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Homological aspects of the normalization of algebraic structures

W.V. Vasconcelos
Department of Mathematics
Rutgers University
Piscataway, NJ 08854
U.S.A.

Homological Aspects of the Normalization of Algebraic Structures

WOLMER V. VASCONCELOS
Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
E-mail: vasconce@math.rutgers.edu

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Preface

These are lecture notes for a set of talks at the *School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics*. The material is taken from a book project in preparation, where other developments will be found. Despite the chopped off character, the author eagerly solicits suggestions from the reader about the addition of new topics and about errors of any type. We add that a bare core of references is mentioned, with a full set to be included in the book. We regret that several authors are fully cited in the book but not in these notes.

We are grateful to the organizers of the *School*, Professors C. Huneke, A. Simis, N. V. Trung, G. Valla and J. Verma, for the opportunity to meet and interact with so many young researchers.

The research of the author was also partially supported by the National Science Foundation (USA).

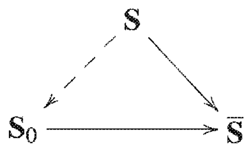
Introduction

Thse notes give a treatment of the notion of integral closure as it applies to ideals, modules and algebras, and associated algebras, emphasizing their relationships. Its main theme are the structures that arise as solutions of collections of equations of integral dependence in an algebra A,

$$\boxed{z^n + a_1 z^{n-1} + \cdots + a_n = 0.}$$

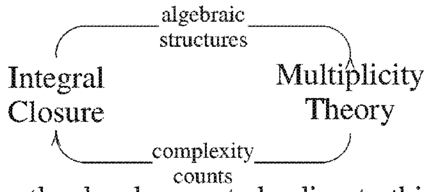
In such equations, the a_i's and z are required to satisfy set theoretic restrictions of various kinds, with the solutions assembled into algebras, ideals, or modules, each process adding a particular flavor to the subject. The study of these equations–including the search for the equations themselves–is a region of convergence of many interests in algebraic geometry, commutative algebra, number theory and computational aspects of each of these fields. The overall goal is to find and understand the equations defining these assemblages. Both challenges and opportunities arise with these issues, the former by the difficulties current models of computation have in dealing with them, the latter in the need to develop new theoretical approaches to its understanding.

For one of these structures, $\mathbf{S}$, the analysis and/or construction of its integral closure $\overline{\mathbf{S}}$ usually passes through the study of its so–called reductions $\mathbf{S}_0$:



The $\mathbf{S}_0$ are structures similar to $\mathbf{S}$, with the same closure as $\mathbf{S}$, possibly providing a pathway to the closure which does not pass through $\mathbf{S}$. They are geometrically and computationally simpler than $\mathbf{S}$ and therefore provide for a convenient platform from which to examine $\overline{\mathbf{S}}$ and give rise to many structures that ultimately bear on $\mathbf{S}$. It is not too far-fetched to make an equivalence between the study of all the $\mathbf{S}_0$ and of $\overline{\mathbf{S}}$. Rather than a source of frustration, this diversity is a mine of opportunities to examine $\mathbf{S}$, and often it is the springboard to the examination of other properties of $\mathbf{S}$ besides its closure.

An algebraic structure–a ring, an ideal or even a module–is often susceptible to smoothing processes that enhance their properties. One major process is the *integral closure* of the structure. This often enable them to support new constructions, including analytic ones. In the case of algebras, the divisors acquire a group structure, the cohomology tends to slim down. To make this more viable, multiplicity theory–broadly seen as the assignment of measures of size to an structure–must be built up with the introduction of new families of degree functions suitable for tracking the processes through their complexity costs. The synergy between these two regions is illustrated in the diagram:



The overall goal is to describe the developments leading to this picture, and, hopefully, of setting the stage for further research.

There is an obvious organization for the several problems that arise. Without emphasizing relationships one has:

- □ *Membership Test*: $f \in \overline{\mathbf{S}}$?
- □ *Completeness Test*: $\mathbf{S} = \overline{\mathbf{S}}$?
- □ *Construction Task*: $\mathbf{S} \rightsquigarrow \overline{\mathbf{S}}$?
- □ *Complexity Cost*: $\mathrm{cx}(\mathbf{S} \rightsquigarrow \overline{\mathbf{S}})$?

None of these problems has an optimal solution yet, that would permit check offs, certainly not for the various structures treated. In addition to these issues, predicting properties of $\overline{\mathbf{S}}$ from $\mathbf{S}$–such as number of generators and their degrees as the case may be–becomes rather compelling and can also be stated in terms of complexities.

