united nations
educational, scientific
and cultural
organization

international atomic
energy agency

the
abdus salam
international centre for theoretical physics

ICTP 40th Anniversary

SMR1563/8

School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics

(24 May - 11 June 2004)

On projective curves of maximal regularity

P. Schenzel
Martin-Luther-Universität
Institut für Informatik
D-06120 Halle
Germany

ON PROJECTIVE CURVES OF MAXIMAL REGULARITY

PETER SCHENZEL

This talk is based on a joint work with M. Brodmann (cf. [2]) (University of Zürich), supported by Swiss National Science Foundation (Projects No. 20-52762.97 and 20-59237.99).

Abstract. Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ be a non-degenerate projective Curve. Then $\operatorname{reg} \mathcal{C} \leq \deg \mathcal{C} - r + 2$ for the regularity $\operatorname{reg} \mathcal{C}$. A non-degenerate curve $\mathcal{C}$ of maximal regularity and of degree $\deg \mathcal{C} > r+1$ has an extremal secant line $\mathbb{L}$. It will be shown that $\mathcal{C} \cup \mathbb{L}$ is arithmetically Cohen Macaulay if $d < 2r - 1$. Moreover there is an investigation of the Betti numbers and the Hartshorne-Rao module of the curves $\mathcal{C}$ of maximal regularity.

Contents

1. Motivation

Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ denote a non-degenerate projective curve, where K is an algebraically closed field.
Two basic numerical invariants related to $\mathcal{C}$ are the degree $\deg \mathcal{C}$ and the Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{C}$.
In their fundamental paper (cf. [6]) Gruson, Lazarsfeld and Peskine have shown that

$$2 \leq \operatorname{reg} \mathcal{C} \leq \deg \mathcal{C} - r + 2.$$

The rôle of these invariants:

— $\deg \mathcal{C}$ reflects the geometric behavior
— $\operatorname{reg} \mathcal{C}$ reflects the cohomological and homological behavior

Note. The Castelnuovo-Mumford regularity of the curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ defined in terms of the vanishing of local cohomology, can be expressed by the degree of the generators of the higher syzygy modules of the defining ideal $I_{\mathcal{C}}$.

Note. In [1] there is a study of reduced irreducible non-degenerate curves of degree $r+2$ in $\mathbb{P}^r_K$. For $r \geq 4$ there is a list of four different main cases I - IV, according to the structure of the Hartshorne-Rao module of the considered curve.
In geometric terms, the case IV is precisely the case in which an extremal secant occurs or – in (co-)homological terms – the case of maximal regularity.

Problem. Consider non-degenerate projective irreducible curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of degree $d > r+1$ (with $r \geq 3$) whose Castelnuovo-Mumford regularity takes the maximally possible value $d - r + 2$.

Reminder. In the case of maximal regularity $\operatorname{reg} C = d - r + 2$, it follows that $\mathcal{C}$ is smooth and rational and has a $(d - r + 2)$-secant line $\mathbb{L}$ (cf. [6]).

Problems.

- What is the structure of the Hartshorne-Rao module of the curve $\mathcal{C}$? It is possible to describe their Betti numbers?
- Investigate the relation between the two curves $\mathcal{C}$ and $\mathcal{C} \cup \mathbb{L}$ from the cohomological and the homological point of view.
- Quite often $\mathcal{C} \cup \mathbb{L}$ is an arithmetically Cohen-Macaulay (CM) curve. Is it true in general ?
- What happens in the "exceptional case" in which $d = r + 1$.

2. Extremal secants

2.1. **Notation and Preliminaries.** Fix the following
Notation.

— $R = \oplus_{n \geq 0} R_n$, $R_+ = \oplus_{n > 0} R_n$
— $S = K[x_0, \ldots, x_r], r \geq 4$
— $\mathcal{J} = \mathcal{J}_{\mathcal{C}} \subseteq \mathcal{O}_{\mathbb{P}^r_K}$ ideal sheaf

– $I = I_{\mathcal{C}} = \oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{J}(n)) \subseteq S$ vanishing ideal
– $A = A_{\mathcal{C}} := S/I$ denote the homogeneous coordinate ring of $\mathcal{C}$
– d degree of $\mathcal{C}, \overline{d} = \min\{n \in \mathbb{N} | I = (I_{\leq n})S\}$, the generating degree of I

Definition. The (Castelnuovo-Mumford) regularity of the sheaf of vanishing ideals $\mathcal{J}_{\mathcal{C}}$ of $\mathcal{C}$, hence the least integer m such that

$$H^i(\mathbb{P}^r, \mathcal{J}_{\mathcal{C}}(m-i)) = 0 \text{ for all } i > 0.$$

The regularity $\operatorname{reg} M$ of a finitely generated graded R-module M is defined to be the least integer m such that $H^i_{R_+}(M)_{n-i} = 0$ for all $n > m$ and all i, where $H^i_{R_+}(M)$ denotes the i-th local cohomology module of M with respect to R_+, furnished with its natural grading.

