SMR1563/7

School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics

(24 May - 11 June 2004)

On birational Macaulayfications and Cohen-Macaulay canonical modules

P. Schenzel
Martin-Luther-Universität
Institut für Informatik
D-06120 Halle
Germany

ON BIRATIONAL MACAULAYFICATIONS AND COHEN-MACAULAY CANONICAL MODULES

PETER SCHENZEL

Abstract. The aim of this talk is threefold. At first there is a characterization of those local domains admitting a birational Macaulayfication. It turns out that a local domain A, quotient of a Gorenstein ring, possesses a Cohen-Macaulay birational extension B if and only if its canonical module $K(A)$ is a Cohen-Macaulay module. In this situation it follows that B is isomorphic to the endomorphism ring of $K(A)$.

The second part of the talk is devoted to the question when the canonical module of a local ring is a Cohen-Macaulay module. There are applications Cohen-Macaulay filtered rings. A large class of rings whose canonical module are Cohen-Macaulay is given by the simplicial affine semigroup rings. As a technical tools there is a spectral sequence related to the duality with respect to the dualizing complex.

Finally we are interested in results related to the property of being Cohen-Macaulay filtered.

Contents

1. Motivation

1.1. **Question 1.** Let $(A, \mathfrak{m})$ denote a local domain.

Question 1. Does there exists a birational extension ring $A \subseteq B \subseteq Q$, ($Q$ denotes the field of quotients) such that B is finitely generated as an A-module and a Cohen-Macaulay ring?

We call such an extension ring a birational Macaulayfication of A.

- In general it does not exist.
- An affine 2-dimensional local domain possesses a birational Macaulayfication.
- When does exists?

1.2. **Question 2.** Let $(A, \mathfrak{m})$ a local ring, quotient of a local Gorenstein ring $(B, \mathfrak{n})$.

Let M denote a finitely generated A-module. Denote by

$$K(M) := \operatorname{Ext}_B^c(M, B), \quad c = \dim B - \dim M,$$

the canonical module of M.

Question 2. When is $K(M)$ a Cohen-Macaulay module?

- Suppose that M is Cohen-Macaulay. Then $K(M)$ is Cohen-Macaulay.
- The converse is not true.
- Let A denote an affine two-dimensional local ring. Then $K(A)$ is a Cohen-Macaulay module.

Definition 1.1. M is called a canonically Cohen-Macaulay (CCM) module, provided $K(M)$ is a Cohen-Macaulay module.

When M is canonically Cohen-Macaulay module?

Note. In case M is CCM, it is – in a certain sense – a generalized Cohen-Macaulay module.

1.3. **Question 3.** Let $(A, \mathfrak{m})$ a local ring.

Definition 1.2. For a finitely generated A-module M with $d = \dim_A M$ and an integer $0 \leq i \leq d$ define M_i the largest submodule of M such that $\dim_A M_i \leq i$. We call $\mathcal{M} = \{M_i\}_{0 \leq i \leq d}$ the dimension filtration of M.

Definition 1.3. We call an A-module M a Cohen-Macaulay filtered module (CMF) provided all of the quotient modules M_i/M_{i-1} are either zero or i-dimensional Cohen-Macaulay modules.

- In particular, an unmixed A-module M is CMF if and only if it is Cohen-Macaulay.
- One might think of a CMF A-module M as another kind of generalized Cohen-Macaulay modules.
- In his book (cf. [St]) Stanley introduced the notion of sequentially Cohen-Macaulay modules. This was done independently (cf. [S2]) by the notion of Cohen-Macaulay filtered modules (CMF) as above. Here we use both of the notion synonymously.

- Besides of Stanley's approach there are recent applications in combinatorics by Herzog and Sbarra (cf. [HS]).

Definition 1.4. For an A-module M and $i \in \mathbb{Z}$ define

$$K^i(M) := \operatorname{Ext}_B^{b-i}(M, B), b = \dim B,$$

the i-th module of deficiency. Note that $K^i(M) = 0$ for $i < 0$ and $i > \dim M = d$. Moreover $K^d(M)$ is the canonical module of M as defined above.

Note. An finitely generated A-module M is Cohen-Macaulay if and only if $K^i(M) = 0$ for all $0 \leq i < \dim M$.

Theorem 1.5. *(cf.* [St] *and* [S2]*) Let M be a finitely generated A-module with $d = \dim_A M$. Then the following conditions are equivalent:*

(i) *M is a CMF A-module.*
(ii) *For all $0 \leq i < d$ the module of deficiency $K^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module.*
(iii) *For all $0 \leq i \leq d$ the A-modules $K^i(M)$ are either zero or i-dimensional Cohen-Macaulay modules.*

Note. In particular it follows that CMF implies CCM for an A-module M.

Question 3. What are the relation between CMF and CCM, are there more examples of CCM related to the dimension filtration?

2. A few Preliminary Results

2.1. **On dualizing complexes.** Without loss of generality let us assume that $(A, \mathfrak{m})$ possesses a dualizing complex $D_A^{\cdot}$. (By the result of T. Kawasaki (cf. [K]) this is equivalent to the fact that $(A, \mathfrak{m})$ is a quotient of a Gorenstein ring.)
A dualizing complex is a bounded complex of injective A-modules D_A^i whose cohomology modules $H^i(D_A^{\cdot}), i \in \mathbb{Z}$, are finitely generated A-modules.

Note that the natural homomorphism of complexes

$$M \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

induces an isomorphism in cohomology for any finitely generated A-module M.

