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Combinatorial Commutative Algebra

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any term order, because the standard monomials for the initial submodule $\mathsf{in}(N)$ form a vector space basis for both $\mathcal{F}/N$ and $\mathcal{F}/\mathsf{in}(N)$. Since $\mathcal{F}/\mathsf{in}(N)$ is a direct sum of multigraded translates of monomial quotients of S, and K-polynomials as well as Hilbert series are additive on direct sums, it suffices by Lemma 8.40 to know that K-polynomials are preserved under passing to quotients by initial submodules. This last bit is Theorem 8.36. □

Example 8.42 Putting the Hilbert series in Example 8.38 over a common denominator yields the $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-graded K-polynomial of M, namely $\mathcal{K}(M; s, t) = s^{-2}t(1 - st)(1 - s) + 1 - s^{-2}t = 1 - s^{-1}t - s^{-1}t^2 + t^2$. ◇

Two open problems concerning Hilbert series and K-polynomials are

(i) how to represent the kernel of the homomorphism of abelian groups {Hilbert series of modest modules} $\to$ {K-polynomials}; and

(ii) how to write down Hilbert series for immodest modules.

The map {Hilbert series} $\to$ {K-polynomials} in (i) is never injective when the grading is nonpositive: there can be many modest modules, with very different-looking Hilbert series, that nonetheless have equal K-polynomials. Such ambiguity does not occur in the positively graded case by Theorem 8.20, because the rational functions that represent positively graded Hilbert series lie in an ambient power series ring that is an integral domain.

8.5 Multidegrees

We saw in Part I that free resolutions in the finest possible multigrading are essentially combinatorial in nature. For coarser gradings, in contrast, combinatorial data can usually be extracted only after a certain amount of condensation. Although K-polynomials can sometimes suffice for this purpose, even they might end up carrying an overload of information. In such cases we prefer a multigraded generalization of the degree of a $\mathbb{Z}$-graded ideal. The characterizing properties of these *multidegrees* give them enormous potential to encapsulate finely textured combinatorics, as we shall see in the cases of Schubert and quiver polynomials in Chapters 15–17.

The ordinary $\mathbb{Z}$-graded degree of a module is usually defined via the leading coefficient of its Hilbert polynomial (see Exercise 8.13). However, as Hilbert polynomials do not directly extend to multigraded situations, we must instead rely on a different characterization. In the next definition, the symbol $\mathrm{mult}_{\mathfrak{p}}(M)$ denotes the *multiplicity* of a module M at the prime $\mathfrak{p}$, which by definition equals the length of the largest finite-length submodule in the localization of M at $\mathfrak{p}$ [Eis95, Section 3.6].

Definition 8.43 Let S be a polynomial ring multigraded by a subgroup $A \subseteq \mathbb{Z}^d$ (so in particular, A is torsion-free). Let $\mathcal{C} : M \mapsto \mathcal{C}(M; \mathbf{t})$ be a function from finitely generated graded S-modules to $\mathbb{Z}[t_1, \ldots, t_d]$.

1. The function $\mathcal{C}$ is **additive** if for all modules M,
$$\mathcal{C}(M;\mathbf{t}) \quad = \quad \sum_{k=1}^{r} \operatorname{mult}_{\mathfrak{p}_k}(M) \cdot \mathcal{C}(S/\mathfrak{p}_k;\mathbf{t}),$$
where $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ are the maximal dimensional associated primes of M.
2. The function $\mathcal{C}$ is **degenerative** if, whenever $M = \mathcal{F}/K$ is a graded free presentation and $\mathsf{in}(M) := \mathcal{F}/\mathsf{in}(K)$ for some term or weight order,
$$\mathcal{C}(M;\mathbf{t}) \quad = \quad \mathcal{C}(\mathsf{in}(M);\mathbf{t}).$$

The definition of additivity implicitly uses the fact (Proposition 8.11) that $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ are themselves multigraded, given that M is. This subtlety is why we restrict to torsion-free gradings when dealing with multidegrees.

Readers who have previously seen the notion of $\mathbb{Z}$-graded degree will recognize it as being both additive and degenerative. The main goal of this section is to produce a multigraded analogue of degree that also satisfies these properties. The very fact that $\mathbb{Z}$-graded degrees already satisfy them means that we will need extra information to distinguish multidegrees from the usual $\mathbb{Z}$-graded degree. To that end, we have the following uniqueness statement, our main result of the section.

Theorem 8.44 *Exactly one additive degenerative function $\mathcal{C}$ satisfies*
$$\mathcal{C}(S/\langle x_{i_1},\ldots,x_{i_r}\rangle;\mathbf{t}) \quad = \quad \langle \mathbf{a}_{i_1},\mathbf{t}\rangle \cdots \langle \mathbf{a}_{i_r},\mathbf{t}\rangle$$
for all prime monomial ideals $\langle x_{i_1},\ldots,x_{i_r}\rangle$, where $\langle \mathbf{a},\mathbf{t}\rangle = a_1t_1+\cdots+a_dt_d$.

Proof. If M is a finitely generated graded module, then let $M \cong \mathcal{F}/K$ be a graded presentation, and pick a term order on $\mathcal{F}$. Set $M' = \mathcal{F}/\mathsf{in}(K)$, so $\mathcal{C}(M;\mathbf{t}) = \mathcal{C}(M';\mathbf{t})$ because $\mathcal{C}$ is degenerative. Note that M' is a direct sum of A-graded translates of monomial quotients of S. Since all of the associated primes of M' are therefore monomial primes, the values of $\mathcal{C}$ on the prime monomial quotients of S determine $\mathcal{C}(M';\mathbf{t})$ by additivity. This proves uniqueness. Existence will follow from Corollary 8.47, Proposition 8.49, and Theorem 8.53. □

From the perspective of uniqueness, we could as well have put *any* function of $\mathbf{t}$ on the right hand side of Theorem 8.44. However, it takes very special such right hand sides to guarantee that a function $\mathcal{C}$ actually exists. Indeed, $\mathcal{C}$ is severely overdetermined, because most submodules $K \subseteq \mathcal{F}$ have lots of distinct initial submodules $\mathsf{in}(K)$ as the term order varies. This brings us to our main definition: to prove the existence of a function $\mathcal{C}$ satisfying Theorem 8.44, we shall explicitly construct one from K-polynomials.

We work exclusively with torsion-free gradings, and therefore we assume $A \subseteq \mathbb{Z}^d$. The symbol $\mathbf{t}^{\mathbf{a}}$ thus stands for an honest Laurent monomial

$t_1^{a_1} \cdots t_d^{a_d}$, allowing us to substitute $1 - t_j$ for every occurrence of t_j. Doing so yields a rational function $(1-t_1)^{a_1} \cdots (1-t_d)^{a_d}$, which can be expanded (even if some of the a_j are less than 0) as a well-defined power series

$$\prod_{j=1}^{d} (1 - t_j)^{a_j} \quad = \quad \prod_{j=1}^{d} (1 - a_j t_j + \tfrac{a_j(a_j-1)}{2} t_j^2 - \cdots)$$

in $\mathbb{Z}[[t_1, \ldots, t_d]]$. Doing the same for each monomial in an arbitrary Laurent polynomial $K(\mathbf{t})$ results in a power series $K[[\mathbf{1}-\mathbf{t}]]$. If $K(\mathbf{t})$ is a polynomial, so $\mathbf{a} \in \mathbb{N}^d$ whenever $\mathbf{t}^{\mathbf{a}}$ appears in $K(\mathbf{t})$, then $K[[\mathbf{1} - \mathbf{t}]]$ is a polynomial.

Definition 8.45 The **multidegree** of a $\mathbb{Z}^d$-graded S-module M is the sum $\mathcal{C}(M; \mathbf{t}) \in \mathbb{Z}[t_1, \ldots, t_d]$ of all terms in $\mathcal{K}[[M; \mathbf{1} - \mathbf{t}]]$ having total degree $\operatorname{codim}(M) = n - \dim(M)$. When $M = S/I$ is the coordinate ring of a subvariety $X \subseteq \mathbb{k}^n$, we may also write $[X]_A$ or $\mathcal{C}(X; \mathbf{t})$ to mean $\mathcal{C}(M; \mathbf{t})$.

Example 8.46 Let $S = \mathbb{k}[a, b, c, d]$ be multigraded by $\mathbb{Z}^2$, with

$$\deg(a) = (2, -1), \;\; \deg(b) = (1, 0), \;\; \deg(c) = (0, 1), \;\; \text{and } \deg(d) = (-1, 2).$$

If M is the module $S/\langle b^2, bc, c^2 \rangle$, then

$$\begin{aligned} \mathcal{K}(M; \mathbf{t}) &= 1 - \mathbf{t}^{\deg(b^2)} - \mathbf{t}^{\deg(bc)} - \mathbf{t}^{\deg(c^2)} + \mathbf{t}^{\deg(b^2c)} + \mathbf{t}^{\deg(bc^2)} \\ &= 1 - t_1^2 - t_1 t_2 - t_2^2 + t_1^2 t_2 + t_1 t_2^2 \end{aligned}$$

because of the Scarf resolution. Gathering $-t_1^2 + t_1^2 t_2 = -t_1^2(1 - t_2)$, we get

$$\begin{aligned} \mathcal{K}[[M; \mathbf{1} - \mathbf{t}]] &= 1 - (1 - t_1)^2 t_2 - t_1(1 - t_2)^2 - (1 - t_1)(1 - t_2) \\ &= 3t_1t_2 - t_1^2 t_2 - t_1 t_2^2, \quad \text{so} \\ \mathcal{C}(M; \mathbf{t}) &= 3t_1t_2 \end{aligned}$$

is the sum of degree $2 = \operatorname{codim}(M)$ terms in $\mathcal{K}[[M; \mathbf{1} - \mathbf{t}]]$. ◇

Immediately from Definition 8.45 and the invariance of K-polynomials under Gröbner degeneration in Theorem 8.36, we get the analogous invariance of multidegrees under Gröbner degeneration.

Corollary 8.47 *The multidegree function $M \mapsto \mathcal{C}(M; \mathbf{t})$ is degenerative.*

Next let us verify that the multidegrees of quotients by monomial primes satisfy the formula in Theorem 8.44. For this we need a lemma.

Lemma 8.48 *Let $\mathbf{b} \in \mathbb{Z}^d$. If $K(\mathbf{t}) = 1 - \mathbf{t}^{\mathbf{b}} = 1 - t_1^{b_1} \cdots t_d^{b_d}$ then substituting $1 - t_j$ for each occurrence of t_j yields $K[[\mathbf{1} - \mathbf{t}]] = b_1 t_1 + \cdots + b_d t_d + O(\mathbf{t}^2)$, where $O(\mathbf{t}^2)$ denotes a sum of terms each of which has total degree at least 2.*

Proof. $K[[\mathbf{1}-\mathbf{t}]] \;=\; 1-\prod_{j=1}^{d}(1-t_j)^{b_j} \;=\; 1-\prod_{j=1}^{d}\Big(1-b_jt_j+O(t_j^2)\Big)$, and this equals $1-\Big(1-\sum_{j=1}^{d}(b_jt_j)+O(\mathbf{t}^2)\Big) \;=\; \Big(\sum_{j=1}^{d}b_jt_j\Big)+O(\mathbf{t}^2)$. □

The linear form $b_1t_1+\cdots+b_dt_d$ in Lemma 8.48 can also be expressed as the inner product $\langle \mathbf{b},\mathbf{t}\rangle$ of the vector $\mathbf{b}$ with the vector $\mathbf{t}=(t_1,\ldots,t_d)$. It can be useful to think of this as the logarithm of the Laurent monomial $\mathbf{t}^{\mathbf{b}}$.

Proposition 8.49 *K-polynomials of prime monomial quotients satisfy*

$$\mathcal{K}[[S/\langle x_{i_1},\ldots,x_{i_r}\rangle;\mathbf{1}-\mathbf{t}]] \;=\; \Big(\prod_{\ell=1}^{r}\langle \mathbf{a}_{i_\ell},\mathbf{t}\rangle\Big)+O(\mathbf{t}^{r+1}),$$

where $O(\mathbf{t}^{r+1})$ is a sum of forms each of which has total degree at least $r+1$. In particular, the multidegree of a prime monomial quotient of S is

$$\mathcal{C}(S/\langle x_{i_1},\ldots,x_{i_r}\rangle;\mathbf{t}) \;=\; \langle \mathbf{a}_{i_1},\mathbf{t}\rangle\cdots\langle \mathbf{a}_{i_r},\mathbf{t}\rangle.$$

Proof. Using the Koszul complex, the K-polynomial of the quotient module $M=S/\langle x_{i_1},\ldots,x_{i_r}\rangle$ is computed to be $\mathcal{K}(M;\mathbf{t})=(1-\mathbf{t}^{\mathbf{a}_1})\cdots(1-\mathbf{t}^{\mathbf{a}_r})$. Now apply Lemma 8.48 to each of the r factors in this product. □

Example 8.50 Consider the ring $S=\Bbbk[a,b,c,d]$ multigraded by $\mathbb{Z}^2$ with

$$\deg(a) \;=\; (3,0),\ \deg(b) \;=\; (2,1),\ \deg(c) \;=\; (1,2),\ \text{and}\ \deg(d) \;=\; (0,3)$$

Then, using variables $\mathbf{s}=s_1,s_2$, we have multidegrees

$$\begin{aligned} \mathcal{C}(S/\langle a,b\rangle,\mathbf{s}) &= (3s_1)(2s_1+s_2) &&= 6s_1^2+3s_1s_2,\\ \mathcal{C}(S/\langle a,d\rangle,\mathbf{s}) &= (3s_1)(3s_2) &&= 9s_1s_2,\\ \text{and}\quad \mathcal{C}(S/\langle c,d\rangle,\mathbf{s}) &= (s_1+2s_2)(3s_2) &&= 3s_1s_2+6s_2^2. \end{aligned}$$

Note that these multidegrees all lie inside the ring

$$\mathbb{Z}[3s_1,s_1+2s_2,2s_1+s_2,3s_2] \;=\; \mathbb{Z}[s_1+2s_2,2s_1+s_2]$$

and not just $\mathbb{Z}[s_1,s_2]$, since the group A is the proper subgroup of $\mathbb{Z}s_1\oplus\mathbb{Z}s_2$ generated by s_1+2s_2 and $2s_1+s_2$. Nonetheless, the definition of multidegree via $\mathcal{K}[[M;\mathbf{1}-\mathbf{s}]]$ still works verbatim. ◇

Remark 8.51 Warning: Proposition 8.49 does not say that the product $\langle \mathbf{a}_{i_1},\mathbf{t}\rangle\cdots\langle \mathbf{a}_{i_r},\mathbf{t}\rangle$ is nonzero. Indeed it can very easily be zero, if $\mathbf{a}_{i_\ell}=0$ for some ℓ. However, this is the only way to get zero, as the product of linear forms takes place in a polynomial ring over $\mathbb{Z}$, which has no zerodivisors.

Proposition 8.49 implies that multidegrees of quotients of S by monomial prime ideals are insensitive to multigraded shifts.

Corollary 8.52 *If* $M = S/\langle x_{i_1}, \ldots, x_{i_r}\rangle$ *then* $\mathcal{C}(M(-\mathbf{b}); \mathbf{t}) = \mathcal{C}(M; \mathbf{t})$.

Proof. Shifting by $\mathbf{b}$ multiplies the K-polynomial by $\mathbf{t}^{\mathbf{b}}$, so $\mathcal{K}(M(-\mathbf{b}); \mathbf{t}) = \mathbf{t}^{\mathbf{b}}\mathcal{K}(M; \mathbf{t})$. The degree r form in $\mathcal{K}[[M(-\mathbf{b}); \mathbf{1} - \mathbf{t}]]$ is the product of the lowest degree forms in $\mathcal{K}[[M; \mathbf{1} - \mathbf{t}]]$ and $(\mathbf{1} - \mathbf{t})^{\mathbf{b}}$, the latter of which is 1. □

In view of Corollary 8.47 and Proposition 8.49, our final result of the chapter completes the characterization of multidegrees in Theorem 8.44.

Theorem 8.53 *The multidegree function* $\mathcal{C} : M \mapsto \mathcal{C}(M; \mathbf{t})$ *is additive.*

Proof. Let $M = M_\ell \supset M_{\ell-1} \supset \cdots \supset M_1 \supset M_0 = 0$ be a filtration of M in which $M_j/M_{j-1} \cong (S/\mathfrak{p}_j)(-\mathbf{b}_j)$ for multigraded primes $\mathfrak{p}_j$ and vectors $\mathbf{b}_j \in A$. Such a filtration exists because we can choose a homogeneous associated prime $\mathfrak{p}_1 = \operatorname{ann}(m_1)$ by Proposition 8.11, set $M_1 = \langle m_1\rangle \cong (S/\mathfrak{p}_1)(-\deg(m_1))$, and then continue by noetherian induction on M/M_1.

The quotients $S/\mathfrak{p}_j$ all have dimension at most $\dim(M)$, and if $S/\mathfrak{p}$ has dimension exactly $\dim(M)$, then $\mathfrak{p} = \mathfrak{p}_j$ for exactly $\operatorname{mult}_{\mathfrak{p}}(M)$ values of j (localize the filtration at $\mathfrak{p}$ to see this). Also, additivity of K-polynomials on short exact sequences implies that $\mathcal{K}(M; \mathbf{t}) = \sum_{j=1}^{\ell} \mathcal{K}(M_j/M_{j-1}; \mathbf{t})$.

Assume for the moment that M is a direct sum of multigraded shifts of quotients of S by monomial ideals. Then all the primes $\mathfrak{p}_j$ are monomial primes. Therefore the only power series $\mathcal{K}[[M_j/M_{j-1}; \mathbf{1} - \mathbf{t}]]$ contributing terms of degree $\operatorname{codim}(M)$ to $\mathcal{K}[[M; \mathbf{1} - \mathbf{t}]]$ are those for which $M_j/M_{j-1} \cong (S/\mathfrak{p}_j)(-\mathbf{b}_j)$ has maximal dimension, by Proposition 8.49 and Corollary 8.52. Hence the theorem holds for such M.

Before continuing with the case of general modules M, let us generalize Proposition 8.49, and hence Corollary 8.52, to arbitrary modules.

Claim 8.54 *If* M *has codimension* r*, then* $\mathcal{K}[[M; \mathbf{1} - \mathbf{t}]] = \mathcal{C}(M; \mathbf{t}) + O(\mathbf{t}^{r+1})$. *In particular,* $\mathcal{C}(M(-\mathbf{b}); \mathbf{t}) = \mathcal{C}(M; \mathbf{t})$ *for arbitrary modules* M.

Proof. We have just finished showing that the first statement holds for direct sums of multigraded shifts of monomial quotients of S. By Corollary 8.47, every module $M \cong \mathcal{F}/K$ of codimension r has the same multidegree as such a direct sum, namely $\mathcal{F}/\mathsf{in}(K)$, whose codimension is also r. The second statement follows as in the proof of Corollary 8.52. □

Now the argument before Claim 8.54 works for arbitrary modules M and primes $\mathfrak{p}_j$, using the Claim in place of Proposition 8.49 and Corollary 8.52. □

Example 8.55 Let $S = \Bbbk[a, b, c, d]$ and let $I = \langle b^2 - ac, bc - ad, c^2 - bd\rangle$ be the *twisted cubic* ideal. Then I has initial ideal $\mathsf{in}(I) = \langle b^2, bc, c^2\rangle$ under the reverse lexicographic term order with $a > b > c > d$. Since $\mathsf{in}(I)$ is supported on $\langle b, c\rangle$ with multiplicity 3, the multidegree of S/I under the $\mathbb{Z}^2$-grading from Example 8.46 is $\mathcal{C}(S/I; t_1, t_2) = 3\langle\deg(b), \mathbf{t}\rangle\langle\deg(c), \mathbf{t}\rangle = 3t_1t_2$. This agrees with the multidegree in Example 8.46, as it should by

Theorem 8.53. It also equals the multidegree $\mathcal{C}(S/I;\mathbf{t})$ of the twisted cubic, by Corollary 8.47.

On the other hand, the twisted cubic ideal I has initial ideal $\mathsf{in}(I) = \langle ac, ad, bd\rangle = \langle a,b\rangle \cap \langle a,d\rangle \cap \langle c,d\rangle$ under the lexicographic term order with $a > b > c > d$. Using the multigrading and notation from Example 8.50, additivity in Theorem 8.53 implies that

$$\begin{aligned}\mathcal{C}(S/\langle ac, ad, bd\rangle;\mathbf{s}) &= \mathcal{C}(S/\langle a,b\rangle;\mathbf{s}) + \mathcal{C}(S/\langle a,d\rangle;\mathbf{s}) + \mathcal{C}(S/\langle c,d\rangle;\mathbf{s}) \\ &= 6s_1^2 + 15s_1s_2 + 6s_2^2.\end{aligned}$$

We conclude by Corollary 8.47 that $\mathcal{C}(S/I;\mathbf{s}) = 6s_1^2 + 15s_1s_2 + 6s_2^2$. This multidegree also equals $3\langle \deg(b),\mathbf{s}\rangle\langle \deg(c),\mathbf{s}\rangle = 3(s_1+2s_2)(2s_1+s_2)$. $\diamond$

Exercises

8.1 Prove that $S_{\mathbf{a}}$ is generated as a module over $S_{\mathbf{0}}$ by any set of monomials that generates the ideal $\langle S_{\mathbf{a}}\rangle$ inside of S.

8.2 Let Q be a pointed affine semigroup in $A \cong \mathbb{Z}^d$, and let $A' \subseteq A$ be the subgroup generated by Q. Write $\mathbb{Z}[[Q]][A] = \mathbb{Z}[[Q]] \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[A]$. Note that when $A' = A$, the ring $\mathbb{Z}[[Q]][A]$ equals the localization $\mathbb{Z}[[Q]][\mathbf{t}^{-\mathbf{a}_1},\ldots,\mathbf{t}^{-\mathbf{a}_n}]$.

(a) Show that if $A' = A$, every element in $\mathbb{Z}[[Q]][A]$ can be represented uniquely by a series $\sum_{\mathbf{a}\in A} c_{\mathbf{a}}\mathbf{t}^{\mathbf{a}}$ supported on a union of finitely many translates of Q.
(b) In the situation of part (a), prove that every series supported on a union of finitely many translates of Q lies in $\mathbb{Z}[[Q]][A]$.
(c) Use Exercise 7.10 to verify parts (a) and (b) when A' does not equal A.

8.3 Consider the twisted cubic ideal I in Example 8.55, and let $w = (0,1,3,2)$.

(a) Prove that the homogenization of I with respect to the weight vector w is the ideal $\tilde{I} = \langle ac - b^2y, bc - ady^2, c^2 - bdy^3, b^3 - a^2dy\rangle$ in $S[y]$.
(b) Compute a minimal free resolution of $\tilde{I}$ graded by $\mathbb{Z}^2 \times \mathbb{Z}$, where the multigrading of $\Bbbk[a,b,c,d]$ by $\mathbb{Z}^2$ is as in either Example 8.46 or Example 8.50.
(c) Verify Proposition 8.28, Theorem 8.29, and Corollary 8.31 in this case by plugging $y = 0$ and $y = 1$ into matrices for the maps in the resolution from (b), and exhibiting the consecutive pairs as described in Remark 8.30.

8.4 Let $S = \Bbbk[\mathbf{x}]$ for $\mathbf{x} = \{x_{ij} \mid i,j = 1,\ldots,4\}$. With $|\cdot| = \det(\cdot)$, set

$$I = \left\langle \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{vmatrix}, \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32}\end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32}\end{vmatrix}, \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34}\end{vmatrix}, \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34}\end{vmatrix} \right\rangle.$$

Compute the K-polynomials and multidegrees of the quotient S/I in the multigradings by $\mathbb{Z}^4$ in which (i) $\deg(x_{ij}) = t_i$ and (ii) $\deg(x_{ij}) = s_j$.

8.5 Make arbitrarily long lists of polynomials generating the ideal $\langle xy-1\rangle \subset \Bbbk[x,y]$, none of which can be left off. (Example 8.35 has lists of length 1 and 2.) Confirm that the corresponding free resolutions all give the same K-polynomial.

8.6 Write down formulas for the K-polynomial and multidegree of the quotient of S by an irreducible monomial ideal (i.e. generated by powers of variables).

8.7 Show that arbitrary multidegrees are nonnegative, in the following sense: The multidegree of any module of dimension $n-r$ over S is a nonnegative sum of "squarefree" homogeneous forms $\langle \mathbf{a}_{i_1}, \mathbf{t}\rangle \cdots \langle \mathbf{a}_{i_r}, \mathbf{t}\rangle$ of degree r with $i_1 < \cdots < i_r$.

8.8 If the linear forms $\langle \mathbf{a}_1, \mathbf{t}\rangle, \ldots, \langle \mathbf{a}_n, \mathbf{t}\rangle$ are nonzero and generate a pointed affine semigroup in $\mathbb{Z}^d$, deduce that no product $\langle \mathbf{a}_{i_1}, \mathbf{t}\rangle \cdots \langle \mathbf{a}_{i_r}, \mathbf{t}\rangle$ for $i_1 < \cdots < i_r$ is zero, and that all of these forms together generate a pointed semigroup in $\mathbb{Z}^R$, where $R = \binom{n+r-1}{n}$ is the number of monomials of degree r in n variables.

8.9 Prove that the positivity in Exercise 8.7 holds for positive multigradings in the stronger sense that any nonempty nonnegative sum of forms $\langle \mathbf{a}_{i_1}, \mathbf{t}\rangle \cdots \langle \mathbf{a}_{i_r}, \mathbf{t}\rangle$ is nonzero. Conclude that $\mathcal{C}(M;\mathbf{t}) \neq 0$ if $M \neq 0$ is positively graded.

8.10 Let M and M' be two $\mathbb{Z}^n$-graded S-modules. Give examples demonstrating that the product of the multidegrees of M and M' need not be expressible as the multidegree of a $\mathbb{Z}^n$-graded module. Can you find sufficient conditions on M and M' to guarantee that $\mathcal{C}(M;\mathbf{t})\mathcal{C}(M';\mathbf{t}) = \mathcal{C}(M \otimes_S M';\mathbf{t})$?

8.11 Let M be a multigraded module and $z \in S$ a homogeneous nonzerodivisor on M of degree $\mathbf{b}$. Prove that

(a) $\mathcal{K}(M/zM;\mathbf{t}) = (1-\mathbf{t}^{\mathbf{b}})\mathcal{K}(M;\mathbf{t})$, and
(b) $\mathcal{C}(M/zM;\mathbf{t}) = \langle \mathbf{b}, \mathbf{t}\rangle \mathcal{C}(M;\mathbf{t})$.

8.12 Let $I \subseteq S$ be multidgraded for a positive $\mathbb{Z}^d$-grading. Suppose that J is a $\mathbb{Z}^d$-graded radical ideal contained inside I, and that J is equidimensional (also known as *pure*: all of its associated primes have the same dimension). If S/I and S/J have equal multidegrees, deduce that $I = J$. Hint: Use Exercise 8.9.

8.13 Let S be $\mathbb{Z}$-graded in the usual way, with $\deg(x_i) = 1$ for $i = 1, \ldots, n$. The $\mathbb{Z}$-graded degree $e(M)$ of a graded module M is usually defined as $(r-1)!$ times the leading coefficient of the Hilbert polynomial of M, where $r = \dim(M)$. Prove that the multidegree of M is $e(M)t^{n-r}$.

8.14 Suppose S is multigraded by A, with $\deg(x_i) = \mathbf{a}_i \in A$, and suppose $A \to A'$ is a homomorphism of abelian groups sending $\mathbf{a}_i$ to $\mathbf{a}'_i \in A'$. Prove the following.

(a) The homomorphism $A \to A'$ induces a new multigrading $\deg'$ on S, in which $\deg'(x_i) = \mathbf{a}'_i$.
(b) If a module M is multigraded by A, then M is also multigraded by A'.
(c) If $\mathcal{K}(M;\mathbf{t})$ and $\mathcal{K}(M;\mathbf{s})$ are the K-polynomials of M under the multigradings by A and A', then $\mathcal{K}(M;\mathbf{t})$ maps to $\mathcal{K}(M;\mathbf{s})$ under the homomorphism $\mathbb{Z}[A] \to \mathbb{Z}[A']$ of group algebras. In particular, this sends $\mathbf{t}^{\mathbf{a}_i}$ to $\mathbf{s}^{\mathbf{a}'_i}$.
(d) If $\mathcal{C}(M;\mathbf{t})$ and $\mathcal{C}(M;\mathbf{s})$ are the multidegrees of M under the multigradings by A and A', then $\mathcal{C}(M;\mathbf{t})$ maps to $\mathcal{C}(M;\mathbf{s})$ under the homomorphism $\mathbb{Z}[\mathbf{t}] \to \mathbb{Z}[\mathbf{s}]$ of polynomial rings, which sends $\langle \mathbf{a}_i, \mathbf{t}\rangle$ to $\langle \mathbf{a}'_i, \mathbf{s}\rangle$.

8.15 Verify functoriality of K-polynomials and multidegrees for the twisted cubic $\Bbbk[a,b,c,d]/\langle b^2-ac, ad-bc, c^2-bd\rangle$ under the two multigradings in Examples 8.46, 8.50, and 8.55. The morphism of gradings sends $t_1 \mapsto 2s_1 - s_2$ and $t_2 \mapsto 2s_2 - s_1$.

8.16 Prove that an ideal I inside an a *priori* ungraded polynomial ring $\Bbbk[\mathbf{x}]$ is homogeneous for a weight vector $w \in \mathbb{Z}^n$ if and only if some (and hence every) reduced Gröbner basis for I is homogeneous for w. Conclude that there is a unique finest **I-universal grading** on $\Bbbk[\mathbf{x}]$ in which the ideal I is homogeneous.

8.17 For any polynomial $g \in \Bbbk[\mathbf{x}]$, let $\log(g)$ be the set of exponent vectors on monomials having nonzero coefficient in g. Suppose that $\mathcal{G}$ is the reduced Gröbner basis of I for some term order. If L is the sublattice of $\mathbb{Z}^n$ generated by the sets $\log(g) - \log(\mathsf{in}(g))$ for $g \in \mathcal{G}$, show that the $\mathbb{Z}^n/L$-grading on $\Bbbk[\mathbf{x}]$ is universal for I.

8.18 Prove that if $L \subseteq \mathbb{Z}^n$ is a sublattice, then the universal grading for the lattice ideal I_L is the multigrading by $\mathbb{Z}^n/L$.

8.19 Define the **universal K-polynomial** and **universal multidegree** of the quotient $\Bbbk[\mathbf{x}]/I$ to be its K-polynomial and multidegree in the I-universal grading. Compute the universal K-polynomial and multidegree of S/I from Exercise 8.4.

Notes

Geometrically, a multigrading on a polynomial ring comes from the action of an algebraic torus times a finite abelian group. The importance of this point of view has surged in recent years due to its connections with toric varieties (see Chapter 10). Multigraded $\Bbbk[x_1, \ldots, x_n]$-modules correspond to *torus-equivariant sheaves* on the vector space $\Bbbk^n$. The K-polynomial of a module is precisely the class represented by the corresponding sheaf in the *equivariant K-theory* of $\Bbbk^n$; this is the content of Theorem 8.34. The degenerative property of K-polynomials in Theorem 8.36 is an instance of the constancy of K-theory classes in flat families. See [BG05] for more on K-theory in the toric context.

The increase of Betti numbers in Theorem 8.29 can be interpreted in terms of associated graded modules for filtrations [Vas98, Section B.2], or as an instance of a more general upper-semicontinuity for flat families [Har77, Theorem III.12.8].

The notion of multidegree, essentially in the form of Definition 8.45, seems to be due to Borho and Brylinski [BB82, BB85] as well as to Joseph [Jos84]. The *equivariant multiplicities* used by Rossmann [Ros89] in complex-analytic contexts are equivalent. Multidegrees are called *T-equivariant Hilbert polynomials* in [CG97, Section 6.6], where they are proved to be additive as well as homogeneous of degree equal to the codimension (the name is confusing when compared with usual Hilbert polynomials). Elementary proofs of these facts appear also in [BB82]. Multidegrees are algebraic reformulations of the geometric *torus-equivariant Chow classes* (or *equivariant cohomology classes* when $\Bbbk = \mathbb{C}$) of varieties in $\Bbbk^n$ [Tot99, EG98]; this is proved in [KMS04, Proposition 1.19]. The transition from K-polynomials to multidegrees is a manifestation of the Grothendieck–Riemann–Roch theorem.

Exercises 8.7–8.9 come from [KnM04b, Section 1.7], where the positively multigraded case of the characterization in Theorem 8.44 (that is, including the degenerative property) was noted. Exercise 8.12 appears in [Mar04, Section 5] and [KnM04b, Lemma 1.7.5]. The $\mathbf{t}$-multidegree in Exercise 8.4 is a Schur function, since I is a grassmannian Schubert determinantal ideal (Exercises 15.2 and 16.9).

Chapter 15

Matrix Schubert varieties

In the previous chapter, we saw how Plücker coordinates parametrize flags in vector spaces. We found that the list of Plücker coordinates is, up to scale, invariant on the set of invertible matrices mapping to a single flag. Here we focus on bigger sets of matrices, called *matrix Schubert varieties*, that map not to single points in the flag variety, but to subvarieties called *Schubert varieties.* Matrix Schubert varieties are sets of rectangular matrices satisfying certain constraints on the ranks of their submatrices. Commutative algebra enters the picture through their defining ideals, which are generated by minors in the generic rectangular matrix of variables.