Secants. Let $\mathbb{L} \subseteq \mathbb{P}^r_K$ be a line, let $L \subseteq S$ be the vanishing ideal of $\mathbb{L}$ and let

$$J = J_{\mathcal{C}\cup\mathbb{L}} = L \cap I \subseteq S$$

be the vanishing ideal of the union $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$.
Let μ denote the degree (thus the length) of the scheme $\mathcal{C}\cap\mathbb{L} = \operatorname{Proj} S/(I + L) \subseteq \mathbb{P}^r_K$, so that $\mathbb{L}$ is a μ-secant of $\mathcal{C}$. As S/L is isomorphic to a polynomial ring in two variables, the vanishing ideal $(I+L)^{\text{sat}} = \cup_{n\in\mathbb{N}}(I+L) :_S (S_+)^n \subseteq S$ of the intersection $\mathcal{C} \cap \mathbb{L} \subseteq \mathbb{P}^r_K$ can be written in the following form

$$(I + L)^{\text{sat}} = L + fS \text{ for some } f \in S_\mu.$$

Lemma 2.1. *In the previous notation we have*

$$\mu \leq \overline{d} \leq \operatorname{reg} \mathcal{C} \leq d - r + 2.$$

Moreover, if $\mu = \overline{d}$, we may choose $f \in I_\mu = I \cap S_\mu$. Finally, if $f \in I_\mu$, then

$$(I + L)^{\text{sat}} = L + fS = I + L \text{ and } I = J + fS.$$

Proof. As L is a prime ideal with $I, S_+ \nsubseteq L$, we have $I_{\overline{d}} \nsubseteq L$. As $I_{\overline{d}} \subseteq L + fS$ it follows $\mu \leq \overline{d}$. Moreover, if $\mu = \overline{d}$, then

$$L_\mu \subsetneqq I_\mu + L_\mu \subseteq (L + fS)_\mu = L_\mu + fK,$$

thus $I_\mu + L_\mu = L_\mu + fK$, and this allows to choose $f \in I_\mu$. Finally, whenever $f \in I_\mu$, we have $L + fS \subseteq I + L \subseteq (I + L)^{\text{sat}} = L + fS$. This proves the stated equalities. □

2.2. **The Main Lemma.**

Lemma 2.2. *If $\mu \geq 2$, then $h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)) \leq d - \mu + 3$.*

Proof. Let $\delta := h^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1))$ and consider the graded K-algebra

$$D := \oplus_{n\geq 0} H^0(\mathcal{C} \cup \mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(n)),$$

so that $\delta = \dim_K D_1 \geq r+1$. As $S/J \subseteq K[D_1] \subseteq D$ and $(S/J)_n = D_n$ for all $n \gg 0$, the inclusion $S/J \hookrightarrow K[D_1]$ yields an isomorphism of schemes

$$\varepsilon : \operatorname{Proj} K[D_1] \xrightarrow{\sim} \operatorname{Proj}(S/J) = \mathcal{C} \cup \mathbb{L}.$$

Now, set $\mathbb{P}^{\delta-1}_K = \operatorname{Proj} T$, where T is the polynomial ring $S[x_{r+1}, \ldots, x_{\delta-1}]$ and let π be a surjective homomorphism of graded K-algebras, which appears in the commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & T \\ \downarrow & & \downarrow \pi \\ S/J & \hookrightarrow & K[D_1]. \end{array}$$

Consider $\operatorname{Proj} K[D_1]$ as a closed non-degenerate subscheme of $\mathbb{P}^{\delta-1}_K$ by means of π and let $Z = \operatorname{Proj}(T/S_+T)$. Then, $Z \cap \operatorname{Proj} K[D_1] = \emptyset$ and we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Proj} K[D_1] & \xrightarrow[\varepsilon]{\sim} & \mathcal{C} \cup \mathbb{L} \\ \downarrow & & \downarrow \\ \mathbb{P}^{\delta-1}_K \setminus Z & \overset{p}{\twoheadrightarrow} & \mathbb{P}^r \end{array}$$

in which p is the projection centered at Z. So, for any closed subscheme $Y \subseteq \mathcal{C} \cup \mathbb{L}$, we know that $\varepsilon^{-1}(Y)$ is a closed subscheme of $\mathbb{P}^{\delta-1}_K$, isomorphic to Y and of the same degree as Y. Therefore $\mathcal{C}' := \varepsilon^{-1}(\mathcal{C}) \subseteq \mathbb{P}^{\delta-1}_K$ is a reduced irreducible curve of degree d, $\mathbb{L}' := \varepsilon^{-1}(\mathbb{L}) \subseteq \mathbb{P}^{\delta-1}_K$ is a line and $\mathcal{C}' \cup \mathbb{L}' = \operatorname{Proj} K[D_1]$. As

$$\mathcal{C}' \cap \mathbb{L}' = \varepsilon^{-1}(\mathcal{C} \cap \mathbb{L}) \simeq \mathcal{C} \cap \mathbb{L},$$