For the modules of deficiencies $K^i(M), i \in \mathbb{Z}$, defined in 1.4 there are the following isomorphisms

$$K^i(M) \simeq H^{-i}(\operatorname{Hom}_A(M, D_A^{\cdot})).$$

Definition 2.1. Let M be a finitely generated A-module and $d = \dim_M$. Because of
$(\mathrm{Hom}_A(M, D_A^{\cdot}))^i = 0$ for all $i < -d$ and $H^{-d}(\mathrm{Hom}_A(M, D_A^{\cdot})) = K(M)$ there is a short exact sequence of complexes

$$0 \to K(M)[d] \to \mathrm{Hom}_A(M, D_A^{\cdot}) \to C^{\cdot}(M) \to 0,$$

where $C^{\cdot}(M)$ denotes the cokernel of the natural embedding. The complex $C^{\cdot}(M)$ carries as cohomology modules the modules of deficiencies, i.e.

$$H^i(C^{\cdot}(M)) \simeq \begin{cases} K^{-i}(M) & \text{for } -d < i \leq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

So we define $C^{\cdot}(M)$ as the *complex of deficiency* of the R-module M.

Note that the A-module M is a Cohen-Macaulay module if and only if the complex of deficiency $C^{\cdot}(M)$ is homologically trivial.

2.2. **Spectral sequences.** For some technical applications we need to compute the cohomology of complexes of the form $\mathrm{Hom}_A(C^{\cdot}, D_A^{\cdot})$. To this end there is the following spectral sequence

$$E_1^{pq} = H^q(\mathrm{Hom}_A(C^{\cdot}, D_A^p)) \Longrightarrow E_{\infty}^{p+q} = H^{p+q}(\mathrm{Hom}_A(C^{\cdot}, D_A^{\cdot})).$$

Because D_A^p is an injective A-module the corresponding E_2-term has the following form

$$E_2^{pq} = H^p(\mathrm{Hom}_A(H^{-q}(C^{\cdot}), D_A^{\cdot})) \Longrightarrow E_{\infty}^{p+q} = H^{p+q}(\mathrm{Hom}_A(C^{\cdot}, D_A^{\cdot})).$$

In the particular case of $C^{\cdot} = \mathrm{Hom}_A(M, D_A^{\cdot})$, the complex of deficiency, the $E_2^{p,q}$-terms are $K^{-p}(K^q(M))$, the modules of deficiencies of the modules of deficiencies.

2.3. **Technical Preliminaries.**

Proposition 2.2. *Let M denote a d-dimensional A-module. Then the following results are true:*

a) $\dim_A K^i(M) \leq i$ *for all* $0 \leq i < d$ *and* $\dim_A K(M) = d$.
b) $\mathrm{Ass}_A K(M) = (\mathrm{Ass}_A M)_d$.
c) $(\mathrm{Ass}_A K^i(M))_i = (\mathrm{Ass}_A M)_i$ *for all* $0 \leq i < d$.
d) $K(M)$ *satisfies* S_2.

In the following we will consider the behavior of the modules of deficiency by generic hyper-surface sections. That is we will consider the relation of $K^i(M)$ and $K^i(M/xM)$ for a certain element $x \in \mathfrak{m}$. To this end we put $\overline{M} := M/H_{\mathfrak{m}}^0(M)$, where $H_{\mathfrak{m}}^0(M)$ denotes the 0-th local cohomology module of M.

Definition 2.3. An element $x \in \mathfrak{m}$ is called an M-strongly filter regular element provided it $\overline{K^i(M)}$-regular for all $0 < i < \dim M$.

Proposition 2.4. *Let $x \in \mathfrak{m}$ denote an M-strongly filter regular element. Then there are short exact sequences*

$$0 \to K^{i+1}(M)/xK^{i+1}(M) \to K^i(M/xM) \to 0 :_{K^i(M)} x \to 0$$

for all integers $i \geq 0$.

Proof. There is the following short exact sequence

$$0 \to 0 :_M x \to M \stackrel{x}{\to} M \to M/xM \to 0$$

induced by the multiplication by x. Since x is $\overline{M}$-regular it follows that $0 :_M x \subseteq H^0_{\mathfrak{m}}(M)$. Therefore $0 :_M x$ has support contained in $V(\mathfrak{m})$. The above exact sequence splits into two short exact sequences

$$0 \to 0 :_M x \to M \to M/0 :_M x \to 0 \quad \text{and}$$
$$0 \to M/0 :_M x \stackrel{x}{\to} M \to M/xM \to 0.$$

The first one induces isomorphisms $K^i(M) \simeq K^i(M/0 :_M x)$ for all $i \geq 1$ and a short exact sequence

$$0 \to K^0(M/0 :_M x) \to K^0(M) \to K^0(0 :_M x) \to 0.$$

By view of the previous isomorphisms the second short exact sequence induces a short exact sequence

$$0 \to K^{i+1}(M)/xK^{i+1}(M) \to K^i(M/xM) \to 0 :_{K^i(M)} x \to 0$$

for all $i \geq 1$. That is, in order to finish the proof we have to prove the existence of such a sequence in the case $i = 0$. Moreover it provides an exact sequence

$$K^1(M) \stackrel{x}{\to} K^1(M) \to K^0(M/xM) \to K^0(M) \stackrel{x}{\to} K^0(M/0 :_M x) \to 0.$$

Putting together both of the exact sequences a simple consideration completes the argument for $i = 0$. □

Note. The importance of the Definition 2.3 has to do with the short exact sequence in the Proposition. In the case of a strong M-filter regular sequence it follows that $0 :_{K^i(M)} x, 0 \leq i < \dim M$, is always a module of finite length. So there is a good control of the local cohomology of $K^{i+1}(M)/xK^{i+1}(M)$ in relation to that of $K^i(M/xM)$ for all $i \geq 0$.