Given an affine algebra $\mathbf{S}$ over a constructible field, the usual approach to the construction of $\overline{\mathbf{S}}$ starting from $\mathbf{S}$, or from some $\mathbf{S}_0$, involves a procedure P which when applied to $\mathbf{S}$, produces an extension

$$\mathbf{S} \subset P(\mathbf{S}) \subset \overline{\mathbf{S}},$$

properly containing $\mathbf{S}$, unless it is already the desired closure.

The presence of Noetherian conditions will guarantee the existence of an integer r such that $P^r(\mathbf{S}) = \overline{\mathbf{S}}$. Interestingly though, under appropriate control conditions obtained by judicious choices for P, it is possible to estimate the order r from $\mathbf{S}_0$. More delicate issues are those regarding the number of generators of $P^i(\mathbf{S})$ (their degrees in the graded case):

□ Determine r such that $P^r(\mathbf{S}) = \overline{\mathbf{S}}$.

□ Determine the number of (module or algebra) generators of $\overline{\mathbf{S}}$.

□ In the graded case, determine the degrees of the (module or algebra) generators of $\overline{\mathbf{S}}$.

If P allows for answering the second and third problems for $P(\mathbf{S})$ in terms of $\mathbf{S}$, then we would achieve a good understanding of the whole closure process.

There has been progress throughout the whole front of problems outlined above. For the other algebraic structures we treat–ideals and modules–the picture is not this rosy. While each of them–to wit, the construction or understanding of the integral closure of an ideal or module $\mathbf{S}$–can be converted into another involving an associated algebra, the process is not robust under two important criteria: The conversion process makes the complexity intractable and brings no understanding about the nature of $\overline{\mathbf{S}}$. Instead we must look for direct pathways, sensitive to the nature of the ideal or module.

Our model is Section 1, where for a reduced affine algebra A we focus on:

- Effective methods to compute the integral closure $\overline{A}$.
- Development and analysis of processes $P(\cdot)$ that associate to an intermediate extension $A \subset B \subset \overline{A}$, another extension $B \subset P(B) \subset \overline{A}$, guaranteed to be distinct from B when $B \neq \overline{A}$.
- Establishing *a priori* bounds for the 'order' of P, that is for the integer n such that $P^n(A) = \overline{A}$.
- Estimation of the number of generators (and of their degrees in the graded case) of $\overline{A}$.
- Basic treatment of the tracking number of an algebra.

Section 2 dealing with the integral closure of ideals might at first resemble a special case of the problems treated in the previous section. Thus, given an ideal I of the normal domain (for simplicity) A, the integral closure $\overline{I}$ of the ideal is to be found as the degree 1 component of the algebra

$$\overline{R[It]} = R + \boxed{\overline{I}t} + \overline{I^2}t^2 + \cdots,$$

so to find $\overline{I}$ passes through the construction of $\overline{R[It]}$! One would rather have direct constructions $I \rightsquigarrow \overline{I}$ taking place entirely in R. The difficulty of realizing this lies in the absence of *conductors*, such as Jacobian ideals in the algebra case: Given A by generators and relations (at least in characteristic zero) the Jacobian ideal J of A has the property

$$J \cdot \overline{A} \subset A,$$

in other words, $\overline{A} \subset A : J$. This fact lies at the base of all current algorithms to build $\overline{A}$. There is no known corresponding *annihilator* for $\overline{I}/I$. The main topics here are:

- Properties of integrally closed ideals.
- Monomial ideals, when the theory is fairly complete and efficiently implemented already.
- Computation of multiplicities.
- Normalization of ideals which may be considered one of the main problems of the theory, particularly the examination of certain conjectures pointing to a decisive role of the Hilbert polynomial in controlling the computation.

Part of Section 2 is a treatment of integral closure for modules and some algebras arising from them. There are two threads running in parallel: First, the emphasis on techniques that seek to assemble such issues with those techniques already discussed in previous sections on algebras and ideals. There are also novel phenomena to modules with a concomitant development appropriate methods.

1 Integral Closure of Algebras

The main theme of this chapter will be a reduced affine algebra A over a field k and its relationship to its integral closure $\overline{A}$. Setting up processes to construct $\overline{A}$, analyzing their complexities and making predictions about properties and descriptions of $\overline{A}$, constitute a significant area of questions, with individualized issues but also with an array of related connections. Those links are the main focus of our discussion.