we see that $\mathbb{L}'$ is a μ-secant of $\mathcal{C}'$.
Moreover, $\mathcal{C}'$ is non-degenerated embedded into $\mathbb{P}^{\delta-1}_K$. Otherwise, we could find a hyperplane $H \subseteq \mathbb{P}^{\delta-1}_K$ with $\mathcal{C}' \subseteq H$. As $\mathcal{C}' \cup \mathbb{L}' \subseteq \mathbb{P}^{\delta-1}_K$ is non-degenerate, this would imply $\mathbb{L}' \nsubseteq H$ and hence $\mathcal{C}' \cap \mathbb{L}' \subseteq H \cap \mathbb{L}' \simeq \operatorname{Spec}(K)$, a contradiction to the assumption $\mu \geq 2$.
But now, by 2.1 we have $\mu \leq d - (\delta - 1) + 2$, thus $\delta \leq d - \mu + 3$. □

2.3. **Extremal Secants.**

Remark 2.3. A) Say that $\mathbb{L}$ is an *extremal secant* of $\mathcal{C}$ if $\mu = \operatorname{reg} \mathcal{C}$. (This is justified in view of 2.1.)
B) On use of [6, p. 504] say:
If $d > r+1$ and if $\mathcal{C}$ is of maximal regularity, then $\mathcal{C}$ is smooth, rational and has an extremal secant line.

If $d > r+1$, the curve $\mathcal{C}$ is of maximal regularity if and only if it has a $(d-r+2)$-secant line. In this case $\mathcal{C}$ is smooth and rational.

Remark 2.4. A) In view of 2.1 we now can write

$$I = J + fS \quad \text{with } f \in I_{d-r+2} \setminus L.$$

L denotes the ideal defining the secant line. As $L \subseteq S$ is a prime ideal, we have $J :_S f = (I \cap L) :_S f = L :_S f = L$ and hence get graded isomorphisms

$$I/J \simeq fS/f(J :_S f) \simeq (S/L)(-d+r-2)$$

which yield the short exact sequence of graded S-modules

$$0 \to S/L(-d+r-2) \to S/J \to A \to 0.$$

B) The curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity is smooth and rational. Therefore, the graded K-algebra

$$\Gamma(\mathcal{C}) := \oplus_{n\geq 0} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n))$$

may be viewed as the homogeneous coordinate ring of a rational normal curve $\mathcal{C}' \subseteq \mathbb{P}^d_K$ of degree d. So, if $K[s,t]$ is a polynomial ring, we have an isomorphism of graded K-algebras, $\Gamma(\mathcal{C}) \simeq K[s,t]^{(d)}$, where $R^{(d)}$ is used to denote the d-th Veronesean subring $\oplus_{n\geq 0} R_{nd}$ of the graded ring $R = \oplus_{n\geq 0} R_n$.

2.4. **Hartshorne-Rao module and Hilbert function.** For i an integer, let $H^i(M) = H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the ideal R_+.
$H^i(M)$ is a graded R-module. If M is finitely generated, each graded component of $H^i(M)$ is of finite dimension over K. In this case, we set

$$h^i(M)_n = h^i_{R_+}(M)_n := \dim_K H^i(M)_n.$$

Definition. The *Hartshorne-Rao module* $\oplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^r_K, \mathcal{J}_{\mathcal{C}}(n)) \simeq H^1(A_{\mathcal{C}})$.

Proposition 2.5. *We have the following results*

a) $h^1(A)_n = \begin{cases} 0, & \text{for } n \notin \{1,\dots,d-r\}, \\ d-r, & \text{for } n = 1, \\ 1, & \text{for } n = d-r. \end{cases}$

b) $h^2(A)_n = \max\{0, -dn-1\}$ *for all* $n \in \mathbb{Z}$.

c) $h^1(S/J)_n = \begin{cases} 0, & \text{for } n \notin \{2,\dots,d-r-1\}, \\ h^1(A)_n - d + r + n - 1, & \text{for } 2 \leq n \leq d-r-1. \end{cases}$

d) $h^2(S/J)_n = \begin{cases} 0, & \text{for all } n > 0, \\ d-r+1, & \text{for } n = 0, \\ -n(d+1)+d-r, & \text{for all } n < 0. \end{cases}$

3. On the Structure of S/J

3.1. **Basic Facts.** The aim is to study the homogeneous coordinate ring S/J of the union $\mathcal{C} \cup \mathbb{L} \subseteq \mathbb{P}^r_K$ and to relate it to the curve $\mathcal{C}$ and its Hartshorne-Rao module $H^1(A)$.