This chapter and the two after it offer a self-contained introduction to determinantal ideals. Our presentation complements the existing extensive literature (see the Notes to Chapter 16), concerning both quantitative and qualitative attributes, such as dimension, degree, primality, and Cohen–Macaulayness, of varieties of matrices with rank constraints. We consider the finest possible multigrading, which demands the refined toolkit of a new generation of combinatorialists. Besides primality and Cohen–Macaulayness, the main results are that the essential minors form a Gröbner basis, and that the multidegree equals a *double Schubert polynomial.*

Harvesting combinatorics from algebraic fields of study requires sowing combinatorial seeds. In our case, the seed is a partial permutation matrix from which the submatrix ranks are determined. Partial permutations lead us naturally in this chapter to the *Bruhat* and *weak orders* on the symmetric group. Part of this story is the notion of *length* for partial permutations, which is characterized in our algebraic context in terms of operations interchanging pairs of rows in matrices. These combinatorial considerations are fertilized by the geometry of Borel group orbits, on which the rank conditions are fixed. This geometry under row exchanges allows us to reap our reward: the multidegrees of matrix Schubert varieties satisfy the *divided difference* recurrence, characterizing them as double Schubert polynomials.

15.1 Schubert determinantal ideals

Throughout this chapter, $M_{k\ell}$ will denote the vector space of matrices with k rows and ℓ columns over the field $\Bbbk$, which we assume is algebraically closed, for convenience. Denote the coordinate ring of $M_{k\ell}$ by $\Bbbk[\mathbf{x}]$, where

$$\mathbf{x} \quad = \quad (x_{\alpha\beta} \mid \alpha = 1, \ldots, k \text{ and } \beta = 1, \ldots, \ell)$$

is a set variables filling the *generic* $k \times \ell$ *matrix*. Our interests in this chapter lie with the following loci inside $\Bbbk^{m \times \ell}$.

Definition 15.1 Let $w \in M_{k\ell}$ be a **partial permutation**, meaning that w is a $k \times \ell$ matrix having all entries equal to 0 except for at most one entry equal to 1 in each row and column. The **matrix Schubert variety** $\overline{X}_w$ inside $M_{k\ell}$ is the subvariety

$$\overline{X}_w \quad = \quad \{Z \in M_{k\ell} \mid \operatorname{rank}(Z_{p\times q}) \leq \operatorname{rank}(w_{p\times q}) \text{ for all } p \text{ and } q\},$$

where $Z_{p\times q}$ is the upper-left $p \times q$ rectangular submatrix of Z. Let $r(w)$ be the $k \times \ell$ array whose entry at (p, q) is $r_{pq}(w) = \operatorname{rank}(w_{p\times q})$.

Example 15.2 The **classical determinantal variety** is the set of all $k \times \ell$ matrices over $\Bbbk$ of rank at most r. This variety is the matrix Schubert variety $\overline{X}_w$ for the partial permutation matrix w with r nonzero entries

$$w_{11} \; = \; w_{22} \; = \; \cdots \; = \; w_{rr} \; = \; 1$$

along the diagonal, and all other entries $w_{\alpha\beta}$ equal to zero. The **classical determinantal ideal**, generated by the set of all $(r+1) \times (r+1)$ minors of the $k \times \ell$ matrix of variables, vanishes on this variety. In Definition 15.5 this ideal will be called the *Schubert determinantal ideal* I_w for the special partial permutation w above. We shall see in Corollary 16.29 that in fact I_w is the prime ideal of $\overline{X}_w$. In Example 15.39 we show that the multidegree of this classical determinantal ideal is a *Schur polynomial*. Our results in Chapter 16 imply that the set of all $(r+1) \times (r+1)$ minors is a Gröbner basis and its determinantal variety is Cohen–Macaulay.

Some readers may wonder whether the machinery developed below is really the right way to prove the Gröbner basis property in this classical case. Our answer to this question is emphatically 'yes'. The local transitions in Section 15.3–15.5 provide the steps for an elementary and self-contained proof, by an induction that involves *all* matrix Schubert varieties, starting from the case of a coordinate subspace in Example 15.3. An essential feature of this induction is that it reflects the combinatorics inherent in the universal grading making all minors homogeneous (see Exercise 15.1).

For the classical determinantal ideal of *maximal* minors, where $r = \min(k, \ell)$, the induction is particularly simple and explicit, as is the combinatorial multidegree formula; see Exercises 15.4, 15.5, and 15.12. $\diamond$

Partial permutation matrices w are sometimes called *rook placements*, because rooks placed on the 1 entries in w are not attacking one another. The number $\mathrm{rank}(w_{p\times q})$ appearing in Definition 15.1 is simply the number of 1 entries (rooks) in the northwest $p\times q$ submatrix of w. A partial permutation can be viewed as a correspondence that takes some of the integers $1, \ldots, k$ to distinct integers from $\{1, \ldots, \ell\}$. We write $w(i) = j$ if the partial permutation w has a 1 in row i and column j. This convention results from viewing matrices in $M_{k\ell}$ as acting on row vectors from the right; it is therefore transposed from the more common convention for writing permutation matrices using columns.

When w is an honest square permutation matrix of size n, so $k = n = \ell$ and there are exactly n entries of w equal to 1, then we can express w in one-line notation: the permutation $w = w_1 \ldots w_n$ of $\{1, \ldots, n\}$ sends $i \mapsto w_i$. This is not to be confused with cycle notation, where (for instance) the permutation $\sigma_i = (i, i+1)$ is the *adjacent transposition* switching i and $i+1$. The number $\mathrm{rank}(w_{p\times q})$ can alternatively be expressed as

$$r_{pq}(w) \quad = \quad \mathrm{rank}(w_{p\times q}) \quad = \quad \#\{(i,j) \leq (p,q) \mid w(i) = j\}$$

for permutations w. The group of permutations of $\{1, \ldots, n\}$ is denoted by S_n. There is a special permutation $w_0 = n \ldots 321$ called the *long word* inside S_n, which reverses the order of $1, \ldots, n$.

Example 15.3 The variety $\overline{X}_{w_0}$ inside M_{nn} for the long word $w_0 \in S_n$ is just the linear subspace of lower-right-triangular matrices; its ideal is $\langle x_{ij} \mid i+j \leq n\rangle$. As we shall see in Section 15.3, this is the smallest matrix Schubert variety indexed by an honest permutation in S_n. $\diamond$

Example 15.4 Five of the six 3×3 matrix Schubert varieties for honest permutations are linear subspaces:

$$\begin{array}{rclrcl}
I_{123} & = & 0 & \overline{X}_{123} & = & M_{33} \\
I_{213} & = & \langle x_{11}\rangle & \overline{X}_{213} & = & \{Z \in M_{33} \mid x_{11} = 0\} \\
I_{231} & = & \langle x_{11}, x_{12}\rangle & \overline{X}_{231} & = & \{Z \in M_{33} \mid x_{11} = x_{12} = 0\} \\
I_{231} & = & \langle x_{11}, x_{21}\rangle & \overline{X}_{312} & = & \{Z \in M_{33} \mid x_{11} = x_{21} = 0\} \\
I_{321} & = & \langle x_{11}, x_{12}, x_{21}\rangle & \overline{X}_{321} & = & \{Z \in M_{33} \mid x_{11} = x_{12} = x_{21} = 0\}
\end{array}$$

The remaining permutation, $w = 132$, has matrix $\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline\end{array}$, so that

$$I_{132} \;=\; \langle x_{11}x_{22} - x_{12}x_{21}\rangle \qquad \overline{X}_{132} \;=\; \{Z \in M_{33} \mid \mathrm{rank}(Z_{2\times 2}) \leq 1\}.$$

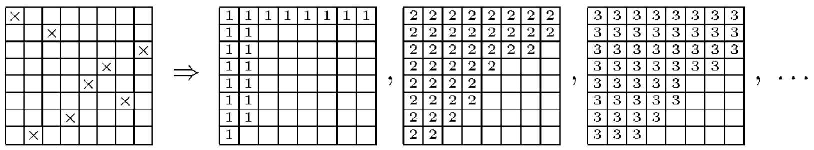
Thus $\overline{X}_{132}$ is the set of matrices whose upper-left 2×2 block is singular. $\diamond$

Since a matrix has rank at most r if and only if its minors of size $r+1$ all vanish, the matrix Schubert variety $\overline{X}_w$ is the (reduced) subvariety of $\Bbbk^{k\times\ell}$ cut out by the ideal I_w defined as follows.

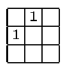
Definition 15.5 Let $w \in M_{k\ell}$ be a partial permutation. The **Schubert determinantal ideal** $I_w \subset \Bbbk[\mathbf{x}]$ is generated by all minors in $\mathbf{x}_{p\times q}$ of size $1 + r_{pq}(w)$ for all p and q, where $\mathbf{x} = (x_{\alpha\beta})$ is the $k\times\ell$ matrix of variables.

It is a nontrivial fact that Schubert determinantal ideals are prime, but we shall not need it in this chapter, where we work exclusively with the zero set $\overline{X}_w$ of I_w. We therefore write $I(\overline{X}_w)$ instead of I_w when we mean the radical of I_w. Chapter 16 gives a combinatorial algebraic primality proof.

Example 15.6 Let $w = 13865742$, so that the matrix for w is given by replacing each $\times$ by 1 in the left matrix below.



Each 8×8 matrix in $\overline{X}_w$ has the property that every rectangular submatrix contained in the region filled with 1's has rank ≤ 1, and every rectangular submatrix contained in the region filled with 2's has rank ≤ 2, and so on. The ideal I_w contains the 21 minors of size 2×2 in the first region and the 144 minors of size 3×3 in the second region. These 165 minors in fact generate I_w; see Theorem 15.15. ◇

Example 15.7 Let w be the 3×3 partial permutation matrix . The matrix Schubert variety $\overline{X}_w$ is the set of 3×3 matrices whose upper-left entry is zero, and whose determinant is zero. The ideal I_w is

$$\left\langle x_{11}, \quad \det\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \right\rangle.$$

The generators of I_w are the same as those of the ideal I_{2143} for the permutation in S_4 sending $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 4$ and $4 \mapsto 3$. ◇

It might seem a bit unhelpful of us to have ignored partial permutations until the very last example above, but in fact there is a general principle illustrated by Example 15.7 that gets us off the hook. Let us say that a partial permutation $\widetilde{w}$ *extends* w if the matrix $\widetilde{w}$ has northwest corner w.

Proposition 15.8 *Every partial permutation matrix w can be extended canonically to a square permutation matrix $\widetilde{w}$ whose Schubert determinantal ideal $I_{\widetilde{w}}$ has the same minimal generating minors as I_w.*

Proof. Suppose that w is not already a permutation, and that by symmetry there is a row (as opposed to a column) of w that has no 1 entries. Define w'

by adding a new column and placing a 1 entry in its highest possible row. Define $\widetilde{w}$ by continuing until there is a 1 entry in every row and column.

The Schubert determinantal ideal of any partial permutation matrix extending w contains the generators of I_w by definition. Therefore it is enough—by induction on the number of rows and columns added to get $\widetilde{w}$—to show that the Schubert determinantal ideal $I_{w'}$ is contained inside I_w. The only generators of $I_{w'}$ that are not obviously in I_w are the minors of size $1+\text{rank}(w'_{p\times(\ell+1)})$ in the generic matrix $\mathbf{x}_{p\times(\ell+1)}$ for $p = 1, \ldots, k$. Now use the next lemma. □

Lemma 15.9 *The ideal generated by all minors of size r in $\mathbf{x}_{p\times q}$ contains every minor of size $r+1$ in $\mathbf{x}_{p\times(q+1)}$ or in $\mathbf{x}_{(p+1)\times q}$.*

Proof. Laplace expand each minor of size $r+1$ along its rightmost column or bottom row, respectively. □

The canonical extension $\widetilde{w}$ in Proposition 15.8 has the property that

- no 1 entry in columns $\ell+1, \ell+2, \ldots$ of $\widetilde{w}$ is northeast of another; and
- no 1 entry in rows $k+1, k+2, \ldots$ of $\widetilde{w}$ is northeast of another.

Although $\widetilde{w}$ has size n for some fixed n, any size $n+n'$ permutation matrix extending w with these properties is a matrix for $\widetilde{w}$, viewed as lying in $S_{n+n'}$. Thus this this size $n+n'$ permutation matrix is $\widetilde{w}$ plus some extra 1 entries on the main diagonal in the southeast corner. Fortunately, Schubert determinantal ideals are insensitive to the choice of n'.

Proposition 15.10 *If $w \in S_n$ and $\widetilde{w}$ extends w to an element of $S_{n+n'}$ fixing $n+1, \ldots, n+n'$, then I_w and $I_{\widetilde{w}}$ have the same minimal generators.*

Proof. Add an extra column to w containing no 1 entries to get a partial permutation matrix w'. Since every row of w contains a 1, the "new" minors generating $I_{w'}$ are of size $1+p$ inside $\mathbf{x}_{p\times(n+1)}$ for each p. These minors are all zero, because they do not fit inside $\mathbf{x}_{p\times(n+1)}$. (If this last statement is unconvincing, think rank-wise: the rank conditions coming from the last column of w' say that the first p rows of matrices in $\overline{X}_{w'}$ have rank at most p; but this is a vacuous condition, since it is always satisfied.) Now apply Proposition 15.8 to w' and repeat n' times to get $\widetilde{w}$. □

Remark 15.11 Geometrically, Propositions 15.8 and 15.10 both say that $\overline{X}_{\widetilde{w}} = \overline{X}_w \times \mathbb{k}^m$, where m is the area of the matrix $\widetilde{w}$ minus the area of w.

As a final note on definitions, let us say what matrix Schubert varieties have to do with flag varieties. Recall from Section 14.1 that every invertible matrix $\Theta \in GL_n$ determines a flag in $\mathbb{k}^n$ by its Plücker coordinates.

Definition 15.12 Let $w \in S_n$ be a permutation. The **Schubert variety** X_w in the flag variety $\mathcal{F}\ell_n$ consists of the flags determined by *invertible* matrices lying in the matrix Schubert variety $\overline{X}_w$.

15.2 Essential sets

As we have seen in Example 15.2, some of the most classical ideals in commutative algebra are certain types of Schubert determinantal deals. To identify other special types, and to reduce the number of generating minors from the set given in Definition 15.5, we use the following tools.

Definition 15.13 The **diagram** $D(w)$ of a partial permutation matrix w consists of all locations (called 'boxes') in the $k \times \ell$ grid neither due south nor due east of a nonzero entry in w. The **length** of w is the cardinality $l(w)$ of its diagram $D(w)$. The **essential set** $\mathcal{E}ss(w)$ consists of the boxes (p, q) in $D(w)$ such that neither $(p, q+1)$ nor $(p+1, q)$ lies in $D(w)$.

The diagram of w determines w up to extension as in Propositions 15.8 and 15.10. The length of w is a fundamental combinatorial invariant that will be used repeatedly, starting in the next section. The diagram can be described more graphically by crossing out all the locations due south and east of nonzero entries in w; this leaves precisely the diagram $D(w)$ remaining. The essential set $\mathcal{E}ss(w)$ consists of the "southeast corners" in $D(w)$.

Example 15.14 Consider the 8×8 square matrix for the permutation $w = 48627315$, whose 1 entries are indicated by $\times$ in the following array:

4	□	□	□	×	·	·	·	·
8	□	□	□	·	□	□	[1]	×
6	□	□	[0]	·	[1]	×	·	·
2	□	×	·	·	·	·	·	·
7	□	·	[1]	·	[2]	·	×	·
3	[0]	·	×	·	·	·	·	·
1	×	·	·	·	·	·	·	·
5	·	·	·	·	×	·	·	·

Locations in the diagram of w are indicated by boxes, and its essential set consists of the subset of boxes with numbers in them. The number in the box $(p, q) \in \mathcal{E}ss(w)$ is $\text{rank}(w_{p\times q})$. ◇

Theorem 15.15 *The Schubert determinantal ideal $I_w \subset \Bbbk[\mathbf{x}]$ is generated by minors coming from ranks in the essential set of w:*

$$I_w \quad = \quad \langle \text{minors of size } 1 + \text{rank}(w_{p\times q}) \text{ in } \mathbf{x}_{p\times q} \mid (p, q) \in \mathcal{E}ss(w) \rangle.$$

Proof. Suppose (p, q) does not lie in $\mathcal{E}ss(w)$. Either (p, q) lies outside $D(w)$, or one of the two locations $(p, q+1)$ and $(p+1, q)$ lies in $D(w)$.

In the former case we demonstrate that the ideal generated by the minors of size $1 + r_{pq}(w)$ in $\mathbf{x}_{p\times q}$ is contained either in the ideal generated by the minors in I_w from $\mathbf{x}_{(p-1)\times q}$ or in the ideal generated by the minors in I_w from $\mathbf{x}_{p\times(q-1)}$. Suppose by symmetry that a nonzero entry of w lies due north of (p, q). Using Lemma 15.9, the minors of size $1 + r_{pq}(w)$ in $\mathbf{x}_{p\times q}$ are stipulated by the rank condition at $(p, q-1)$. Continuing in this way, we can move north and/or west until we get to a box in $D(w)$, or else to a location outside the matrix. The first possibility reduces to the case $(p, q) \in D(w)$. For the other possibility, we find that $r_{pq}(w) = \min\{p, q\}$, so there are no minors of size $1 + r_{pq}(w)$ in $\mathbf{x}_{p\times q}$.

Now we treat the case $(p, q) \in D(w)$, where we assume by symmetry that $D(w)$ has a box at $(p, q+1)$. The rank at (p, q) equals the rank at $(p, q+1)$ in this case, so the minors of size $1+\mathrm{rank}(w_{p\times q})$ in $\mathbf{x}_{p\times q}$ are contained (as a set) inside the set of minors of size $1+\mathrm{rank}(w_{p\times(q+1)})$ in $\mathbf{x}_{p\times(q+1)}$. Now continue east and/or south until a box in $\mathcal{E}ss(w)$ is reached. □

Example 15.16 Let w be the partial permutation of Example 15.2, so $\overline{X}_w$ is the variety of all matrices of rank $\leq r$. Then the essential set $\mathcal{E}ss(w)$ is the singleton $\{(k, \ell)\}$ consisting of a box in the southeast corner of w. ◇

Example 15.17 Suppose the (partial) permutation w has essential set

$$\mathcal{E}ss(w) \quad = \quad \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$$

in which the α's weakly decrease and the β's weakly increase:

$$\alpha_1 \geq \cdots \geq \alpha_m \quad \text{and} \quad \beta_1 \leq \cdots \leq \beta_m.$$

Thus $\mathcal{E}ss(w)$ lies along a path snaking its way east and north. Such (partial) permutations are called **vexillary**. Various classes of vexillary Schubert determinantal ideals have been objects of study in recent years, under the name "ladder determinantal ideals" (usually with prepended adjectives). ◇

15.3 Bruhat and weak orders

Given a determinantal ideal I, one can analyze I by examining the relations between its generating minors. On the other hand, a main theme of this chapter is that one can also learn a great deal by examining the relations between ideals in a combinatorially structured family containing I as a member. In our case, the set of matrix Schubert varieties inside $M_{k\ell}$ forms a poset under inclusion. The first goal of this section is to analyze this poset, to get a criterion on a permutation v and a partial permutation w for when $\overline{X}_v$ contains $\overline{X}_w$ (Proposition 15.23). Then we define a related but weaker order on the partial permutations in $M_{k\ell}$ (Definition 15.24). These partial orders will result in our being able (in Section 15.4) to calculate the dimensions of matrix Schubert varieties, and to show that they are irreducible, meaning that the radical of I_w is a prime ideal.

Definition 15.18 For $k\times\ell$ partial permutations v and w, write $v \leq w$ and say that v **precedes** w **in Bruhat order** if $\overline{X}_v$ contains $\overline{X}_w$.

Thus the Bruhat partial order reverses the partial order by containment.

Lemma 15.19 *In Bruhat order, $v \leq w$ if and only if w lies in $\overline{X}_v$. In other words, $v \leq w$ if and only if* $\mathrm{rank}(v_{p\times q}) \geq \mathrm{rank}(w_{p\times q})$ *for all p, q.*

Proof. Clearly $v \leq w$ implies $w \in \overline{X}_v$. For the other direction, note that $w \in \overline{X}_v$ implies $\mathrm{rank}(Z_{p\times q}) \leq \mathrm{rank}(w_{p\times q}) \leq \mathrm{rank}(v_{p\times q})$ for all $Z \in \overline{X}_w$. □

This motivates us to consider the array $r(v)-r(w)$, whose entry at (p,q) is the integer $r_{pq}(v) - r_{pq}(w)$ that is nonnegative if $v \leq w$. Let us say that (p,q) is a *southeast corner* in $r(v)-r(w)$ if $v \leq w$, the entry $r_{pq}(v)-r_{pq}(w)$ is strictly positive, and $r_{ij}(v) - r_{ij}(w) = 0$ whenever $(p,q) < (i,j)$.

Lemma 15.20 *Let v be an $n\times n$ permutation, and $(p,q) \neq (n,n)$ a southeast corner in $r(v)-r(w)$. Then $(p+1, q+1) \leq (n,n)$, and $v(p+1) = q+1$.*

Proof. Suppose $r_{in}(v) > r_{in}(w)$ for some row index i, so w has less than i nonzero entries in its first i rows. Then w has less than i' nonzero entries in its top i' rows for all $i' \geq i$. Since $r_{in}(v) = i$ for all $i = 1, \ldots, n$, the nonzero entries of $r(v) - r(w)$ run all the way down the right column from (i,n) to (n,n). Hence $q+1 \leq n$; transposing the argument shows $p+1 \leq n$.

In general, $\mathrm{rank}(w_{(i+1)\times j})$ and $\mathrm{rank}(w_{i\times(j+1)})$ can each equal either $r_{ij}(w)$ or $1+r_{ij}(w)$; the same of course holds for v. Therefore, (p,q) being a southeast corner means that $r(w)$ increases as it passes south one unit from (p,q) to $(p+1,q)$, as well as east one unit from (p,q) to $(p,q+1)$, while $r(v)$ remains constant. The south passage implies that $w(p+1) = j$ for some $j \leq q$, while $v(p+1) \geq q+1$; the east passage implies that $w(i) = q+1$ for some $i \leq p$, while $v^{-1}(q+1) \geq p+1$. Since $r_{p+1,q+1}(v) - r_{p+1,q+1}(w) \geq 0$, we must have $v(p+1) = q+1$. □

Let $\tau_{i,i'}$ be the operator switching rows i and i' in partial permutations.

Lemma 15.21 *Fix a $k\times \ell$ partial permutation matrix v with nonzero entries $v(i) = j$ and $v(i') = j'$. If $(i,j) \leq (i',j')$, then the following hold.*

1. *$l(\tau_{i,i'}v) = l(v) + 1 +$ twice the number of nonzero entries of v strictly inside the rectangle enclosed by (i,j) and (i',j').*
2. *$r_{pq}(\tau_{i,i'}v) = r_{pq}(v)$ unless (p,q) lies inside the rectangle enclosed by (i,j) and $(i'-1, j'-1)$, in which case $r_{pq}(\tau_{i,i'}v) = r_{pq}(v) - 1$.*

Proof. Outside the mentioned rectangle, the diagram stays the same after the row switch. Inside the rectangle, nothing changes except that before the switch, no boxes in the diagram lie across the top edge or down the left edge, whereas after the switch, no boxes in the diagram lie on the bottom or right edges. In the process, new boxes appear at the upper-left corner, as well as above the top of and to the left of every nonzero entry inside the rectangle. Boxes already along the bottom row or right column of the rectangle before the switch move instead to the top or left. See Fig. 15.1 for an illustrative example, where the notation follows that of Example 15.14.

The claim concerning $r_{pq}(v)$ is easier, and left as an exercise. □

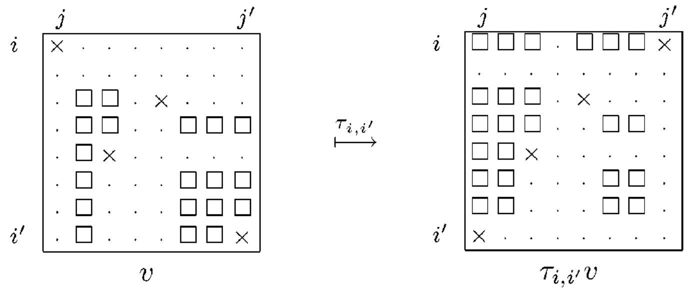


Figure 15.1: Change in length under switching rows

Lemma 15.22 *Fix a $k \times \ell$ partial permutation matrix w with a nonzero entry $w(i) = j$, and a zero row i'. If $i < i'$ then the following hold.*

1. *$l(\tau_{i,i'}w) = l(w) + 1 + 2e + f$, where e is the number of nonzero entries of w inside the rectangle enclosed by $(i+1, j+1)$ and $(i'-1, \ell)$, whereas f is the number of zero rows of w indexed by $i+1, \ldots, i'-1$.*
2. *$r_{pq}(\tau_{i,i'}w) = r_{pq}(w)$ unless (p,q) lies inside the rectangle enclosed by (i,j) and $(i'-1, \ell)$, in which case $r_{pq}(\tau_{i,i'}w) = r_{pq}(w) - 1$.*

Proof. Essentially the same as that of Lemma 15.21, except that the zero rows have no boxes in column $\ell + 1$ to move back to column j. □

For $k \times \ell$ partial permutations w and w', locations of nonzero entries in w that are also locations of nonzero entries in w' are said to be *shared* with v. Write $w \subseteq w'$ if all locations of nonzero entries in w are shared with w'.

Proposition 15.23 *Fix an $n \times n$ permutation v and an $n \times n$ partial permutation w. If $v < w$, then at least one of the following must hold.*

1. *$v < \tau_{i,i'}v \leq w$ for some transposition $\tau_{i,i'}$; or*
2. *$v \leq \tau_{i,i'}w < w$ for some transposition $\tau_{i,i'}$; or*
3. *a nonzero entry can be added to w, yielding $w \subset w'$ with $v \leq w' < w$.*

Proof. Suppose that v agrees with w in rows $1, \ldots, i-1$, but not in row i. Setting $j = v(i)$ we find, by comparing rank conditions along row i, that either (a) row i of w has a nonzero entry in some column strictly to the right of column j, or (b) row i of w is zero. In case (a), there is guaranteed to be a nonzero entry of v strictly southeast of (i,j), because there must be such an entry in column $w(i)$. In case (b), one of two things can happen: (b′) there is some nonzero entry of v strictly southeast of (i,j); or (b″) not. In all cases, $r_{ij}(v) - r_{ij}(w)$ is greater than zero by construction.

Treat cases (a) and (b′) together, by choosing a nonzero entry (i', j') in v with i' minimal. Since v is a permutation, it has nonzero entries in rows $i+1, \ldots, i'-1$, and these lie west of column j by minimality of i'. Given that v and w agree in rows $1, \ldots, i-1$, the function $r_{pq}(v)$ therefore increases at least as fast as $r_{pq}(w)$ does, for (p, q) in the rectangle bounded by (i, j) and $(i' - 1, j' - 1)$. It follows that $r_{pq}(v) - r_{pq}(w)$ is nonzero in this rectangle, because $r_{ij}(v)$ is nonzero. Part 2 of Lemma 15.20 allows us to conclude that $v < \tau_{i,i'}v$ and $\tau_{i,i'}v \leq w$.

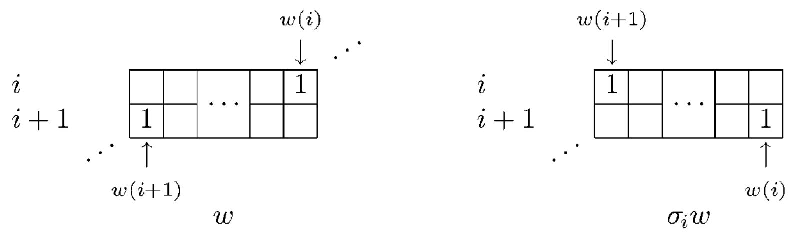
In case (b$''$), the only southeast corner of $r(v) - r(w)$ that lies southeast of (i, j) is (n, n), by Lemma 15.20, so $r(v) - r(w)$ is strictly positive in the rectangle enclosed by (i, j) and (n, n). If w has a nonzero entry in row i' of column j, then use part 2 of Lemma 15.22. Otherwise, add a nonzero entry to w at (i, j), which adds 1 to $r_{pq}(w)$ when (p, q) lies in the rectangle. □

The Bruhat order is important in combinatorics, geometry, and representation theory. Usually, it is applied only when both partial permutations are honest square permutations. Restricting to that case would have made the statement and proof of Proposition 15.23 substantially simpler. However, the extra generality will come in handy in the process of calculating the dimensions of matrix Schubert varieties, particularly in Lemma 15.29.

That being said, we shall be even more concerned with an equally important partial order that has fewer relations than Bruhat order.

Definition 15.24 Let v and w be $k \times \ell$ partial permutations. The **adjacent transposition** σ_i for $i < k$ takes w to the result $\sigma_i w$ of switching rows i and $i + 1$ of w. If $l(v) = l(w) - 1$ and either $v = \sigma_i w$ or v differs from w only in row k, then v is **covered** by w **in weak order**. More generally, v **precedes** w **in weak order** if $v = v_0 < v_1 < \cdots < v_{r-1} < v_r = w$ is a sequence of covers, so in particular, $l(v) = l(w) - r$.

The operator σ_i should be thought of as simply the $k \times k$ permutation matrix for σ_i, whose only off-diagonal unit entries are at $(i, i + 1)$ and $(i + 1, i)$. As such, σ_i acts on all of $M_{k\ell}$, as well as on its coordinate ring $\mathbb{k}[\mathbf{x}] = \mathbb{k}[M_{k\ell}]$. Lemmas 15.21 and 15.22 imply that $l(\sigma_i w)$ must equal either $l(w)+1$ or $l(w)-1$, as long as rows i and $i + 1$ of w are not both zero (in which case $\sigma_i w = w$). When rows i and $i + 1$ of w are both nonzero, the hypothesis $\sigma_i w < w$ means that w and $\sigma_i w$ look heuristically like



(15.1)

between columns $w(i + 1)$ and $w(i)$ in rows i and $i + 1$.

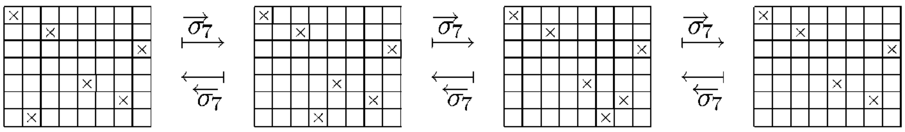
Remark 15.25 Weak order is usually considered only as a partial order on honest $n \times n$ permutation matrices, where the only covers are given by transposition of adjacent rows. Indeed, distinct permutation matrices must differ in at least two rows.

The other kind of cover, that of altering only row k, also has a concrete (though longer) description, through operators with suggestive names $\overleftarrow{\sigma}_k$ and $\overrightarrow{\sigma}_k$. Roughly, these move nonzero entries in row k to the left and right, respectively, as little as possible. More precisely, they act on $k\times\ell$ partial permutations w with the convention that

- if w has zero last row, then $\overrightarrow{\sigma}_k$ has no effect, while $\overleftarrow{\sigma}_k$ adds a nonzero entry in the last zero column (if there is one); otherwise,
- $\overrightarrow{\sigma}_k$ moves the nonzero entry in row k of w to the next column (to the right) in which w has a zero column, unless there is no such next column, in which case $\overrightarrow{\sigma}_k$ sets the last row to zero; and
- $\overleftarrow{\sigma}_k$ moves the nonzero entry of w in its last row to the bottom of the previous zero column (if it has one).

When parenthesized 'if' clauses are not satisfied, $\overleftarrow{\sigma}_k$ and $\overrightarrow{\sigma}_k$ have no effect. Note that σ_i for $i < k$ can also have no effect, if rows i and $i+1$ are zero.

Example 15.26 The operators $\overrightarrow{\sigma}_7$ and $\overleftarrow{\sigma}_7$ move the bottom $\times$ as follows:



If we had chosen a tall and thin ambient rectangle, then it would be possible to have a blank last row and no blank columns (this would fail to satisfy the first of the two parenthesized 'if' clauses). To understand why these operators must work the way they have been defined, draw the diagrams of these partial permutations and compare the numbers of boxes in them. $\diamond$

We have written $v < w$ for covers in weak order because these are also relations in Bruhat order; this follows from Lemmas 15.21 and 15.22 along with an easy calculation for row k covers (by $\overleftarrow{\sigma}_k$ or $\overrightarrow{\sigma}_k$).

15.4 Borel group orbits

Matrix Schubert varieties are clearly stable under separate rescaling of each row or column. Moreover, since they only impose rank conditions on submatrices that are as far north and west as possible, any operation that adds a multiple of some row to a row below it ("sweeping downward"), or that adds a multiple of some column to another column to its right ("sweeping to the right") preserves every matrix Schubert variety.

In terms of group theory, let B denote the Borel group of invertible *lower* triangular $k\times k$ matrices and B_+ the invertible upper triangular $\ell\times\ell$ matrices. (Borel groups appeared briefly in Chapter 2.) The previous paragraph says exactly that matrix Schubert varieties $\overline{X}_w$ are preserved by the action of $B \times B_+$ on $M_{k\ell}$ in which $(b, b_+) \cdot Z = bZb_+^{-1}$. This is a left group action, in the sense that $(b, b_+) \cdot ((b', b'_+) \cdot Z)$ equals $((b, b_+) \cdot (b', b'_+)) \cdot Z$ instead of $((b', b'_+) \cdot (b, b_+)) \cdot Z$, even though—in fact because—the b_+ acts via its inverse on the *right*. We will get a lot of mileage out of the following fact, which implies that $B \times B_+$ has finitely many orbits on $M_{k\ell}$.