We develop frameworks that allow for discussions of various complexities associated with the construction of $\overline{A}$:

$$A = k[x_1,\ldots,x_n]/I \stackrel{\mathbb{P}}{\mapsto} \overline{A} = k[y_1,\ldots,y_m]/J,$$

where $\mathbb{P}$ is some algorithm. The process is often characterized by iterations of a basic procedure P outputting integral, rational extensions of the affine ring A terminating at its integral closure $\overline{A}$:

- $A = A_0 \mapsto A_1 \mapsto A_2 \mapsto \cdots \mapsto A_n = \overline{A}$

Short of a direct description of the integral closure of A by a single operation-a situation achieved in some cases-the construction of $\overline{A}$ is usually achieved by a 'smoothing' procedure: An operation P on affine rings with the properties

- $A \subset P(A) \subset \overline{A}$;
- If $A \neq \overline{A}$ then $A \neq P(A)$.

The general style of the operation P is of the form

$$A \rightsquigarrow I(A) \rightsquigarrow P(A) = \mathrm{Hom}_A(I(A), I(A)),$$

where $I(A)$ is an ideal somewhat related to the conductor of A. The ring

$$P(A) = \mathrm{Hom}_A(I(A), I(A))$$

is called the *idealizer of* $I(A)$. Two examples of such methods, using Jacobian ideals, are featured in [Va91b] and [Jo98], respectively: Let J be the Jacobian ideal of A. The corresponding smoothing operations are the following:

(i) $P(A) = \mathrm{Hom}_A(J^{-1}, J^{-1})$

(ii) $P(A) = \mathrm{Hom}_A(\sqrt{J}, \sqrt{J})$

The 'order' of the construction is the smallest integer n such that $P^n(A) = \overline{A}$. The 'cost' of the computation $C(\overline{A})$ however will consist of $\sum_{i=1}^{n} c(i)$, where $c(i)$ is the complexity of the operation P on the data set represented by $P^{i-1}(A)$.

Obviously, in a Gröbner basis setting, different iterations of P may carry noncomparable costs. This holds true particularly if each iteration uses its own local variables.

Almost irresponsibly one could define the astronomical complexity of a smoothing operation by the order of P:

$$\boxed{n = C_P(\overline{A}).}$$

It is not yet clear what significance is carried by this number. The fact however is that P usually acts not on the full set $B(A)$ of integral birational extensions of A,

$$P : B(A) \mapsto B(A),$$

but also on much smaller subsets of extensions (containing $\overline{A}$)

$$P : B_0(A) \mapsto B_0(A).$$

We will discuss the following issues:

- Descriptions of several such P.
- How long these chains might be?
- How long is the description of $\overline{A}$ in terms of a description of A?

The first subsection is a compiling of several resources that appear in the actual construction of integral closures of rings. To describe the results of the next subsections, we introduce some notation and terminology. The two most basic invariants of an affine ring A (assumed reduced, equidimensional throughout) are its dimension $\dim A = d$ and its 'degree', the rank of A with respect to one of its Noether normalizations $k[x_1, \ldots, x_d]$. If k has characteristic zero and A is a standard graded algebra, we can find an embedding

$$S = k[x_1, \ldots, x_{d+1}]/(f) \hookrightarrow A,$$

with f monic in x_{d+1}, where $\deg f = \deg(A)$. In the non graded case, pick f of least possible total degree. Observe

$$\overline{S} = \overline{A},$$

so we could always assume that A is a hypersurface ring. Often however, we may want to use information on A that is not shared by S.

In subsection 2 we will describe an approach to the issue of complexity by developing a setting for analyzing the efficiency of algorithms that compute the integral closure of affine rings. It gives quadratic (cubic in the non-homogeneous case) multiplicity based but dimension independent bounds for the number of passes any construction of a broad class will make, thereby leading to the notion of astronomical complexity.

Unlike the treatment of subsection 4, that seeks to estimate the complexity of $\overline{A}$ in terms of the number of generators that will be required to give a presentation of $\overline{A}$ (i.e. its *embedding dimension*), here we are going to estimate the number of passes by any method that progressively builds $\overline{A}$ by taking larger integral extensions.

Here we will argue that while the finite generation of $\overline{A}$ as an A-module guarantees that chains of integral extensions,

$$A = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subset \overline{A},$$

are stationary, there are no bounds for n if the Krull dimension of A is at least 2. We show that if the extensions A_i are taken satisfying the S_2-condition of Serre, then n can be bound by data essentially contained in the Jacobian ideal of A (in characteristic zero at least).

A preferred approach to the computation of the integral closure should have the following general properties:

- All calculations are carried out in the same ring of polynomials. If the ring A is not Gorenstein, it will require one Noether normalization.
- It uses the Jacobian ideal of A, or of an appropriate hypersurface subring, only theoretically to control the length of the chains of the extensions.
- There is an explicit quadratic bound (cubic in the non-homogeneous case) on the multiplicity for the number of passes the basic operation has to be carried out. In particular, and surprisingly, the bound is independent of the dimension.
- It can make use of known properties of the ring A.