Note. Of course one might extend this definition to strong M-filter regular sequences. This was independently introduced in the paper [CMN] by the name strict f-sequences.

3. Canonically Cohen-Macaulay Modules

3.1. **Examples.** Remind the definition of CCM of M being Cohen-Macaulay (cf. 1.2).

Example 3.1. a) Any Cohen-Macaulay module M is also a CCM module.
b) Let M be a finitely generated A-module with $\dim M \leq 2$. Then M is a CCM module. This follows since depth $K(M) = \dim M$.
c) By the result of Stanley (cf. [St, p. 88]) resp. by the result (cf. [S2, Theorem 5.5]) any sequentially Cohen-Macaulay module is a CCM module.
d) The approximately Cohen-Macaulay rings studied by Gôto (cf. [G]) are sequentially Cohen-Macaulay of depth $A \geq \dim A - 1$ and therefore CCM rings.
e) Let $(A, \mathfrak{m})$ denote a complete Cohen-Macaulay local ring with $d = \dim A$. Let M be an A-module with $\dim M < d$. Then we consider $A \ltimes M$, the idealization of M over A. That is, the additive group of $A \ltimes M$ coincides with the direct sum of the abelian groups A and M. The multiplication is given by

$$(a, m) \cdot (b, n) := (ab, an + bm).$$

Then $A \ltimes M$ is a d-dimensional local ring with the maximal ideal $\mathfrak{m} \ltimes M$. So there is the following short exact sequence

$$0 \to 0 \ltimes M \to A \ltimes M \to A \to 0.$$

It is easily seen that A as an $A \ltimes M$-module is a Cohen-Macaulay module. By the change of ring theorem for local cohomology there are the isomorphisms

$$H^i_{\mathfrak{m} \ltimes M}(0 \ltimes M) \simeq H^i_{\mathfrak{m}}(M) \text{ and } H^i_{\mathfrak{m} \ltimes M}(A) \simeq H^i_{\mathfrak{m}}(A)$$

for all $i \in \mathbb{Z}$.
Applying the local cohomology functor $H^{\cdot}_{\mathfrak{m} \ltimes M}(\cdot)$ provides the following isomorphisms

$$H^d_{\mathfrak{m} \ltimes M}(A \ltimes M) \simeq H^d_{\mathfrak{m}}(A) \text{ and } H^i_{\mathfrak{m} \ltimes M}(A \ltimes M) \simeq H^i_{\mathfrak{m}}(M) \text{ for } 0 \leq i < d.$$

Therefore $A \ltimes M$ is a CCM ring, while the modules of deficiency $K^i(A \ltimes M), 0 \leq i < d$, may vary rather wide depending on the A-module M. In particular let M denote t-dimensional Cohen-Macaulay module with $t < d$. Then $A \ltimes M$ is a CCM module of dimension d and of depth t.
e) Let $(A, \mathfrak{m})$ a local ring as above. A finitely generated A-module M is a CCM module if and only if $M[[x]]$ is a CCM module over $A[[x]]$. Here x denotes a variable over A.

3.2. **Permanence Properties.**

Lemma 3.2. *Let M be a finitely generated A-module. Let $x \in \mathfrak{m}$ be an M-strongly filter regular element. Then the following statements are true:*

a) *Let M be a CCM module. Let $\mathfrak{p} \in \operatorname{Supp} M$ a prime ideal with $\dim M = \dim M_{\mathfrak{p}} + \dim A/\mathfrak{p}$. Then $M_{\mathfrak{p}}$ is also CCM.*

b) *Let M be a CCM module and $d = \dim M \geq 3$. Then M/xM is also a CCM module.*

c) *Suppose that M/xM is a CCM module and* $\operatorname{depth} K(M) \geq 3$. *Then M is a CCM module.*

Proof. The statement in a) is easy. Recall that the Cohen-Macaulay property of $K(M)$ localizes under the assumption.
For the proof of b) consider the short exact sequence of Proposition 2.4 in the case $i = d - 1$. That is

$$0 \to K(M)/xK(M) \to K(M/xM) \to 0 :_{K^{d-1}(M)} x \to 0.$$

The module $K(M)/xK(M)$ CM with $\dim K(M)/xK(M) \geq 2$ because of $d \geq 3$. By the choice of $x \in \mathfrak{m}$ the module $0 :_{K^{d-1}(M)} x$ is an A-module of finite length. So the long exact local cohomology sequence provides $0 :_{K^{d-1}(M)} x = 0$. Recall that $\operatorname{depth} K(M)/xK(M) \geq 2$. So there is an isomorphism $K(M)/xK(M) = K(M/xM)$ and $K(M/xM)$ is a Cohen-Macaulay module.
The proof of c) follows the same arguments as in the proof of b). □

It is worth to notice that in the case of M a CCM module with $\dim M \geq 3$ it follows that $\operatorname{depth} K^{d-1}(M) > 0, d = \dim M$. It will be not sufficient as one might see by the following example.