Proposition 15.27 *In each orbit of $B \times B_+$ on $M_{k\ell}$ lies a unique partial permutation w, and the orbit $\mathcal{O}_w$ through w is contained inside $\overline{X}_w$.*

Proof. Row and column operations that sweep down and to the right can get us from an arbitrary matrix Z to a partial permutation matrix w. Such sweeping preserves the ranks of northwest $p \times q$ submatrices. This proves uniqueness of the partial permutation w in its orbit, and also shows that the minors cutting out $\overline{X}_w$ vanish on $\mathcal{O}_w$. □

Lemma 15.28 *Set $w' = \tau_{i,i'} w$, where $i < i'$. If $w < w'$, then the closure $\overline{\mathcal{O}}_{w'}$ of the orbit through w' in $M_{k\ell}$ is properly contained inside $\overline{\mathcal{O}}_w$. The same proper inclusion of orbit closures holds for weak order covers $w < w'$.*

Proof. Let t be an invertible parameter. View each of the following equations as a possible scenario occurring in the two rows i, i' and two columns j, j' in equations $b(t) \cdot w \cdot b_+(t)^{-1} = w(t)$, by inserting appropriate extra identity rows and columns into $b(t)$ and $b_+(t)^{-1}$ while completing the middle matrices to w:

$$\begin{bmatrix} 1 & 0 \\ t^{-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 1 & 0 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 \\ t^{-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t & 1 \\ 0 & -t^{-1} \end{bmatrix} = \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix}.$$

Each equation yields a 1-parameter family of matrices in $\mathcal{O}_w = BwB_+$. The limit at $t = 0$ is w' in both cases, as seen from the right hand sides.

For weak order covers $w < w' = \overrightarrow{\sigma}_k w$ moving a nonzero entry in column j of row k to column j', simply "sweep w to the right" by first adding column j to column j' and then multiplying column j by t. Again, taking the limit at $t = 0$ yields w'.

By the previous paragraphs, w' lies in $\overline{\mathcal{O}}_w$. Hence we get $\overline{\mathcal{O}}_{w'} \subset \overline{\mathcal{O}}_w$ because $\overline{\mathcal{O}}_w$ is stable under the action of $B \times B_+$ and closed inside $M_{k\ell}$. The containment $\overline{\mathcal{O}}_{w'} \subset \overline{\mathcal{O}}_w$ is proper because $r_{pq}(w') < r_{pq}(w)$ for some pair (p, q), so the corresponding minor vanishes on $\overline{\mathcal{O}}_{w'}$ but not on $\overline{\mathcal{O}}_w$. □

Lemma 15.29 *Given a $k \times \ell$ partial permutation w, there exists a chain $v_0 < v_1 < v_2 < \cdots < v_{k\ell-1} < v_{k\ell}$ of covers in the weak order, in which $l(v_i) = i$ for all i, and $v_{l(w)} = w$.*

Proof. It is enough to show that if $0 < l(w)$ then there exists a cover $v < w$, and if $l(w) < k\ell$ then there exists a cover $w < v$. In the former case, choose

$v = \sigma_i w$ for the row index i on any box in the essential set $\mathcal{E}ss(w)$, using $v = \overleftarrow{\sigma}_k w$ if $i = k$. In the latter case we have $w \neq 0$, so choose $v = \sigma_i w$ for the row index i on the lowest nonzero entry of w, using $v = \overrightarrow{\sigma}_k w$ if $i = k$. □

Proposition 15.30 *If w is a $k \times \ell$ partial permutation, then the orbit closure $\overline{\mathcal{O}}_w$ is an irreducible variety of dimension* $\dim(\overline{\mathcal{O}}_w) = k\ell - l(w)$.

Proof. The map $B \times B_+ \to \mathcal{O}_w$ that expresses $\mathcal{O}_w$ as an orbit of $B \times B_+$ takes $(b, b_+) \mapsto bwb_+^{-1}$. This map of varieties induces a homomorphism $\Bbbk[M_{k\ell}] \to \Bbbk[B \times B_+]$ in which the target is a domain and the kernel is the ideal of $\overline{\mathcal{O}}_w$. Hence the ideal of $\overline{\mathcal{O}}_w$ is prime so $\overline{\mathcal{O}}_w$ is irreducible.

A weak order chain as in Lemma 15.29 gives a corresponding chain of prime ideals of orbits, properly containing one another, as in the second sentence of Lemma 15.28. Since the polynomial ring $\Bbbk[M_{k\ell}]$ itself has dimension $k\ell$, the part of this chain of primes consisting of those containing $I(\overline{\mathcal{O}}_w)$ must have maximal length among all chains of primes containing $I(\overline{\mathcal{O}}_w)$. Lemma 15.29 therefore implies that $\overline{\mathcal{O}}_w$ has Krull dimension $k\ell - l(w)$. □

Next comes the result toward which we have been building for the last two sections. To say that w is a *smooth point* means that localizing at

$$\mathfrak{m}_w \quad = \quad \langle x_{\alpha\beta} \mid w(\alpha) \neq \beta \rangle + \langle x_{\alpha\beta} - 1 \mid w(\alpha) = \beta \rangle, \qquad (15.2)$$

its maximal ideal, yields a regular local ring [Eis95, Section 10.3].

Theorem 15.31 *Let w be a $k \times \ell$ partial permutation. The matrix Schubert variety $\overline{X}_w$ is the closure $\overline{\mathcal{O}}_w$ of the $B \times B_+$ orbit through $w \in M_{k\ell}$, and is irreducible of dimension $k\ell - l(w)$. The matrix w is a smooth point of $\overline{X}_w$.*

Proof. Every point on an orbit $\mathcal{O}$ of an algebraic group is a smooth point of $\mathcal{O}$, because $\mathcal{O}$ has a smooth point [Har77, Theorem I.5.3], and the group action is transitive on $\mathcal{O}$. Hence the smoothness at w follows from the rest.

Let $\widetilde{w}$ be the extension of w to a permutation as in Proposition 15.8. If the theorem holds for $\widetilde{w}$, then the irreducibility and dimension count hold for $\overline{X}_w$ by Remark 15.11. Since the orbit closure $\overline{\mathcal{O}}_w$ is closed and has dimension $k\ell - l(w)$ by Proposition 15.30, the containment in Proposition 15.27 implies the whole theorem for w. Hence we assume that $w = \widetilde{w}$.

The irreducibility and dimension count follow from Proposition 15.30 as soon as we show that $\overline{X}_w = \overline{\mathcal{O}}_w$. For this, Proposition 15.27 plus the stability of $\overline{X}_w$ under $B \times B_+$ imply that $\overline{X}_w$ is the union of the orbits $\overline{\mathcal{O}}_{w'}$ through the partial permutations $w' \in \overline{X}_w$, so we need only show that these partial permutations all lie in $\overline{\mathcal{O}}_w$. Indeed, then we can conclude that $\overline{X}_w$ is contained in $\overline{\mathcal{O}}_w$, and hence equal to $\overline{\mathcal{O}}_w$ because $\mathcal{O}_w \subseteq \overline{X}_w \subseteq \overline{\mathcal{O}}_w$.

Since $w' \in \overline{X}_w$ if and only if $w \leq w'$ by Lemma 15.19, we must show that $w \leq w'$ implies $w' \in \overline{\mathcal{O}}_w$. Proposition 15.23 says that w' is obtained from w by sequentially applying some length-reducing transpositions and setting entries to zero. The transpositions stay inside $\overline{\mathcal{O}}_w$ by Lemma 15.28.

Setting entries of partial permutation matrices to zero stays inside $\overline{\mathcal{O}}_w$ because $\overline{\mathcal{O}}_w$ is stable under independent scaling of each row or column. □

Corollary 15.32 *If $v < w$ in Bruhat order then $l(v) < l(w)$.*

This section concludes with some results about *boundary components* of matrix Schubert varieties $\overline{X}_v$, meaning components of $\overline{X}_v \smallsetminus \mathcal{O}_v$, to be used in the proof of Proposition 15.37 (which is crucial for Theorem 15.40). Once we see that $\overline{X}_{\sigma_i w}$ for $\sigma_i w < w$ is fixed under multiplication by the permutation matrix σ_i, the subsequent lemmas say that all of its boundary components except $\overline{X}_w$ are fixed by σ_i, and that the variable $x_{i+1,w(i+1)}$ maps to a generator of the maximal ideal in the local ring of $\sigma_i w$ in $\overline{X}_{\sigma_i w}$.

Corollary 15.33 *Let w be a $k \times \ell$ partial permutation and fix $i < k$. If $\sigma_i w < w$, then $\sigma_i(\overline{X}_{\sigma_i w}) = \overline{X}_{\sigma_i w}$.*

Proof. Let's let $B \times B_+$ act on $\sigma_i(\overline{X}_{\sigma_i w})$, and take the closure. As $\sigma_i(\overline{X}_{\sigma_i w})$ is irreducible, its image under the morphism $\mu : B \times B_+ \times \sigma_i(\overline{X}_{\sigma_i w}) \to M_{k\ell}$ is irreducible. Since $B \times B_+$ has only finitely many orbits in $M_{k\ell}$, and the image of μ is stable under the $B \times B_+$ action, the closure of the image of μ is a matrix Schubert variety. By Theorem 15.31 we need only check that its codimension is $l(\sigma_i w)$, because $\sigma_i w \in \sigma_i \overline{X}_{\sigma_i w}$ (apply σ_i to $w \in \overline{X}_{\sigma_i w}$).

Every partial permutation $w' \in \overline{X}_{\sigma_i w}$ other than $\sigma_i w$ has length greater than $l(\sigma_i w)$. Since $l(\sigma_i w')$ is at least $l(w') - 1$, and $l(\sigma_i \sigma_i w) = l(w) > l(\sigma_i w)$, every partial permutation in $\sigma_i \overline{X}_{\sigma_i w}$ has length at least $l(\sigma_i w)$. □

Remark 15.34 Corollary 15.33 is really a combinatorial statement about weak order, as the second paragraph of its proof indicates. It is equivalent to the statement: if $w < \sigma_i w$, then $w < v$ if and only if $w < \sigma_i v$.

Lemma 15.35 *Let v be an $n \times n$ partial permutation and w an $n \times n$ permutation with $\sigma_i w < w$ and $\sigma_i w < v$. If $l(v) = l(w)$, then $\overline{X}_v$ has codimension 1 inside $\overline{X}_{\sigma_i w}$, and $\overline{X}_v$ is mapped to itself by σ_i unless $v = w$.*

Proof. The codimension statement comes from Theorem 15.31. Using Proposition 15.23, we find that v is obtained from $\sigma_i w$ either by switching a pair of rows or deleting a single nonzero entry from $\sigma_i w$.

Any 1 that we delete from $\sigma_i w$ must have no 1's southeast of it, or else the length increases by more than one. Thus the 1 in row i of $\sigma_i w$ cannot be deleted, by (15.1), leaving us in the situation of Corollary 15.33 with $v = w$, and completing the case where an entry of $\sigma_i w$ has been set to zero.

Suppose now that v is obtained by switching rows p and p' of $\sigma_i w$, and assume that $\sigma_i(\overline{X}_v) \neq \overline{X}_v$. Then $v(i) > v(i+1)$ by Corollary 15.33. At least one of p and p' must lie in $\{i, i+1\}$ because moving neither row p nor row p' of $\sigma_i w$ leaves $v(i) < v(i+1)$. On the other hand, it is impossible for exactly one of p and p' to lie in $\{i, i+1\}$; indeed, since switching rows p

and p' increases length, either the 1 at $(i, w(i+1))$ or the 1 at $(i+1, w(i))$ would lie in the rectangle formed by the switched 1's, making $l(v)$ too big by Lemma 15.21. Thus $\{p, p'\} = \{i, i+1\}$ and $v = w$, completing the proof. □

Lemma 15.36 *Let w be an $n \times n$ permutation with $\sigma_i w < w$. If $\mathfrak{m} = \mathfrak{m}_{\sigma_i w}$ is the maximal ideal of $\sigma_i w \in \overline{X}_{\sigma_i w}$, then the variable $x_{i+1,w(i+1)}$ maps to $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ under the natural map $\Bbbk[\mathbf{x}] \to (\Bbbk[\mathbf{x}]/I(\overline{X}_{\sigma_i w}))_{\mathfrak{m}}$.*

Proof. Let v be the permutation $\sigma_i w$, and consider the map $B \times B_+ \to M_{nn}$ sending $(b, b^+) \mapsto bvb^+$. The image of this map is the orbit $\mathcal{O}_v \subset \overline{X}_v$, and the identity $\mathrm{id} := (\mathrm{id}_B, \mathrm{id}_{B_+})$ maps to v. The induced map of local rings the other way thus takes $\mathfrak{m}_v$ to the maximal ideal

$$\mathfrak{m}_{\mathrm{id}} \quad := \quad \langle b_{ii} - 1, b^+_{ii} - 1 \mid 1 \le i \le n \rangle + \langle b_{ij}, b^+_{ji} \mid i > j \rangle$$

in the local ring at the identity $\mathrm{id} \in B \times B_+$. It is enough to demonstrate that the image of $x_{i+1,w(i+1)}$ lies in $\mathfrak{m}_{\mathrm{id}} \smallsetminus \mathfrak{m}^2_{\mathrm{id}}$.

Direct calculation shows that $x_{i+1,w(i+1)}$ maps to the entry

$$b_{i+1,i} b^+_{w(i+1),w(i+1)} + \sum_{p \in P} b_{i+1,p} b^+_{p,w(i+1)}$$

at $(i+1, w(i+1))$ in bvb_+, where $P = \{p < i \mid w(p) < w(i+1)\}$ consists of the row indices of 1's in $\sigma_i w$ northwest of $(i, w(i+1))$. In particular, all of the summands $b_{i+1,p} b^+_{p,w(i+1)}$ lie in $\mathfrak{m}^2_{\mathrm{id}}$. On the other hand, $b^+_{w(i+1),w(i+1)}$ is a unit in the local ring at id, so $b_{i+1,i} b^+_{w(i+1),w(i+1)}$ lies in $\mathfrak{m}_{\mathrm{id}} \smallsetminus \mathfrak{m}^2_{\mathrm{id}}$. □

Certain functions on matrix Schubert varieties are obviously nonzero. For instance, if v has its nonzero entries in rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$, then the minor Δ of the generic matrix $\mathbf{x}$ using those rows and columns is nowhere zero on $\mathcal{O}_v$. Therefore the zero set of Δ inside $\overline{X}_v$ is a union of its boundary components, although the multiplicities may be more than 1. The permutation σ_i acts on the coordinate ring $\Bbbk[\mathbf{x}]$ by switching rows i and $i+1$. Therefore, if Δ uses row i, then $\sigma_i \Delta$ uses row $i+1$ instead.

Proposition 15.37 *Assume $\sigma_i w < w$, set $j = w(i) - 1$, and define Δ as the minor in $\mathbf{x}$ using all rows and columns in which $(\sigma_i w)_{i \times j}$ is nonzero. The images of Δ and $\sigma_i \Delta$ in $\Bbbk[\mathbf{x}]/I(\overline{X}_{\sigma_i w})$ have equal multiplicity along every boundary component of $\overline{X}_{\sigma_i w}$ other than $\overline{X}_w$, and Δ has multiplicity 1 along the component $\overline{X}_w$. In particular, $\sigma_i \Delta$ is not the zero function on $\overline{X}_w$.*

Proof. Lemma 15.35 says that σ_i induces an automorphism of the local ring at the prime ideal of $\overline{X}_v$ inside $\overline{X}_{\sigma_i w}$, for every boundary component $\overline{X}_v$ of $\overline{X}_{\sigma_i w}$ other than $\overline{X}_w$. This automorphism takes Δ to $\sigma_i \Delta$, so these two functions have the same multiplicity along $\overline{X}_v$. The only remaining codimension 1 boundary component of $\overline{X}_{\sigma_i w}$ is $\overline{X}_w$, and we shall now verify that Δ has multiplicity 1 there.

By Theorem 15.31, the local ring of $\sigma_i w$ in $\overline{X}_{\sigma_i w}$ is regular. Since σ_i is an automorphism of $\overline{X}_{\sigma_i w}$ (Corollary 15.33), we find that the localization of $\mathbb{k}[\mathbf{x}]/I(\overline{X}_{\sigma_i w})$ at the maximal ideal $\mathfrak{m}_w$ (15.2) of w is also regular. In this localization, the variables $x_{\alpha\beta}$ corresponding to the locations of nonzero entries in $w_{i\times j}$ are units. This implies that the coefficient of $x_{i,w(i+1)}$ in Δ is a unit in the local ring of $w \in \overline{X}_{\sigma_i w}$. On the other hand, the variables in spots where w has zeros generate $\mathfrak{m}_w$. Therefore, all terms of Δ lie in the square of $\mathfrak{m}_w$ in the localization, except for the unit times $x_{i,w(i+1)}$ term produced above. Hence, to prove multiplicity one, it is enough to prove that $x_{i,w(i+1)}$ itself lies in $\mathfrak{m}_w \smallsetminus \mathfrak{m}_w^2$ or equivalently (after applying σ_i) that $x_{i+1,w(i+1)}$ lies in $\mathfrak{m}_{\sigma_i w} \smallsetminus \mathfrak{m}_{\sigma_i w}^2$. This is Lemma 15.36. □

15.5 Schubert polynomials

Having proved that matrix Schubert varieties are reduced and irreducible, let us begin to unravel their homologically hidden combinatorics. Working with multigradings here instead of the usual $\mathbb{Z}$-grading means that the homological invariants we seek possess algebraic structure themselves: they are polynomials, as opposed to the integers resulting in the $\mathbb{Z}$-graded case. The forthcoming definition will let us mine this algebraic structure to compare the multidegrees of all of the different matrix Schubert varieties by downward induction on weak order.

Definition 15.38 Let R be a commutative ring, and $\mathbf{t} = t_1, t_2, \ldots$ an infinite set of independent variables. The i^{th} **divided difference operator** ∂_i takes each polynomial $f \in R[\mathbf{t}]$ to

$$\partial_i f(t_1, t_2, \ldots) \quad = \quad \frac{f(t_1, t_2, \ldots,) - f(t_1, \ldots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \ldots)}{t_i - t_{i+1}}.$$

Letting $\mathbf{s}$ be another set of variables and $R = \mathbb{Z}[\mathbf{s}]$, the **double Schubert polynomial** for a permutation matrix w is defined recursively by

$$\mathfrak{S}_{\sigma_i w}(\mathbf{t} - \mathbf{s}) \quad = \quad \partial_i \mathfrak{S}_w(\mathbf{t} - \mathbf{s})$$

whenever $\sigma_i w < w$, and the initial conditions

$$\mathfrak{S}_{w_0}(\mathbf{t} - \mathbf{s}) \quad = \quad \prod_{i+j \leq n} (t_i - s_j)$$

for all n, where $w_0 = n \cdots 321$ is the long word in S_n. The **(ordinary) Schubert polynomial** $\mathfrak{S}_w(\mathbf{t})$ is defined by setting $\mathbf{s} = \mathbf{0}$ everywhere. For partial permutations w, define $\mathfrak{S}_w = \mathfrak{S}_{\widetilde{w}}$ as the Schubert polynomial for the minimal extension of w to a permutation (Proposition 15.8).

Example 15.39 Let w be the partial permutation matrix in Example 15.2 with $k \leq \ell$. In this classical case, the double Schubert polynomial $\mathfrak{S}_w$ is the *Schur polynomial* associated to the partition with rectangular Ferrers shape $(k-r)\times(\ell-r)$. The *Jacobi–Trudi formula* expresses $\mathfrak{S}_w(\mathbf{t}-\mathbf{s})$ as the determinant of a Hankel matrix of size $(k-r) \times (k-r)$. The (α,β)-entry in this matrix is the coefficient of $q^{\ell-r+\beta-\alpha}$ in the generating function

$$\frac{\prod_{j=1}^{\ell}(1-s_j q)}{\prod_{i=1}^{k}(1-t_i q)}. \tag{15.3}$$

This formula appears in any book on symmetric functions, e.g. [Macd95]. $\diamond$

In the definition of $\mathfrak{S}_w(\mathbf{t}-\mathbf{s})$, the operator ∂_i acts only on the $\mathbf{t}$ variables and not on the $\mathbf{s}$ variables. Checking monomial by monomial verifies that $t_i - t_{i+1}$ divides the numerator of $\partial_i(f)$, so $\partial_i(f)$ is again a polynomial, homogeneous of degree $d-1$ if f is homogeneous of degree d. Note that only finitely many variables from $\mathbf{t}$ and $\mathbf{s}$ are ever used at once. Also, setting all $\mathbf{s}$ variables to zero commutes with divided differences.

In the literature, double Schubert polynomials are usually written with $\mathbf{x}$ and $\mathbf{y}$ instead of $\mathbf{t}$ and $\mathbf{s}$; but we have used $\mathbf{x}$ throughout this book to mean coordinates on affine space, while $\mathbf{t}$ has been used for multidegrees.

Every $n \times n$ permutation matrix w can be expressed as a product $w = \sigma_{i_r}\cdots\sigma_{i_1}w_0$ of matrices, where the $n\times n$ matrix w_0 is the long word in S_n and $l(w_0)-l(w) = r$. The condition $l(w_0)-l(w) = r$ implies by definition that r is minimal, so $ww_0 = \sigma_{i_r}\cdots\sigma_{i_1}$ is what is known as a *reduced expression* for the permutation matrix ww_0. The recursion for both single and double Schubert polynomials can be summarized as $\mathfrak{S}_w = \partial_{i_r}\cdots\partial_{i_1}\mathfrak{S}_{w_0}$. More generally, if $w = \sigma_{i_r}\cdots\sigma_{i_1}v$ and $l(w) = l(v) - r$, then it holds that

$$\mathfrak{S}_w \quad = \quad \partial_{i_r}\cdots\partial_{i_1}\mathfrak{S}_v. \tag{15.4}$$

Indeed, this reduces to the case where $v = w_0$ by writing $\mathfrak{S}_v = \partial_{j_s}\cdots\partial_{j_1}w_0$.

It is not immediately obvious from Definition 15.38 that $\mathfrak{S}_w$ is well-defined, because we could have used any downward chain of covers in weak order to define $\mathfrak{S}_w$ from $\mathfrak{S}_{w_0}$. However, the well-definedness will follow from our main theorem in this chapter, Theorem 15.40. It is also a consequence of the fact that divided differences satisfy the braid relations in Exercise 15.3, which the reader is encouraged to check directly.

We are interested in a multigrading of $\Bbbk[\mathbf{x}]$ by $\mathbb{Z}^{k+\ell}$, which we take to have basis $\mathbf{t}\cup\mathbf{s}$, where $\mathbf{t} = t_1,\ldots,t_k$ and $\mathbf{s} = s_1,\ldots,s_\ell$.

Theorem 15.40 *If w is a $k\times\ell$ partial permutation and $\Bbbk[\mathbf{x}]$ is $\mathbb{Z}^{k+\ell}$-graded with $\deg(x_{ij}) = t_i - s_j$, then the matrix Schubert variety $\overline{X}_w$ has multidegree*

$$\mathcal{C}(\overline{X}_w;\mathbf{t},\mathbf{s}) \quad = \quad \mathfrak{S}_w(\mathbf{t}-\mathbf{s})$$

equal to the double Schubert polynomial for w.

Example 15.41 The multidegree of the classical determinantal variety $\overline{X}_w$ in Example 15.2 equals the Schur polynomial in Example 15.39. Replacing every t_i by t and every s_j by 0 yields the classical degree of that projective variety. This substitution replaces (15.3) by $1/(1-tq)^k$, and the (α, β)-entry of the Jacobi matrix specializes to $\binom{k+\ell-r+\beta-\alpha-1}{k-1}t^{\ell-r+\beta-\alpha}$. The determinant of this matrix (and hence the classical degree of $\overline{X}_w$) equals the number of semistandard Young tableaux of rectangular shape $(k-r) \times (\ell-r)$. This statement holds more generally for the matrix Schubert varieties associated with Grassmannians; see Exercise 16.9. $\diamond$

The proof of Theorem 15.40 will compare the zero sets of two functions on $\overline{X}_{\sigma_i w} \times \Bbbk$ with equal degrees. The zeros of the first function consist of $\overline{X}_w \times \Bbbk$ plus some boundary components, while the second function has zeros $(\sigma_i \overline{X}_w \times \Bbbk) \cup (\overline{X}_{\sigma_i w} \times \{0\})$ plus the *same* boundary components. When the (equal) multidegrees of the zero sets of our two functions are decomposed by additivity and compared, the extra components cancel.

Proof of Theorem 15.40. As the matrix Schubert varieties for w and its minimal completion to a permutation have equal multidegrees by Proposition 15.8, we assume that w is a permutation. The result for $\mathfrak{S}_{w_0}$ follows immediately from Proposition 8.49 and Example 15.3. For other permutations w we shall use downward induction on weak order.

Consider the polynomials Δ and $\sigma_i\Delta$ from Proposition 15.37 not as elements in $\Bbbk[\mathbf{x}]$, but as elements in the polynomial ring $\Bbbk[\mathbf{x}, y]$ with $k\ell+1$ variables. Setting the degree of the new variable y equal to $\deg(y) = t_i - t_{i+1}$ makes Δ and the product $y\sigma_i\Delta$ in $\Bbbk[\mathbf{x}, y]$ have the *same* degree $\delta \in \mathbb{Z}^{k+\ell}$. Since the affine coordinate ring $\Bbbk[\mathbf{x}]/I(\overline{X}_{\sigma_i w})$ of $\overline{X}_{\sigma_i w}$ is a domain, neither Δ nor $\sigma_i\Delta$ vanishes on $\overline{X}_{\sigma_i w}$, so we get two short exact sequences

$$0 \;\to\; \Bbbk[\mathbf{x}, y](-\delta)/I(\overline{X}_{\sigma_i w}) \;\xrightarrow{\Theta}\; \Bbbk[\mathbf{x}, y]/I(\overline{X}_{\sigma_i w}) \;\longrightarrow\; Q(\Theta) \;\to\; 0,$$

in which Θ equals either Δ or $y\sigma_i\Delta$. The quotients $Q(\Delta)$ and $Q(y\sigma_i\Delta)$ have equal $\mathbb{Z}^{k+\ell}$-graded Hilbert series, and hence equal multidegrees.

Note that $\Bbbk[\mathbf{x}, y]$ is the coordinate ring of $M_{k\ell} \times \Bbbk$. The minimal primes of $Q(\Delta)$ all correspond to varieties $\overline{X}_v \times \Bbbk$ for boundary components $\overline{X}_v$ of $\overline{X}_{\sigma_i w}$. Similarly, almost all minimal primes of $Q(y\sigma_i\Delta)$ correspond by Proposition 15.37 to varieties $\overline{X}_v \times \Bbbk$. The only exceptions are $\overline{X}_{\sigma_i w} \times \{0\}$, because of the factor y, and the image $\sigma_i \overline{X}_w \times \Bbbk$ of $\overline{X}_w \times \Bbbk$ under the automorphism σ_i. As a consequence of Proposition 15.37, the multiplicity of $y\sigma_i\Delta$ along $\sigma_i\overline{X}_w \times \Bbbk$ equals 1, just as Δ has multiplicity 1 along $\overline{X}_w \times \Bbbk$.

Now break up the multidegrees of $Q(\Delta)$ and $Q(y\sigma_i\Delta)$ into sums over components of top dimension, by additivity in Theorem 8.53. Proposition 15.37 implies that almost all terms in the equation $\mathcal{C}(Q(\Delta); \mathbf{t}, \mathbf{s}) = \mathcal{C}(Q(y\sigma_i\Delta); \mathbf{t}, \mathbf{s})$ cancel; the only terms that remain yield the equation

$$\mathcal{C}(\overline{X}_w \times \Bbbk\,; \mathbf{t}, \mathbf{s}) \;=\; \mathcal{C}(\sigma_i\overline{X}_w \times \Bbbk\,; \mathbf{t}, \mathbf{s}) + \mathcal{C}(\overline{X}_{\sigma_i w} \times \{0\}\,; \mathbf{t}, \mathbf{s}) \quad (15.5)$$

on multidegrees. Since the equations in $\Bbbk[\mathbf{x}, y]$ for $\overline{X}_w \times \Bbbk$ are the same as those for $\overline{X}_w$ in $\Bbbk[\mathbf{x}]$, the K-polynomials of $\overline{X}_w$ and $\overline{X}_w \times \Bbbk$ agree. Hence the multidegree on the left side of (15.5) equals $\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s})$. For the same reason, the first multidegree on the right side of (15.5) equals the result $\sigma_i\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s})$ of switching t_i and t_{i+1} in the multidegree of $\overline{X}_w$.

The equations defining $\overline{X}_{\sigma_i w} \times \{0\}$, on the other hand, are those defining $\overline{X}_{\sigma_i w}$ along with the equation $y = 0$. The K-polynomial of $\overline{X}_{\sigma_i w} \times \{0\}$ therefore equals $(t_i/t_{i+1})\mathcal{K}(\overline{X}_{\sigma_i w}; \mathbf{t}, \mathbf{s})$, which is the "exponential weight" t_i/t_{i+1} of y times the K-polynomial of $\overline{X}_{\sigma_i w}$. Therefore the second multidegree on the right side of (15.5) equals $(t_i - t_{i+1})\mathcal{C}(\overline{X}_{\sigma_i w}; \mathbf{t}, \mathbf{s})$.

Substituting these multidegree calculations into (15.5), we find that

$$\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s}) \quad = \quad \sigma_i\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s}) + (t_i - t_{i+1})\mathcal{C}(\overline{X}_{\sigma_i w}; \mathbf{t}, \mathbf{s})$$

as polynomials in $\mathbf{t}$ and $\mathbf{s}$. Subtracting $\sigma_i\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s})$ from both sides and dividing through by $t_i - t_{i+1}$ yields $\partial_i\mathcal{C}(\overline{X}_w; \mathbf{t}, \mathbf{s}) = \mathcal{C}(\overline{X}_{\sigma_i w}; \mathbf{t}, \mathbf{s})$. □

Example 15.42 The first five of the six 3×3 matrix Schubert varieties in Example 15.4 have $\mathbb{Z}^{3+3}$-graded multidegrees that are products of expressions having the form $t_i - s_j$ by Proposition 8.49. They are, in the order they appear in Example 15.4: 1, $t_1 - s_1$, $(t_1 - s_1)(t_1 - s_2)$, $(t_1 - s_1)(t_2 - s_1)$, and $(t_1 - s_1)(t_1 - s_2)(t_2 - s_1)$. This last one is $\mathcal{C}(\overline{X}_{321}; \mathbf{t}, \mathbf{s})$, and applying $\partial_2\partial_1$ to it yields the multidegree

$$\mathcal{C}(\overline{X}_{132}; \mathbf{t}, \mathbf{s}) \quad = \quad t_1 + t_2 - s_1 - s_2$$

of $\overline{X}_{132}$, as the reader should check. ◇

Example 15.43 The ideal I_{2143} from Example 15.7 equals $I(\overline{X}_{2143})$, since it has a squarefree initial ideal $\langle x_{11}, x_{13}x_{22}x_{31}\rangle$ and is therefore a radical ideal. The multidegree of $\overline{X}_{2143}$ is the double Schubert polynomial

$$\begin{aligned}
&\mathfrak{S}_{2143}(\mathbf{t} - \mathbf{s}) \\
&\quad = \quad \partial_2\partial_1\partial_3\partial_2\big((t_1 - s_3)(t_1 - s_2)(t_1 - s_1)(t_2 - s_2)(t_2 - s_1)(t_3 - s_1)\big) \\
&\quad = \quad \partial_2\partial_1\partial_3\big((t_1 - s_3)(t_1 - s_2)(t_1 - s_1)(t_2 - s_1)(t_3 - s_1)\big) \\
&\quad = \quad \partial_2\partial_1\big((t_1 - s_3)(t_1 - s_2)(t_1 - s_1)(t_2 - s_1)\big) \\
&\quad = \quad \partial_2\big((t_1 - s_1)(t_2 - s_1)(t_1 + t_2 - s_2 - s_3)\big) \\
&\quad = \quad (t_1 - s_1)(t_1 + t_2 + t_3 - s_1 - s_2 - s_3).
\end{aligned}$$

Compare this to the multidegree of $\Bbbk[\mathbf{x}_{4\times 4}]/\langle x_{11}, x_{13}x_{22}x_{31}\rangle$. ◇

Setting $\mathbf{s} = \mathbf{0}$ in Theorem 15.40 yields the "ordinary" version.

Corollary 15.44 *If w is a $k \times \ell$ partial permutation and $\Bbbk[\mathbf{x}]$ is $\mathbb{Z}^k$-graded with $\deg(x_{ij}) = t_i$, then the multidegree of the matrix Schubert variety $\overline{X}_w$ equals the ordinary Schubert polynomial for w: $\mathcal{C}(\overline{X}_w; \mathbf{t}) = \mathfrak{S}_w(\mathbf{t})$.*

Remark 15.45 The K-polynomials of matrix Schubert varieties satisfy similarly nice recursions under the so-called *isobaric divided differences* (or *Demazure operators*) $f \mapsto -\partial_i(t_{i+1}f)$; see the Notes to this chapter.

Exercises

15.1 Prove that the unique finest multigrading on $\mathbb{k}[\mathbf{x}]$ in which all Schubert determinantal ideals I_w are homogeneous is the $\mathbb{Z}^{k+l}$-grading here. Prove that this multigrading is also universal for the set of classical determinantal ideals.

15.2 Express the ideal I in Exercise 8.4 as an ideal of the form $I(\overline{X}_w)$. Compute the multidegree of $\mathbb{k}[\mathbf{x}]/I$ for the $\mathbb{Z}^{4+4}$-grading $\deg(x_{ij}) = t_i - s_j$, and show that it specializes to the $\mathbb{Z}^4$-graded multidegrees you computed in Exercise 8.4.

15.3 Verify that divided difference operators ∂_i satisfy the relations $\partial_i\partial_j = \partial_j\partial_i$ for $|i-j| \geq 2$, and the **braid relations**, which say that $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.