Remark 3.1. A) For a finitely generated graded module M over a non-negatively graded Noetherian ring $R = \oplus_{n \geq 0} R_n$ and for $i \in \mathbb{N}_0$ let

$$a_i(M) := \sup\{n \in \mathbb{Z} | H^i_{R_+}(M)_n \neq 0\}.$$

It follows from 2.5 that

$$\operatorname{reg} S/J = \max\{2, a_1(S/J) + 1\} \leq d - r.$$

B) As a consequence of this last observation we have $\operatorname{reg} J \leq d - r + 1$. It follows $I = J + fS$ for some $f \in I_{d-r+2}$. As J is generated in degrees $\leq \operatorname{reg} J$, it implies

$$J = (I_{\leq \operatorname{reg} J})S = (I_{\leq d-r+1})S.$$

3.2. **The Cohen Macaulay Criterion.**

Theorem 3.2. *The following statements are equivalent:*

(i) S/J *is CM.*
(ii) $H^1(S/J) = 0$.
(iii) $h^1(S/J)_2 = 0$.
(iv) $\operatorname{reg} S/J = 2$.
(v) $h^1(A)_n = d - r + 1 - n$, *for* $n = 1, \ldots, d - r$.
(vi) $h^1(A)_2 \leq d - r - 1$.
(vii) $\operatorname{soc} H^1(A) = K(r - d)$.
(viii) *There is an isomorphism of graded* S*-modules*

$$H^1(A) \simeq H^2(S/L)(-d + r - 2)_{\geq 1}.$$

(ix) *There are independent linear forms* $y_0, \ldots, y_r \in S_1$ *and an isomorphism of graded* S*-modules*

$$H^1(A) \simeq \operatorname{Hom}_K(S/((y_0, y_1)^{d-r} + (y_2, \ldots, y_r)), K)(r - d).$$

Now, we are ready to prove the announced Cohen-Macaulay criterion.

Proposition 3.3. *If* $d < 2r - 1$, *then* S/J *is a Cohen-Macaulay ring.*

Proof. Let $\ell \in S_1$ be a generic linear form so that $\operatorname{Proj}(A/\ell A)$ is a scheme of d points $p_1, \ldots, p_d$ in semi-uniform position. Thus these points span the hyperplane $H = \operatorname{Proj}(A/\ell A)$ and the dimension of the linear span of any subset P of $\{p_1, \ldots, p_d\}$ depends only on the cardinality of P (cf. [8, (2.1.1.7)]).
After a linear coordinate transformation we may assume that $\ell = x_r$ and set $S/x_r S = K[x_0, \ldots, x_{r-1}] = T$. Then,

$$\operatorname{Proj}(A/x_r A) = \operatorname{Proj}(T/IT) \subseteq \operatorname{Proj} T = \mathbb{P}^{r-1}_K$$

is a scheme of d points in semi-uniform position with vanishing ideal

$$N = (IT)^{\mathrm{sat}} = \cup_{n\geq 0}(IT :_T (T_+)^n).$$

We know that N is generated by quadrics. In view of the natural isomorphism $H^0(A/x_rA) \simeq N/IT$ it follows that $H^0(A/x_rA)$ is generated in degree two. If we apply cohomology to the exact sequence $0 \to A(-1) \xrightarrow{x_r} A \to A/x_rA \to 0$, we get exact sequences

$$0 \to H^0(A/x_rA)_n \to H^1(A)_{n-1} \to H^1(A)_n \to H^1(A/x_rA)_n$$

for all $n \in \mathbb{Z}$. As

$$H^1(A/x_rA)_n \simeq H^1(T/IT)_n \simeq H^1(T/N)_n = 0$$

for all $n \geq \operatorname{reg} T/N$ yields that $H^1(A/x_rA)_n = 0$ for all $n \geq \left[\frac{d-1}{r-1}\right] = 2$. If we apply the above sequence with $n = d - r + 1$ and observe 2.5 a), it follows $H^0(A/x_rA)_{d-r+1} \neq 0$. As $H^0(A/x_rA)$ is generated in degree 2 we get $H^0(A/x_rA)_2 \neq 0$. Applying the above sequence with $n = 2$ and observing 2.5 a) we get $h^1(A)_2 \leq h^1(A)_1 - 1 = d - r - 1$. So, by 3.2 we get that S/J is CM. □

It is known that $H^1(S/J)$ is generated in degree 2. Here is a result on the number of generators of the module $H^1(S/J)$.

Corollary 3.4. *$H^1(S/J)$ is minimally generated by*

$$\dim_K I_2 - \binom{r+1}{2} + d + 1$$

homogeneous elements of degree 2. In particular we have

$$\dim_K I_2 \geq \binom{r+1}{2} - d - 1$$

with equality if and only if S/J is CM.

4. Betti Numbers

Now relate the Betti numbers of the S-module A to the Betti numbers of the S-modules S/J and $H^1(A)$. Our interest shall be focused to the case in which S/J is CM.