It will thus differ from the algorithms proposed in either [Jo98] or [Va91b], by the fact that it does not require changes in the rings for each of its basic cycles of computation. The primitive operation itself is based on elementary facts of the theory of Rees algebras. To enable the calculation we will introduce the notion of a *proper construction* and discuss instances of it.

Key to our discussion here is the elementary, but somehow surprising, observation that the set $S_2(A)$ of extensions with the condition S_2 of Serre between an equidimensional, reduced affine algebra A and its integral closure $\overline{A}$ satisfy the *ascending* chain condition inherited from the finiteness of $\overline{A}$ over A, but the *descending* condition as well.

The explanation requires the presence of a Gorenstein subalgebra $S \subset A$, over which A is birational and integral. If A is not Gorenstein, S is obtained from a Noether normalization of A and the theorem of the primitive element. One shows (Theorem 1.29) that there is a inclusion reversing one-one correspondence between $S_2(A)$ and a subset of the set $D(\mathfrak{c})$ of divisorial ideals of S that contain the conductor $\mathfrak{c}$ of $\overline{A}$ relative to S. Since $\mathfrak{c}$ is not accessible one uses the Jacobian ideal as an approximation in order to determine the maximal length of chains of elements of $S_2(A)$. In Corollary 1.31, it is shown that if A is a standard graded algebra over a field of characteristic zero, and of multiplicity e, then any chain in $S_2(A)$ has at most $(e-1)^2$ elements. In the non-homogeneous case of a hypersurface ring defined by an equation of degree e, the best bound we know is $e(e-1)^2$.

Next we describe a process to create chains in $S_2(A)$. Actually the focus is on processes that produce shorter chains in $S_2(A)$ and its elements are amenable to computation. A standard construction in the cohomology of blowups provides chains of length at most $\left\lceil \frac{(e-1)^2}{2} \right\rceil$ (correspondingly, $\frac{e(e-1)^2}{2}$, in the non-homogeneous case).

Subsection 3 is an abstract treatment of the length of divisorial extensions between a graded domain and its integral closure, according to [DV3]. It does not improve on the bounds already discussed, for characteristic zero, nevertheless in arbitrary characteristics, it imposes absolute bounds.

In subsection 4, we will discuss, following [UV3], the problem of estimating the number of generators the integral closure $\overline{A}$ of an affine domain A may require. This number, and the degrees of the generators in the graded case, are major measures of costs of the computation. We discuss current developments on this question for various kinds of algebras, particularly algebras with a small singular locus. At the worst, these estimates have the double exponential shape of Gröbner bases computations.

1.1 Normalization Toolbox

We will assemble in this section some resources that intervene in any discussion of the integral closure of an algebra. Most can be found in one of [BH93], [Ma86], or [Va98b], but for convenience will be re-assembled here.

Noether Normalization Module

In nonzero characteristics, when the theorem of the primitive element may fail to apply, the following result is often useful.

Proposition 1.1 *Let* $A = R[y_1, \ldots, y_n]$ *be an integral domain finite over the subring* R, *and let* $\text{rank}_R(A) = e$. *Let* K *be the field of fractions of* R *and define the field extensions*

$$F_0 = K, \quad F_i = K[y_1, \ldots, y_i], \quad i = 1 \ldots n.$$

Then the module

$$E = \sum R y_1^{j_1} \cdots y_n^{j_n}, \quad 0 \le j_i < [F_i : F_{i-1}]$$

is R*-free of rank* e.

Proof. As the rank satisfies the equality $e = \prod_{i=1}^n r_i$, $r_i = [F_i : F_{i-1}]$, e is the number of 'monomials' $y_1^{j_1} \cdots y_n^{j_n}$. Their linear independence over R is a simple verification. □

Remark 1.2 If B is a finitely generated standard graded algebra over the field k,

$$B = k \oplus B_1 \oplus \cdots = k[B_1],$$

the Noether normalization process can described differently. The argument is that establishing the Hilbert polynomial associated to B. More precisely, if B has Krull dimension 0, $A = k$ is a Noether normalization. If $\dim B \geq 1$, one begins by picking a homogeneous element $z \in B_+$ which is not contained in any of the minimal prime ideals of B. When k is infinite, the usual prime avoidance finds $z \in B_1$. If k is finite however one argues differently: If $P_1, \ldots, P_n$ are the minimal primes of B, we must find a form h which is not contained in any P_i. Suppose the best one can do is to find a form $f \notin \bigcup_{i \leq s} P_i$, for s maximum $< n$. This means that $f \in P_{s+1}$. If $g \in \bigcap_{i \leq P_i} \setminus P_{s+1}$ is a form (easy to show), then $h = f^a + g^b$, where $a = \deg g$, $b = \deg f$ is a form not contained in a larger subset of the associated primes, contradicting the definition of s.