15.4 Using cycle notation, let $v = (n \cdots 321)$ be the permutation cycling $n, \ldots, 1$.

(a) Write down generators for the Schubert determinantal ideal I_v.
(b) Calculate that $\mathfrak{S}_v(\mathbf{t}) = t_1^{n-1}$. Hint: Don't use divided differences.

15.5 Let I be the ideal of maximal minors in the generic $k \times \ell$ matrix, where $k \leq \ell$, and let X be the zero set of I in $M_{k\ell}$, so X consists of the singular $k \times \ell$ matrices.

(a) Prove that I has the same minimal generators as I_w for the $(\ell+1) \times (\ell+1)$ permutation $w = \sigma_k \cdots \sigma_2\sigma_1 v$, for v as in Exercise 15.4 with $n = \ell + 1$.
(b) Deduce using Eq. (15.4) that $\mathcal{C}(X;\mathbf{t}) = h_{\ell+1-k}(t_1, \ldots, t_k)$ is the complete homogeneous symmetric function of degree $\ell + 1 - k$ in k variables.
(c) Conclude that X has ordinary $\mathbb{Z}$-graded degree $\binom{\ell}{k-1}$.

15.6 An $n \times n$ permutation w is **grassmannian** if it has at most one descent—that is, if $w(k) > w(k+1)$ for at most one value of $k < n$. Show that a permutation is grassmannian with descent at k if and only if its essential set lies along row k. Describe the Schubert determinantal ideals for grassmannian permutations.

15.7 Consider positive integers $i_1 < \cdots < i_m \leq k$ and $j_1 < \cdots < j_m \leq \ell$, and let $\mathbf{x}$ be the $k \times \ell$ matrix of variables. Find a partial permutation w such that I_w is generated by the size $m+1$ minors of $\mathbf{x}$ along with the union over $r = 1, \ldots, m$ of the minors of size r in the top $i_r - 1$ rows of $\mathbf{x}$ and the minors of size r in the left $j_r - 1$ columns of $\mathbf{x}$. Compute the extension of w to a permutation in $S_{k+\ell}$.

15.8 Prove that the Bruhat poset is a graded poset, with rank function $w \mapsto l(w)$.

15.9 Write down explicitly the degree δ in the proof of Theorem 15.40.

15.10 Let w be a permutation matrix. Show that $\mathfrak{S}_{w^{-1}}(\mathbf{s}-\mathbf{t})$ can be expressed as $(-1)^{l(w)}\mathfrak{S}_w((-\mathbf{t})-(-\mathbf{s}))$. In other words, $\mathfrak{S}_{w^{-1}}(\mathbf{s}-\mathbf{t})$ is obtained by substituting each variable with its negative in the argument of $(-1)^{l(w)}\mathfrak{S}_w(\mathbf{t}-\mathbf{s})$. Hint: consider the rank conditions transpose to those determined by w.

15.11 Consider divided difference operators ∂_i' that act only on $\mathbf{s}$ variables instead of on $\mathbf{t}$ variables. Deduce from Theorem 15.40 applied to the transpose of w that $\mathfrak{S}_w(\mathbf{t}-\mathbf{s})$ can be obtained (with a global sign factor of $(-1)^{l(w)}$) from $\mathfrak{S}_{w_0}(\mathbf{t}-\mathbf{s})$ by using the divided differences ∂_i' in the $\mathbf{s}$ variables.

15.12 As in Exercise 15.5, let X be the variety of singular $k \times \ell$ matrices, where we assume $k \leq \ell$. This time, though, use the multigrading of $\mathbb{k}[\mathbf{x}]$ by $\mathbb{Z}^\ell$ in which $\deg(x_{ij}) = s_j$. Prove that $\mathcal{C}(X;\mathbf{s}) = e_{\ell+1-k}(s_1, \ldots, s_\ell)$ is an elementary symmetric function, and conclude again that X has $\mathbb{Z}$-graded degree $\binom{\ell}{k-1}$.

15.13 Let f and g be polynomials in $R(t_1, \ldots, t_n)$ over a commutative ring R.

(a) Prove that if f is symmetric in t_i and t_{i+1} then $\partial_i(fg) = f\partial_i(g)$.
(b) Deduce that f is symmetric in t_i and t_{i+1} if and only if $\partial_i f = 0$.
(c) Show that $\partial_i f$ is symmetric in t_i and t_{i+1}.
(d) Conclude that $\partial_i^2 = \partial_i \circ \partial_i$ is the zero operator, so $\partial_i^2 f = 0$ for all f.

15.14 For a permutation w, let $m+w$ be the result of letting w act in the obvious way on $m+1, m+2, m+3, \ldots$ instead of $1, 2, 3, \ldots$, so $m+w$ fixes $1, \ldots, m$. Show that $\mathfrak{S}_{m+w}(\mathbf{t} - \mathbf{s})$ is symmetric in $t_1, \ldots, t_m$ as well as (separately) in $s_1, \ldots, s_m$.

Notes

The class of determinantal ideals in Definition 15.1 was identified by Fulton in [Ful92], which is also where the essential set, Example 15.14, and the characterization of vexillary permutations in Example 15.17 come from. A permutation is vexillary precisely when it is '2143-avoiding'. Treatments of various aspects of vexillary (a.k.a. ladder determinantal) ideals include [Mul89, HT92, Ful92, Con95, MS96, CH97, GL97, KP99, BL00, GL00, GM00], and much more can be found by looking at the papers cited in the references to these.

Proposition 15.23, applied in the case where both v and w are permutations, is a characterization of Bruhat order on the symmetric group. As in Remark 15.25, our weak order on partial permutations restricts to the standard definition of weak order on the symmetric group. For readers wishing to see the various characterizations of Bruhat and weak order, their generalizations to other Coxeter groups, and further areas where they arise, we suggest starting with [Hum90].

Schubert polynomials were invented by Lascoux and Schützenberger [LS82a], based on general notions of divided differences developed by Bernstein–Gelfand–Gelfand [BGG73] and Demazure [Dem74]. Their purpose was to isolate representatives for the cohomology classes of Schubert varieties (Definition 15.12) that are polynomials with desirable algebraic and combinatorial properties, some of which we will see in Chapter 16. Our indexing of Schubert polynomials is standard, but paradoxically, it is common practice to index Schubert *varieties* backwards from Definition 15.12, replacing w with $w_0 w$. For an introduction (beyond Chapter 16) to the algebra, combinatorics, and geometry of Schubert polynomials related to flag manifolds, we recommend [Man01]. Other sources include [Macd91] for a more algebraic perspective, and [FP98] for a more global geometric perspective.

The characterization of Schubert polynomials as multidegrees of determinantal varieties in Theorem 15.40 is due to Knutson and Miller [KnM04b, Theorem A]. The original motivation was to geometrically explain the desirable algebraic and combinatorial properties of Schubert polynomials. Theorem 15.40 can be viewed as a statement in the equivariant Chow group of $M_{k\ell}$ [Tot99, EG98]. It is essentially equivalent to the main theorem of [Ful92] expressing double Schubert polynomials as classes of certain degeneracy loci for vector bundle morphisms.

Remark 15.45 means that the K-polynomials of matrix Schubert varieties are the *Grothendieck polynomials* of Lascoux and Schützenberger [LS82b]. The proof of this statement in [KnM04b] does not rely on theory more general than what appears in Chapters 15–16, although it does requires more intricate combinatorics. Viewing the K-polynomial statement as taking place in the equivariant K-theory of $M_{k\ell}$, it is essentially equivalent to a theorem of Buch [Buc02, Theorem 2.1].

The $\mathbb{Z}$-graded result of Exercises 15.5 and 15.12, which follows from work of Giambelli [Gia04], is the most classical of all. The method of starting from Exercise 15.4 and using divided differences to prove Exercise 15.5 by induction on k demonstrates the utility of replacing the integer $\mathbb{Z}$-graded degree with a polynomial multidegree. The more finely graded statements in both of these two exercises are special cases of Exercise 16.9 in Chapter 16. The determinantal ideals described in Exercise 15.7 constitute the class of ideals discussed in [HT92].

We have more references and comments to make on Schubert polynomials and determinantal ideals, but we postpone them until the Notes to Chapter 16.

Chapter 16

Antidiagonal initial ideals

Schubert polynomials have integer coefficients. This, at least, is clear from the algebraic recursion via divided differences in Section 15.5, where we also saw their geometric expression as multidegrees. In contrast, this chapter explores the combinatorial properties of Schubert polynomials, particularly why their integer coefficients are positive.

One of our main goals is to illustrate the combinatorial importance of Gröbner bases and their geometric interpretation. Suppose a polynomial is expressed as the multidegree of some variety. Gröbner degeneration of that variety yields pieces whose multidegrees add up to the given polynomial. This process can provide geometric explanations for positive combinatorial formulas. The example pervading this chapter comes from Theorem 15.40:

Corollary 16.1 *Schubert polynomials have nonnegative coefficients.*

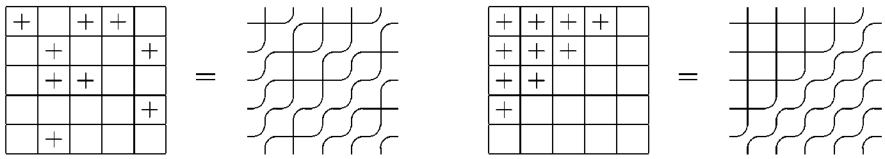
Proof. Write $\mathfrak{S}_w(\mathbf{t}) = \mathcal{C}(\overline{X}_w; \mathbf{t})$ as in Theorem 15.40. Choosing a term order on $\Bbbk[\mathbf{x}]$, Corollary 8.47 implies that $\mathfrak{S}_w(\mathbf{t}) = \mathcal{C}(\Bbbk[\mathbf{x}]/\mathsf{in}(I(\overline{X}_w)); \mathbf{t})$. Now use Theorem 8.53 to write $\mathfrak{S}_w(\mathbf{t})$ as a positive sum of multidegrees of quotients $\Bbbk[\mathbf{x}]/\langle x_{i_1,j_1}, \ldots, x_{i_r,j_r}\rangle$ by monomial primes. Proposition 8.49 says that the multidegree of this quotient of $\Bbbk[\mathbf{x}]$ is the monomial $t_{i_1} \cdots t_{i_r}$. □

The *existence* of a Gröbner basis proves the positivity in Corollary 16.1. After choosing an especially nice term order, our efforts in this chapter will identify the prime components of the initial ideal explicitly as combinatorial diagrams called *reduced pipe dreams*. Hence adding up monomials corresponding to reduced pipe dreams yields Schubert polynomials.

This positive formula will be our motivation for a combinatorial study of reduced pipe dreams, in terms of reduced expressions in the permutation group S_n. Applications include the primality of Schubert determinantal ideals, and the fact that matrix Schubert varieties are Cohen–Macaulay.

16.1 Pipe dreams

Combinatorics of Schubert polynomials—and as it will turn out in Section 16.4, of Schubert determinantal ideals—is governed by certain “drawings” of (partial) permutations. Consider a $k \times \ell$ grid of squares, with the box in row i and column j labeled (i, j), as in a $k \times \ell$ matrix. If each box in the grid is covered with a square tile containing either $+$ or ╯╭, then the tiled grid looks like a network of pipes. Each such tiling corresponds to a subset of the $k \times \ell$ rectangle, namely the set of its crossing tiles:



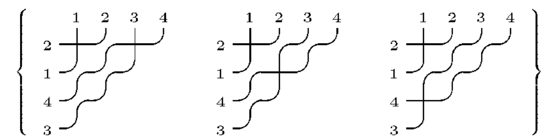
We omit the square tile boundaries in the right sides of these equations.

Definition 16.2 A $k \times \ell$ **pipe dream** is a tiling of the $k \times \ell$ rectangle by **crosses** $+$ and **elbow joints** ╯╭. A pipe dream is **reduced** if each pair of pipes crosses at most once. The set $\mathcal{RP}(w)$ of reduced pipe dreams for a $k \times \ell$ partial permutation w consists of those pipe dreams D with $l(w)$ crossing tiles such that the pipe entering row i exits from column $w(i)$.

Example 16.3 The long permutation $w_0 = n \ldots 321$ in S_n has a unique $n \times n$ reduced pipe dream D_0, whose $+$ tiles fill the region strictly above the main antidiagonal, in spots (i, j) with $i + j \leq n$. The right hand pipe dream displayed before Definition 16.2 is D_0 for $n = 5$. ◇

Example 16.4 The permutation $w = 2143$ has three reduced pipe dreams:

$\mathcal{RP}(2143) =$



The permutation is written down the left edge of each pipe dream; thus each row is labeled with the destination of its pipe. Reduced pipe dreams for permutations are contained in D_0 (Exercise 16.1), so the crossing tiles only occur strictly above the main antidiagonal. Therefore we omit the wavy “sea” of elbow pipes below the main antidiagonal. ◇

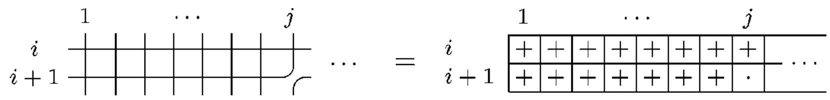
If $\widetilde{w}$ is the minimal-length completion of a $k \times \ell$ partial permutation w to an $n \times n$ permutation, then $\mathcal{RP}(w)$ is the set of $k \times \ell$ pipe dreams to which adding elbow tiles in the region $(n \times n) \smallsetminus (k \times \ell)$ yields a reduced pipe dream for $\widetilde{w}$. In other words, $\mathcal{RP}(w) = \mathcal{RP}(\widetilde{w})_{k \times \ell}$ consists of the restrictions to the northwest $k \times \ell$ rectangle of reduced pipe dreams for $\widetilde{w}$ (Exercise 16.2). Pipes can exit out of the east side of a pipe dream $D \in \mathcal{RP}(w)$, rather than out the top; when w is zero in row i, for example, a pipe traverses row i.

Although we always draw crossing tiles as some sort of cross (either '+' or '┼', the former with square tile boundary and the latter without), we often leave elbow tiles blank or denote them by dots, to minimize clutter.

Here is an easy criterion, to be used in Theorem 16.11, for when removing a ┼ from a pipe dream $D \in \mathcal{RP}(w)$ leaves a pipe dream in $\mathcal{RP}(\sigma_i w)$.

Lemma 16.5 *Suppose $D \in \mathcal{RP}(w)$, and let j be a fixed column index with $(i+1, j) \notin D$, but $(i,p) \in D$ for all $p \leq j$, and $(i+1,p) \in D$ for all $p < j$. Then $l(\sigma_i w) < l(w)$, and if $D' = D \smallsetminus (i,j)$ then $D' \in \mathcal{RP}(\sigma_i w)$.*

The hypotheses of the lemma say precisely that D looks like

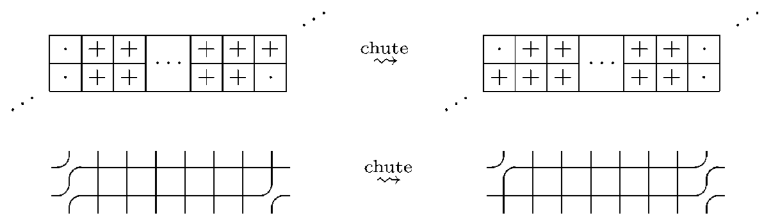


at the left end of rows i and $i+1$ in D, and the ┼ to be deleted sits at (i,j).

Proof. Removing (i,j) only switches the exit points of the two pipes starting in rows i and $i+1$. Thus the pipe starting in row p of D' exits out of column $\sigma_i w(p)$ for every row index p. No pair of pipes can cross twice in D' because there are $l(\sigma_i w) = l(w) - 1$ crossings. $\square$

Definition 16.6 A **chutable rectangle** is a $2 \times r$ block C of ┼ and ⌟⌜ tiles such that $r \geq 2$, and the only elbows in C are its northwest, southwest, and southeast corners. Applying a **chute move** to a pipe dream D is accomplished by placing a ┼ in the southwest corner of a chutable rectangle $C \subseteq D$, and removing the ┼ from the northeast corner of the same C.

Heuristically, chuting looks like:



Lemma 16.7 *Chuting $D \in \mathcal{RP}(w)$ yields another reduced pipe dream for w.*

Proof. If two pipes intersect at the ┼ in the northeast corner of a chutable rectangle C, then chuting that ┼ only relocates the crossing point of those two pipes to the southwest corner of C. No other pipes are affected. $\square$

The rest of this section is devoted to a procedure generating all reduced pipe dreams. Given a $k \times \ell$ pipe dream D, row i of D is filled solidly with ┼ tiles until the first elbow tile (or until the end of the row). In what follows, we shall need a notation for the column index of this first elbow:

$$\text{start}_i(D) \quad = \quad \min\big(\{j \mid (i,j) \text{ is an elbow tile in } D\} \cup \{k+1\}\big).$$

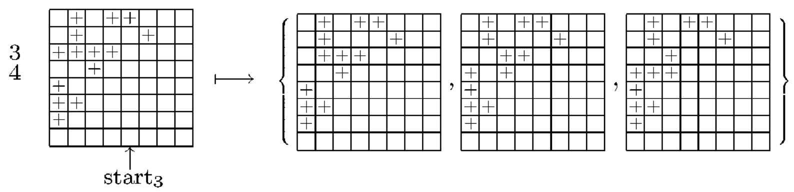
Definition 16.8 Let D be a pipe dream, and fix a row index i. Suppose there is a smallest column index j such that $(i+1, j)$ is an elbow tile, while (i, p) is a $+$ tile in D for all $p \leq j$. Construct the m^{th} **offspring** of D by

1. removing (i, j), and then
2. performing $m-1$ chute moves west of $\text{start}_i(D)$ from row i to row $i+1$.

The i^{th} **mitosis** operator sends a pipe dream $D \in \mathcal{RP}(w)$ to the set $\text{mitosis}_i(D)$ of its offspring. Write $\text{mitosis}_i(\mathcal{P}) = \bigcup_{D\in\mathcal{P}} \text{mitosis}_i(D)$ whenever $\mathcal{P}$ is a set of pipe dreams.

The total number of offspring is the number of $\begin{smallmatrix}+\\ \cdot\end{smallmatrix}$ configurations in rows i and $i+1$ that are west of $\text{start}_i(D)$. This number, which is allowed to equal zero (so D is "barren"), equals 3 in the next example.

Example 16.9 The pipe dream D at left is a reduced pipe dream for $w = 13865742$. Applying mitosis_3 yields the indicated set of pipe dreams:



The three offspring on the right are listed in the order they are produced by successive chute moves. $\diamond$

Mitosis can be reversed. Equivalently: "Parentage can be determined."

Lemma 16.10 *Fix a $k\times\ell$ partial permutation w, and suppose $i < k$ satisfies $\sigma_i w < w$. Then every pipe dream $D' \in \mathcal{RP}(\sigma_i w)$ lies in* $\text{mitosis}_i(D)$ *for some pipe dream $D \in \mathcal{RP}(w)$.*

Proof. In column $\text{start}_{i+1}(D')$, rows i and $i+1$ in D' look like $\begin{smallmatrix}\cdot\\ \cdot\end{smallmatrix}$, because otherwise one of two illegal things must happen: the pipes passing through the row i of column start_{i+1} in D' intersect again at the closest $\begin{smallmatrix}\cdot\\ +\end{smallmatrix}$ column to the left in rows i and $i+1$ of D'; or else the pipe entering row $i+1$ of D' crosses the pipe entering row i of D'. This latter occurrence is illegal because $\sigma_i w < w$, so $\sigma_i w$ has no descent at i.

Consequently, we can perform a sequence of inverse chute moves on D', the first one with its northeast corner at $(i, \text{start}_{i+1}(D'))$, and the last with its west end immediately east of the solid $\begin{smallmatrix}+\\ +\end{smallmatrix}$ part of rows i and $i+1$. These chute moves preserve the property of being in $\mathcal{RP}(\sigma_i w)$ by Lemma 16.7. Now adding the $+$ into row i of the last vacated column yields a pipe dream D whose pipes go to the correct destinations to be in $\mathcal{RP}(w)$ (see Lemma 16.5). That D is reduced follows because it has $l(w)$ crossing tiles. That $D' \in \text{mitosis}_i(D)$ is by construction. $\square$

Theorem 16.11 *If w is a $k \times \ell$ partial permutation, and $i < k$ is a row index that satisfies $\sigma_i w < w$, then the set of reduced pipe dreams for $\sigma_i w$ is the disjoint union* $\dot{\bigcup}_{D \in \mathcal{RP}(w)} \text{mitosis}_i(D)$.

Proof. Lemmas 16.5 and 16.7 imply that $\text{mitosis}_i(D) \subseteq \mathcal{RP}(\sigma_i w)$ whenever $D \in \mathcal{RP}(w)$, and Lemma 16.10 gives the reverse containment. That the union is disjoint, i.e. that $\text{mitosis}_i(D) \cap \text{mitosis}_i(D') = \varnothing$ if $D \neq D'$ are reduced pipe dreams for w, is easy to deduce directly from Definition 16.8. □

Corollary 16.12 *Let w be an $n \times n$ permutation. If $w = \sigma_{i_1} \cdots \sigma_{i_m} w_0$ with $m = l(w_0) - l(w)$, then* $\mathcal{RP}(w) = \text{mitosis}_{i_m} \cdots \text{mitosis}_{i_1}(D_0)$.

The previous corollary says that mitosis (irredundantly) generates all reduced pipe dreams for honest permutations. By replacing a partial permutation w with an extension $\widetilde{w}$ to a permutation, this implies that mitosis generates all reduced pipe dreams for w, with no restriction on w.

16.2 A combinatorial formula

The manner in which mitosis generates reduced pipe dreams has substantial algebraic structure, to be exploited in this section. In particular, we shall prove the following positive combinatorial formula for Schubert polynomials. (The corresponding formula for double Schubert polynomials will appear in Corollary 16.30, below.) Recall that $k \times \ell$ pipe dreams are identified with their sets of $+$ tiles in the $k \times \ell$ grid.

Theorem 16.13 $\mathfrak{S}_w(\mathbf{t}) = \displaystyle\sum_{D \in \mathcal{RP}(w)} \mathbf{t}^D$, *where* $\mathbf{t}^D = \displaystyle\prod_{(i,j) \in D} t_i$.

The proof, at the end of this section, comes down to an attempt at calculating $\partial_i(\mathbf{t}^D)$ directly. Fixing the loose ends in this method requires the involution in Proposition 16.16, below, to gather terms together in pairs. The involution is defined by first partitioning rows i and $i+1$.

Definition 16.14 Let D be a pipe dream and i a fixed row index. Order the tiles in rows i and $i+1$ of D as in the following diagram:

	1	2	3	4	$\cdots$
i	1	3	5	7	$\cdots$
$i+1$	2	4	6	8	

An **intron** in these two adjacent rows is a height 2 rectangle C such that

1. the first and last tiles in C (the northwest and southeast corners) are elbow tiles; and
2. no elbow tile in C is strictly northeast or strictly southwest of another elbow (so due north, due south, due east, or due west are all okay).

Ignoring all $\genfrac{}{}{0pt}{}{+}{+}$ columns in rows i and $i+1$, an intron is thus just a sequence of $\genfrac{}{}{0pt}{}{\cdot}{+}$ columns in rows i and $i+1$, followed by a sequence of $\genfrac{}{}{0pt}{}{+}{\cdot}$ columns, possibly with one $\genfrac{}{}{0pt}{}{\cdot}{\cdot}$ column in between. (Columns $\genfrac{}{}{0pt}{}{+}{+}$ with two crosses can be ignored for the purpose of the proof of the next result.)

An intron C is **maximal** if it satisfies the following extra condition:

3. the elbow with largest index before C (if there is one) lies in row $i+1$, and the elbow with smallest index after C (if there is one) lies in row i.

Lemma 16.15 *Let C be an intron in a* reduced *pipe dream. There is a unique intron $\tau(C)$ satisfying the following two conditions.*

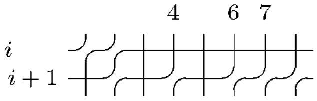
1. *The sets of $\genfrac{}{}{0pt}{}{+}{+}$ columns are the same in C and $\tau(C)$.*
2. *The number c_i of $+$ tiles in row i of C equals the number of $+$ tiles in row $i+1$ of $\tau(C)$, and the same holds with i and $i+1$ switched.*

The involution τ, called ***intron mutation****, can always be accomplished by a sequence of chute moves or inverse chute moves.*

Proof. First assume $c_i > c_{i+1}$ and work by induction on $c = c_i - c_{i+1}$. If $c = 0$ then $\tau(C) = C$ and the lemma is obvious. If $c > 0$ then consider the leftmost $\genfrac{}{}{0pt}{}{+}{\cdot}$ column. Moving to the left from this column there must be a column not equal to $\genfrac{}{}{0pt}{}{+}{+}$, since the northwest entry of C is an elbow. The rightmost such column must be $\genfrac{}{}{0pt}{}{\cdot}{\cdot}$, because its row i entry is an elbow (by construction) and its row $i+1$ entry cannot be a $+$ (for then the pipes crossing there would also cross in the $\genfrac{}{}{0pt}{}{+}{\cdot}$ column). This means we can chute the $+$ in $\genfrac{}{}{0pt}{}{+}{\cdot}$ into the $\genfrac{}{}{0pt}{}{\cdot}{\cdot}$ column, and proceed by induction.

Flip the argument 180° if $c_i < c_{i+1}$, so the chute move becomes an inverse chute move. □

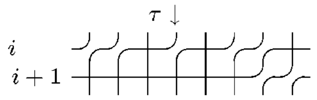
For example, here is an intron mutation accomplished by chuting the crossing tiles in columns 4, 6, and then 7 of row i. The zigzag shapes formed by the dots in these introns are typical.



=

				4		6	7	
i	·	·	+	+	+	+	+	+
$i+1$	+	·	+	·	+	·	·	·

$\tau \downarrow$



=

i	·	·	+	·	+	·	·	+
$i+1$	+	+	+	+	+	+	·	·

Proposition 16.16 *For each i there is an involution $\tau_i : \mathcal{RP}(w) \to \mathcal{RP}(w)$ such that $\tau_i^2 = 1$, and for all $D \in \mathcal{RP}(w)$:*

1. *$\tau_i D$ agrees with D outside rows i and $i+1$.*
2. *$\text{start}_i(\tau_i D) = \text{start}_i(D)$, and $\tau_i D$ equals D strictly west of this column.*
3. *The number of $+$ tiles in $\tau_i D$ from row i in columns $\geq \text{start}_i(\tau_i D)$ equals the number of $+$ tiles in D from row $i+1$ in these columns.*

Proof. Let $D \in \mathcal{RP}(w)$. Consider the union of all columns in rows i and $i+1$ of D that are east of or coincide with column $\text{start}_i(D)$. Since the first and last tiles in this region (numbered as in Definition 16.14) are elbows, this region breaks uniquely into a disjoint union of height 2 rectangles, each of which is either a maximal intron or completely filled with $+$ tiles. Indeed, this follows from the definition of start_i and Definition 16.14. Applying intron mutation to each maximal intron therein leaves a pipe dream that breaks up uniquely into maximal introns and solid regions of $+$ tiles in the same way. Therefore the proposition comes down to verifying that intron mutation preserves the property of being in $\mathcal{RP}(w)$. This is an immediate consequence of Lemmas 16.7 and 16.15. $\square$

Proof of Theorem 16.13. It suffices to prove the result for honest permutations, so we use downward induction on weak order in S_n. The result for the long permutation $w = w_0$ holds because $\mathcal{RP}(w_0) = \{D_0\}$ (Example 16.3).

Fix $D \in \mathcal{RP}(w)$, write $\mathbf{t}^D = \prod_{(i,j)\in D} t_i$, and let $m = |\text{mitosis}_i(D)|$ be the number of mitosis offspring of D. This number m equals the number of $+$ tiles in $\begin{smallmatrix}+\\ \cdot\end{smallmatrix}$ configurations located west of $\text{start}_i(D)$ in rows i and $i+1$ of D. Let D' be the pipe dream (*not* reduced) that results after deleting these $+$ tiles from D. The monomial $\mathbf{t}^D$ is then the product $t_i^m \mathbf{t}^{D'}$. Definition 16.8 immediately implies that

$$\sum_{E\in\text{mitosis}_i(D)} \mathbf{t}^E \;=\; \sum_{d=1}^{m} t_i^{m-d} t_{i+1}^{d-1} \cdot \mathbf{t}^{D'} \;=\; \partial_i(t_i^m)\cdot \mathbf{t}^{D'}. \qquad (16.1)$$

If $\tau_i D = D$, then $\mathbf{t}^{D'}$ is symmetric in t_i and t_{i+1} by Proposition 16.16, so

$$\partial_i(t_i^m)\cdot \mathbf{t}^{D'} \;=\; \partial_i(t_i^m \cdot \mathbf{t}^{D'}) \;=\; \partial_i(\mathbf{t}^D)$$

in this case (see Exercise 15.13). On the other hand, if $\tau_i D \neq D$, then letting the transposition σ_i act on polynomials by switching t_i and t_{i+1}, Proposition 16.16 implies that adding the sums in (16.1) for D and $\tau_i D$ yields

$$\partial_i(t_i^m)\cdot(\mathbf{t}^{D'} + \sigma_i \mathbf{t}^{D'}) \;=\; \partial_i\big(t_i^m(\mathbf{t}^{D'} + \sigma_i\mathbf{t}^{D'})\big) \;=\; \partial_i(\mathbf{t}^D + \mathbf{t}^{\tau_i D}).$$

Pairing off the elements of $\mathcal{RP}(w)$ not fixed by τ_i, we conclude that

$$\partial_i\Bigg(\sum_{D\in\mathcal{RP}(w)} \mathbf{t}^D\Bigg) \;=\; \sum_{E\in\text{mitosis}_i(\mathcal{RP}(w))} \mathbf{t}^E.$$

The left side is $\mathfrak{S}_{\sigma_i w}(\mathbf{t})$ by induction and the recursion for $\mathfrak{S}_w(\mathbf{t})$ in Definition 15.38, while the right side is $\sum_{E\in\mathcal{RP}(\sigma_i w)} \mathbf{t}^E$ by Theorem 16.11. $\square$

16.3 Antidiagonal simplicial complexes

It should come as no surprise that we wish to reduce questions about determinantal ideals to computations with monomials, since this is a major theme in combinatorial commutative algebra. The rest of this chapter is devoted to deriving facts about sets of minors defined by the rank conditions $r_{pq}(w) = \operatorname{rank}(w_{p\times q})$ by exploring (and exploiting) the combinatorics of their antidiagonal terms.

As in the previous chapter, let $Z_{p\times q}$ be the northwest $p \times q$ subarray of any rectangular array Z (such as a matrix or a pipe dream).

Definition 16.17 Let $\mathbf{x} = (x_{\alpha\beta})$ be the $k\times\ell$ matrix of variables. An **antidiagonal** of size r in $\Bbbk[\mathbf{x}]$ is the antidiagonal term of a minor of size r, i.e. the product of entries along the antidiagonal of an $r \times r$ submatrix of $\mathbf{x}$. For a $k\times\ell$ partial permutation w, the **antidiagonal ideal** $J_w \subset \Bbbk[\mathbf{x}]$ is generated by all antidiagonals in $\mathbf{x}_{p\times q}$ of size $1 + r_{pq}(w)$ for all p, q. Denote by $\mathcal{L}_w$ the Stanley–Reisner simplicial complex of the antidiagonal ideal J_w.

Observe that J_w is indeed a squarefree monomial ideal. This section is essentially a complicated verification that two Stanley–Reisner ideals are Alexander dual. These ideals are the antidiagonal ideal J_w and the ideal whose generators are the monomials $\mathbf{x}^D$ for reduced pipe dreams $D \in \mathcal{RP}(w)$. Here is an equivalent, more geometric statement.

Theorem 16.18 *The facets of the antidiagonal complex $\mathcal{L}_w$ are the complements of the reduced pipe dreams for w, yielding the prime decomposition*

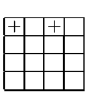
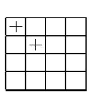
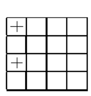
$$J_w \quad = \bigcap_{D\in\mathcal{RP}(w)} \langle x_{ij} \mid (i,j) \textit{ is a crossing tile in } D\rangle.$$

It is convenient to identify each antidiagonal $\underline{a} \in \Bbbk[\mathbf{x}]$ with the subset of the $k\times\ell$ array of variables dividing $\underline{a}$, just as we identify pipe dreams with their sets of $+$ tiles. Then Theorem 16.18 can be equivalently rephrased as saying that a pipe dream D meets every antidiagonal in J_w, and is minimal with this property, if and only if D lies in $\mathcal{RP}(w)$. This is the statement that we shall actually be thinking of in our proofs.

Example 16.19 The antidiagonal ideal J_{2143} for the 4×4 permutation 2143 equals $\langle x_{11}, x_{13}x_{22}x_{31}\rangle$. The antidiagonal complex $\mathcal{L}_{2143}$ is the union of three coordinate subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$, with ideals

$$I(L_{11,13}) = \langle x_{11}, x_{13}\rangle, \quad I(L_{11,22}) = \langle x_{11}, x_{22}\rangle, \quad \text{and} \quad I(L_{11,31}) = \langle x_{11}, x_{31}\rangle$$

whose intersection yields the prime decomposition of J_{2143}. Pictorially, represent the subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$ by pipe dreams

$D_{L_{11,13}} =$ , $D_{L_{11,22}} =$ , and $D_{L_{11,31}} =$ 

inside the 4×4 grid that have $+$ entries wherever the corresponding subspace is required to be zero. These three pipe dreams coincide with the reduced pipe dreams in $\mathcal{RP}(2143)$ from Example 16.4. ◇

The next lemma is a key combinatorial observation. Its proof (in each case, check that each rank condition is still satisfied) is omitted.