4.1. **The general case.**

Proposition 4.1. *Let $t := \operatorname{reg} S/J$. Then $t \leq d - r$ and for all $i \in \{1, \ldots, r\}$ we have*

$$\operatorname{Tor}_i^S(K, A) \simeq \operatorname{Tor}_i^S(K, S/J) \oplus K^{\binom{r-1}{i-1}}(-i - d + r + 1).$$

Proof. As depth $S/J > 0$ we have

$$\operatorname{Tor}_i^S(K, S/J)_{i+j} = 0 \text{ if } (i, j) \notin \{1, \ldots, r\} \times \{1, \ldots, t\}.$$

Moreover we have $\operatorname{Tor}_i^S(K, S/L) \simeq K^{\binom{r-1}{i}}(-i)$ for all $i \in \mathbb{N}_0$. By the sequence of 2.4 A), for all $i, j \in \mathbb{N}$, we get an exact sequence

$$K^{\binom{r-1}{i}}(-i-d+r-2)_{i+j} \to \operatorname{Tor}_i^S(K, S/J)_{i+j} \to$$
$$\operatorname{Tor}_i^S(K, A)_{i+j} \to K^{\binom{r-1}{i-1}}(-i-d+r-1)_{i+j} \to \operatorname{Tor}_{i-1}^S(K, S/J)_{i+j}.$$

It follows that $\operatorname{Tor}_i^S(K, S/J)_{i+j} \simeq \operatorname{Tor}_i^S(K, A)_{i+j}$ for all $j \notin \{d-r+1, d-r+2\}$. As $t = \operatorname{reg} S/J \leq d-r$ (cf. 3.1 a)), we have $\operatorname{Tor}_i^S(K, S/J)_{i+j} = \operatorname{Tor}_{i-1}^S(K, S/J)_{i+j} = 0$ for all $j \geq d-r+1$. Now, our claim follows easily. □

Concerning the Betti modules of the S-module $D(A)$ we have the following auxiliary result, which shall be used later to determine the Betti numbers of A_C.

Lemma 4.2. *For $i \in \{1, \dots, r-1\}$ let $c_i := (d-1)\binom{r-1}{i} - \binom{r-1}{i-1}$. Then*

$$\operatorname{Tor}_i^S(K, D(A)) = \begin{cases} K(0) \oplus K^{d-r}(-1), & \text{for } i = 0 \\ K^{c_i}(-i-1), & \text{for } i \in \{1, \dots, r-1\}. \end{cases}$$

Now, we consider the Betti numbers of the Hartshorne-Rao module $H^1(A)$.

Proposition 4.3. *We have with $H := H^1(A)$:*

a) $\operatorname{Tor}_0^S(K, H) \simeq K^{d-r}(-1)$.
b) $\operatorname{Tor}_1^S(K, H) \simeq K^{c_1 - \dim_K I_2}(-2)$.
c) $\operatorname{Tor}_i^S(K, H) \simeq K^{a_i}(-i-1) \oplus \operatorname{Tor}_{i-1}^S(K, A)_{\geq i+2}$ *with* $a_i \leq \binom{r+1}{i}(d-r)$ *for* $i \in \{2, \dots, r\}$.
d) $\operatorname{Tor}_{r+1}^S(K, H) \simeq \operatorname{Tor}_r^S(K, A)_{\geq r+3}$.

4.2. **The Cohen-Macaulay case.** In case S/J is CM, the Betti modules of the Hartshorne-Rao module $H^1(A)$ can be determined precisely:

Proposition 4.4. *For $i \in \{0, \dots, r+1\}$ let $a_i := (d-r+1)\binom{r-1}{i-1} + (d-r)\binom{r-1}{i}$ and $b_i := \binom{r-1}{i-2}$. Assume that S/J is CM. Then, for all $i \in \{0, \dots, r+1\}$ we have*

$$\operatorname{Tor}_i^S(K, H^1(A)) \simeq K^{a_i}(-i-1) \oplus K^{b_i}(-i-d+r).$$

Proof. For $a \in \mathbb{N}, m \in \{1, \dots, r\}$ and $i \in \{0, \dots, m\}$ we set

$$M_m := S/((x_0, x_1)^a + (x_2, \dots, x_m)).$$

Choose $m = r, a = d-r$ and set $M := M_r$. After a linear change of coordinates, we may assume by Theorem 3.2 that

$$H := H^1(A) = \operatorname{Hom}_K(M, K)(r-d).$$

An inductive argument r proves the claim. □

Finally, if S/J is CM, the Betti numbers of A can be approximated as follows:

Theorem 4.5. *Assume that S/J is CM and that $d > r+1$. Then, for each $i \in \{1,2,\dots,r\}$ we have*

$$\mathrm{Tor}_i^S(K,A) \simeq K^{u_i}(i-1) \oplus K^{v_i}(-i-2) \oplus K^{\binom{r-1}{i-1}}(-i-d+r-1),$$

where u_i and v_i are given resp. bounded according to the following table

i	1	$2 \le i \le r-2$	$r-1$	r
u_i	$\binom{r+1}{2} - d - 1$	$\le c_i$	$\le d-1$	0
v_i	$\le (d-1)\binom{r}{2} + (r-1)$	$\le a_{i+1}$	$d-r+1$	0

in which c_i and a_{i+1} are defined according to 4.4. Moreover $u_i - v_{i-1} = c_i - a_i$ for all $i \in \{2,\dots,r-1\}$.

Proof. The stated general shape of the Betti module $\mathrm{Tor}_i^S(K,A)$ follows from Proposition 4.1, as I is generated in degree ≥ 2 and as reg $S/J = 2$ (cf. Theorem 3.2). □

5. Examples

We introduce the notation

$$\beta_{ij} := \dim_K \mathrm{Tor}_i^S(K,A)_{i+j}$$

for the Betti numbers of $\mathcal{C}$.