Example 3.3. Let $(A, \mathfrak{m})$ denote a d-dimensional local Buchsbaum ring with $\operatorname{depth} A = d - 1 > 2$. Then there are the following isomorphisms

$$H^i_{\mathfrak{m}}(K(A)) \simeq K^{d+1-i}(A) \quad \text{for} \quad 2 \leq i < d$$

(cf. [S1, 3.1.3]). So A is a non CCM ring, while $K^{d-1}(K(A)) = 0$.

3.3. **The CCM criterion.**

Proposition 3.4. *Let M denote a finitely generated A-module M. Then M is a CCM module if and only if $H^i(\operatorname{Hom}(C^{\cdot}(M), D^{\cdot})) = 0$ for all $i \geq 2$, where $C^{\cdot}(M)$ denotes the complex of deficiency of M.*

The criterion gives a complete answer to Question 2. For practical purposes it is difficult to apply. So one might derive some more intrinsic, sufficient conditions.

Proof. By the definition of the complex of deficiency there is the following short exact sequence of complexes

$$0 \to K(M)[d] \to \operatorname{Hom}_A(M, D^{\cdot}_A) \to C^{\cdot}(M) \to 0.$$

Now we apply the dualizing functor $\operatorname{Hom}(\cdot, D^{\cdot}_A)$. Because the complex $D^{\cdot}_A$ is a bounded complex of injective modules it induces a short exact sequence of complexes

$$0 \to \operatorname{Hom}(C^{\cdot}(M), D^{\cdot}_A) \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, D^{\cdot}_A), D^{\cdot}_A)) \to$$
$$\to \operatorname{Hom}_A(K(M), D^{\cdot}_A)[-d] \to 0.$$

Since $D_A^{\cdot}$ is a dualizing complex the canonical homomorphism

$$M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, D_A^{\cdot}), D_A^{\cdot})$$

induces an isomorphism in cohomology. So the previous short exact sequence of complexes induces an exact sequence

$$(1)\quad 0 \to H^0(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot})) \to M \to K(K(M)) \to \\ \to H^1(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot})) \to 0$$

and the isomorphisms

$$H^{-d+i}(\mathrm{Hom}(K(M), D_A^{\cdot})) \simeq H^{i+1}(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot}))$$

for $i \geq 1$. Finally the canonical module $K(M)$ is a Cohen-Macaulay module if and only if the modules of deficiency

$$K^{d-i}(K(M)) \simeq H^{-d+i}(\mathrm{Hom}(K(M), D_A^{\cdot}))$$

vanish for all $i \geq 1$. By the above isomorphisms this holds if and only if the cohomology modules $H^i(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot}))$ vanish for all $i \geq 2$. By the definition of CCM this proves the equivalence of the statement. □

4. A few applications

4.1. **Large depth (I).**

Theorem 4.1. *Let M be a finitely generated A-module. Suppose that* depth $K^i(M) \geq i - 1$ *for all* $2 \leq i <$ dim M. *Then M is a CCM module, that is $K(M)$ is a Cohen-Macaulay module.*

Proof. By the criterion it will be enough to show that $H^i(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot}))$ vanishes for all $i \geq 2$. To this end we investigate the spectral sequence

$$E_2^{pq} = H^p(\mathrm{Hom}_A(H^{-q}(C^{\cdot}(M)), D_A^{\cdot})) \Longrightarrow \\ \Longrightarrow E_\infty^{p+q} = H^{p+q}(\mathrm{Hom}_A(C^{\cdot}(M), D_A^{\cdot})).$$

By the definition we have the isomorphisms $K^{-p}(K^q(M)) \simeq E_2^{pq}$ for $q \neq d$ and the vanishing $E_2^{p,d} = 0$ for all p. By the assumption on the depth depth $K^q(M) \geq q - 1$ the initial terms $E_2^{p,q}$ vanish for all $-p < q - 1$. This proves the claim by considering the subsequent stages. Then apply the criterion 3.4. □

4.2. **Relation to Cohen-Macaulay filtered modules.**

Theorem 4.2. *Let M be an equi-dimensional A-module and $d =$* dim M. *Suppose that* depth $K^i(M) \geq i - 1$. *Then the following is true:*

a) *$K^i(M), 0 \leq i < d$, is either zero or an $(i-1)$-dimensional Cohen-Macaulay module and $K(M)$ is a Cohen-Macaulay module.*
b) *There is a short exact sequence*

$$0 \to M \to K(K(M)) \to H^1(\mathrm{Hom}(C^{\cdot}(M), D_A^{\cdot})) \to 0.$$

c) $H^1(\text{Hom}(C^\cdot(M), D_A^\cdot))$ *possesses a filtration* $\{M_i\}$

$$0 \subseteq M_0 \subseteq \cdots \subseteq M_{d-1} = H^1(\text{Hom}(C^\cdot(M), D_A^\cdot))$$

such that $M_i/M_{i-1} \simeq K^{i-1}(K^i(M)), 0 \leq i < d$, *is either zero or an* $(i-1)$*-dimensional Cohen-Macaulay module.*