Lemma 16.20 *If $\underline{a}$ and $\underline{a}'$ are antidiagonals in $\Bbbk[\mathbf{x}]$, with $\underline{a} \in J_w$, then $\underline{a}'$ also lies in J_w if it is obtained from $\underline{a}$ by one of the following operations:*

(W) moving west one or more of the variables in $\underline{a}$;
(E) moving east any variable except *the northeast one in $\underline{a}$;*
(N) moving north one or more of the variables in $\underline{a}$; or
(S) moving south any variable except *the southwest one in $\underline{a}$.*

For each subset L of the $k \times \ell$ grid, let D_L be its complement. Thus the subspace of the $k \times \ell$ matrices corresponding to L is $\langle x_{ij} \mid (i,j) \in D_L \rangle$.

Lemma 16.21 *The set of complements D_L of faces $L \in \mathcal{L}_w$ is closed under chutes and inverse chutes.*

Proof. A pipe dream D is equal to D_L for some face $L \in \mathcal{L}_w$ if and only if D meets every antidiagonal in J_w. Suppose that C is a chutable rectangle in D_L for $L \in \mathcal{L}_w$. For chutes it is enough to show that the intersection $\underline{a} \cap D_L$ of any antidiagonal $\underline{a} \in J_w$ with D_L does not consist entirely of the single $+$ in the northeast corner of C, unless $\underline{a}$ also contains the southwest corner of C. So assume that $\underline{a}$ contains the $+$ in the northeast corner (p,q) of C, but not the $+$ in the southwest corner, and split into cases:

(i) $\underline{a}$ does not continue south of row p.
(ii) $\underline{a}$ continues south of row p but skips row $p+1$.
(iii) $\underline{a}$ intersects row $p+1$, but strictly east of the southwest corner of C.
(iv) $\underline{a}$ intersects row $p+1$, but strictly west of the southwest corner of C.

Letting $(p+1,t)$ be the southwest corner of C, construct antidiagonals $\underline{a}'$ that are in J_w, and hence intersect D_L, by moving the $+$ at (p,q) in $\underline{a}$ to:

(i) (p,t), using Lemma 16.20(W);
(ii) $(p+1,q)$, using Lemma 16.20(S);
(iii) (p,q), so $\underline{a} = \underline{a}'$ trivially; or
(iv) (p,t), using Lemma 16.20(W).

Observe that in case (iii), $\underline{a}$ already shares a box in row $p+1$ where D_L has a $+$. Each of the other antidiagonals $\underline{a}'$ intersects *both $\underline{a}$ and D_L* in some box that is not (p,q), since the location of $\underline{a}' \smallsetminus \underline{a}$ has been constructed not to be a crossing tile in D_L.

The proof for inverse chutes is just as easy, and left to the reader. □

Call a pipe dream *top-justified* if no $+$ tile is due south of a ╯╭ tile, in other words if the configuration $\begin{smallmatrix}\square\\ +\end{smallmatrix}$ does not occur.

Lemma 16.22 *Given a face $L \in \mathcal{L}_w$, there is a sequence $L_0, \ldots, L_m$ of faces of $\mathcal{L}_w$ in which $L_0 = L$, the face L_m is top-justified, and L_{e+1} is obtained from L_e by either deleting a $+$ tile or performing an inverse chute.*

Proof. Suppose that D_L for some face $L \in \mathcal{L}_w$ is not top-justified, and has no inverse-chutable rectangles. Consider a configuration $\begin{array}{|c|}\hline \cdot \\ \hline + \\ \hline\end{array}$ in the most eastern column containing one. To the east of this configuration, the first 2×1 configuration that is not $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$ must be $\begin{array}{|c|}\hline + \\ \hline \cdot \\ \hline\end{array}$ because of the absence of inverse-chutable rectangles. The union of the original $\begin{array}{|c|}\hline \cdot \\ \hline + \\ \hline\end{array}$ configuration along with this $\begin{array}{|c|}\hline + \\ \hline \cdot \\ \hline\end{array}$ configuration and all intervening $\begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$ configurations is a chutable rectangle with an extra crossing tile in its southwest corner. Reasoning exactly as in the proof of Lemma 16.21 shows that we may delete the crossing tile at the northeast corner of this rectangle. $\square$

Given a $k\times\ell$ pipe dream D, denote by L_D the coordinate subspace inside the $k\times\ell$ matrices whose ideal in $\Bbbk[\mathbf{x}]$ is $\langle x_{ij} \mid (i,j) \text{ is a } + \text{ tile in } D\rangle$.

Proposition 16.23 *Let D and E be pipe dreams, with E obtained from D by a chute move. Then L_D is a facet of $\mathcal{L}_w$ if and only if L_E is.*

Proof. Suppose that L_D is not a facet. This means that deleting from D some $+$, let us call it $\boxplus$, yields a pipe dream D' whose subspace $L_{D'}$ is still a face of $\mathcal{L}_w$. We shall show that some $+$ may be deleted from E to yield a pipe dream whose face still lies in $\mathcal{L}_w$. Let C be the chutable rectangle on which the chute move acts.

If $\boxplus$ lies outside of the rectangle C, then deleting it from E yields the result E' of chuting C in D'; that $L_{E'}$ still lies in $\mathcal{L}_w$ is by Lemma 16.21. If $\boxplus$ is the northeast corner of C, then deleting the southwest corner of C from E yields again D'. Thus we may assume $\boxplus$ lies in C, and not at either end. Every antidiagonal $\underline{a} \in J_w$ contains a $+$ in D in some row other than that of $\boxplus$. If $\boxplus$ lies in the top row of C, then this other $+$ lies in E, as well; thus deleting $\boxplus$ from E has the desired effect. Finally, if $\boxplus$ lies in the bottom row of C, then let $\boxplus'$ be the crossing tile immediately due north of it in D. Chuting $\boxplus'$ and subsequently the northeast corner of C in $D \smallsetminus \boxplus$ yields $D' \smallsetminus \boxplus'$, and $L_{D' \smallsetminus \boxplus'}$ lies in $\mathcal{L}_w$ by Lemma 16.21 again.

We have shown that L_D is a facet if L_E is; the converse is similar. $\square$

The previous result implies that the facets of $\mathcal{L}_w$ constitute the nodes of a graph whose edges connect pairs of facets related by chute moves. The main hurdle to jump before the proof of Theorem 16.18 is the connectedness of this graph. By Lemma 16.22, this amounts to the uniqueness of a top-justified facet complement for $\mathcal{L}_w$, which we shall show in Proposition 16.26. In general, top-justified pipe dreams enjoy some desirable properties.

Proposition 16.24 *Every top-justified pipe dream is reduced, and $\mathcal{RP}(w)$ contains a unique one, called the* ***top reduced pipe dream*** $\mathrm{top}(w)$*. Every pipe dream $D \in \mathcal{RP}(w)$ can be reached by a sequence of chutes from* $\mathrm{top}(w)$.

Proof. Replacing w with $\widetilde{w}$ if necessary, it suffices to consider honest permutations, as usual. Next we show that reduced pipe dreams that aren't top-justified always admit inverse chute moves. Consider a configuration $\begin{smallmatrix}\cdot\\+\end{smallmatrix}$ in the most eastern column containing one. To the east of this configuration, the first 2×1 configuration that is not $\begin{smallmatrix}+\\+\end{smallmatrix}$ must either be $\begin{smallmatrix}+\\\cdot\end{smallmatrix}$ or $\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}$. The former is impossible because the pipes passing through the $+$ in $\begin{smallmatrix}+\\\cdot\end{smallmatrix}$ would also intersect at the $+$ in $\begin{smallmatrix}\cdot\\+\end{smallmatrix}$. Hence the union of the original $\begin{smallmatrix}\cdot\\+\end{smallmatrix}$ along with this $\begin{smallmatrix}\cdot\\\cdot\end{smallmatrix}$ and all intervening $\begin{smallmatrix}+\\+\end{smallmatrix}$ configurations is an inverse-chutable rectangle.

Now simply count: there are $n!$ top pipe dreams contained in the long permutation pipe dream D_0 for S_n, and we've just finished showing that there must be at least $n!$ distinct top-justified reduced pipe dreams inside D_0. The result follows immediately. □

Let us say that a rank condition $r_{pq} \leq r$ *causes* an antidiagonal $\underline{a}$ of the generic matrix $\mathbf{x}$ if $\mathbf{x}_{p \times q}$ contains $\underline{a}$ and $\underline{a}$ has size at least $r+1$. For instance, when the rank condition comes from $r(w)$, the antidiagonals it causes include those antidiagonals $\underline{a} \in J_w$ that are contained in $\mathbf{x}_{p \times q}$ but not in any smaller northwest rectangular submatrix of $\mathbf{x}$.

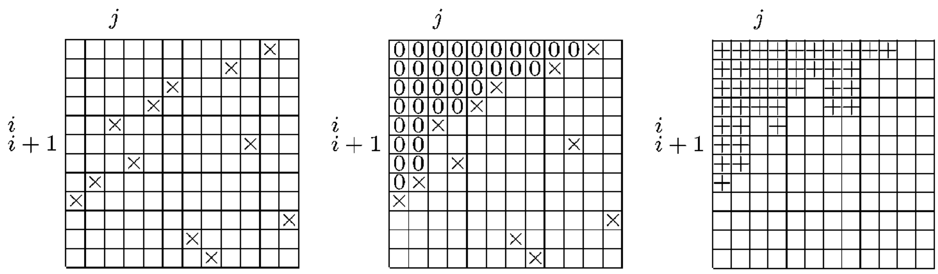
Lemma 16.25 *Antidiagonals in $J_w \smallsetminus J_{\sigma_i w}$ are contained in $\mathbf{x}_{i \times w(i)-1}$ and intersect row i.*

Proof. If an antidiagonal in J_w is either contained in $\mathbf{x}_{i-1 \times w(i)}$ or not contained in $\mathbf{x}_{i \times w(i)}$, then some rank condition causing it is in both $r(w)$ and $r(\sigma_i w)$. Indeed, it is easy to check that the rank matrices $r(\sigma_i w)$ and $r(w)$ differ only in row i between columns $w(i+1)$ and $w(i)-1$, inclusive. □

Proposition 16.26 *For each partial permutation w, there is a unique facet $L \in \mathcal{L}_w$ whose complementary pipe dream D_L is top-justified, and in fact $D_L = \mathrm{top}(w)$ is the top reduced pipe dream for w.*

Proof. This is clearly true for $w = w_0$. Assuming it for all $n \times n$ permutations of length at least l, we prove it for $n \times n$ permutations of length $l-1$.

Let $v \in S_n \smallsetminus w_0$ be a permutation. Then v has an ascent, $v(i) < v(i+1)$. Choose i minimal with this property, and set $j = v(i)$. Then let $w = \sigma_i v$, so that $v = \sigma_i w < w$, as usual. Since i is minimal, the northwest $i+1 \times j$ rectangle $r(\sigma_i w)_{i+1 \times j}$ of the rank matrix of $\sigma_i w$ is zero except at (i, j) and $(i+1, j)$, where $r(\sigma_i w)$ takes the value 1. Every variable x_{pq} sitting on one of these zero entries is an antidiagonal of size 1 in $J_{\sigma_i w}$. Therefore *every* pipe dream D_L that is the complement of a facet $L \in \mathcal{L}_{\sigma_i w}$ has $+$ tiles in these locations. See the figure below for a medium-sized example.



$\sigma_i w$ zeros in $r(\sigma_i w)$ $\text{top}(\sigma_i w)$

The same statements hold with w in place of $\sigma_i w$, except that the only nonzero entry of $r(w)_{i+1\times j}$ is $r_{i+1,j}(w) = 1$. We shall need the consequence $J_w \supseteq J_{\sigma_i w}$ of the componentwise inequality $r(w) \leq r(\sigma_i w)$.

Let D be a top-justified facet complement for $\mathcal{L}_{\sigma_i w}$. These exist by Lemma 16.22. If $\underline{a}$ is an antidiagonal in $J_w \smallsetminus J_{\sigma_i w}$, then either $\underline{a}$ intersects D, or else $\underline{a}$ contains x_{ij}. Indeed, suppose that $\underline{a}$ misses D and that also x_{ij} does not lie in a. Since $\underline{a}$ lies in $J_w \smallsetminus J_{\sigma_i w}$, Lemma 16.25 implies that $\underline{a}$ is caused by a row i rank condition $r_{iq}(w)$ in some column q between $w(i+1)$ and $w(i) - 1$. This rank matrix entry is one less than the entry due south of it by (15.1): $r_{i+1,q}(w) = r_{iq}(w) + 1$. Hence $x_{i+1,j}\underline{a}$ is an antidiagonal in $J_{\sigma_i w}$ that misses D, which is impossible. Therefore adding a $+$ tile at (i, j) to D yields a pipe dream D' whose complement is a face $L \in \mathcal{L}_w$.

It remains to show that L is a facet, and that $D = D' \smallsetminus (i, j)$. Indeed, using the former we deduce that D' must be the top reduced pipe dream for w by induction, and using the latter we conclude that $D = \text{top}(\sigma_i w)$ by Lemma 16.5 and Proposition 16.24. First we show that L is a facet.

Every variable in $\mathbf{x}_{i+1\times j}$ except for $x_{i+1,j}$ itself is actually an antidiagonal in J_w, so no $+$ tile in $D'_{i+1\times j}$ can be deleted. Suppose $\boxplus$ is one of the remaining $+$ tiles in D'. Then $\boxplus$ equals the unique intersection of some antidiagonal $\underline{a} \in J_{\sigma_i w}$ with D. If $\underline{a}$ misses x_{ij}, then $\{\boxplus\} = \underline{a} \cap D'$, so $\boxplus$ can't be deleted from D'.

On the other hand, suppose $\underline{a}$ contains x_{ij}. If $\underline{a}$ continues southwest of x_{ij}, then $\underline{a}$ skips row $i+1$ because we assumed $\boxplus$ does not lie in $D'_{i+1\times j}$. Hence we can replace $\underline{a}$ with $(x_{i+1,j}/x_{ij})\underline{a}$, which misses x_{ij}, using Lemma 16.20(S). Finally, if $\underline{a}$ has its southwest end at (i, j), suppose the northeast end of $\underline{a}$ lies in column q. Using (15.1) as a guide, calculate that $r_{i-1,q}(\sigma_i w) = r_{iq}(\sigma_i w) - 1$. Thus $\underline{a}/x_{ij}$ lies in $J_{\sigma_i w}$ and misses x_{ij}.

Finally, note that the arguments in the previous two paragraphs can also be used to show that deleting (i, j) from D' yields a face complement for $\mathcal{L}_{\sigma_i w}$. Hence $D = D' \smallsetminus (i, j)$, as required. □

Proof of Theorem 16.18. The set $\mathcal{RP}(w)$ of reduced pipe dreams for w is characterized by Proposition 16.24 as the set of pipe dreams obtained from $\text{top}(w)$ by applying chute moves. Lemma 16.22, Proposition 16.23 and Proposition 16.26 imply that the set of complements D_L of facets $L \in \mathcal{L}_w$ is characterized by the same property. □

16.4 Minors form Gröbner bases

Theorem 16.18 immediately implies some useful statements about Schubert determinantal ideals. As we shall see, the next result will be enough to conclude that the minors generating Schubert determinantal ideals I_w form Gröbner bases, and therefore that the ideals I_w are prime.

Corollary 16.27 *The antidiagonal simplicial complex $\mathcal{L}_w$ is pure. In the multigrading with $\deg(x_{ij}) = t_i$, it has multidegree $\mathcal{C}(\Bbbk[\mathbf{x}]/J_w; \mathbf{t}) = \mathfrak{S}_w(\mathbf{t})$.*

Proof. Purity of $\mathcal{L}_w$ is immediate from Theorem 16.18 and the fact that all reduced pipe dreams for w have the same number of $+$ tiles. Using purity, Theorem 8.53 and Proposition 8.49 together imply that $\mathcal{C}(\Bbbk[\mathbf{x}]/J_w; \mathbf{t})$ is the sum of monomials $\mathbf{t}^{D_L}$ for complements D_L of facets $L \in \mathcal{L}_w$. As these facet complements are precisely the reduced pipe dreams for w by Theorem 16.18, the result follows from the formula in Theorem 16.13. □

A term order on $\Bbbk[\mathbf{x}]$ is called *antidiagonal* if the leading term of every minor of $\mathbf{x}$ is its antidiagonal term. Thus if $[i_1 \cdots i_r | j_1 \cdots j_r]$ is the determinant of the square submatrix of the generic matrix $\mathbf{x}$ whose rows and columns are indexed by $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$, respectively, then

$$\mathsf{in}([i_1 \cdots i_r | j_1 \cdots j_r]) = x_{i_r j_1} x_{i_{r-1} j_2} \cdots x_{i_2 j_{r-1}} x_{i_1 j_r}.$$

There are numerous antidiagonal term orders (Exercise 16.10).

Theorem 16.28 *The minors inside the Schubert determinantal ideal I_w constitute a Gröbner basis under any antidiagonal term order:*

$$\mathsf{in}(I_w) \quad = \quad J_w.$$

Proof. The multidegree $\mathcal{C}(\Bbbk[\mathbf{x}]/\mathsf{in}(I_w); \mathbf{t})$ equals the Schubert polynomial $\mathfrak{S}_w(\mathbf{t})$ by Theorem 15.40 and Corollary 8.47. Since J_w is obviously contained inside the initial ideal $\mathsf{in}(I_w)$ under any antidiagonal term order, Exercise 8.12 applies with $I = \mathsf{in}(I_w)$ and $J = J_w$, by Corollary 16.27. □

Geometrically, Theorem 16.28 exhibits a *Gröbner degeneration* of each matrix Schubert variety to the antidiagonal simplicial complex. This is a certain flat family of subvarieties, parametrized by the affine line, whose fiber at 1 is $\overline{X}_w$ and whose fiber at 0 is $\mathcal{L}_w$ (realized as a union of coordinate subspaces). Actually, Gröbner degenerations are only defined once suitable weight vectors are chosen; see Definition 8.25 and do Exercise 16.10.

Here is the first important consequence of the Gröbner basis statement.

Corollary 16.29 *Schubert determinantal ideals I_w are prime.*

Proof. The zero set of I_w is the matrix Schubert variety $\overline{X}_w$, which is irreducible by Theorem 15.31. Hence the radical of I_w is prime. But Theorem 16.28 says that I_w has a squarefree initial ideal, which automatically implies that I_w is a radical ideal. □

The primality of Schubert determinantal ideals means that I_w equals the radical ideal $I(\overline{X}_w)$ of polynomials vanishing on the matrix Schubert variety for w. Therefore the multidegree calculation for matrix Schubert varieties in Theorem 15.40 holds for $\Bbbk[\mathbf{x}]/I_w$. This enables us to deduce the double version of Theorem 16.13.

Corollary 16.30 *The double Schubert polynomial for w satisfies*

$$\mathfrak{S}_w(\mathbf{t}-\mathbf{s}) = \sum_{D \in \mathcal{RP}(w)} (\mathbf{t}-\mathbf{s})^D, \quad \textit{where} \quad (\mathbf{t}-\mathbf{s})^D = \prod_{(i,j)\in D} (t_i - s_j).$$

Proof. The multidegree $\mathcal{C}(\Bbbk[\mathbf{x}]/J_w; \mathbf{t}, \mathbf{s})$ equals the double Schubert polynomial by Theorem 15.40, Corollary 8.47, and Theorem 16.28, using the fact that $I_w = I(\overline{X}_w)$ (Corollary 16.29). Now apply additivity of multidegrees on components (Theorem 8.53) and the explicit calculation of multidegrees for coordinate subspaces (Proposition 8.49), using Theorem 16.18 to get the sum to be over reduced pipe dreams. □

Generally speaking, the minors generating I_w fail to be Gröbner bases for other term orders, although these can still be used to get formulas for double Schubert polynomials.

Example 16.31 Consider the Schubert determinantal ideal I_{2143} for the 4×4 permutation 2143. This ideal has the same generators as the ideal I_w in Example 15.7, although in a bigger polynomial ring. We discussed the antidiagonal ideal $J_w = \mathsf{in}(I_w)$ in Example 16.19. Note that the two minors generating I_{2143} never form a Gröbner basis for a *diagonal* term order, because x_{11} divides the diagonal term $x_{11}x_{22}x_{33}$.

In the multigrading where $\deg(x_{ij}) = t_i - s_j$, the multidegree of $L_{i_1 j_1, i_2 j_2}$ equals $(t_{i_1} - s_{j_1})(t_{i_2} - s_{j_2})$. The formula in Corollary 16.30 says that

$$\mathfrak{S}_{2143}(\mathbf{t}-\mathbf{s}) \quad = \quad (t_1-s_1)(t_1-s_3) + (t_1-s_1)(t_2-s_2) + (t_1-s_1)(t_3-s_1),$$

which agrees with the calculation of this double Schubert polynomial in Example 15.43. On the other hand, there is a diagonal term order under which $\langle x_{11}, x_{13}x_{21}x_{32}\rangle = \langle x_{11}, x_{13}\rangle \cap \langle x_{11}, x_{21}\rangle \cap \langle x_{11}, x_{32}\rangle$ is the initial ideal of I_{2143}. Thus we can also calculate

$$\mathfrak{S}_{2143}(\mathbf{t}-\mathbf{s}) \quad = \quad (t_1-s_1)(t_1-s_3) + (t_1-s_1)(t_2-s_1) + (t_1-s_1)(t_3-s_2),$$

using additivity and the explicit calculation for subspaces. ◇

The title of this section alludes to that of Section 14.3, where antidiagonals are initial terms of Plücker coordinates. The differences are that Theorem 14.11 works in a sagbi (subalgebra) context and speaks only of top-justified minors, whereas Theorem 16.28 works in a Gröbner basis (ideal) context, and allows certain more general collections of minors.

16.5 Subword complexes

Our goal in this section is to prove that Schubert determinantal rings $\mathbb{k}[\mathbf{x}]/I_w$ are Cohen–Macaulay. We shall in fact show that antidiagonal complexes are Cohen–Macaulay. The argument involves some satisfying combinatorics of reduced expressions in permutation groups.

Every $n \times n$ permutation matrix can be expressed as a product of elements in the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$ of simple $n \times n$ reflection matrices—that is, permutation matrices for adjacent transpositions (see Definition 15.24 and the paragraph after it). Simple reflections σ_i are allowed to appear more than once in such an expression.

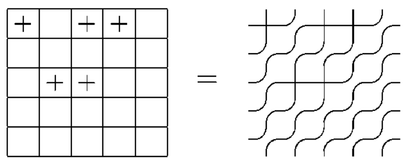
Definition 16.32 A **reduced expression** for a permutation matrix w is an expression $w = \sigma_{i_m} \cdots \sigma_{i_1}$ as a product of $m = l(w)$ simple reflections.

Lemma 16.33 *The minimal number of matrices required to express a permutation matrix w as a product of simple reflections is $l(w)$.*

Proof. For the identity matrix this is obvious, since it has length zero. Multiplying an arbitrary permutation matrix on the left by a simple reflection either increases length by 1 or decreases it by 1; this is a special case of Lemma 15.21. Ascending in weak order from the identity to a permutation matrix w therefore requires at least $l(w)$ many simple reflections. □

It is easy to see that reduced expressions exist, or in other words that the minimum $l(w)$ in Lemma 16.33 is actually attained. In fact, we are about to produce a number of reduced expressions explicitly, using pipe dreams. For notation, let us say that a $+$ tile at (p, q) in a pipe dream D sits on the i^{th} *antidiagonal* if $p + q - 1 = i$.

Let $Q(D)$ be the ordered sequence of simple reflections σ_i corresponding to the antidiagonals on which the $+$ tiles of D sit, starting from the southwest corner of D and reading left to right in each row, snaking up to the northeast corner. For a random example, the pipe dream



yields the ordered sequence $\sigma_4\sigma_5\sigma_1\sigma_3\sigma_4$. We should mention that compared to conventions in the literature (see the Notes), this convention looks like the sequence is read backward. But this is only because our permutation matrices here have corresponding abstract permutations gotten by reading the column indices of the nonzero entries instead of the rows. Transposing matrices inverts the permutations and reverses the reduced expressions.

Example 16.34 The unique pipe dream D_0 for the $n \times n$ long permutation (antidiagonal matrix) w_0 corresponds to the ordered sequence

$$Q(D_0) \quad = \quad \underbrace{\sigma_{n-1}}\underbrace{\sigma_{n-2}\sigma_{n-1}} \cdots\cdots \underbrace{\sigma_2\sigma_3\ldots\sigma_{n-1}}\underbrace{\sigma_1\sigma_2\ldots\sigma_{n-1}},$$

the **reverse triangular** reduced expression for w_0. The part of $Q(D_0)$ arising from each row of D_0 has its own underbrace. When $n = 4$, the above expression simplifies to $Q_0 = \sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$. ◇

Example 16.35 The ordered sequence constructed from the pipe dream whose crossing tiles entirely fill the $n \times n$ grid is the **reverse square word**

$$Q_{n\times n} \quad = \quad \underbrace{\sigma_n\sigma_{n+1}\ldots\sigma_{2n-2}\sigma_{2n-1}}_{\text{bottom row}} \quad \cdots \quad \underbrace{\sigma_2\sigma_3\ldots\sigma_n\sigma_{n+1}}_{\text{second row}} \; \underbrace{\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n}_{\text{top row}}.$$

This sequence necessarily involves reflections $\sigma_1, \ldots, \sigma_{2n-1}$, which lie in S_{2n}, even though reduced expressions for permutation matrices $w \in S_n$ never involve reflections σ_i with $i \geq n$. ◇

Lemma 16.36 *Suppose that the pipe entering row i of an $n \times n$ pipe dream D exits column $w(i)$ for some $n \times n$ permutation w. Multiplying the reflections in $Q(D)$ yields the permutation matrix w. Thus $Q(D)$ is a reduced expression for w if and only if $D \in \mathcal{RP}(w)$.*

Proof. Use induction on the number of crossing tiles: adding a $+$ in the i^{th} antidiagonal at the start of the list switches the destinations of the pipes entering through rows i and $i+1$. □

In other words, pipe dreams in the $n \times n$ grid are naturally 'subwords' of the reverse square word, while reduced pipe dreams are naturally *reduced* subwords. This explains the adjective 'reduced' for pipe dreams.

Definition 16.37 A **word** of size m is a sequence $Q = (\sigma_{i_m} \ldots, \sigma_{i_1})$ of simple reflections. An ordered subsequence P of Q is a **subword** of Q.

1. P **represents** an $n \times n$ permutation matrix w if the ordered product of the simple reflections in P is a reduced expression for w.
2. P **contains** w if some subsequence of P represents w.

The **subword complex** $\Delta(Q, w)$ is the set of subwords whose complements contain w: $\Delta(Q, w) = \{Q \smallsetminus P \mid P \text{ contains } w\}$.

In other words, deleting a face of $\Delta(Q, w)$ from Q leaves a reduced expression for w as a subword of what remains. If $Q \smallsetminus D$ is a facet of the subword complex $\Delta(Q, w)$, then the reflections in D constitute a reduced expression for w. Note that subwords of Q come with their embeddings into Q, so two subwords P and P' involving reflections at different positions in Q are unequal, even if the sequences of reflections in P and P' are equal.

Usually we write Q as a string without parentheses or commas, and abuse notation by saying that Q is a word in S_n, without explicit reference to the set of simple reflections. Note that Q need not itself be a reduced expression. The following lemma is immediate from the definitions and the fact that all reduced expressions for $w \in S_n$ have the same length.

Lemma 16.38 *$\Delta(Q, w)$ is a pure simplicial complex whose facets are the subwords $Q \smallsetminus P$ such that $P \subseteq Q$ represents w.* $\square$

Example 16.39 Consider the subword complex $\Delta = \Delta(\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3, 1432)$ for the 4×4 permutation $w = 1432$. This permutation has two reduced expressions, namely $\sigma_3\sigma_2\sigma_3$ and $\sigma_2\sigma_3\sigma_2$. Labeling the vertices of a pentagon with the reflections in $Q = \sigma_3\sigma_2\sigma_3\sigma_2\sigma_3$ (in cyclic order), the facets of Δ are the pairs of adjacent vertices. Thus Δ is the boundary of the pentagon. $\diamond$

Proposition 16.40 *Antidiagonal complexes $\mathcal{L}_w$ are subword complexes.*

Proof. When w is a permutation matrix, the fact that

$$\mathcal{L}_w \quad = \quad \Delta(Q_{n \times n}, w)$$

is a subword complex for the $n \times n$ reverse square word is immediate from Theorem 16.18 and Lemma 16.33. When w is an arbitrary $k \times \ell$ partial permutation, simply replace w by a minimal extension to a permutation $\widetilde{w}$, and replace $Q_{n \times n}$ by the word corresponding to tiles in a $k \times \ell$ rectangle. $\square$

We will show that subword complexes are Cohen–Macaulay via Theorem 13.45 by proving that they are shellable. In fact, we shall verify a substantially stronger, but less widely known criterion. Recall from Definition 1.38 the notion of the *link* of a face in a simplicial complex.

Definition 16.41 Let Δ be a simplicial complex and $F \in \Delta$ a face.

1. The **deletion** of F from Δ is $\mathrm{del}_\Delta(F) = \{G \in \Delta \mid F \cap G = \varnothing\}$.
2. The simplicial complex Δ is **vertex-decomposable** if Δ is pure and either (i) $\Delta = \{\varnothing\}$, or (ii) for some vertex $v \in \Delta$, both $\mathrm{del}_\Delta(v)$ and $\mathrm{link}_\Delta(v)$ are vertex-decomposable.

The definition of vertex-decomposability is not circular, but rather inductive on the number of vertices in Δ. Here is a typical example of how this inductive structure can be mined.

Proposition 16.42 *Vertex-decomposable complexes are shellable.*

Proof. Use induction on the number of vertices by first shelling $\mathrm{del}_\Delta(v)$ and then shelling the cone from v over $\mathrm{link}_\Delta(v)$ to get a shelling of Δ. $\square$

Theorem 16.43 *Antidiagonal complexes are shellable, and hence Cohen–Macaulay. More generally, subword complexes are vertex-decomposable.*

Proof. By Proposition 16.40 and Proposition 16.42, it is enough to prove the second sentence. With $Q = (\sigma_{i_m}, \sigma_{i_{m-1}}, \ldots, \sigma_{i_1})$, it suffices by induction on the number of vertices to demonstrate that both the link and the deletion of σ_{i_m} from $\Delta(Q, w)$ are subword complexes. By definition, both consist of subwords of $Q' = (\sigma_{i_{m-1}}, \ldots, \sigma_{i_1})$. The link is naturally identified with the subword complex $\Delta(Q', w)$. For the deletion, there are two cases. If $\sigma_{i_m} w$ is longer than w, then the deletion of σ_{i_m} equals its link because no reduced expression for w has σ_{i_m} at its left end. On the other hand, when $\sigma_{i_m} w$ is shorter than w, the deletion is $\Delta(Q', \sigma_{i_m} w)$. □

Corollary 16.44 *Schubert determinantal rings are Cohen–Macaulay.*

Proof. Apply Theorems 16.43 and 8.31 to Theorem 16.28. □

Exercises

16.1 Use the length condition in Definition 16.2 to show that crossing tiles in reduced pipe dreams for permutations all occur strictly above the main antidiagonal.

16.2 Suppose that a $k \times \ell$ partial permutation matrix w is given, and that $\widetilde{w}$ is a permutation matrix extending w. Prove that the crossing tiles in every reduced pipe dream for $\widetilde{w}$ all fit inside the northwest $k \times \ell$ rectangle.

16.3 What permutation in S_4 has the most reduced pipe dreams? In S_5? In S_n?

16.4 Prove directly, using the algebra of antidiagonals and without using Theorem 16.11 or Theorem 16.18, that if L is a facet of $\mathcal{L}_w$ and $\sigma_i w < w$, then $\text{mitosis}_i(D_L)$ consists of pipe dreams $D_{L'}$ for facets L' of $\mathcal{L}_{\sigma_i w}$.

16.5 Change each box in the diagram of w (Definition 15.13) into a $+$ tile, and then push all of these tiles due north as far as possible. Show that the resulting top-justified pipe dream is $\text{top}(w)$.

16.6 Given a $+$ tile in the top reduced pipe dream $\text{top}(w)$, construct an explicit antidiagonal in J_w whose intersection with $\text{top}(w)$ is precisely the given $+$ tile. Hint: Consider the ╯╭ tiles along the pipe passing vertically through the $+$ tile.

16.7 Show that each partial permutation has a unique **bottom** reduced pipe dream in which no ╯╭ is due west of a $+$ in the same row.

16.8 Prove that the bottom reduced pipe dream for a grassmannian permutation (Exercise 15.6) forms the Ferrers shape (in 'French' position) of a partition. Verify that the resulting map from the set of $n \times n$ grassmannian permutations with descent at k to the set of partitions that fit into the $k \times (n-k)$ grid is bijective.

16.9 Each semistandard tableau T (Definition 14.12) determines a monomial $\mathbf{t}^T$ whose degree in t_i is the number of entries of T equal to i. Given a grassmannian permutation w, let $\lambda(w)$ be the partition from Exercise 16.8. Exhibit a bijection from reduced pipe dreams for w to semistandard Young tableaux with shape $\lambda(w)$ that is *monomial-preserving*, in the sense that $\mathbf{t}^D = \mathbf{t}^T$ when $D \mapsto T$. Conclude that Schubert polynomials for grassmannian permutations are Schur polynomials.

16.10 Find a total ordering of the variables in the $k \times \ell$ array $\mathbf{x} = (x_{ij})$ whose reverse lexicographic term order is antidiagonal. Do the same for lexicographic order. Find an explicit weight vector inducing an antidiagonal partial term order.