5.1. **The exceptional case.** . We first consider the "exceptional case" in which $d = r+1$, a case which has been excluded previously.

Remark 5.1. In this case we know that (cf. [1, (4.7) B)]) $\mathcal{C} \subseteq \mathbb{P}_K^r$ is either

- an elliptic normal curve,
- the projection of a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}_K^{r+1}$ from a generic point or else
- a singular rational curve, obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}_K^{r+1}$ from a point which lies precisely on one secant line of $\tilde{\mathcal{C}}$.

Conclusion. In the first and the third case, $\mathcal{C}$ is of arithmetic genus 1, so that $H^2(A)_0 \neq 0$. In the second case we have $H^1(A)_1 \neq 0$. So, in all three cases we have reg $\mathcal{C}$ = reg $A + 1 \geq 3$ and hence reg $\mathcal{C} = 3$, (cf. 2.1). So $\mathcal{C}$ is of maximal regularity in any case.

Trisecants. If $\mathcal{C}$ has a 3-secant line $\mathbb{L}$, then by 2.1 I is generated by quadrics and one cubic. According to [6, p. 504] this only may occur in the case where $\mathcal{C}$ is smooth and rational. It follows that the curve $\mathcal{C}$ always lies on a rational surface scroll $\mathbb{S}_{r-1-2a,a} \subseteq \mathbb{P}_K^r, (0 \leq a \leq \frac{r-1}{2})$. Moreover, by [6, Remark (2), p. 504], $\mathcal{C}$ has a trisecant line if and only if $a = 1$.

Conclusion. Assume that $\mathcal{C}$ has a trisecant line $\mathbb{L}$. Then, for some $f \in S_3 \setminus L$ we have $I = J + fS, I + L = L + fS$ (cf. 2.1) and the resulting exact sequence

$$0 \to S/J \to S/L \oplus A \to S/(L + fS) \to 0$$

together with the fact that $H^1(A)_n = 0$ for all $n \neq 1$ shows the vanishing $H^1(S/J)_n = 0$ for all $n \neq 1$. If we apply 2.2 with $\mu = 3$ we also obtain

$$r+1+h^1(S/J)_1 = \dim(S/J)_1+h^1(S/J)_1 = h^0(\mathcal{C}\cup\mathbb{L}, \mathcal{O}_{\mathcal{C}\cup\mathbb{L}}(1)) \leq r+1,$$

hence $h^1(S/J)_1 = 0$. Therefore $H^1(S/J) = 0$ and S/J becomes CM, too. So, the statement of 3.2 remains valid.

Examples 5.2. We consider the two non-degenerate rational curves $\mathcal{C}_k \subseteq \mathbb{P}^{10}_K$ of degree $11, (k = 1, 2)$ given parametrically by

$$\mathcal{C}_1 :(s^{11} : s^{10}t : s^9t^2 : s^7t^4 : s^6t^5 : s^5t^6 : s^4t^7 : s^3t^8 : s^2t^9 : st^{10} : t^{11}),$$
$$\mathcal{C}_2 :(s^{11} : s^{10}t : s^8t^3 : s^7t^4 : s^6t^5 : s^5t^6 : s^4t^7 : s^3t^8 : s^2t^9 : st^{10} : t^{11}).$$

Facts:

- $\mathcal{C}_k$ lies on the rational surface scroll $\mathbb{S}_{9-2k,k}$ for $k = 1, 2$,
- the curves are obviously smooth and obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ from a point (which avoids all secant lines),
- both of the curves are of regularity 3.

Moreover for the Betti numbers β_{ij} of $A = A_{\mathcal{C}_k}$ we have

k	i	1	2	3	4	5	6	7	8	9	10
1	β_{i1}	43	221	550	812	742	398	91	8	0	0
	β_{i2}	0	1	8	28	56	70	84	45	11	1
2	β_{i1}	43	222	558	840	798	468	147	8	0	0
	β_{i2}	1	9	36	84	126	126	84	45	11	1

k=1 $\beta_{12} = 0$, so that I is generated by quadrics, by 5.1 B) $\mathcal{C}_1$ has no trisecant line,

k=2 $\beta_{12} = 1$, so that I is generated by quadrics and one cubic, by [6, p. 504] has a trisecant line, S/J must be CM .

For the Betti numbers

$$\gamma_{ij} := \dim_K \operatorname{Tor}_i^S(K, S/J)_{i+j}$$

of S/J we get the following values

i	1	2	3	4	5	6	7	8	9	10
γ_{i1}	43	222	558	840	798	468	147	8	0	0
γ_{i2}	0	0	0	0	0	0	0	9	2	0

5.2. **The non-exceptional case.** It is easy to see that for each $r \geq 3$ and each $d > r+1$ there are reduced irreducible non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity and of degree d lying on a rational surface scroll $\mathbb{S} \subseteq \mathbb{P}^r_K$. But in general, curves of maximal regularity need not lie on a scroll.