Proof. In the case M is an unmixed A-module it follows that dim $K^i(M) \leq i-1$. So the assumption implies that $K^i(M), 0 \leq i < d$, is either zero or an $(i-1)$-dimensional Cohen-Macaulay module. Since depth $K^i(M) \geq i-1$ for $2 \leq i <$ dim M it follows that $K(M)$ is Cohen-Macaulay module. This proves the statement in a).
For the proof of b) it will be enough to show that $H^0(\text{Hom}(C^\cdot(M), D_A^\cdot)) = 0$ (cf. the four term exact sequence in the proof of Prop. 3.4 (1)). The subsequent stages of the spectral sequence are

$$E_r^{-q-r,q+r-1} \to E_r^{-q,q} \to E_r^{-q+r,q-r+1}.$$

The module in the middle is a subqoutient of $E_2^{-q,q} = K^q(K^q(M)) = 0$. That means the limit term is zero. So the proof of b) is shown.
For the proof of c) consider the filtration induced by the spectral sequence on

$$H^1(\text{Hom}(C^\cdot(M), D_A^\cdot)).$$

The subsequent stages are derived by the homology of the complex are

$$E_r^{-q+1-r,q+r-1} \to E_r^{-q+1,q} \to E_r^{-q+1+r,q-r+1}.$$

The module at the left hand side is a subqoutient of $K^{q-1+r}(K^{q+r-1}(M)) = 0$ because of the fact dim $K^{q+r-1}(M) < q+r-1$. Moreover the module on the right hand side is a subqoutient of $K^{q-1-r}(K^{q-r+1}(M)) = 0$ since depth $K^{q-r+1}(M) \geq q-r$. Therefore it follows that

$$K^{q-1}(K^q(M)) \simeq E_2^{-q+1,q} \simeq E_\infty^{-q+1,q}.$$

Moreover the spectral sequence induces a filtration

$$0 \subseteq M_0 \subseteq \cdots \subseteq M_{d-1} = H^1(\text{Hom}(C^\cdot(M), D_A^\cdot))$$

of $H^1(\text{Hom}(C^\cdot(M), D_A^\cdot))$ such that $M_q/M_{q-1} \simeq E_\infty^{-q+1,q}$ for all $0 \leq q < d$. By view of a) $K^q(M)$ is either zero or a $(q-1)$-dimensional Cohen-Macaulay module it follows that $K^{q-1}(K^q(M))$ is either zero or a $(q-1)$-dimensional Cohen-Macaulay module. This completes the proof of c). □

It is a surprising fact that under the assumptions the cokernel of the canonical embedding $M \to K(K(M))$ is a CMF module. Its dimension depends upon the local cohomology of M. In fact, we have

$$\dim H^1(\text{Hom}(C^\cdot(M), D_A^\cdot)) = \max\{i < \dim M | H_{\mathfrak{m}}^i(M) \neq 0\} - 1.$$

This follows easily by the exact sequence and the fact that the dimension of the cohomology module is at most $d-2$.

In order to continue let us prove a general result about the cokernel of the natural embedding $0 \to M \to K(K(M))$. It gives a description in terms of intrinsic data of the A-module M.
To this end let us recall the following definition. Let N denote a submodule of a finitely generated A-module M. Let $N = N_1 \cap \ldots \cap N_r$ denote the reduced co-primary decomposition of N. Then $u_M(N)$ denotes the intersection of all those N_i such that $\dim M/N_i = \dim M/N$.

Lemma 4.3. *Let M denote an unmixed finitely generated A-module. Then there is a short exact sequence*

$$0 \to M \to K(K(M)) \to H^1(\operatorname{Hom}(C^{\cdot}(M), D_A^{\cdot})) \to 0$$

and an isomorphism

$$H^1(\operatorname{Hom}(C^{\cdot}(M), D_A^{\cdot})) \simeq u_M(xM)/xM,$$

where $x \in \operatorname{Ann} K(K(M))/M$ denotes an M-regular element. Moreover

$$\dim H^1(\operatorname{Hom}(C^{\cdot}(M), D_A^{\cdot})) \leq \dim M - 2.$$

Proof. The existence of the short exact sequence is shown above. Recall that $\dim K^q(M) < q$ for all $q \neq \dim M$ since M is unmixed.
For the proof of the isomorphism we abbreviate $H^1(\operatorname{Hom}(C^{\cdot}(M), D_A^{\cdot}))$ by H. First of all we show that $\dim H \leq d - 2$, where $d = \dim M$. Since M is unmixed it follows that $\dim M = \dim M_{\mathfrak{p}} + \dim A/\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in \operatorname{Supp} M$. Let $\mathfrak{p} \in \operatorname{Supp} M$ denote a prime ideal such that $\dim M_{\mathfrak{p}} \leq 1$. Then $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module as it is a consequence of the unmixedness of M. Therefore $M_{\mathfrak{p}} \simeq K(K(M_{\mathfrak{p}}))$ since $M_{\mathfrak{p}}$ is Cohen-Macaulay. By the functoriality of the above short exact sequence it follows that $H_{\mathfrak{p}} = 0$. But this means $\dim H \leq d - 2$.
Because of $\dim H \leq d - 2$ there exists an M-regular element $x \in \operatorname{Ann} H$ as follows by prime avoidance arguments. Therefore there is a natural isomorphism $H \simeq xK(K(M))/xM$. But now $xK(K(M)) \cap M = xK(K(M))$ as follows by the choice of $x \in \operatorname{Ann} H$. The multiplication map with respect to x induces an injection

$$0 \to M/xK(K(M)) \cap M \to K(K(M))/xK(K(M)).$$

Since $K(K(M))$ satisfies the condition S_2 it follows that $M/xK(K(M)) \cap M$ is unmixed. Finally a localization argument provides that $u_M(xM) = xK(K(M)) \cap M$. This finishes the proof. □