16.11 Let P be obtained from a pipe dream in $\mathcal{RP}(w)$ by adding a single extra $+$ tile. Explain why there is at most one other $+$ tile that can be deleted from P to get a reduced pipe dream for w.

16.12 By a general theorem, shellable complexes whose ridges (codimension 1 faces) each lie in at most 2 facets is a ball or sphere [BLSWZ99, Proposition 4.7.22]. Use Exercise 16.11 to deduce that antidiagonal complexes are balls or spheres.

16.13 (For those who know about Coxeter groups.) Define subword complexes for arbitrary Coxeter groups. Show they are pure and vertex-decomposable. Prove that if Δ is a subword complex for a Coxeter group, then no ridge is contained in more than two facets. Conclude that Δ is homeomorphic to a ball or sphere.

16.14 What conditions on a word Q and an element w guarantee that $\Delta(Q, w)$ is (i) Gorenstein or (ii) spherical?

16.15 Recall the notation from Exercise 15.14. If w is a permutation of length $l(w) = m$, prove that the coefficient on $t_1 t_2 \cdots t_m$ in the Schubert polynomial $\mathfrak{S}_{m+w}(\mathbf{t})$ equals the number of reduced expressions for w.

Notes

There are many important ways of extracting combinatorics from determinantal ideals other than via pipe dreams. For example, there are vast literatures on this topic concerned with *straightening laws* [DRS74, DEP82, Hib86], and the *Robinson–Schensted–Knuth correspondence* [Stu90, HT92, BC01]. The former is treated in [BV88], as well as more briefly in [BH98, Chapter 7] and [Hib92, Part III], while the state of the art in RSK methods is explained in the excellent expository article [BC03].

Reduced pipe dreams are special cases of the curve diagrams invented by Fomin and Kirillov [FK96]. Our notation follows Bergeron and Billey [BB93], who called them *rc-graphs*; the corresponding objects in [FK96] are rotated by 135°. The definition of chute move comes from [BB93], as does the characterization of reduced pipe dreams in Proposition 16.24, which is [BB93, Theorem 3.7].

The mitosis recursion in Theorem 16.11 is [KnM04b, Theorem C]. Our proof here is approximately the one in [Mil03a], although Lemma 16.10 is new, as is the resulting argument proving the formula in Theorem 16.13. This result was first proved by Billey, Jockusch, and Stanley [BJS93], although independently(!) and almost simultaneously, Fomin and Stanley gave a shorter, more elegant combinatorial proof [FS94]. Corollary 16.30 is due to Fomin and Kirillov [FK96].

Theorem 16.18 and Theorem 16.28 together with the shellability of antidiagonal simplicial complexes is due to Knutson and Miller [KnM04b, Theorem B]. Our proof here is much simplified, because we avoid proving any K-polynomial statements along the way, focusing instead on multidegrees. Mitosis is a shadow of the combinatorial transitions in the weak order on S_n that governs the standard monomials of Schubert determinantal ideals and antidiagonal ideals, which are necessary for proving Hilbert series formulas as in Remark 15.45. The use of Exercise 8.12 in Theorem 16.28 is the same as in [KnM04b], and similar to

[Mar04]. Martin's applications are to *picture spaces* [Mar03], which parametrize drawings of graphs in the projective plane, and particularly to *slope varieties*, which record the edge slopes.

The appearances of antidiagonal initial terms in Theorem 14.11 and Theorem 16.28 are not coincidentally similar: there is a direct geometric connection [KoM04], in which each reduced pipe dream subspace from the Gröbner degeneration maps to a face of the Gelfand–Tsetlin toric variety.

Primality and Cohen–Macaulayness of Schubert determinantal ideals is due to Fulton [Ful92], who originally defined them. His elegant proof relied on related statements for Schubert varieties in flag manifolds [Ram85] that use positive characteristic methods and vanishing theorems for sheaf cohomology. These Schubert variety statements follow from Corollary 16.29 and Corollary 16.44.

Subword complexes were introduced in [KnM04b] for the same purpose as they appear in this chapter; their vertex-decomposability is Theorem E of that paper. The notion of vertex-decomposability was introduced by Billera and Provan, who proved that it implies shellability [BP79]. Further treatment of subword complexes, in the generality of Exercise 16.13, can be found in [KnM04a]. Included there are explicit characterizations of when balls and spheres occur, and Hilbert series calculations. In addition, several down-to-earth open problems on combinatorics of reduced expressions in Coxeter groups appear there.

Exercise 16.3 is inspired by a computation due to Woo [Woo02]. We learned the bijection in Exercise 16.9 from Kogan [Kog00]. The last sentence of Exercise 16.9 is essentially the statement that the Schubert classes on Grassmannians are represented by the Schur polynomials. The result of Exercise 16.9 holds more generally for double Schubert polynomials and supersymmetric Schur polynomials. Explicit weight orders as in Exercise 16.10 are crucial in some applications of Schubert determinantal ideals and the closely related quiver ideals of Chapter 17 [KoM04, KMS04]. Exercise 16.15 leads into the theory of *stable Schubert polynomials*, which are also known as *Stanley symmetric functions*. This important part of the theory surrounding Schubert polynomials is due to Stanley [Sta84], along with subsequent positivity results and connections to combinatorics contributed by [LS85, EG87, LS89, Hai92, RS95], among others.

Chapter 17

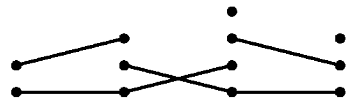
Minors in matrix products

Chapters 14–16 dealt with minors inside a single matrix. In this chapter, we consider *quiver ideals*, which are generated by minors in products of matrices. The zero sets of these ideals are called *quiver loci.* Surprisingly, we can reduce questions about quiver ideals and quiver loci to questions about Schubert determinantal ideals, by using the *Zelevinsky map*, which embeds a sequence of matrices as blocks in a single larger matrix. As a consequence, we deduce that quiver ideals are prime, and that quiver loci are Cohen–Macaulay. In addition, we get a glimpse of how the combinatorics of quivers, pipe dreams, and Schubert polynomials are reflected in formulas for *quiver polynomials*, which are the multidegrees of quiver loci.

17.1 Quiver ideals and quiver loci

The questions we asked about ideals generated by minors in a matrix of variables—concerning primality, Cohen–Macaulayness, and explicit formulas for multidegrees—also make sense for ideals generated by minors in products of two or more such matrices. Whereas the former correspond geometrically to varieties of linear maps with specified ranks between two fixed vector spaces, the latter correspond to varieties of *sequences* of linear maps. Naturally, if we hope to get combinatorics out of this situation, then we should first isolate the combinatorics that goes into it: what kinds of conditions on the ranks of composite maps is it reasonable for us to request?

Example 17.1 The sequence of three matrices in Fig. 17.1 constitutes an element in the vector space $M_{23} \times M_{34} \times M_{43}$ of sequences of linear maps $\Bbbk^2 \to \Bbbk^3 \to \Bbbk^4 \to \Bbbk^3$ (so $\Bbbk^r$ consists of row vectors for each r). Note that these matrices—call them w_1, w_2, and w_3—can be multiplied in the order they are given. Since they are all partial permutations (Definition 15.1), we can represent the sequence $\mathbf{w} = (w_1, w_2, w_3)$ by the graph above it, called its *lacing diagram.* When w_i has a 1 entry in row α and column β, the



$$\left(\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)$$

Figure 17.1: A sequence of partial permutations and its lacing diagram

lacing diagram has a segment directly above w_i joining the dot at height α to the dot in the next column to the right at height β.

Let $\Bbbk[\mathbf{f}]$ denote the coordinate ring of $M_{23} \times M_{34} \times M_{43}$. Thus $\Bbbk[\mathbf{f}]$ is a polynomial ring $6 + 12 + 12 = 30$ variables $\mathbf{f} = \{f^i_{\alpha\beta}\}$, arranged in three rectangular *generic matrices* $\Phi_1 = (f^1_{\alpha\beta})$, $\Phi_2 = (f^2_{\alpha\beta})$, and $\Phi_3 = (f^3_{\alpha\beta})$. We would like to think of the sequence $\mathbf{w}$ as lying in the zero set of an ideal generated by minors in the products of these generic matrices Φ_i.

What size minors should we take? The ranks of the three maps w_1, w_2, and w_3 are all 2, as this is the number of nonzero entries in each matrix. Hence $\mathbf{w}$ lies in the zero set of the ideal generated by all 3×3 minors Φ_1, Φ_2, and Φ_3. But $\mathbf{w}$ satisfies additional conditions: the composite maps $\Bbbk^2 \xrightarrow{w_1 w_2} \Bbbk^4$ and $\Bbbk^3 \xrightarrow{w_2 w_3} \Bbbk^3$ both have rank 1. One way to see this without multiplying the matrices is to count the number of length 2 laces (one each) spanning the first three or the last three columns of dots. Therefore $\mathbf{w}$ also lies in the ideal generated by the 2×2 minors of $\Phi_1\Phi_2$ and $\Phi_2\Phi_3$. Finally, the composite map $\Bbbk^2 \xrightarrow{w_1 w_2 w_3} \Bbbk^3$ is zero, since no laces span all the columns, so $\mathbf{w}$ lies in the zero set of the entries of the product $\Phi_1\Phi_2\Phi_3$. Hence the rank conditions that best describe $\mathbf{w}$ are the bounds determined by $\mathbf{w}$ on the ranks of the $6 = 3+2+1$ consecutive products of generic matrices Φ_i. $\diamond$

Sequences of partial permutations given by lacing diagrams are in many ways fundamental. In particular, our goal in this section is to show in Proposition 17.9 that they are the *only examples* of matrix lists, up to changes of basis. Therefore let us formalize the notion of lacing diagram.

Definition 17.2 Fix $r_0, \ldots, r_n \in \mathbb{N}$. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a list of partial permutations, with w_i of size $r_{i-1} \times r_i$. The **lacing diagram** of $\mathbf{w}$ is a graph having r_i vertices in column i for $i = 0, \ldots, n$, and an edge from the α^{th} dot in column $i-1$ to the β^{th} dot in column i whenever $w_i(\alpha) = \beta$. We identify $\mathbf{w}$ with its lacing diagram and call its connected components **laces**.

To describe the general framework for ideals in products of matrices, fix nonnegative integers $r_0, \ldots, r_n$. Denote by $Mat = M_{r_0 r_1} \times \cdots \times M_{r_{n-1} r_n}$ the variety of *quiver representations* over the field $\Bbbk$ with *dimension vector* $(r_0, \ldots, r_n)$. That is, Mat equals the vector space of sequences

$$\phi : \quad \Bbbk^{r_0} \xrightarrow{\phi_1} \Bbbk^{r_1} \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} \Bbbk^{r_{n-1}} \xrightarrow{\phi_n} \Bbbk^{r_n}$$

of linear transformations. By convention, set $\phi_0 = 0 = \phi_{n+1}$. As the above notation suggests, we have fixed a basis for each of the vector spaces $\Bbbk^{r_i}$, and we express elements of $\Bbbk^{r_i}$ as row vectors of length r_i. Each map ϕ_i in the quiver representation ϕ therefore gets identified with a matrix over $\Bbbk$ of size $r_{i-1} \times r_i$. The coordinate ring of *Mat* is a polynomial ring $\Bbbk[\mathbf{f}]$ in variables $\mathbf{f} = \{f^i_{\alpha\beta}\} = (f^1_{\alpha\beta}), \ldots, (f^n_{\alpha\beta})$, where the i^{th} index β and the $(i{+}1)^{\text{st}}$ index α run from 1 to r_i. Let Φ be the *generic* quiver representation, in which the entries in the matrices Φ_i are the variables $f^i_{\alpha\beta}$.

Definition 17.3 For an array $\mathbf{r} = (r_{ij})_{0\leq i\leq j\leq n}$ of nonnegative integers with $r_{ii} = r_i$, the **quiver ideal** $I_{\mathbf{r}} \subseteq \Bbbk[\mathbf{f}]$ is generated by the union over $i < j$ of the size $1{+}r_{ij}$ minors in the product $\Phi_{i+1} \cdots \Phi_j$ of generic matrices:

$$I_{\mathbf{r}} \quad = \quad \langle \text{minors of size } 1 + r_{ij} \text{ in } \Phi_{i+1} \cdots \Phi_j \text{ for } i < j \rangle.$$

The **quiver locus** $\Omega_{\mathbf{r}} \subseteq Mat$ is the zero set of the quiver ideal $I_{\mathbf{r}}$.

The quiver locus $\Omega_{\mathbf{r}}$ consists exactly of those ϕ satisfying $r_{ij}(\phi) \leq r_{ij}$ for all $i < j$, where $r_{ij}(\phi)$ is the rank of the composite map $\Bbbk^{r_i} \to \Bbbk^{r_j}$:

$$r_{ij}(\phi) \quad = \quad \text{rank}(\phi_{i+1} \cdots \phi_j) \quad \text{for } i < j. \tag{17.1}$$

Example 17.4 Whenever $0 \leq k \leq \ell \leq n$, there is a quiver representation

$$\mathbf{w}(k, \ell): \qquad 0 \to \cdots \to 0 \to \underset{k}{\Bbbk} = \cdots = \underset{\ell}{\Bbbk} \to 0 \to \cdots \to 0$$

having copies of the field $\Bbbk$ in spots between k and ℓ, with identity maps between them and zeros elsewhere. The array $\mathbf{r} = \mathbf{r}(\mathbf{w}(k,\ell))$ in this case has entry $r_{ij} = 1$ if $k \leq i \leq j \leq \ell$, and $r_{ij} = 0$ otherwise. Quiver representations of this form are called **indecomposable**. The matrices in $\mathbf{w}(k,\ell)$ are all 1×1, filled with either 0 or 1, so $\mathbf{w}(k,\ell)$ is a lacing diagram with one lace stretching from the dot in column k to the dot in column ℓ. ◇

It was particularly simple to determine the array $\mathbf{r}$ for the lacing diagrams $\mathbf{w}(k,\ell)$. As it turns out, it is not much harder to do so for arbitrary lacing diagrams. The (easy) proof of the following is left to Exercise 17.2.

Lemma 17.5 *If $\mathbf{w} \in Mat$ is a lacing diagram with precisely $q_{k\ell}$ laces beginning in column k and ending in column ℓ, for each $k \leq \ell$, then $r_{ij}(\mathbf{w})$ equals the number of laces passing through both column i and column j:*

$$r_{ij}(\mathbf{w}) \quad = \quad \sum_{k=0}^{i} \sum_{\ell=j}^{n} q_{k\ell}.$$

Definition 17.6 Let $\mathbf{q} = (q_{ij})$ be a **lace array**, filled with arbitrary nonnegative integers for $0 \leq i \leq j \leq n$. The associated **rank array** is the nonnegative integer array $\mathbf{r} = (r_{ij})$ for $0 \leq i \leq j \leq n$ defined by Lemma 17.5. The **rectangle array** of $\mathbf{r}$ (or of $\mathbf{q}$) is the array $\boldsymbol{R} = (R_{ij})$ of rectangles for $0 \leq i < j \leq n$, such that R_{ij} has height $r_{i,j-1} - r_{ij}$ and width $r_{i+1,j} - r_{ij}$.

The point is that for a lacing diagram, we could just as well specify the ranks $\mathbf{r}$ by giving the lace array $\mathbf{q}$. We defined the rectangle array here because it fits naturally with $\mathbf{r}$ and $\mathbf{q}$, but we will not use it until Lemma 17.13.

Example 17.7 The lacing diagram from Example 17.1 has rank array $\mathbf{r} = (r_{ij})$, lace array $\mathbf{q} = (q_{ij})$, and rectangle array $\boldsymbol{R} = (R_{ij})$ as follows.

$$\mathbf{r} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & 2 & 0 \\ & & 3 & 2 & 1 \\ & 4 & 2 & 1 & 2 \\ 3 & 2 & 1 & 0 & 3 \end{array} \qquad \mathbf{q} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & 0 & 0 \\ & & 0 & 1 & 1 \\ & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \end{array} \qquad \boldsymbol{R} = \begin{array}{cccc|c} 3 & 2 & 1 & 0 & i/j \\ \hline & & & & 0 \\ & & & \_ & 1 \\ & & \square\!\square & \square & 2 \\ & \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \square & \square & 3 \end{array}$$

Lemma 17.5 says that each entry of $\mathbf{r}$ is the sum of the entries in $\mathbf{q}$ weakly southeast of the corresponding location. The height of R_{ij} is obtained by subtracting the entry r_{ij} from the one above it, while the width of R_{ij} is obtained by subtracting the entry r_{ij} from the one to its left. The reason for writing the arrays in this orientation will come from the Zelevinsky map; compare $\mathbf{q}$ and $\boldsymbol{R}$ here to the illustration in Example 17.14, below. $\diamond$

The reason why the ranks of lacing diagrams decompose as sums is because the lacing diagrams themselves decompose into sums. In general, if ϕ and ψ are two quiver representations with dimension vectors $(r_0, \ldots, r_n)$ and $(r'_0, \ldots, r'_n)$, then the *direct sum* of ϕ and ψ is the quiver representation $\phi \oplus \psi = (\phi_1 \oplus \psi_1, \ldots, \phi_n \oplus \psi_n)$, whose i^{th} vector space is $\Bbbk^{r_i} \oplus \Bbbk^{r'_i}$. Every direct sum of indecomposables is represented by a sequence of partial permutations, and hence is a lacing diagram; but not every lacing diagram is equal to a such a direct sum (try the lacing diagram in Example 17.1). On the other hand, with the right notion of isomorphism, every lacing diagram is isomorphic to such a direct sum, after permuting the dots (basis vectors) in each column. To make a precise statement, two quiver representations ϕ and ψ are called *isomorphic* if there are invertible $r_i \times r_i$ matrices η_i for $i = 0, \ldots, n$ such that $\phi_i \eta_i = \eta_{i-1} \psi_i$. In other words, η gives invertible maps $\Bbbk^{r_i} \to \Bbbk^{r_i}$ making every square in the diagram $\phi \xrightarrow{\eta} \psi$ commute.

Lemma 17.8 *Every lacing diagram* $\mathbf{w} \in Mat$ *is isomorphic to the direct sum of the indecomposable lacing diagrams corresponding to its laces. Two lacing diagrams are isomorphic if and only if they have the same lace array.*

The (easy) proof is left to Exercise 17.2; note that the second sentence is a consequence of the first. The lemma brings us to the main result of the section. It is the sequences-of-maps analogue of the fact that every linear map between two vector spaces can be written as a diagonal matrix with only zeros and ones, after changing bases in both the source and target.

Proposition 17.9 *Every quiver representation* $\phi \in Mat$ *is isomorphic to a lacing diagram* $\mathbf{w}$*, whose lace array* $\mathbf{q}$ *is independent of the choice of* $\mathbf{w}$.

Proof. It suffices by Lemma 17.8 to show that ϕ is isomorphic to a direct sum of indecomposables. We may as well assume that $r_0 \neq 0$, and let j be the largest index for which the composite $\Bbbk^{r_0} \to \Bbbk^{r_j}$ is nonzero. Choose a linearly independent set $B_0 \subset \Bbbk^{r_0}$ whose span maps isomorphically to the image of $\Bbbk^{r_0}$ in $\Bbbk^{r_j}$ under the composite $\Bbbk^{r_0} \to \Bbbk^{r_j}$. The image $B_i \subset \Bbbk^{r_i}$ of B_0 under $\Bbbk^{r_0} \to \Bbbk^{r_i}$ for $i \leq j$ is independent. Setting $B_i = \varnothing$ for $i > j$, let ψ be the induced quiver representation on $(\mathrm{span}(B_0), \ldots, \mathrm{span}(B_n))$.

Choose a splitting $\Bbbk^{r_j} = V'_j \oplus \mathrm{span}(B_j)$. This induces for each $i \leq j$ a splitting $\Bbbk^{r_i} = V'_i \oplus \mathrm{span}(B_i)$, where V'_i is the preimage of V'_j under the composite map $\Bbbk^{r_i} \to \Bbbk^{r_j}$. Set $V'_i = \Bbbk^{r_i}$ for $i > j$, so we get a quiver representation ϕ' on $V' = (V'_0, \ldots, V'_n)$ by restriction of ϕ.

By construction, $\phi = \psi \oplus \phi'$. Since ψ is isomorphic to a direct sum of $\#B_0$ copies of $\mathbf{w}(0, j)$, induction on $r_0 + \cdots + r_n$ completes the proof. □

Definition 17.3 made no assumptions about the array $\mathbf{r}$ of nonnegative integers, and the ranks r_{ij} there are only upper bounds. Unless every rank r_{ij} equals zero, there will always be matrix lists $\phi \in \Omega_{\mathbf{r}}$ whose composite maps have strictly smaller rank than $\mathbf{r}$; and if some rank r_{ij} for $i < j$ is very big, then *all* matrix lists $\phi \in \Omega_{\mathbf{r}}$ will have strictly smaller rank $r_{ij}(\phi)$. The point is that only certain arrays $\mathbf{r}$ can actually occur as ranks of quiver representations: nontrivial restrictions on the array $\mathbf{r}(\phi)$ are imposed by Proposition 17.9. In more detail, after choosing an isomorphism $\phi \cong \mathbf{w}$ with a lacing diagram $\mathbf{w}$, inverting Lemma 17.5 yields

$$q_{ij} \quad = \quad r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1}$$

for $i \leq j$, where $r_{ij} = 0$ if i and j do not both lie between 0 and n. Therefore the array $\mathbf{r}$ can occur as in (17.1) if and only if $r_{ii} = r_i$ for $i = 0, \ldots, n$, and

$$r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \quad \geq \quad 0 \qquad (17.2)$$

for $i \leq j$, since the left hand side is simply q_{ij}. Here, finally, is the answer to what kinds of rank conditions can we reasonably request.

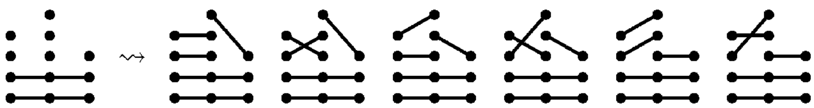
Convention 17.10 Starting in Section 17.2, *we consider only rank arrays* $\mathbf{r}$ *that occur as in (17.1)*, and we call these rank arrays **irreducible** if we need to emphasize this point. Thus we can interchangeably use a lace array $\mathbf{q}$ or its corresponding rank array $\mathbf{r}$ to specify a quiver ideal or locus.

We close this section with an example to demonstrate how the irreducibility of a rank array $\mathbf{r}$ can detect good properties of the quiver ideal $I_{\mathbf{r}}$.

Example 17.11 (Minors of fixed size in a product of two matrices) Consider two matrices of variables, Φ_1 and Φ_2, where Φ_1 has size $r_0 \times r_1$ and Φ_2 has size $r_1 \times r_2$. We are interested in the ideal I generated by all of the minors of size $\rho + 1$ in the product $\Phi_1\Phi_2$, so the quiver locus Ω consists of the pairs (ϕ_1, ϕ_2) such that $\phi_1\phi_2$ has rank at most ρ. This rank condition is automatically satisfied unless $\rho < \min\{r_0, r_1, r_2\}$, so we assume this inequality. The question is whether I is prime. Suppose that $I = I_{\mathbf{r}}$ for some rank array $\mathbf{r}$. In order that the only equations generating I be the minors in $\Phi_1\Phi_2$, there must be no rank conditions on Φ_1 individually, and also none on Φ_2 individually. That is, we must stipulate that $r_{01} = \min(r_0, r_1)$ and $r_{12} = \min(r_1, r_2)$ are as large as possible. Suppose this is the case, and consider a quiver representation $\phi \in \Omega_{\mathbf{r}}$.

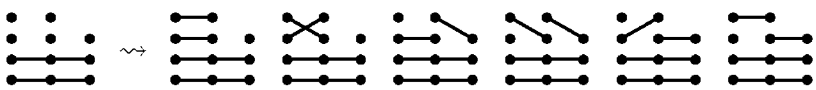
In terms of elementary linear algebra, ϕ_1 and ϕ_2 are matrices of maximal rank, and we want $\operatorname{rank}(\phi_1\phi_2) \leq \rho$. However, if the middle vector space in $\Bbbk^{r_0} \xrightarrow{\phi_1} \Bbbk^{r_1} \xrightarrow{\phi_2} \Bbbk^{r_2}$ is the smallest of the three, so $r_0 \geq r_1$ and $r_1 \leq r_2$, then ϕ_1 is surjective and ϕ_2 is injective. Hence $\phi_1\phi_2$ has rank precisely r_1 in this case, and we require that $\rho < r_1$. We conclude that r_1 cannot be too small.

How large must r_1 be? Answering this question from first principles is possible, but with lacing diagrams it becomes easy. So suppose our ϕ is actually a lacing diagram $\mathbf{w}$. Then $\mathbf{w}$ has ρ laces spanning all three columns of dots because $r_{02} = \rho$, so $q_{02} = \rho$. Next, to make w_1 of maximal rank r_{01}, we must have $r_{01} - \rho$ laces from column 0 to column 1; that is, $q_{01} = r_{01} - \rho$, where we recall that $r_{01} = \min(r_0, r_1)$. Similarly, $q_{12} = r_{12} - \rho$. Graph-theoretically, we must find a matching on the set of dots above height ρ that saturates the two outside columns of dots, because no endpoint of the q_{01} laces can be shared with one of the q_{12} laces. In the following diagrams,


(17.3)

$(r_0, r_1, r_2) = (4, 5, 3)$ and $\rho = 2$. We conclude that the array $\mathbf{r}$ is irreducible if and only if $r_1 \geq \rho + q_{01} + q_{12} = r_{01} + r_{12} - \rho$. We shall see in Theorem 17.23 that in this case $\Omega_{\mathbf{r}}$ is an irreducible variety, and in fact $I_{\mathbf{r}}$ is prime.

What happens if $\mathbf{r}$ is not irreducible? Take the lacing diagrams below:



Here, $(r_0, r_1, r_2) = (4, 4, 3)$ and $\rho = 2$. In contrast to (17.3), the six choices of matchings (edges of length 1) in this case give rise to lacing diagrams with *two different rank arrays.* It follows that $\Omega_{\mathbf{r}}$ contains the quiver loci for both, so $\Omega_{\mathbf{r}}$ is reducible as a variety. Therefore $I_{\mathbf{r}}$ is not prime. $\diamond$

17.2 Zelevinsky map

The rank conditions given by arrays in Definition 17.3 are essentially forced upon us by naturality: they are the only rank conditions that are invariant under arbitrary changes of basis, by Proposition 17.9. In this section we shall see how the irreducible rank conditions in Convention 17.10 can be transformed into the rank conditions for a Schubert determinantal ideal.

Given a rank array $\mathbf{r}$ or equivalently a lace array $\mathbf{q}$, we shall construct a permutation $v(\mathbf{r})$ in the symmetric group S_d, where $d = r_0 + \cdots + r_n$. In general, any matrix in the space M_d of $d \times d$ matrices comes with a decomposition into block rows of heights $r_0, \ldots, r_n$ (block rows listed from top to bottom) and block columns of widths $r_n, \ldots, r_0$ (block columns listed from left to right). Note that our indexing convention may be unexpected, with the square blocks lying along the main block *anti*diagonal rather than on the diagonal as usual. With these conventions, the i^{th} block column refers to the block column of width r_i, which sits i blocks from the *right*.

Draw the matrix for each permutation $v \in S_d$ by placing a symbol $\times$ (instead of a 1) at each position $(k, v(k))$, and zeros elsewhere.

Proposition–Definition 17.12 *Given a rank array* $\mathbf{r}$*, there is a unique* ***Zelevinsky permutation*** $v(\Omega_{\mathbf{r}}) = v(\mathbf{r})$ *in* S_d*, satisfying the following conditions. Consider the block in the* i^{th} column *and* j^{th} row.

1. *If* $i \leq j$ *(that is, the block sits on or below the main block antidiagonal) then the number of* $\times$ *entries in that block equals* q_{ij}*.*
2. *If* $i = j+1$ *(that is, the block sits on the main block superantidiagonal) then the number of* $\times$ *entries in that block equals* $r_{j,j+1}$*.*
3. *If* $i \geq j+2$ *(that is, the block lies strictly above the main block superantidiagonal) then there are no* $\times$ *entries in that block.*
4. *Within every block row or block column, the* $\times$ *entries proceed from northwest to southeast, that is, no* $\times$ *entry is northeast of another.*

Proof. We need the number of $\times$ entries in any block row, as dictated by conditions 1–3, to equal the height of that block row (and transposed for columns), since condition 4 then stipulates uniquely how to arrange the $\times$ entries within each block. In other words the height $r_j = r_{jj}$ of the j^{th} block row must equal the number $r_{j,j+1}$ of $\times$ entries in the superantidiagonal block in that block row, plus the sum $\sum_{i \leq j} q_{ij}$ of the number of $\times$ entries in the rest of the blocks in that block row (and a similar statement must hold for block columns). These statements follow from Lemma 17.5. □

The diagram (Definition 15.13) of a Zelevinsky permutation refines the data contained in the rectangle array $\boldsymbol{R}$ for the corresponding ranks (Definition 17.6) . The next lemma is a straightforward consequence of the definition of Zelevinsky permutation, and we leave it to Exercise 17.2.

Lemma 17.13 *In each block of the diagram of a Zelevinsky permutation* $v(\mathbf{r})$ *that is on or below the superantidiagonal, the boxes form a rectangle justified in the southeast corner of the block. Moreover the rectangle in the* i^{th} *block column and* j^{th} *block row is the rectangle* $R_{i-1,j+1}$ *in the array* $\boldsymbol{R}$*.*

Example 17.14 Let $\mathbf{r}, \mathbf{q}, \boldsymbol{R}$ be as in Example 17.7. The Zelevinsky permutation for this data is

$$v(\mathbf{r}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 9 & 4 & 5 & 11 & 1 & 2 & 6 & 12 & 3 & 7 & 10 \end{pmatrix},$$

whose permutation matrix is indicated by the $\times$ entries in the array

8	*	*	*	*	*	*	*	×	·	·	·	·
9	*	*	*	*	*	*	*	·	×	·	·	·
4	*	*	*	×	·	·	·	·	·	·	·	·
5	*	*	*	·	×	·	·	·	·	·	·	·
11	*	*	*	·	·	□	□	·	·	□	×	·
1	×	·	·	·	·	·	·	·	·	·	·	·
2	·	×	·	·	·	·	·	·	·	·	·	·
6	·	·	□	·	·	×	·	·	·	·	·	·
12	·	·	□	·	·	·	□	·	·	□	·	×
3	·	·	×	·	·	·	·	·	·	·	·	·
7	·	·	·	·	·	·	×	·	·	·	·	·
10	·	·	·	·	·	·	·	·	·	×	·	·

and whose diagram $D(v(\mathbf{r}))$ is indicated by the set of all $*$ and $\square$ entries. $\diamond$

The locations in the diagram of $v(\mathbf{r})$ strictly above the block superantidiagonal are drawn as $*$ entries instead of boxes because they are contained in the diagram of the Zelevinsky permutation $v(\mathbf{r})$ for every rank array $\mathbf{r}$ with fixed dimension vector $(r_0, \ldots, r_n)$. In fact the $*$ entries form the diagram of the Zelevinsky permutation $v(Mat)$ corresponding to the quiver locus that equals the entire quiver space Mat. We shall henceforth denote this unique Zelevinsky permutation of minimal length by $v_* = v(Mat)$.

It is clear from the combinatorics of Zelevinsky permutations that we can read off the rank array $\mathbf{r}$ and the lace array $\mathbf{q}$ from $v(\mathbf{r})$. We next demonstrate that the combinatorial encoding of $\mathbf{r}$ by its Zelevinsky permutation reflects a simple geometric map that translates between quiver loci and matrix Schubert varieties.

Definition 17.15 The **Zelevinsky map** $\mathcal{Z} : Mat \to M_d$ takes

$$(\phi_1, \phi_2, \ldots, \phi_n) \quad \stackrel{\mathcal{Z}}{\longmapsto} \quad \left[\begin{array}{ccccc} 0 & & 0 & \phi_1 & \mathbf{1} \\ 0 & & \phi_2 & \mathbf{1} & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \mathbf{1} & 0 & 0 \\ \phi_n & \cdot^{\cdot^{\cdot}} & 0 & 0 & 0 \\ \mathbf{1} & & 0 & 0 & 0 \end{array}\right], \qquad (17.4)$$

so $\mathcal{Z}(\phi)$ is a block matrix of total size $d \times d$. If $\Bbbk[\mathbf{x}]$ denotes the coordinate ring of M_d, then denote the kernel of the induced map $\Bbbk[\mathbf{x}] \twoheadrightarrow \Bbbk[\mathbf{f}]$ by $\mathfrak{m}_{\mathbf{f}}$.

Indexing for the $d \times d$ matrix $\mathbf{x}$ of variables in $\Bbbk[\mathbf{x}] = \Bbbk[M_d]$ does not arise in this section, so it will be introduced later, as necessary. The ideal $\mathfrak{m}_{\mathbf{f}}$ is generated by equations setting the appropriate variables in $\Bbbk[\mathbf{x}]$ to 0 or 1. To be more precise, $\mathfrak{m}_{\mathbf{f}}$ contains every $\mathbf{x}$ variable except

- those in superantidiagonal blocks, as they correspond to the coordinates $\mathbf{f}$ on Mat and map isomorphically to their images in $\Bbbk[\mathbf{f}]$; and
- those on the diagonals of the antidiagonal blocks. For each such variable x, the ideal $\mathfrak{m}_{\mathbf{f}}$ contains $x - 1$ instead.

The proof of Theorem 17.17 will use the following handy general lemma.