Example 5.3. Let $\mathcal{C} \subseteq \mathbb{P}^8_K$ be the non-degenerate rational curve given parametrically by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : s^6t^5 : s^5t^6 : (st^{10} + s^2t^9) : t^{11}).$$

Calculating the Betti numbers β_{ij} with the help of SINGULAR (cf. [4]) we get

i	1	2	3	4	5	6	7	8
β_{i1}	24	84	126	84	20	0	0	0
β_{i2}	0	0	0	20	36	21	4	0
β_{i3}	0	0	0	0	0	0	0	0
β_{i4}	1	7	21	35	35	21	7	1

In particular, we have :

- $\operatorname{reg} A = 4$, thus $\operatorname{reg} \mathcal{C} = 5 = 11 - 8 - 2$,
- $\mathcal{C}$ is of maximal regularity,
- $\beta_{61} = \dim_K \operatorname{Tor}^6_K(K, A)_7 = 0$, Green's $K_{p,1}$-Theorem shows that $\mathcal{C}$ does not lie on a surface scroll (cf. [3]),
- S/J is CM (cf. 3.2),
- the first two lines of the previous table describe the Betti numbers of S/J.

5.3. **Scrolls.** It is easy to see that for each $r \geq 3$ and each $d > r+1$ there are reduced irreducible non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity and of degree d lying on a rational surface scroll $\mathbb{S} \subseteq \mathbb{P}^r_K$. But in general, curves of maximal regularity need not lie on a scroll.

Example 5.4. Let $\mathcal{C} \subseteq \mathbb{P}^8_K$ be the non-degenerate rational curve given parametrically by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : s^6t^5 : s^5t^6 : (st^{10} + s^2t^9) : t^{11}).$$

Calculating the Betti numbers β_{ij} we get

i	1	2	3	4	5	6	7	8
β_{i1}	24	84	126	84	20	0	0	0
β_{i2}	0	0	0	20	36	21	4	0
β_{i3}	0	0	0	0	0	0	0	0
β_{i4}	1	7	21	35	35	21	7	1

In particular:

- $\operatorname{reg} A = 4$ and $\operatorname{reg} \mathcal{C} = 5 = 11 - 8 - 2$, so that $\mathcal{C}$ is of maximal regularity,
- $\beta_{61} = \dim_K \operatorname{Tor}^6_K(K, A)_7 = 0$,

- Green's $K_{p,1}$-Theorem shows that $\mathcal{C}$ does not lie on a surface scroll (cf. [3]),
- S/J is CM (cf. 3.2 (iv)),
- the first two lines of the previous table describe the Betti numbers of S/J.

5.4. **The non Cohen-Macaulay case of S/J.** By 3.3 we know that S/J is CM if $d < 2r - 1$. The next example illustrates that this result is sharp: There are reduced irreducible non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r_K$ of maximal regularity and of degree $d = 2r - 1$ such that S/J is not a CM ring.

Example 5.5. Let $\mathcal{C} \subseteq \mathbb{P}^6_K$ be given by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : st^{10} : t^{11}),$$

so that $\mathcal{C}$ is non-degenerate and of degree $11 = 2 \cdot 6 - 1$. Here, the Betti numbers β_{ij} take the following values

i	1	2	3	4	5	6
β_{i1}	10	20	15	4	0	0
β_{i2}	3	10	10	0	0	0
β_{i3}	1	5	10	15	7	1
β_{i4}	0	0	0	0	0	0
β_{i5}	0	0	0	0	0	0
β_{i6}	1	5	10	10	5	1

- $\operatorname{reg} A = 6$, so that $\operatorname{reg} \mathcal{C} = 7 = 11 - 6 + 2 = d - r + 2$,
- $\mathcal{C}$ is of maximal regularity,
- S/J is not a CM-ring (apply 3.2 (iv),
- the first three lines of the above table provide the Betti numbers of S/J (see 4.1),
- $H^1(S/J)$ is minimally generated by one element of degree 2 (cf. 3.4),
- the socle formula shows that $H^1(S/J)_n = 0$ for all $n > 2$,
- S/J is a Buchsbaum ring with $H^1(S/J) = K(-2)$.

Question. Is the converse of 3.3 true.

No. We shall give an example showing that this is not the case in general: There are reduced irreducible non-degenerate curves $\mathcal{C} \subseteq \mathbb{P}^r$ of maximal regularity of degree $d \geq 2r - 1$ and such that S/J is a CM ring.