4.3. **Large depth (II).**

Theorem 4.4. *Let M be an unmixed A-module. Suppose that* $\operatorname{depth} M \geq d - 1$, *where $d = \dim M$.*

a) *There are a short exact sequence*

$$0 \to M \to K(K(M)) \to K^{d-2}(K^{d-1}(M)) \to 0$$

and isomorphisms $K^{d-i}(K^d(M)) \simeq K^{d-2-i}(K^{d-1}(M))$ for all $1 < i < d$.

b) *There is an isomorphism*

$$K^{d-2}(K^{d-1}(M)) \simeq u_M(xM)/xM,$$

where $x \in \operatorname{Ann} K(K(M))/M$ *denotes an* M*-regular element.*

c) *The canonical module* $K^d(M)$ *is Cohen-Macaulay if and only if* $K^{d-1}(M)$ *is either zero or a* $(d-2)$*-dimensional Cohen-Macaulay module.*

Proof. As it was shown above there is the following short exact sequence

$$0 \to M \to K(K(M)) \to H^1(\operatorname{Hom}(C^{\cdot}(M), D^{\cdot}_A)) \to 0$$

and for $i \geq 2$ isomorphisms

$$H^{-d+i}(\operatorname{Hom}(K(M), D^{\cdot}_A)) \simeq H^{i+1}(\operatorname{Hom}(C^{\cdot}(M), D^{\cdot}_A)).$$

Moreover the complex of deficiency $C^{\cdot}(M)$ has cohomology concentrated in degree $-d+1$. Note that depth $M \geq d-1$. So the corresponding spectral sequence degenerates to isomorphisms

$$H^i(\operatorname{Hom}(C^{\cdot}(M), D^{\cdot}_A)) \simeq K^{d-1-i}(K^{d-1}(M))$$

for all $i \in \mathbb{Z}$. By virtue of the previous isomorphisms this proves the statement in a).
The proof of the statement in b) follows by view of Lemma 4.3. Recall that the cokernel of the embedding $M \to K(K(M))$ is isomorphic to $K^{d-2}(K^{d-1}(M))$ as shown in a).
The conclusion in c) is a consequence of a) and the Cohen-Macaulay characterization by the vanishing of the corresponding modules of deficiency for $K^d(M)$ and $K^{d-1}(M)$. □

It would be of a certain interest to give an interpretation of the cohomology modules $H^i(\operatorname{Hom}(C^{\cdot}(M), D^{\cdot}))$ in an intrinsic way similar to the case of $i = 1$ in the previous result.

5. Birational Macaulayfication

5.1. **S_2-fication.** Recall that S_2-fications have been studied in several papers starting with the work of Aoyama and Gotô (cf. [AG]) and [S1]. Moreover, there is a generalization of the construction of the S_2-fication with respect to a certain unconditioned strong d-sequence done by Gotô and Yamagishi (cf. [GY, Section 5]).

Definition 5.1. Let$(A, \mathfrak{m})$ denote a local domain. An extension ring $A \subseteq B \subseteq Q(A)$ that is finitely generated and satisfies S_2 is called an S_2-fication.

Remark 5.2. a) Suppose that A possesses a birational Macaulayfication, then

$$\operatorname{Supp}_A B = \operatorname{Spec} A.$$

Therefore A is catenary.
b) Since B is a finitely generated A-module it is an integral extension. So B is a semi local ring and equi-codimensional, i.e. all the maximal ideals of B have the same height.
c) Let $(A, \mathfrak{m})$ denote a local Nagata domain of dimension two. Then $(A, \mathfrak{m})$ possesses a birational Macaulayfication. To this end let $B = \varinjlim \operatorname{Hom}(\mathfrak{m}^n, A)$ denote the global transform of A. It follows that B is a finitely generated A-module and depth $B = 2$.
d) Nagata's example (cf. [N, Example 2]) is an example of a local two-dimensional domain $(A, \mathfrak{m})$ which is not catenary. So A does not admit a S_2-fication. Therefore there does not exist a birational Macaulayfication.

Lemma 5.3. *Let $(A, \mathfrak{m})$ denote a local d-dimensional domain possessing a dualizing complex $D_A^{\cdot}$.*

a) *The endomorphism ring $K(K(A)) \simeq \operatorname{Hom}(K(A), K(A))$ is a finitely generated birational extension of A.*
b) *The extension ring $K(K(A))$ satisfies the condition S_2. There is a short exact sequence $0 \to A \to K(K(A)) \to C \to 0$, where the map $A \to K(K(A))$ is the natural map and $\dim C \leq d - 2$.*
c) *Let $A \subseteq B \subseteq Q$ denote a birational extension such that B is a finitely generated A-module and satisfies S_2. Then $B \simeq K(K(A))$.*

Proof. First note that by the definition of $K(A)$ there is an exact sequence

$$0 \to K(A) \to D_A^{-d} \to D_A^{-d+1}.$$

Applying the functor $\operatorname{Hom}(K(A), \cdot)$ provides the exact sequence

$$0 \to \operatorname{Hom}(K(A), K(A)) \to \operatorname{Hom}(K(A), D_A^{-d}) \to \operatorname{Hom}(K(A), D_A^{-d+1}).$$

