{i-1} is either zero or an i-dimensional Cohen-Macaulay module. Note that

$$R_i/R_{i-1} \simeq \begin{cases} A & \text{for} \quad i = d, \\ 0 & \text{for} \quad i = t+1,\dots,d-1, \text{ and} \\ M_i/M_{i-1} & \text{for} \quad i = 1,\dots,t. \end{cases}$$

By view of 3.3 this proves the claim.
d) Let $A[[x]]$ denote the formal power series ring in one variable x over the local ring $(A,\mathfrak{m})$. Then a finitely A-module M is a CMF module if and only if $M[[x]]$ is a CMF module over the ring $A[[x]]$.
e) Let M be a finitely generated A-module such that $H^i_{\mathfrak{m}}(M), i \neq \dim_A M$, is a finitely generated A-module. Then M is a CMF module if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $0 < i < \dim_A M$. In particular, under these circumstances M is a Cohen-Macaulay module if and only if M is a CMF module with $\operatorname{depth}_A M > 0$.
f) Every 1-dimensional A-module M is a CMF module. Therefore for any d-dimensional Cohen-Macaulay ring with $d \ge 2$ and a 1-dimensional A-module M the idealization $A \ltimes M$ is a d-dimensional CMF ring.

Question. It would be of some interest to understand the descend of the CMF property from $M \otimes_A \hat{A}$ to M. What are sufficient condition on A? The Example 5.1 does not has Cohen-Macaulay formal fibres. Is it enough to suppose that the homomorphism $A \to \hat{A}$ has Cohen-Macaulay formal fibres ?

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