Example 5.6. Let $\mathcal{C} \subseteq \mathbb{P}^6_K$ be the curve of degree 11 defined parametrically by

$$\mathcal{C} : (s^{11} : s^{10}t : s^9t^2 : s^8t^3 : s^7t^4 : (s^2t^9 + st^{10}) : t^{11}).$$

Here, the Betti numbers of $\mathcal{C}$ are as listed below:

i	1	2	3	4	5	6
β_{i1}	9	16	9	0	0	0
β_{i2}	6	24	36	25	6	0
β_{i3}	0	0	0	0	0	0
β_{i4}	0	0	0	0	0	0
β_{i5}	0	0	0	0	0	0
β_{i6}	1	5	10	10	5	1

- reg $\mathcal{C}$ = reg $A + 1 = 7 = 11 - 6 + 2 = d - r + 2$ so that $\mathcal{C}$ is of maximal regularity,
- the number of generating quadrics is $9 = \binom{6+1}{2} - 11 - 1 = \binom{r+1}{2} - d - 1$ so that S/J is a CM-ring by 3.4,
- we have $d = 11 = 2 \cdot 6 - 1 = 2r - 1$,
- the first two lines of the above diagram furnishes the Betti numbers of $\mathcal{C} \cup \mathbb{L}$.

5.5. **The socle of $H^1(S/J)$.** We know that S/J is CM if and only if soc $H^1(A) \simeq K(r-d)$ (cf. 3.2), while soc $H^1(A) \simeq$ soc$(H^1(S/J)) \oplus K(r-d)$ in general. Here there are examples which illustrate how much soc $H^1(A)$ may vary in general.

Example 5.7. Consider the two curves $\mathcal{C}_k \subseteq \mathbb{P}^6_K$ of degree $13, (k = 1, 2)$ given parametrically by

$$\mathcal{C}_1 : (s^{13} : s^{12}t : s^{11}t^2 : s^{10}t^3 : s^9t^4 : st^{12} : t^{13}),$$
$$\mathcal{C}_2 : (s^{13} : s^{12}t : s^{11}t^2 : s^{10}t^3 : s^9t^4 : (st^{12} - s^2t^{11}) : t^{13}).$$

The Betti numbers are listed below

k	i	1	2	3	4	5	6
1	β_{i1}	10	20	15	4	0	0
	β_{i2}	1	0	0	0	0	0
	β_{i3}	0	10	20	15	4	0
	β_{i4}	0	0	0	0	0	0
	β_{i5}	1	5	10	10	5	1
	β_{i6}	0	0	0	0	0	0
	β_{i7}	0	0	0	0	0	0
	β_{i8}	1	5	10	10	5	1
2	β_{i1}	9	16	9	0	0	0
	β_{i2}	2	4	0	1	0	0
	β_{i3}	2	14	36	34	14	2
	β_{i4}	0	0	0	0	0	0
	β_{i5}	0	0	0	0	0	0
	β_{i6}	0	0	0	0	0	0
	β_{i7}	0	0	0	0	0	0
	β_{i8}	1	5	10	10	5	1

- reg $\mathcal{C}_k = 9 = 13 - 6 + 2 = d - r + 2$, so that $\mathcal{C}_k$ is of maximal regularity,

- reg $S/J = 6$ resp. 4 in the case $k = 1$ resp. $k = 2$, i.e. S/J is not CM ,
-
$$\operatorname{soc} H^1(A) \simeq \oplus_{j\geq 3} K^{\beta_{rj}}(-j+1).$$

Conclusion.

$$\operatorname{soc} H^1(A) \simeq \begin{cases} K(-4) \oplus K(-7), & \text{if } k = 1, \\ K^2(-2) \oplus K(-7), & \text{if } k = 2. \end{cases}$$

It follows that S/J is a Buchsbaum ring with $H^1(S/J) \simeq K^2(-2)$ in the case $k = 2$ while it is not a Buchsbaum ring in the case $k = 1$.

Note. Further examples in higher degrees show that soc $H^1(A)$ may indeed vary rather strongly. These were found with the aid of Singular (cf. [4]).

References

[1] Brodmann, M., Schenzel, P.: *Curves of Degree $r+2$ in $\mathbb{P}^r$: Cohomological, Geometric, and Homological Aspects.* J. of Algebra **242** (2001), 577 - 623.

[2] Brodmann, M., Schenzel, P.: *On projective curves of maximal regularity.* Math. Z. **244** (2003), 271-289.

[3] Green, M.: *Koszul Homology and the Geometry of Projective Varieties.* J. of Differential Geometry **19** (1984), 125 - 171.

[4] Green, M., Lazarsfeld, R.: *Some Results on the Syzygies of Finite Sets and Algebraic Curves.* Compositio Math. **67** (1988), 301 - 314.

[4] Greuel, G.-M., Pfister, G. et al.: *Singular 2.0, A Computer Algebra System for Polynomial Computations.* Center for Computer Algebra, Univ. of Kaiserslautern (2001) (http://www.singular.uni-kl.de).

[6] Gruson, L., Lazarsfeld, R., Peskine, C.: *On a Theorem of Castelnuovo and the Equations defining Space Curves.* Inventiones Math. **72** (1983), 491 - 506.

[7] Hoa, T., Stückrad, J., Vogel, W.: *Towards a structure theory of projective varieties of degree = codimension +2.* J. Pure Appl. Algebra **71** (1991), 203 - 231.

[8] Nagel, U.: *Über Gradschranken für Syzygien und komologische Hilbertfunktionen.* Dissertation, Univ. Paderborn (1990).

Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, D — 06 099 Halle (Saale), Germany

E-mail address: schenzel@mathematik.uni-halle.de