By the definition of $K(K(A))$ this proves the isomorphism of the statement in a). The canonical module $K(A)$ of A is isomorphic to a fractional ideal. So it follows that the ring of endomorphisms $\operatorname{Hom}(K(A), K(A))$ is a commutative birational extension of A.
For the proof of b) first note that $K(K(A))$ satisfies S_2. The short exact sequence of the statement is shown above. By virtue of the duality homomorphisms it follows easily that the first homomorphism is given by

$$A \to K(K(A)), \quad a \mapsto f_a,$$

where f_a denotes the multiplication map on $K(A)$. Moreover the co-kernel C is isomorphic to $H^1(\operatorname{Hom}(C^{\cdot}(A), D^{\cdot}))$. So the result about the dimension is shown above.
Now let $A \subseteq B \subseteq Q$ a birational Macaulayfication of A. Then the cokernel of the embedding has dimension $\leq d - 2$. So there are isomorphisms $K(A) \simeq K(B)$ and $K(K(A)) \simeq K(K(B))$. But now $K(K(B)) \simeq B$ since B satisfies S_2 by the assumption. This completes the proof. □

5.2. **The Criterion for birational Macaulayfication.** Now we are prepared for the answer of question 1. There will be a characterization of the CCM property for a local ring in terms of the S_2-fication.

Theorem 5.4. *Let $(A, \mathfrak{m})$ denote a local ring, quotient of a local Gorenstein ring. Then A is a CCM ring (i.e. the canonical module $K(A)$ is a Cohen-Macaulay module) if and only if the S_2-fication is a Cohen-Macaulay ring (i.e. there is a birational Macaulayfication).*

Proof. Suppose that $K(A)$ is a Cohen-Macaulay module. Then $K(K(A))$ is a Cohen-Macaulay module. Because of the isomorphism of $K(K(A))$ to the S_2-fication and their uniqueness it follows that it is a Cohen-Macaulay ring.
Conversely let $A \subseteq B \subseteq Q$ be an S_2-fication that is a Cohen-Macaulay ring. Then it follows that $B \simeq K(K(A))$. In particular $K(K(A))$ is a Cohen-Macaulay module. Therefore $K(K(K(A)))$ is also a Cohen-Macaulay module. Since $K(A)$ satisfies the S_2-condition the natural homomorphism $K(A) \to K(K(K(A)))$ is an isomorphism. Finally this means that $K(A)$ is a Cohen-Macaulay module. □

There is a comment in general without assuming the existence of a dualizing complex concerning the S_2-fication.

Lemma 5.5. *Let $(A, \mathfrak{m})$ denote a local domain possessing an S_2-fication $A \subseteq B \subset Q$. Then there is an element $x \neq 0$ such that $B \simeq u(xA)$. Moreover there is an isomorphism $B/A \simeq u(xA)/xA$. This A-module is of dimension $\leq \dim A - 2$.*

Proof. Because $A \subseteq B \subseteq Q$ is a birational extension $\dim B/A \leq \dim A - 1$. So there exists a non-zero element $x \in \mathfrak{m}$ such that $xB \subseteq A$. Furthermore the multiplication by x induces an isomorphism $B/A \simeq xB \cap A/xA$. Now it will be easily seen that $xB \cap A = u(xA)$. So it follows that $\dim B/A \leq \dim A - 2$ and $B \simeq u(xA)$ as required. □

6. Simplicial affine semigroup rings

6.1. **Definitions.** Let k denote a field. Let S denote a finitely generated submonoid of the additive monoid $\mathbb{N}^n, n$ a positive integer. The affine semigroup ring $k[S]$ of S over k is defined as the subring of $k[x_1, \ldots, x_n]$ generated by all monomials $\underline{x}^{\underline{s}}$, $\underline{s} \in S$. Let $G(S)$ denote the group generated by S in $\mathbb{Z}^n$. An affine semigroup S is called standard provided the following conditions are satisfied:

a) The normalization $\overline{S}$ of S is given by $\overline{S} = G(S) \cap \mathbb{N}^n$.
b) For $i = 1, \ldots, n$ the image of the projection π_i on the i-th component is a numerical semigroup.
c) For $i = 1, \ldots, n$ the semigroups $S_i = S \cap \ker \pi_i$ are pairwise disjunct with $\operatorname{rank} G(S_i) = \operatorname{rank} G(S) - 1$.

Moreover we put $S' = \cap_{i=1}^{m}(S - S_i)$. Recall that $S' = S''$.
By the result of Hochster (cf. [Ho]) any semigroup is isomorphic to a standard semigroup. In the rest of the talk we assume S to be a standard semigroup.

Proposition 6.1. *Let S denote a affine semigroup. The S_2-fication of $k[S]$ is given by $k[S']$. So there is a short exact sequence*

$$0 \to k[S] \to k[S'] \to k[S' \setminus S] \to 0, \text{ and } \dim k[S' \setminus S] \leq k[S] - 2.$$

The proof is well-known, see e.g. [SS]. In his paper (cf. [Ho]) Hochster has shown that an affine semigroup ring is Cohen-Macaulay if $k[S]$ is normal, i.e. $S = \overline{S}$. By Serre's criterion $k[S]$ is normal if and only if it satisfies S_2 and R_1. The S_2-condition is not sufficient for the Cohen-Macaulayness.

Example 6.2. (cf. [Ho]) Let $S = \{(a,b,c,d) \in \mathbb{N}^4 | a+b = c+d, a,b \neq 1\}$. Then $k[S]$ is a three-dimensional non-Cohen-Macaulay ring that satisfies S_2.

Definition 6.3. The affine semigroup S is called simplicial provided there is a homogeneous system of parameters $f_1, \ldots, f_n, n = \dim k[S]$, for $k[S]$. There are various equivalent conditions (cf. [St]) for S a simplicial affine semigroup.