Lemma 17.16 *Let $\Gamma\Phi$ be the product of two matrices with entries in a commutative ring R. If Γ is square and* $\det(\Gamma)$ *is a unit, then for each fixed $u \in \mathbb{N}$, the ideals generated by the size u minors in Φ and in $\Gamma\Phi$ coincide.*

Proof. The result is easy when $u = 1$. The case of arbitrary u reduces to the case $u = 1$ by noting that the minors of size u in a matrix for a map $R^k \to R^\ell$ of free modules are simply the entries in a particular choice of matrix for the associated map $\bigwedge^u R^k \to \bigwedge^u R^\ell$ between u^{th} exterior powers. □

Here now is our comparison connecting the algebra of quiver ideals, which are generated by minors of fixed size in products of generic matrices, to that of Schubert determinantal ideals, which are generated by minors of varying sizes in a single generic matrix. We remark that it does *not* imply that the generators in Definition 17.3 form a Gröbner basis; see the Notes.

Theorem 17.17 *Let $\mathbf{r}$ be a rank array and $v(\mathbf{r})$ its Zelevinsky permutation. Under the map $\Bbbk[\mathbf{x}] \twoheadrightarrow \Bbbk[\mathbf{f}]$, the image of the Schubert determinantal ideal $I_{v(\mathbf{r})}$ equals the quiver ideal $I_{\mathbf{r}}$. Equivalently, $\Bbbk[\mathbf{f}]/I_{\mathbf{r}} \cong \Bbbk[\mathbf{x}]/(I_{v(\mathbf{r})} + \mathfrak{m}_{\mathbf{f}})$.*

Example 17.18 For a generic 4×5 matrix Φ_1 and 5×3 matrix Φ_2, let I be the ideal of 3×3 minors in $\Phi_1\Phi_2$. Thus $I = I_{\mathbf{r}}$ for the rank array $\mathbf{r}$ of the lacing diagrams on the right side of (17.3). The essential set of $v(\mathbf{r})$ consists of the two boxes at $(9, 8)$ and $(4, 3)$, so $I_{v(\mathbf{r})}$ is generated by the 7×7 minors in $\mathbf{x}_{9\times 8}$ and the entries of $\mathbf{x}_{4\times 3}$ (Exercise 17.4). Using the generators of $\mathfrak{m}_{\mathbf{f}}$ to set $\mathbf{x}$ variables equal to 0 or 1 yields the block 2×2 matrix

$$\Phi \quad = \quad \left[\begin{array}{ccc|ccccc} 0&0&0 & & & & & \\ 0&0&0 & & & \Phi_1 & & \\ 0&0&0 & & & & & \\ 0&0&0 & & & & & \\ \hline & & & 1 & & & & \\ & & & & 1 & & & \\ & \Phi_2 & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{array}\right]$$

in the northwest 9×8 corner. It follows from the Binet–Cauchy formula for the minors of $\Phi_1\Phi_2$ as sums of products of minors in Φ_1 and in Φ_2 that the ideal generated by all of the 7×7 minors in the block matrix Φ equals I. Compare the above block matrix with the general version in (17.6). ◇

Proof of Theorem 17.17. By Lemma 17.13 the essential set $\mathcal{E}ss(v(\mathbf{r}))$ consists of boxes (k, ℓ) at the southeast corners of blocks. Therefore by Theorem 15.15, $I_{v(\mathbf{r})}$ is generated by the minors of size $1 + \text{rank}(v(\mathbf{r})_{k\times\ell})$ in $\mathbf{x}_{k\times\ell}$ for the southeast corners (k, ℓ) of blocks on or below the superantidiagonal, along with all variables strictly above the block superantidiagonal.

Consider a box (k, ℓ) at the southeast corner of $B_{i+1,j-1}$, the intersection of block column $i + 1$ and block row $j - 1$, so that

$$\text{rank}(v(\mathbf{r})_{k\times\ell}) \quad = \quad \sum_{\substack{\alpha>i \\ \beta<j}} q_{\alpha\beta} + \sum_{m=i+1}^{j} r_{m-1,m} \qquad (17.5)$$

by definition of Zelevinsky permutation. We pause to prove the following.

Lemma 17.19 *The number* $\mathrm{rank}(v(\mathbf{r})_{k\times\ell})$ *in (17.5) is* $r_{ij} + \sum_{m=i+1}^{j-1} r_m$.

Proof. The coefficient on $q_{\alpha\beta}$ in $r_{ij} + \sum_{m=i+1}^{j-1} r_m$ is the number of elements in $\{r_{ij}\} \cup \{r_{i+1,i+1}, \ldots, r_{j-1,j-1}\}$ that are weakly northwest of $r_{\alpha\beta}$ in the rank array $\mathbf{r}$ (when the array $\mathbf{r}$ is oriented so that its southeast corner is r_{0n}). This number equals the number of elements in $\{r_{i,i+1}, \ldots, r_{j-1,j}\}$ that are weakly northwest of $r_{\alpha\beta}$, unless $r_{\alpha\beta}$ happens to lie strictly north and strictly west of r_{ij}, in which case we get one fewer. This one fewer is exactly made up by the sum of entries from $\mathbf{q}$ in (17.5). □

Resuming the proof of Theorem 17.17, consider the minors in $I_{v(\mathbf{r})}$ coming from the northwest $k\times\ell$ submatrix $\mathbf{x}_{k\times\ell}$, for (k,ℓ) in the southeast corner of $B_{i+1,j-1}$. Taking their images in $\Bbbk[\mathbf{f}]$ has the effect of setting the appropriate $\mathbf{x}$ variables to 0 or 1, and then changing the block superantidiagonal $\mathbf{x}$ variables into the corresponding $\mathbf{f}$ variables. Therefore the minors in $I_{v(\mathbf{r})}$ become minors of the matrix in (17.4), if each ϕ_i is replaced by the generic matrix Φ_i of $\mathbf{f}$ variables. In particular, using Lemma 17.19, the equations in $\Bbbk[\mathbf{f}]$ from $\mathbf{x}_{k\times\ell}$ are the minors of size $1+u+r_{ij}$ in the generic $(u+r_i)\times(u+r_i)$ block matrix

$$\begin{bmatrix} 0 & 0 & & 0 & \Phi_{i+1} \\ 0 & 0 & & \Phi_{i+2} & \mathbf{1} \\ 0 & 0 & \iddots & \mathbf{1} & 0 \\ 0 & \Phi_{j-1} & \iddots & 0 & 0 \\ \Phi_j & \mathbf{1} & & 0 & 0 \end{bmatrix}, \tag{17.6}$$

where $u = \sum_{m=i+1}^{j-1} r_m$ is the sum of the ranks of the subantidiagonal identity blocks. The ideal generated by these minors of size $1+u+r_{ij}$ is preserved under multiplication of (17.6) by any determinant 1 matrix with entries in $\Bbbk[\mathbf{f}]$, by Lemma 17.16. Now multiply (17.6) on the left by

$$\begin{bmatrix} \mathbf{1} & -\Phi_{i+1} & \Phi_{i+1}\Phi_{i+2} & \cdots & \pm\Phi_{i+1,j-2} & \mp\Phi_{i+1,j-1} \\ & \mathbf{1} & & & & \\ & & \mathbf{1} & & & \\ & & & \ddots & & \\ & & & & \mathbf{1} & \\ & & & & & \mathbf{1} \end{bmatrix},$$

where $\Phi_{i+1,m} = \Phi_{i+1}\cdots\Phi_m$ for $i+1 \leq m$. The result agrees with (17.6) except in its top block row, which has left block $(-1)^{j-1-i}\Phi_{i+1}\cdots\Phi_j$ and all other blocks zero. Crossing out the top block row and the left block column leaves a block upper-left-triangular matrix that is square of size u, so the size $1+u+r_{ij}$ minors in (17.6) generate the same ideal in $\Bbbk[\mathbf{f}]$ as the size $1+r_{ij}$ minors in $\Phi_{i+1}\cdots\Phi_j$. This holds for all $i \leq j$, completing the proof. □

Corollary 17.20 *The Zelevinsky map takes the quiver locus $\Omega_{\mathbf{r}} \subseteq Mat$ isomorphically to the intersection $\mathcal{Z}(\Omega_{\mathbf{r}}) = \overline{X}_{v(\mathbf{r})} \cap \mathcal{Z}(Mat)$ of the matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ with the affine space $\mathcal{Z}(Mat)$ inside of M_d.*

17.3 Primality and Cohen–Macaulayness

Theorem 17.17 shows how to get the equations for $I_{\mathbf{r}}$ directly from those for $I_{v(\mathbf{r})}$: set the appropriate $\mathbf{x}$ variables to 0 or 1. Its proof never needed that Schubert determinantal ideals are prime or Cohen–Macaulay (Corollaries 16.29 and 16.44). Our next goal is to put these assertions to good use, to reach the same conclusions for quiver ideals. This involves a more detailed study of the group theory surrounding the geometry in Corollary 17.20.

Let P be the *parabolic subgroup* of block lower triangular matrices in GL_d, where the diagonal blocks have sizes $r_0, \ldots, r_n$ (proceeding from left to right). The quotient $P\backslash GL_d$ of the general linear group GL_d by the parabolic subgroup P is called a *partial flag variety*. By definition, the *Schubert variety* $X_{v(\mathbf{r})}$ is the image of $\overline{X}_{v(\mathbf{r})} \cap GL_d$ in $P\backslash GL_d$. The Zelevinsky image $\mathcal{Z}(\Omega_{\mathbf{r}})$ of the quiver locus maps isomorphically to its image inside of $X_{v(\mathbf{r})}$, and this image is often called the *opposite Schubert cell* in $X_{v(\mathbf{r})}$, even though $\mathcal{Z}(\Omega_{\mathbf{r}})$ is usually not isomorphic to an affine space (that is, $\Omega_{\mathbf{r}}$ is not a cell).

Note that the block structure on P is block-column reversed from the one considered earlier in this chapter. The coordinate ring $\Bbbk[P]$ is obtained from the polynomial ring $\Bbbk[\mathbf{p}]$ in the variables from the block lower triangle by inverting the determinant polynomial. In particular, the square blocks $\mathbf{p}^{00}, \ldots, \mathbf{p}^{nn}$ in (17.7) below have inverses filled with regular functions on P.

Lemma 17.21 *The multiplication map $P \times Mat \to P \cdot \mathcal{Z}(Mat)$ that sends (π, ϕ) to the product $\pi\mathcal{Z}(\phi)$ of matrices in M_d is an isomorphism of varieties that takes $P \times \Omega_{\mathbf{r}}$ isomorphically to $P \cdot \mathcal{Z}(\Omega_{\mathbf{r}})$ for each rank array $\mathbf{r}$.*

Proof. It is enough to treat the case where $\Omega_{\mathbf{r}} = Mat$. Denote by Φ the generic matrix obtained from (17.4) after replacing its blocks ϕ_i by Φ_i, and let $\mathbf{x}_{v_*}$ be the block matrix of coordinate variables on $\overline{X}_{v_*}$, at left below.

$$\begin{bmatrix} 0 & 0 & 0 & \mathbf{x}^{01} & \mathbf{x}^{00} \\ 0 & 0 & \mathbf{x}^{12} & \mathbf{x}^{11} & \mathbf{x}^{10} \\ 0 & . \cdot ^{\cdot} & \mathbf{x}^{22} & \mathbf{x}^{21} & \mathbf{x}^{20} \\ \mathbf{x}^{n-1,n} & & \vdots & \vdots & \vdots \\ \mathbf{x}^{nn} & \cdots & \mathbf{x}^{n2} & \mathbf{x}^{n1} & \mathbf{x}^{n0} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{p}^{00} & 0 & 0 & 0 & 0 \\ \mathbf{p}^{10} & \mathbf{p}^{11} & 0 & 0 & 0 \\ \mathbf{p}^{20} & \mathbf{p}^{21} & \mathbf{p}^{22} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \mathbf{p}^{n0} & \mathbf{p}^{n1} & \mathbf{p}^{n2} & \cdots & \mathbf{p}^{nn} \end{bmatrix} \cdot \Phi \qquad (17.7)$$

Direct calculation show that $P \cdot \mathcal{Z}(Mat) \subseteq \overline{X}_{v_*}$. Therefore the morphism $P \times Mat \to P \cdot \mathcal{Z}(Mat)$ is determined by (and is basically equivalent to—see

Exercise 17.12) the map (17.7) of algebras $\Bbbk[\mathbf{x}_{v_*}] \to \Bbbk[\mathbf{p}, \mathbf{f}]$, which sends

$$\begin{array}{rcll} \mathbf{x}^{i,i+1} & \mapsto & \mathbf{p}^{ii}\Phi_{i+1} & \text{for } i = 0, \ldots, n-1; \\ \mathbf{x}^{j0} & \mapsto & \mathbf{p}^{j0} & \text{for } j = 0, \ldots, n; \text{ and} \\ \mathbf{x}^{ji} & \mapsto & \mathbf{p}^{ji} + \mathbf{p}^{j,i-1}\Phi_i & \text{for } 1 \leq i \leq j \leq n. \end{array} \qquad (17.8)$$

Observe that the inverse of $\mathbf{x}^{00} = \mathbf{p}^{00}$ is regular on $P \cdot \mathcal{Z}(Mat)$, so we can recover $\Phi_1 = (\mathbf{x}^{00})^{-1}\mathbf{x}^{01}$. Then we can recover the first column $\mathbf{p}^{j1}$ of $\mathbf{p}$ by subtracting (the zeroth column of $\mathbf{p}$) $\cdot\, \Phi_1$ from (the penultimate column of $\mathbf{x}_{v_*}$). Continuing similarly, we can produce Φ and $\mathbf{p}$ as regular functions on $P \cdot \mathcal{Z}(Mat)$ to construct the inverse map $\Bbbk[P \times Mat] \to \Bbbk[P \cdot \mathcal{Z}(Mat)]$. □

Proposition 17.22 *Multiplication by P on the left preserves the matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$. In fact, $\overline{X}_{v(\mathbf{r})}$ is the closure in M_d of $P \cdot \mathcal{Z}(\Omega_\mathbf{r})$.*

Proof. Definition 17.12.4 and Corollary 15.33 imply that the matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ is stable under the action of $S_{r_0} \times \cdots \times S_{r_n}$, the block diagonal permutation matrices whose blocks have sizes $r_0, \ldots, r_n$ acting on the left. This finite group and the lower-triangular Borel group B together generate P. Combining this with the stability of $\overline{X}_{v(\mathbf{r})}$ under the left action of B in Theorem 15.31 completes the proof of the first statement.

Theorem 17.17 says that $\Bbbk[\mathbf{f}]/I_\mathbf{r} \cong \Bbbk[\mathbf{x}]/(I_{v(\mathbf{r})} + \mathfrak{m}_\mathbf{f})$. But $I_{v(\mathbf{r})}$ already contains the variables in $\mathfrak{m}_\mathbf{f}$ above the block antidiagonal, and only $\dim(P)$ many generators of $\mathfrak{m}_\mathbf{f}$ remain. Thus the codimension of $\mathcal{Z}(\Omega_\mathbf{r})$ inside $\overline{X}_\mathbf{r}$ is at most $\dim(P)$. But $\overline{X}_{v(\mathbf{r})}$ contains $P \cdot \mathcal{Z}(\Omega_\mathbf{r})$ by the stability of $\overline{X}_{v(\mathbf{r})}$ under P, and $\dim(P \cdot \mathcal{Z}(\Omega_\mathbf{r})) = \dim(\mathcal{Z}(\Omega_\mathbf{r})) + \dim(P)$ by Lemma 17.21. Thus the codimension of $\mathcal{Z}(\Omega_\mathbf{r})$ inside $\overline{X}_{v(\mathbf{r})}$ is at least $\dim(P)$. We conclude that $\mathcal{Z}(\Omega_\mathbf{r})$ has codimension exactly $\dim(P)$ inside $\overline{X}_{v(\mathbf{r})}$, so $\dim(P \cdot \mathcal{Z}(\Omega_\mathbf{r})) = \dim(\overline{X}_{v(\mathbf{r})})$. Since $\overline{X}_{v(\mathbf{r})}$ is an irreducible variety, it follows that $P \cdot \mathcal{Z}(\Omega_\mathbf{r})$ is Zariski dense inside $\overline{X}_{v(\mathbf{r})}$, proving the second statement. □

Theorem 17.23 *Given an irreducible rank array $\mathbf{r}$, the quiver ideal $I_\mathbf{r}$ inside $\Bbbk[\mathbf{f}]$ is prime, and the quiver locus $\Omega_\mathbf{r}$ is Cohen–Macaulay.*

Proof. The matrix Schubert variety $\overline{X}_{v(\mathbf{r})}$ is Cohen–Macaulay by Corollary 16.44, and $P \cdot \mathcal{Z}(\Omega_\mathbf{r})$ is a dense subvariety of $\overline{X}_{v(\mathbf{r})}$ by Proposition 17.22. Since being Cohen–Macaulay is a local property [BH98, Definition 2.1.1], we conclude that $P \cdot \mathcal{Z}(\Omega_\mathbf{r})$ is Cohen–Macaulay. By Lemma 17.21, the coordinate ring $\Bbbk[P \cdot \mathcal{Z}(\Omega_\mathbf{r})]$, which we have just seen is Cohen–Macaulay, is isomorphic to the localization by $\det(\mathbf{p})$ of the polynomial ring $\Bbbk[\Omega_\mathbf{r}][\mathbf{p}]$ over the coordinate ring of $\Omega_\mathbf{r}$. This localization is Cohen–Macaulay if and only if $\Bbbk[\Omega_\mathbf{r}][\mathbf{p}]$ is; see Exercise 17.16. As the equations setting the off-diagonal $\mathbf{p}$ variables to 0 and the diagonal $\mathbf{p}$ variables to 1 constitute a regular sequence on $\Bbbk[\Omega_\mathbf{r}][\mathbf{p}]$, we conclude by Criterion 2 of Theorem 13.37 and repeated application of Lemma 8.27 that $\Bbbk[\Omega_\mathbf{r}]$ is Cohen–Macaulay.

The variety $\Omega_\mathbf{r}$ is irreducible by Lemma 17.21 and Proposition 17.22, because $\overline{X}_{v(\mathbf{r})}$ is, so the radical of $I_\mathbf{r}$ is prime; but it still remains to prove that $I_{\mathbf{r}}$ is itself prime. By Theorem 17.17, we need that the image of $\mathfrak{m}_{\mathbf{f}}$ in $\Bbbk[\mathbf{x}]/I_{v(\mathbf{r})}$ is prime. As the homomorphism $\Bbbk[\mathbf{x}]/I_{v(\mathbf{r})} \to \Bbbk[P \cdot \mathcal{Z}(\Omega_{\mathbf{r}})]$ is injective by Proposition 17.22, we only need the image of $\mathfrak{m}_{\mathbf{f}}$ in $\Bbbk[P \cdot \mathcal{Z}(\Omega_{\mathbf{r}})]$ to generate a prime ideal. To that end, we identify the ideal generated by the image of $\mathfrak{m}_{\mathbf{f}}$ in $\Bbbk[P \times \Omega_{\mathbf{r}}]$ under the isomorphism with $\Bbbk[P \cdot \mathcal{Z}(\Omega_{\mathbf{r}})]$ in Lemma 17.21, given by (17.7) and (17.8). The generators of $\mathfrak{m}_{\mathbf{f}}$ set $\mathbf{x}^{ji} = 0$ in (17.8) for $i < j$ and $\mathbf{x}^{ii} = \mathbf{1}$. By induction on i, the images in $\Bbbk[\mathbf{p}, \mathbf{f}]$ of these equations imply the equations setting $\mathbf{p}^{ii} = \mathbf{1}$ and $\mathbf{p}^{ji} = 0$ for $i < j$. Hence $\mathbf{p}^{ii}\Phi_{i+1} = \Phi_{i+1}$ modulo $\mathfrak{m}_{\mathbf{f}}$, so the image of $\mathfrak{m}_{\mathbf{f}}$ generates the kernel of the homomorphism $\Bbbk[P \times \Omega_{\mathbf{r}}] \to \Bbbk[\Omega_{\mathbf{r}}]$ coming from the inclusion $\Omega_{\mathbf{r}} \cong \mathrm{id} \times \Omega_{\mathbf{r}} \hookrightarrow P \times \Omega_{\mathbf{r}}$. The result follows because $\Bbbk[\Omega_{\mathbf{r}}]$ is a domain. □

17.4 Quiver polynomials

Having exploited the algebra and geometry of matrix Schubert varieties to deduce qualitative statements about quiver ideals and loci, we now turn to more quantitative data, namely multidegrees and K-polynomials. For this, we (finally) need full details on the indexing of all the variables involved.

Again setting $d = r_0 + \cdots + r_n$, the coordinate ring $\Bbbk[\mathbf{f}]$ of Mat is graded by $\mathbb{Z}^d$. To describe this grading efficiently, write

$$\mathbb{Z}^d \;=\; \mathbb{Z}^{r_0} \oplus \cdots \oplus \mathbb{Z}^{r_n}, \quad \text{where} \quad \mathbb{Z}^{r_i} \;=\; \mathbb{Z} \cdot \{\mathbf{e}^i_1, \ldots \mathbf{e}^i_{r_i}\}.$$

Thus the basis of $\mathbb{Z}^d$ splits into a sequence of $n+1$ subsets $\mathbf{e}^0, \ldots, \mathbf{e}^n$ of sizes $r_0, \ldots, r_n$. We declare the variable $f^i_{\alpha\beta} \in \Bbbk[\mathbf{f}]$ to have degree

$$\deg(f^i_{\alpha\beta}) \;=\; \mathbf{e}^{i-1}_\alpha - \mathbf{e}^i_\beta \tag{17.9}$$

in $\mathbb{Z}^d$ for each $i = 1, \ldots, n$. Under this multigrading, the quiver ideal $I_{\mathbf{r}}$ is homogeneous. Indeed, the entries in products $\Phi_{i+1} \cdots \Phi_j$ of consecutive matrices are $\mathbb{Z}^d$-graded (check this!), so minors in such products are, too. When we write multidegrees and K-polynomials for this $\mathbb{Z}^d$-grading, we similarly split the list $\mathbf{t}$ of d variables into a sequence of $n+1$ alphabets $\mathbf{t}^0, \ldots, \mathbf{t}^n$ of sizes $r_0, \ldots, r_n$, so that the i^{th} alphabet is $\mathbf{t}^i = t^i_1, \ldots, t^i_{r_i}$.

Definition 17.24 Under the above $\mathbb{Z}^d$-grading on $\Bbbk[\mathbf{f}]$, the multidegree

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathring{\mathbf{t}}) \;=\; \mathcal{C}(\Omega_{\mathbf{r}}; \mathbf{t})$$

of $\Bbbk[\mathbf{f}]/I_{\mathbf{r}}$ is the **(ordinary) quiver polynomial** for the rank array $\mathbf{r}$.

For the moment, the argument $\mathbf{t} - \mathring{\mathbf{t}}$ of $\mathcal{Q}_{\mathbf{r}}$ can be regarded as a formal symbol, denoting that $n+1$ alphabets $\mathbf{t} = \mathbf{t}^0, \ldots, \mathbf{t}^n$ are required as input. Later in this section we shall define 'double quiver polynomials', with arguments $\mathbf{t} - \mathring{\mathbf{s}}$ indicating two sequences of alphabets as input. Then, in Theorem 17.34, the symbol $\mathring{\mathbf{t}}$ will take on additional meaning as the reversed sequence $\mathbf{t}^n, \ldots, \mathbf{t}^0$ of alphabets constructed from $\mathbf{t}$.

Example 17.25 The quiver ideal $I_{\mathbf{r}}$ for the rank array determined by the lacing diagram  is a complete intersection of codimension 2. It is generated by the two entries in the product $\Phi_1\Phi_2\Phi_3$ of the generic 2×3, 3×3, and 3×1 matrices. These two entries have degree $\mathbf{e}_1^0 - \mathbf{e}_1^3$ and $\mathbf{e}_2^0 - \mathbf{e}_1^3$, so the multidegree of $\mathbb{k}[\mathbf{f}]/I_{\mathbf{r}}$ is $\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathring{\mathbf{t}}) = (t_1^0 - t_1^3)(t_2^0 - t_1^3)$. $\diamond$

Example 17.26 Consider a sequence of $2n$ vector spaces with dimensions $1, 2, 3 \ldots, n-1, n, n, n-1, \ldots, 3, 2, 1$. For a size $n+1$ permutation matrix w, let $\mathbf{q}_w$ be the lace array whose entries are zero outside of the southeast $n \times n$ corner, which is filled with 1s and 0s by rotating $w_{n\times n}$ around 180°. Exercise 17.14 explores the combinatorics of the arrays $\mathbf{q}_w$. Quiver polynomials for the associated rank arrays $\mathbf{r}_w$ are called **Fulton polynomials**. $\diamond$

The algebraic connection from quiver ideals to Schubert determinantal ideals will allow us to compute quiver polynomials in terms of double Schubert polynomials. For this, the coordinate ring $\mathbb{k}[\mathbf{x}] = \mathbb{k}[M_d]$ of the $d \times d$ matrices is multigraded by the group $\mathbb{Z}^{2d} = (\mathbb{Z}^{r_0} \oplus \cdots \oplus \mathbb{Z}^{r_n})^2$, which we take to have basis $\{\mathbf{e}_\alpha^i, \dot{\mathbf{e}}_\alpha^i \mid i = 0, \ldots, n \text{ and } \alpha = 1, \ldots, r_i\}$; note the dot over the second $\mathbf{e}_\alpha^i$. In our context, it is most natural to index the variables

$$\mathbf{x} \quad = \quad (x_{\alpha\beta}^{ji} \mid i, j = 0, \ldots, n \text{ and } \alpha = 1, \ldots, r_i \text{ and } \beta = 1, \ldots, r_j)$$

in the generic $d \times d$ matrix in a slightly unusual manner: $x_{\alpha\beta}^{ji} \in \mathbb{k}[\mathbf{x}]$ occupies the spot in row α and column β within the rectangle at the intersection of the i^{th} block column and the j^{th} block row, where we label block columns starting from the *right*. Declare the variable $x_{\alpha\beta}^{ji}$ to have degree

$$\deg(x_{\alpha\beta}^{ji}) \quad = \quad \mathbf{e}_\alpha^j - \dot{\mathbf{e}}_\beta^i. \qquad (17.10)$$

To write multidegrees and K-polynomials we use two sets of $n+1$ alphabets

$$\mathbf{t} = \mathbf{t}^0, \ldots, \mathbf{t}^n \quad \text{and} \quad \mathring{\mathbf{s}} = \mathbf{s}^n, \ldots, \mathbf{s}^0, \qquad (17.11)$$

$$\text{where} \quad \mathbf{t}^j = t_1^j, \ldots, t_{r_j}^j \quad \text{and} \quad \mathbf{s}^j = s_1^j, \ldots, s_{r_j}^j. \qquad (17.12)$$

We rarely see the degree (17.10) directly; more often, we see the polynomial $t_\alpha^j - s_\beta^i = \mathcal{C}(\mathbb{k}[\mathbf{x}]/\langle x_{\alpha\beta}^{ji}\rangle; \mathbf{t}, \mathbf{s})$, which we call the *ordinary weight* of $x_{\alpha\beta}^{ji}$. Ordinary weights are the building blocks for multidegrees because of Theorem 8.44. (The analogous building block for K-polynomials, namely the ratio $t_\alpha^j/s_\beta^i = \mathcal{K}(\mathbb{k}[\mathbf{x}]/\langle x_{\alpha\beta}^{ji}\rangle; \mathbf{t}, \mathbf{s})$, is called the *exponential weight* of $x_{\alpha\beta}^{ji}$.) Pictorially, label the rows of the $d \times d$ grid with the $\mathbf{t}$ variables in the order they are given, from top to bottom, and similarly label the columns with $\mathring{\mathbf{s}} = \mathbf{s}^n, \ldots, \mathbf{s}^0$, from left to right. The ordinary weight of the variable $x_{\alpha\beta}^{ji}$ is then its row $\mathbf{t}$-label minus its column $\mathbf{s}$-label.

For notational clarity in examples, it is convenient to use alphabets $\mathbf{t}^0 = \mathbf{a}$ and $\mathbf{t}^1 = \mathbf{b}$ and $\mathbf{t}^2 = \mathbf{c}$, and so on, rather than upper indices, where

$$\mathbf{a} = a_1, a_2, a_3, \ldots \quad \text{and} \quad \mathbf{b} = b_1, b_2, b_3, \ldots \quad \text{and} \quad \mathbf{c} = c_1, c_2, c_3, \ldots$$

The quiver polynomial in Example 17.25 is $(a_1-d_1)(a_2-d_1)$ in this notation. For the $\mathbf{s}$ alphabets we use $\mathbf{s}^0 = \dot{\mathbf{a}}$ and $\mathbf{s}^1 = \dot{\mathbf{b}}$ and $\mathbf{s}^2 = \dot{\mathbf{c}}$, and so on, where

$$\dot{\mathbf{a}} = \dot{a}_1, \dot{a}_2, \dot{a}_3, \ldots \quad \text{and} \quad \dot{\mathbf{b}} = \dot{b}_1, \dot{b}_2, \dot{b}_3, \ldots \quad \text{and} \quad \dot{\mathbf{c}} = \dot{c}_1, \dot{c}_2, \dot{c}_3, \ldots$$

are the same as $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ but with dots on top. All of the notation above should be made clearer by the following example.

Example 17.27 If $(r_0, r_1, r_2) = (2, 3, 1)$, then $\Bbbk[\mathbf{x}]$ has variables $x^{ji}_{\alpha\beta}$ as they appear in the matrices below (the x variables are the same in both):

$$\begin{array}{c|c|ccc|cc|}
 & s^2_1 & s^1_1 & s^1_2 & s^1_3 & s^0_1 & s^0_2 \\ \hline
t^0_1 & x^{02}_{11} & x^{01}_{11} & x^{01}_{12} & x^{01}_{13} & x^{00}_{11} & x^{00}_{12} \\
t^0_2 & x^{02}_{21} & x^{01}_{21} & x^{01}_{22} & x^{01}_{23} & x^{00}_{21} & x^{00}_{22} \\ \hline
t^1_1 & x^{12}_{11} & x^{11}_{11} & x^{11}_{12} & x^{11}_{13} & x^{10}_{11} & x^{10}_{12} \\
t^1_2 & x^{12}_{21} & x^{11}_{21} & x^{11}_{22} & x^{11}_{23} & x^{10}_{21} & x^{10}_{22} \\
t^1_3 & x^{12}_{31} & x^{11}_{31} & x^{11}_{32} & x^{11}_{33} & x^{10}_{31} & x^{10}_{32} \\ \hline
t^2_1 & x^{22}_{11} & x^{21}_{11} & x^{21}_{12} & x^{21}_{13} & x^{20}_{11} & x^{20}_{12} \\ \hline
\end{array}
\quad = \quad
\begin{array}{c|c|ccc|cc|}
 & \dot{c}_1 & \dot{b}_1 & \dot{b}_2 & \dot{b}_3 & \dot{a}_1 & \dot{a}_2 \\ \hline
a_1 & x^{02}_{11} & x^{01}_{11} & x^{01}_{12} & x^{01}_{13} & x^{00}_{11} & x^{00}_{12} \\
a_2 & x^{02}_{21} & x^{01}_{21} & x^{01}_{22} & x^{01}_{23} & x^{00}_{21} & x^{00}_{22} \\ \hline
b_1 & x^{12}_{11} & x^{11}_{11} & x^{11}_{12} & x^{11}_{13} & x^{10}_{11} & x^{10}_{12} \\
b_2 & x^{12}_{21} & x^{11}_{21} & x^{11}_{22} & x^{11}_{23} & x^{10}_{21} & x^{10}_{22} \\
b_3 & x^{12}_{31} & x^{11}_{31} & x^{11}_{32} & x^{11}_{33} & x^{10}_{31} & x^{10}_{32} \\ \hline
c_1 & x^{22}_{11} & x^{21}_{11} & x^{21}_{12} & x^{21}_{13} & x^{20}_{11} & x^{20}_{12} \\ \hline
\end{array}$$

The ordinary weight of each x variable equals its row label minus its column label. For example, the variable x^{01}_{23} has ordinary weight $t^0_2 - s^1_3 = a_2 - \dot{b}_3$. $\diamond$

The coordinate ring $\Bbbk[\mathcal{Z}(Mat)] = \Bbbk[\mathbf{x}]/\mathfrak{m}_\mathbf{f}$ of the image of the Zelevinsky map is not naturally multigraded by all of $\mathbb{Z}^{2d}$, but only by $\mathbb{Z}^d$, with the variable $x^{ji}_{\alpha\beta} \in \Bbbk[\mathcal{Z}(Mat)]$ having ordinary weight $t^j_\alpha - t^i_\beta$. This convention is consistent with the multigrading on $\Bbbk[\mathbf{f}]$ in (17.9) under the isomorphism to $\Bbbk[\mathbf{x}]/\mathfrak{m}_\mathbf{f}$ induced by the Zelevinsky map. Indeed, the $\mathbf{x}$ variable $x^{i-1,i}_{\alpha\beta} \in \Bbbk[\mathbf{x}]/\mathfrak{m}_\mathbf{f}$ maps to $f^i_{\alpha\beta} \in \Bbbk[\mathbf{f}]$, and their ordinary weights $t^{i-1}_\alpha - t^i_\beta$ agree. In what follows, we need to consider not only the Zelevinsky image of Mat, but also the variety of all block upper-left triangular matrices.

Definition 17.28 The **opposite big cell** is the variety Y inside M_d obtained by setting $\mathbf{x}^{ji} = 0$ for $i < j$ and $\mathbf{x}^{ii} = \mathbf{1}$ for all i. Denote the remaining nonconstant coordinates on Y by $\mathbf{y} = \{y^{ji}_{\alpha\beta} \mid i > j\}$, so $\Bbbk[Y] = \Bbbk[\mathbf{y}]$.