Moreover there is the following Cohen-Macaulay criterion for simplicial affine semigroup rings.

Proposition 6.4. *Let S denote a simplicial affine semigroup. Then $k[S]$ is a Cohen-Macaulay ring if and only if $S = S'$, i.e. S satisfies the condition S_2.*

Proof. (cf. [St]) It will be enough to show that the S_2-condition is sufficient for the Cohen-Macaulayness of $R = k[S]$. Suppose that $f_1, \ldots, f_n$ is a homogeneous system of parameters. Then it will be sufficient to show that $f_1, \ldots, f_n$ forms a regular sequence for the case $n \geq 3$. To this end let $g \in (f_1, \ldots, f_{i-1})R :_R f_i$ for a certain $2 \leq i \leq n-1$. Then $gf_i = hf_j$ for some $1 \leq j \leq i-1$. Recall that there should be a homogeneous relation among the elements. Two homogeneous elements coincide up to a non-zero scalar if and only if the are of the same multi degree. That means $g \in f_jR :_R f_i$ with $j < i$. Since R is by assumption S_2 it follows that $f_jR :_R f_i = f_jR$ and $g \in (f_1, \ldots, f_{i-1})R$. □

6.2. **Birational Macaulayfication.** With these preparations we are ready to prove the main result for affine semigroup rings.

Theorem 6.5. *Let S denote a simplicial affine semigroup. Then $k[S]$ is a CCM ring. In particular, the birational Macaulayfication of $k[S]$ is given by $k[S']$.*

Proof. It will be enough to show that the canonical module of $k[S]$ is a Cohen-Macaulay module. To this end consider the short exact sequence

$$0 \to k[S] \to k[S'] \to k[S' \setminus S] \to 0.$$

Since $\dim k[S' \setminus S] \leq \dim k[S] - 2$ it induces an isomorphism $K(k[S]) \simeq K(k[S'])$. Furthermore S' is a simplicial affine semigroup. Therefore $k[S']$ is a Cohen-Macaulay ring. Recall that $S' = S''$. Whence $K(k[S'])$ is a Cohen-Macaulay module. By the above isomorphism this completes the proof. So finally $k[S']$ is a Cohen-Macaulay ring. □

6.3. **Codimension 2.** We want to continue with a few more examples. To this end we may write $k[S] \simeq k[\underline{x}]/I(S)$ as a quotient of a polynomial ring $k[\underline{x}], \underline{x} = x_1, \ldots, x_e$, where e denotes the minimal numbers of generators of S and $I(S)$ denotes the ideal of vanishing of $k[S]$. Let m denote the minimal numbers of generators of the ideal $I(S)$.

Theorem 6.6. *Let $k[S]$ denote an affine simplicial semigroup ring of dimension d. Suppose that $k[S]$ is of codimension two, i.e. $e = n + 2$. Then $K^i(K[S]) = 0$ for all $0 \leq i < n - 1$. Then $k[S]$ is a Cohen-Macaulay ring if and only if $m \leq 3$. Suppose that $m > 3$. Then $K^{n-1}(k[S])$ is a $(n-2)$-dimensional Cohen-Macaulay module.*

Proof. Under the assumptions it was shown (cf. [Mo, 3.5] resp. [PS, Theorem 6.1]) that $k[S]$ is a Cohen-Macaulay ring if and only if $m \leq 3$. Moreover it is shown (cf. [PS, Theorem 2.3]) that $\operatorname{depth} k[S] \geq n - 1$. This provides the vanishing of the modules of deficiencies in the range indicated above. Now we have to show that $K^{n-1}(k[S])$ is a $(n-2)$-dimensional Cohen-Macaulay module. Since $k[S]$ is an affine simplicial semigroup ring $K(k[S])$ is a Cohen-Macaulay module. This finally implies that $K^{d-1}(k[S])$ is a $(n-2)$-dimensional Cohen-Macaulay module (cf. 4.4). □

The fact that $K^{n-1}(k[S])$ is a $(n-2)$-dimensional Cohen-Macaulay module was shown independently by M. Morales in an unpublished note by a different argument.
By the result of Peeva and Sturmfels (cf. [PS]) it follows that $\operatorname{depth} A \geq \dim A - 1$ is more general true for the case of lattice ideals. So one might ask whether the above result remains true more general for the case of lattice ideals. This is not the case.

Example 6.7. (cf. [PS, Example 5.10]) Let $R = k[a, b, c, d, e, f]$ and $I = (b^{11}c^6 - ade^8 f^7, ab^5c - d^2e^3 f^2, bc^4d^3 - a^3e^2 f^3, c^3d^5e - a^4b^4 f, b^6c^5d - a^2e^5 f^5, a^5b^9 - c^2d^7e^4 f, c^7d^8 - a^7b^3ef^4)$. Then I is a lattice ideal (cf. [PS]). For the ring $A = R/I$ it turns out that $\dim A = 4, \operatorname{depth} A = 3$ by a computation with Singular. This was communicated by M. Morales, who used his own program for affine rings to compute the depth. Moreover the canonical module $K(A)$ is a four dimensional module with depth $K(A) = 3$. This is in accordance with $\dim K^3(A) = 2$ and depth $K^3(A) = 1$.

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Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, D — 06 099 Halle (Saale), Germany

E-mail address: `schenzel@mathematik.uni-halle.de`