Using language at the end of Section 17.3, Y is the opposite cell in the Schubert subvariety of $P\backslash GL_d$ consisting of the whole space. Note that Y is actually a cell—that is, isomorphic to an affine space. The $\mathbb{Z}^d$-grading of $\Bbbk[\mathbf{x}]$ descends to the $\mathbb{Z}^d$-grading of $\Bbbk[\mathbf{y}]$, which is positive (check this!).

Example 17.29 In the situation of Example 17.27, the coordinate ring $\Bbbk[\mathbf{y}]$ has only the variables $y^{ji}_{\alpha\beta}$ that appear in the matrices below.

$$\begin{array}{c|c|ccc|cc|}
 & t^2_1 & t^1_1 & t^1_2 & t^1_3 & t^0_1 & t^0_2 \\ \hline
t^0_1 & y^{02}_{11} & y^{01}_{11} & y^{01}_{12} & y^{01}_{13} & 1 & \\
t^0_2 & y^{02}_{21} & y^{01}_{21} & y^{01}_{22} & y^{01}_{23} & & 1 \\ \hline
t^1_1 & y^{12}_{11} & 1 & & & & \\
t^1_2 & y^{12}_{21} & & 1 & & & \\
t^1_3 & y^{12}_{31} & & & 1 & & \\ \hline
t^2_1 & 1 & & & & & \\ \hline
\end{array}
\quad = \quad
\begin{array}{c|c|ccc|cc|}
 & c_1 & b_1 & b_2 & b_3 & a_1 & a_2 \\ \hline
a_1 & y^{02}_{11} & y^{01}_{11} & y^{01}_{12} & y^{01}_{13} & 1 & \\
a_2 & y^{02}_{21} & y^{01}_{21} & y^{01}_{22} & y^{01}_{23} & & 1 \\ \hline
b_1 & y^{12}_{11} & 1 & & & & \\
b_2 & y^{12}_{21} & & 1 & & & \\
b_3 & y^{12}_{31} & & & 1 & & \\ \hline
c_1 & 1 & & & & & \\ \hline
\end{array}$$

In this case, the variable y^{01}_{23} has ordinary weight $t^0_2 - t^1_3 = a_2 - b_3$. $\diamond$

Definition 17.30 The **double quiver polynomial** $\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathring{\mathbf{s}})$ is the ratio

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathring{\mathbf{s}}) \quad = \quad \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{t}-\mathring{\mathbf{s}})}{\mathfrak{S}_{v_*}(\mathbf{t}-\mathring{\mathbf{s}})}$$

of double Schubert polynomials in the concatenations of the two sequences of finite alphabets described in (17.11) and (17.12).

The denominator $\mathfrak{S}_{v_*}(\mathbf{t}-\mathring{\mathbf{s}})$ should be regarded as a fudge factor, being simply the product of all ordinary weights $(t_* - s_*)$ of variables lying strictly above the block superantidiagonal. These variables lie in locations corresponding to $*$ entries in the diagram of every Zelevinsky permutation, so $\mathfrak{S}_{v_*}$ obviously divides $\mathfrak{S}_{v(\mathbf{r})}$ (see Corollary 16.30).

The simple relation between double and ordinary quiver polynomials, to be presented in Theorem 17.34, justifies the notation $\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathring{\mathbf{t}})$ for the ordinary case: quiver polynomials are the specializations of double quiver polynomials obtained by setting $\mathbf{s}^i = \mathbf{t}^i$ for all i. For this purpose, write

$$\mathring{\mathbf{t}} \quad = \quad \mathbf{t}^n, \ldots, \mathbf{t}^0$$

to mean the reverse of the finite list $\mathbf{t}$ of alphabets from (17.11). That way, setting $\mathbf{s}^i = \mathbf{t}^i$ for all i is simply setting $\mathring{\mathbf{s}} = \mathring{\mathbf{t}}$. In the case where every block has size 1, so $P = B$ is the Borel subgroup of lower-triangular matrices in GL_d, each alphabet in the list $\mathbf{t}$ consists of just one variable (as opposed to there being only one alphabet in the list), so the reversed list $\mathring{\mathbf{t}}$ is really just a globally reversed alphabet in that case.

Our goal is to relate double quiver polynomials to ordinary quiver polynomials. At first, we work with K-polynomials, for which we need a lemma.

Proposition 17.31 *Let $\mathcal{F}_{\bullet}$ be a $\mathbb{Z}^{2d}$-graded free resolution of $\Bbbk[\mathbf{x}]/I_{v(\mathbf{r})}$ over $\Bbbk[\mathbf{x}]$. If $\mathfrak{m}_{\mathbf{y}}$ is the ideal of Y in $\Bbbk[\mathbf{x}]$, then the complex $\mathcal{F}_{\bullet} \otimes_{\Bbbk[\mathbf{x}]} \Bbbk[\mathbf{y}] = \mathcal{F}_{\bullet}/\mathfrak{m}_{\mathbf{y}}\mathcal{F}_{\bullet}$ is a $\mathbb{Z}^d$-graded free resolution of $\Bbbk[\mathcal{Z}(\Omega_{\mathbf{r}})]$ over $\Bbbk[\mathbf{y}]$.*

Proof. Note that $\mathcal{F}_{\bullet}/\mathfrak{m}_{\mathbf{y}}\mathcal{F}_{\bullet}$ is complex of $\mathbb{Z}^d$-graded free modules over $\Bbbk[\mathbf{y}]$. Indeed, coarsening the $\mathbb{Z}^{2d}$-grading on $\Bbbk[\mathbf{x}]$ to the grading by $\mathbb{Z}^d$ in which $x^{ji}_{\alpha\beta}$ has ordinary weight $t^j_\alpha - t^i_\beta$ (by setting $s^i_\beta = t^i_\beta$) makes the generators of $\mathfrak{m}_{\mathbf{y}}$ homogeneous, because the variables set equal to 1 have degree zero.

The $\mathbf{x}$ variables in blocks strictly above the block superantidiagonal already lie inside $I_{v(\mathbf{r})}$, so $I(\mathcal{Z}(\Omega_{\mathbf{r}})) = I_{v(\mathbf{r})} + \mathfrak{m}_{\mathbf{f}} = I_{v(\mathbf{r})} + \mathfrak{m}_{\mathbf{y}}$ by Theorem 17.17. What we would like is that the generators of $\mathfrak{m}_{\mathbf{y}}$ form a regular sequence on $\Bbbk[\mathbf{x}]/I_{v(\mathbf{r})}$, because then repeated application of Lemma 8.27 would complete the proof. What we will actually show is almost as good: we will check that the generators of $\mathfrak{m}_{\mathbf{y}}$ form a regular sequence on the localization of $\Bbbk[\mathbf{x}]/I_{v(\mathbf{r})}$ at every maximal ideal $\mathfrak{p}$ of $\Bbbk[\mathbf{x}]$ containing $I_{v(\mathbf{r})} + \mathfrak{m}_{\mathbf{y}}$.

This suffices because (i) the complex $\mathcal{F}_\bullet/\mathfrak{m}_\mathbf{y}\mathcal{F}_\bullet$ is exact if and only if its localization at every maximal ideal of $\Bbbk[\mathbf{y}]$ is exact [Eis95, Lemma 2.8], and (ii) if $\mathfrak{p}$ does not contain $I_{v(\mathbf{r})}$ then $(\mathcal{F}_\bullet)_\mathfrak{p}$, and hence also $(\mathcal{F}_\bullet/\mathfrak{m}_\mathbf{y}\mathcal{F}_\bullet)_\mathfrak{p}$, is a free resolution of 0, which is split exact.

For the local regular sequence property, we use [BH98, Theorem 2.1.2]: if N is a Cohen–Macaulay module over a local ring, and $z_1, \ldots, z_r$ is any sequence of elements, then $N/\langle z_1, \ldots, z_r\rangle N$ has dimension $\dim(N)-r$ if and only if $z_1, \ldots, z_r$ is a regular sequence on N. Noting that $\mathfrak{m}_\mathbf{y}$ is generated by $\dim(\overline{X}_{v(\mathbf{r})}) - \dim(\Omega_\mathbf{r})$ elements, we are done by Corollary 16.44. □

If $\Bbbk[\mathbf{x}]/I(\overline{X})$ is a $\mathbb{Z}^{2d}$-graded coordinate ring of a subvariety $\overline{X}$ inside M_d, write $\mathcal{K}_M(\overline{X}; \mathbf{t}, \mathring{\mathbf{s}})$ for its K-polynomial. Similarly, write $\mathcal{K}_Y(Z; \mathbf{t})$ for the K-polynomial of a $\mathbb{Z}^d$-graded quotient $\Bbbk[\mathbf{y}]/I(Z)$, if $Z \subseteq Y$. The geometry in Corollary 17.20 has the following interpretation in terms of K-polynomials.

Corollary 17.32 $\mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{t}) = \mathcal{K}_M(\overline{X}_{v(\mathbf{r})}; \mathbf{t}, \mathring{\mathbf{t}})$.

Proof. This is immediate from Proposition 17.31, by Definition 8.32. □

Lemma 17.33 *The K-polynomial $\mathcal{K}_{Mat}(\Omega_\mathbf{r}; \mathbf{t})$ of $\Omega_\mathbf{r}$ inside Mat is*

$$\mathcal{K}_{Mat}(\Omega_\mathbf{r}; \mathbf{t}) \quad = \quad \frac{\mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{t})}{\mathcal{K}_Y(\mathcal{Z}(Mat); \mathbf{t})}.$$

Proof. The equality $H(\Omega_\mathbf{r}; \mathbf{t}) = \mathcal{K}_{Mat}(\Omega_\mathbf{r}; \mathbf{t}) H(Mat; \mathbf{t})$ of Hilbert series (which are well-defined by positivity of the grading of $\Bbbk[\mathbf{f}]$ by $\mathbb{Z}^d$) follows from Theorem 8.20. For the same reason, we have

$$\begin{aligned} H(Mat; \mathbf{t}) &= \mathcal{K}_Y(\mathcal{Z}(Mat); \mathbf{t}) H(Y; \mathbf{t}) \\ \text{and also} \quad H(\Omega_\mathbf{r}; \mathbf{t}) &= \mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{t}) H(Y; \mathbf{t}). \end{aligned}$$

Thus $\mathcal{K}_Y(\mathcal{Z}(\Omega_\mathbf{r}); \mathbf{t}) H(Y; \mathbf{t}) = \mathcal{K}_{Mat}(\Omega_\mathbf{r}; \mathbf{t}) \mathcal{K}_Y(\mathcal{Z}(Mat); \mathbf{t}) H(Y; \mathbf{t})$. □

Theorem 17.34 *The ordinary quiver polynomial $\mathcal{Q}_\mathbf{r}(\mathbf{t} - \mathring{\mathbf{t}})$ is the $\mathring{\mathbf{s}} = \mathring{\mathbf{t}}$ specialization of the double quiver polynomial $\mathcal{Q}_\mathbf{r}(\mathbf{t} - \mathring{\mathbf{s}})$. In other words, the quiver polynomial $\mathcal{Q}_\mathbf{r}(\mathbf{t} - \mathring{\mathbf{t}})$ for a rank array $\mathbf{r}$ equals the ratio*

$$\mathcal{Q}_\mathbf{r}(\mathbf{t} - \mathring{\mathbf{t}}) \quad = \quad \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{t} - \mathring{\mathbf{t}})}{\mathfrak{S}_{v_*}(\mathbf{t} - \mathring{\mathbf{t}})}$$

of double Schubert polynomials in the two alphabets $\mathbf{t}$ and $\mathring{\mathbf{t}}$.

Proof. After clearing denominators in Lemma 17.33, substitute using Corollary 17.32 to get

$$\mathcal{K}_{Mat}(\Omega_\mathbf{r}; \mathbf{t})\, \mathcal{K}_M(\overline{X}_{v_*}; \mathbf{t}, \mathring{\mathbf{t}}) \quad = \quad \mathcal{K}_M(\overline{X}_{v(\mathbf{r})}; \mathbf{t}, \mathring{\mathbf{t}}).$$

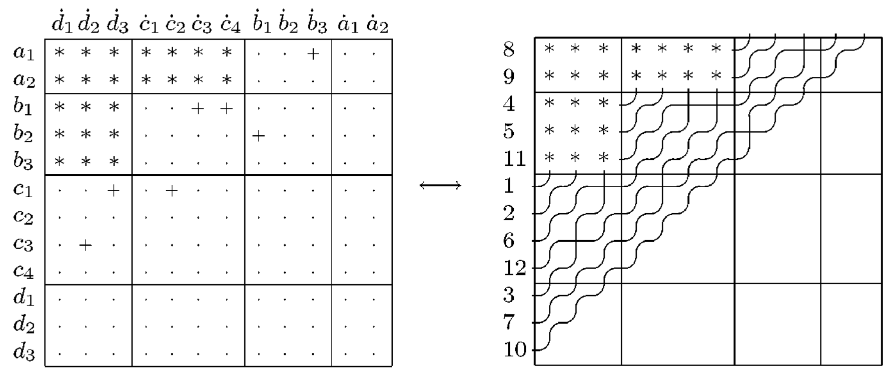


Figure 17.2: Zelevinsky pipe dream

Now substitute $1-t$ for every occurrence of each variable t, and take lowest degree terms to get

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathring{\mathbf{t}})\,\mathfrak{S}_{v_*}(\mathbf{t}-\mathring{\mathbf{t}}) \quad = \quad \mathfrak{S}_{v(\mathbf{r})}(\mathbf{t}-\mathring{\mathbf{t}}).$$

The polynomial $\mathfrak{S}_v(\mathbf{t}-\mathring{\mathbf{t}})$ is nonzero, being simply the product of the $\mathbb{Z}^d$-graded ordinary weights $t^j_\alpha - t^i_\beta$ of the $\mathbf{y}$ variables $y^{ji}_{\alpha\beta}$ with $i > j$. Therefore we may divide through by $\mathfrak{S}_{v_*}(\mathbf{t}-\mathring{\mathbf{t}})$. □

17.5 Pipes to laces

Having a formula for quiver polynomials in terms of Schubert polynomials produces a formula in terms of pipe dreams, given the simplicity of the denominator polynomial $\mathfrak{S}_{v_*}$. Let us begin unraveling the structure of pipe dreams for Zelevinsky permutations with an example.

Example 17.35 A typical reduced pipe dream for the Zelevinsky permutation v in Example 17.14 looks like the one in Fig. 17.2, when we leave the $*$'s as they are in the diagram $D(v(\mathbf{r}))$. Although each $*$ represents a $+$ in every pipe dream for $v(\mathbf{r})$, the $*$'s will be just as irrelevant here as they were for the diagram of $v(\mathbf{r})$. The left pipe dream in Fig. 17.2 is labeled on the side and top with the row and column variables for ordinary weights. ◇

Given a set D of $+$ entries in the square $d\times d$ grid, let $(\mathbf{t}-\mathring{\mathbf{s}})^D$ be its *monomial*, defined as the product over all $+$ entries in D of $(t_+ - s_+)$, where t_+ sits at the left end of the row containing $+$, and s_+ sits atop the column containing $+$.

Theorem 17.36 *The double quiver polynomial for ranks* $\mathbf{r}$ *equals the sum*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathring{\mathbf{s}}) \quad = \quad \sum_{D\in\mathcal{RP}(v(\mathbf{r}))} (\mathbf{t}-\mathring{\mathbf{s}})^{D\smallsetminus D(v_*)}$$

of the monomials for the complement of $D(v_*)$ in all reduced pipe dreams for the Zelevinsky permutation $v(\mathbf{r})$.

Proof. This follows from Definition 17.30 and Corollary 16.30, using the fact that every pipe dream $D \in \mathcal{RP}(v(\mathbf{r}))$ contains the subdiagram $D(v_*)$, and that $\mathcal{RP}(v_*)$ consists of the single pipe dream $D(v_*)$. □

Double quiver polynomials $\mathcal{Q}_\mathbf{r}(\mathbf{t} - \mathring{\mathbf{s}})$ are thus sums of all monomials for "skew reduced pipe dreams" $D \smallsetminus D(v_*)$ with $D \in \mathcal{RP}(v(\mathbf{r}))$. That is why we only care about crosses in D occupying the block antidiagonal and superantidiagonal. The monomial $(\mathbf{t} - \mathring{\mathbf{s}})^{D \smallsetminus D(v_*)}$ for the pipe dream in Fig. 17.2 is

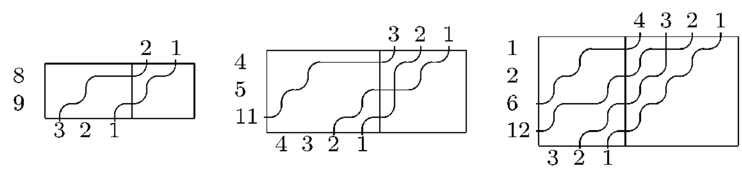
$$(a_1 - \dot{b}_3)(b_1 - \dot{c}_3)(b_1 - \dot{c}_4)(b_2 - \dot{b}_1)(c_1 - \dot{d}_3)(c_1 - \dot{c}_2)(c_3 - \dot{d}_2),$$

ignoring all $*$ entries as required. Removing the dots yields this pipe dream's contribution to the ordinary quiver polynomial.

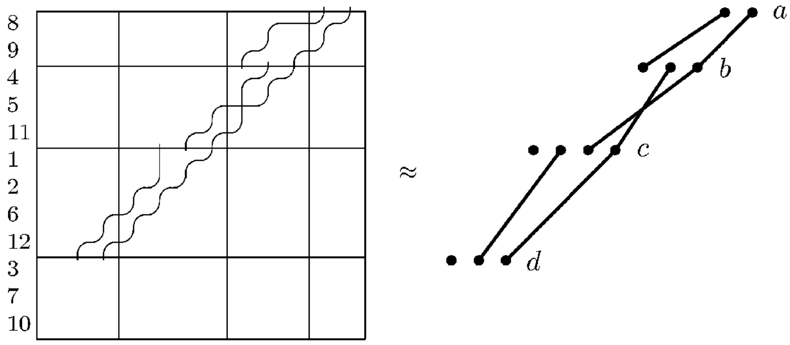
Recall that we started in Section 17.1 analyzing quiver representations by decomposing them as direct sums of laces, as in Example 17.1. Although we have by now taken a long detour, here we come back again to some concrete combinatorics: pipe dreams for Zelevinsky permutations give rise to lacing diagrams.

Definition 17.37 The j^{th} antidiagonal block is the block of size $r_j \times r_j$ along the main antidiagonal in the j^{th} block row. Given a reduced pipe dream D for the Zelevinsky permutation $v(\mathbf{r})$, define the partial permutation $w_j = w_j(D)$ sending k to ℓ if the pipe entering the k^{th} column from the *right* of the $(j-1)^{\text{st}}$ antidiagonal block enters the j^{th} antidiagonal block in its ℓ^{th} column from the *right*. Set $\mathbf{w}(D) = (w_1, \ldots, w_n)$, so that $\mathbf{w}(D)$ is the lacing diagram determined by D.

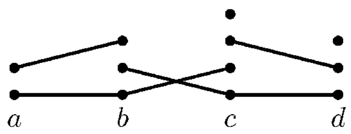
Example 17.38 The partial permutations arising from the pipe dream in Example 17.35 come from the following partial reduced pipe dreams:



These send each number along the top either to the number along the bottom connected to it by a pipe (if such a pipe exists), or to nowhere. It is easy to see the pictorial lacing diagram $\mathbf{w}(D)$ from these pictures. Indeed, removing all segments of all pipes not contributing to one of the partial permutations leaves some pipes



that can be interpreted directly as the desired lacing diagram

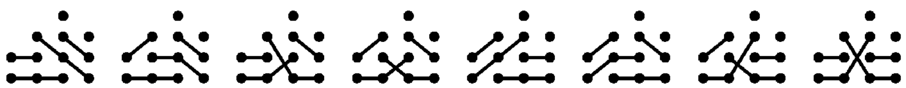


by shearing to make the rightmost dots in each row line up vertically, and then reflecting through the diagonal line $\diagdown$ of slope -1. $\diamond$

Proposition 17.39 *Every reduced pipe dream $D \in \mathcal{RP}(v(\mathbf{r}))$ gives rise to a lacing diagram $\mathbf{w}(D)$ representing a partial permutation list with ranks $\mathbf{r}$.*

Proof. Each $\times$ entry in the permutation matrix for $v(\mathbf{r})$ corresponds to a pipe in D entering due north of it and exiting due west of it. The permutation $v(\mathbf{r})$ was specifically constructed to have exactly q_{ij} entries $\times$ (for $i \leq j$) in the intersection of the i^{th} block row and the j^{th} block column from the right, where $\mathbf{q}$ is the lace array from Lemma 17.5. $\square$

The fact that lacing diagrams popped out of pipe dreams for Zelevinsky permutations suggests that lacing diagrams control the combinatorics of quiver polynomials as deeply as they controlled the algebra in Section 17.1. This turns out to be true: there is a different, more intrinsic combinatorial formula for quiver polynomials in terms of Schubert polynomials. To state it, define the *length* of a lacing diagram $\mathbf{w} = (w_1, \ldots, w_n)$ to be the sum $l(\mathbf{w}) = l(w_1) + \cdots + l(w_n)$ of the lengths of its constituent partial permutations. For an irreducible rank array $\mathbf{r}$, we are interested in the set $W(\mathbf{r})$ of *minimal* lacing diagrams for $\mathbf{r}$, that is, with minimal length. For instance, with $\mathbf{r}$ as in Examples 17.1, 17.7, 17.14, 17.35, and 17.38, the set $W(\mathbf{r})$ is:



Theorem 17.40 *The quiver polynomial $\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathring{\mathbf{t}})$ equals the sum*

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathring{\mathbf{t}}) \quad = \quad \sum_{\mathbf{w} \in W(\mathbf{r})} \mathfrak{S}_{w_1}(\mathbf{t}^0 - \mathbf{t}^1)\mathfrak{S}_{w_2}(\mathbf{t}^1 - \mathbf{t}^2) \cdots \mathfrak{S}_{w_n}(\mathbf{t}^{n-1} - \mathbf{t}^n)$$

of products of double Schubert polynomials indexed by minimum length lacing diagrams $\mathbf{w} = (w_1, \ldots, w_n)$ with rank array $\mathbf{r}$.

This statement was discovered by Knutson, Miller, and Shimozono, who at first proved only that the expansion on the right hand side has positive coefficients. After publicizing their weaker statement and conjecturing the precise statement above, independent (and quite different) proofs of the conjecture were given by the conjecturers [KMS04] and by Rimányi [BFR03]. For information on the motivation, consequences, and variations that have appeared and could in the future appear, see the Notes.

Exercises

17.1 Given the rank array $\mathbf{r} = \begin{array}{ccc|c} 2 & 1 & 0 & i/j \\ \hline & & 3 & 0 \\ & 4 & 2 & 1 \\ 2 & 2 & 0 & 2 \end{array}$ with $n = 2$, compute the lace array $\mathbf{q}$ and rectangle array $\boldsymbol{R}$. Find all the minimal lacing diagrams with rank array $\mathbf{r}$.

17.2 Prove Lemma 17.5, Lemma 17.8, and Lemma 17.13.

17.3 What conditions on a dimension vector (r_0, r_1, r_2, r_3) and a rank ρ guarantee that the minors of size $\rho+1$ in the product $\Phi_1\Phi_2\Phi_3$ generate a prime ideal, where Φ_i is a generic matrix of size $r_{i-1} \times r_i$?

17.4 For the general data in Example 17.11 (and the particular case in (17.3) and Example 17.18), show that the Zelevinsky permutation has essential set of size 2. Use the Binet–Cauchy formula to prove Theorem 17.17 directly in this case.

17.5 Work out the lace array $\mathbf{q}$, rank array $\mathbf{r}$, rectangle diagram $\boldsymbol{R}$, and Zelevinsky permutation $v(\mathbf{r})$ for the data in Example 17.25. Check the degree calculations there. Find all six minimal lacing diagrams sharing the rank array $\mathbf{r}$. Verify the pipe dream and lacing diagram formulas in Theorems 17.36 and 17.40 for $\mathbf{r}$.

17.6 Set $d = r_0 + \cdots + r_n$ as usual, and fix an irreducible rank array $\mathbf{r}$. Consider the set $S_d(\mathbf{r})$ of permutations in S_d whose permutation matrices have the same number of nonzero entries as $v(\mathbf{r})$ does in every $r_j \times r_i$ block. Prove that $v = v(\mathbf{r})$ if and only if $v \in S_d$ and every other permutation $v' \in S_d$ satisfies $l(v') > l(v)$.

17.7 Interpret (17.2) as a statement about the rectangles in the rectangle array $\boldsymbol{R}$.

17.8 A **variety of complexes** is a quiver locus $\Omega_\mathbf{r}$ such that for all $\phi \in \Omega_\mathbf{r}$, $\phi_{i-1}\phi_i = 0$ for $i = 2, \ldots, n$. Which varieties of complexes $\Bbbk^{r_0} \to \cdots \to \Bbbk^{r_n}$ are irreducible as varieties? What is the multidegree of a variety of complexes?

17.9 Pick a random quiver representation ϕ with dimension vector $(2, 3, 3, 1)$, and compute an isomorphism $\phi \cong \mathbf{w}$ with a lacing diagram $\mathbf{w}$. Could you have predicted the lace array $\mathbf{q}$ and the rank array $\mathbf{r}$ of $\mathbf{w}$? What is $\mathcal{E}ss(v(\mathbf{r}))$?

17.10 Calculate the dimension of $\Omega_\mathbf{r}$ in terms of the rectangle array $\boldsymbol{R}$ of $\mathbf{r}$.

17.11 Suppose that $\mathbf{r}$ is a rank array that is *not* irreducible. Must it always be the case that $\Omega_\mathbf{r}$ has more than one component? Can $\Omega_\mathbf{r}$ be nonreduced?

17.12 Let $\overline{P}$ be the closure of P in M_d. Verify that (17.7) and (17.8) correspond to a morphism $\overline{P}\times Mat \to \overline{X}_{v_*}$ that happens to take the subset $P\times Mat$ to the subset $P\cdot\mathcal{Z}(Mat) \subset \overline{X}_{v_*}$. Use Proposition 17.22 to show that (17.7) and (17.8) define the *only* algebra map $\Bbbk[\mathbf{x}_{v_*}] \to \Bbbk[\mathbf{p}, \mathbf{f}]$ inducing the morphism $P\times Mat \to P\cdot\mathcal{Z}(Mat)$.

17.13 Let $1+\mathbf{r}$ be obtained by adding 1 to every entry of a rank array $\mathbf{r}$. Compare the lace arrays of $\mathbf{r}$ and $1+\mathbf{r}$. What is the difference between the Zelevinsky permutations $v(\mathbf{r})$ and $v(1+\mathbf{r})$? How about the rectangle arrays of $\mathbf{r}$ and $1+\mathbf{r}$?

17.14 Let w be a permutation matrix of size $n+1$, and consider the rank array $\mathbf{r}_w$ in Example 17.26.

(a) Prove that the Zelevinsky permutation $v(\mathbf{r}_w)$ has as many diagonal $\times$ entries as will fit in each superantidiagonal block.
(b) Show that every rectangle in the rectangle array $\boldsymbol{R}$ has size 1×1, and explain how $\boldsymbol{R}$ can be naturally identified with the diagram $D(w)$.
(c) If $\mathbf{r} = \mathbf{r}_w$, then the ordinary quiver polynomial $\mathcal{Q}_\mathbf{r}$ takes $2n$ alphabets for its argument. Suppose that the first n of these alphabets are specialized to $\{t_1\}, \{t_1, t_2\}, \ldots, \{t_1, \ldots, t_n\}$, and that the last n of these alphabets are specialized to $\{s_1, \ldots, s_n\}, \ldots, \{s_1, s_2\}, \{s_1\}$. Prove that $\mathcal{Q}_\mathbf{r}$ evaluates at these alphabets to the double Schubert polynomial $\mathfrak{S}_w(\mathbf{t} - \mathbf{s})$.

17.15 Give a direct proof of Lemma 17.16, without using exterior powers.

17.16 Let $\mathbf{y}$ be a set of variables, and fix a $\Bbbk$-algebra R. Using any definition of Cohen–Macaulay that suits this generality, prove that for every polynomial $f \in \Bbbk[\mathbf{y}]$, the ring $R[\mathbf{y}][f^{-1}]$ is Cohen–Macaulay if and if $R[\mathbf{y}]$ is Cohen–Macaulay.

17.17 Prove that the minimum length for a lacing diagram with rank array $\mathbf{r}$ is the length $l(v(\mathbf{r}))$ of its Zelevinsky permutation. Hint: Given a minimal lacing diagram $\mathbf{w}$, exhibit a reduced pipe dream D for $v(\mathbf{r})$ such that $\mathbf{w}(D) = \mathbf{w}$.

17.18 Show by example that Theorem 17.40 fails for double quiver polynomials, when all $\mathbf{t}$ alphabets with minus signs are replaced by corresponding $\mathbf{s}$ alphabets.

Notes

The use of laces to denote indecomposable quiver representations as in Definition 17.2 is due to Abeasis and Del Fra [AD80], who identified unordered sets of laces (called *strands* there) as giving rank conditions. The refinement of this notion to include the partial permutations between columns in a lacing diagram is due to Knutson, Miller, and Shimozono [KMS04], who needed it for the statement of Theorem 17.40. Quiver ideals, quiver loci, (indecomposable) quiver representations, and Proposition 17.9 are part of a much larger theory of representations of finite type quivers; see below. The rectangle arrays in Definition 17.6 were invented by Buch and Fulton [BF99].

The Zelevinsky map originated in Zelevinsky's two-page paper [Zel85], where he proved the set-theoretic (as opposed to the scheme-theoretic) version of Corollary 17.20. Zelevinsky's original big block matrix, being essentially the inverse matrix of (17.4), visibly contained all of the consecutive products $\Phi_{i+1}\cdots\Phi_j$ for $i < j$. Theorem 17.17 and the concept of Zelevinsky permutation appeared in [KMS04], from which much of Section 17.2 has been lifted with few changes. The primality in Theorem 17.23 is due to Lakshmibai and Magyar [LM98], as is the Cohen–Macaulayness of quiver loci over fields of arbitrary characteristic, although earlier, Abeasis, Del Fra, and Kraft had proved (without primality) that the underlying reduced variety is Cohen–Macaulay in characteristic zero [ADK81].

Quiver polynomials were defined by Buch and Fulton [BF99]. Double quiver polynomials as ratios of Schubert polynomials, as well as the subsequent ratio and pipe dream formulas for ordinary quiver polynomials in Theorem 17.34 and Theorem 17.36, were discovered by Knutson, Miller, and Shimozono [KMS04]. That paper also contains the combinatorial connections between lacing diagrams and reduced pipe dreams for Zelevinsky permutations in Proposition 17.39. Attributions for Theorem 17.40 appear in the text, after its statement.

In contrast to the situation for minors in Chapters 15–16, it is not known whether there is a term order under which the generators of $I_{\mathbf{r}}$ in Definition 17.3 form a Gröbner basis. Although the degeneration to pipe dreams at the level of Schubert determinantal ideals, which results in Theorem 17.36, descends to a degeneration of the Zelevinsky image $\mathcal{Z}(\Omega_{\mathbf{r}})$, this degeneration fails to be Gröbner. Indeed, some of the variables are set equal to 1, so the resulting flat family of ideals in $\Bbbk[\mathbf{x}]$ is not obtained by scaling the variables. On the other hand, there is still a *partial* Gröbner degeneration [KMS04, Section 4 and Theorem 6.16]; the components in its special fiber are indexed by lacing diagrams, so it gives rise to the positive formula in Theorem 17.40 in the manner of Corollary 16.1.

We have drawn Exercise 17.6 from [Yon03]. Exercise 17.16 was used in the proof of Theorem 17.23; it follows from [BH98, Theorems 2.1.3 and 2.1.9]. What we have called Fulton polynomials in Example 17.26 and Exercise 17.14 were originally called *universal Schubert polynomials* by Fulton because they specialize to quantum and double Schubert polynomials [Ful99]. Treatments of combinatorial aspects of Fulton polynomials and their K-theoretic analogues appear in [BKTY04a, BKTY04b].

The topics in this chapter have historically developed in the contexts of algebraic geometry and representation theory. On the algebraic geometry side, the direct motivation comes from [BF99] and its predecessors, which deal with *degeneracy loci* for vector bundle morphisms; see [FP98, Man01] for background on the long history of this perspective. In particular, the three formulas for $\mathcal{Q}_{\mathbf{r}}$ in this chapter (Theorems 17.34, 17.36, and 17.40) were originally aimed at a solution in [KMS04] of the main conjecture in [BF99], which is a positive combinatorial formula for $\mathcal{Q}_{\mathbf{r}}$ as a sum of products of double Schur functions. Further topics in this active area of research include new proofs of Theorem 17.40 or steps along the way [BFR03, Yon03], relations between quiver polynomials and symmetric functions [Buc01, BSY03], and K-theoretic versions [Buc02, Buc03, Mil03b].

The representation theory motivation comes from general quivers. The term *quiver* is a synonym for *directed graph.* In our *equioriented type A* case, the quiver is a directed path. The definition of quiver representation makes sense for arbitrary quivers (attach a vector space to each vertex and a matrix of variables to each directed edge), and the notion of quiver locus can be extended, as well (to orbit closures for the general linear group that acts by changing bases); see [ARS97] or [GR97] for background. The extent to which we understand the multidegrees of quiver loci for orientations of *Dynkin diagrams* of type A, D, or E comes from the topological perspective of Fehér and Rimányi [FR02], but as yet there are no known analogues of the positivity in Theorem 17.40 for other types. This open problem is only a sample of the many relations of quiver representations with combinatorial commutative algebra. Other connections include the work of Bobiński and Zwara on normality and rational singularities [BZ02], and Derksen and Weyman on *semi-invariants* [DW00].

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