Krull-Serre Normality Criterion

Throughout we will assume that A is a reduced Noetherian ring, and denote by K its total ring of fractions. For each prime ideal $\mathfrak{p}$, the localization $A_\mathfrak{p}$ can be naturally identified to a subring of K so that intersections of such subrings may be considered. A itself may be represented as

$$A = \bigcap_{\mathfrak{p}} A_\mathfrak{p}, \quad \text{grade } \mathfrak{p}A_\mathfrak{p} \leq 1.$$

It is then easy to describe, using these localizations, when A is integrally closed:

Proposition 1.3 (Krull-Serre) *Let A be a reduced Noetherian ring. The following conditions are equivalent:*

(a) *A is integrally closed.*

(b) *For each prime ideal $\mathfrak{p}$ of A associated to a principal ideal, $A_\mathfrak{p}$ is a discrete valuation domain.*

(c) *A satisfies:*

  (i) *The condition S_2: For each prime ideal $\mathfrak{p}$ associated to a principal ideal,* $\dim A_\mathfrak{p} \leq 1$.

  (ii) *The condition R_1: For each prime ideal $\mathfrak{p}$ of codimension* 1, *$A_\mathfrak{p}$ is a discrete valuation domain.*

Remark 1.4 If A is an integral domain with a finite integral closure $\overline{A}$ and f is a nonzero element in ann $(\overline{A}/A)$, these conditions can be simply stated as: (i) (f) has no embedded primes, and (ii) for each minimal prime $\mathfrak{p}$ of (f) $R_\mathfrak{p}$ is a discrete valuation ring. We leave the proof as an exercise.

Canonical Module and S_2-ification

We are going to use a notion of "canonical ideal" of a reduced algebra A that is appropriate for our treatment of normality. A general reference is [BH93, Chapter 4]; we will also make use of [Va98b, Chapter 6].

Definition 1.5 Let A be a reduced affine algebra. A *weak* canonical ideal of A is an ideal L of A with the following properties:

(i) L satisfies the condition S_2 of Serre.

(ii) For any ideal I that satisfies the condition S_2 of Serre, the canonical mapping

$$I \mapsto \mathrm{Hom}_A(\mathrm{Hom}_A(I, L), L)$$

is an isomorphism.

There are almost always many choices for L, but once fixed we will denote it by ω_A, or simply ω if no confusion can arise. Here are some of its properties:

Proposition 1.6 *Let A be a reduced affine algebra with a canonical ideal ω.*

(i) *For any ideal I,*

$$\mathrm{Hom}_A(\mathrm{Hom}_A(I, \omega_A), \omega_A))$$

is the smallest ideal containing I that satisfies the condition S_2.

(ii) *A satisfies the condition S_2 if and only if*

$$A = \mathrm{Hom}_A(\omega, \omega).$$

Definition 1.7 If A is a Noetherian ring of total ring of fractions K, the smallest finite extension $A \to A' \subset K$ with the S_2 property is called the S_2-ification of A.

An extension with this property is unique (prove it!). It may not always exist, even when A is an integral domain. In most of the rings we discuss however its existence is guaranteed simply from the theory of the canonical module. The extension

$$A \subset \mathrm{Hom}_A(\omega, \omega) \subset \overline{A}$$

is the S_2*-ification* of A.

Sometimes one can avoid the direct use of canonical modules, as in the following:

Proposition 1.8 *Let R be a Noetherian domain and let A be a torsionfree finite R-algebra. If $R_\mathfrak{p}$ is a Gorenstein ring for every prime ideal of height 1, then $A^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(A, R), R)$ is the S_2-ification of A.*

Proof. Consider the natural embedding

$$0 \to A \longrightarrow A^{**} \longrightarrow C \to 0.$$

For each prime $\mathfrak{p} \subset R$ of height 1, $A_\mathfrak{p}$ is a reflexive module since it is torsionfree over the Gorenstein ring $R_\mathfrak{p}$. This shows that C has codimension at least two. We have

$$A' = \bigcap A_\mathfrak{p} \subset \bigcap A_\mathfrak{p}^{**},$$

over all such primes of R. The second module is just A^{**}, from elementary properties of dual modules. Since each $A_{\mathfrak{p}}$ contains A^{**}, we have $A' = A^{**}$. □

Two lazy ways to find an integral extension of a ring with the property S_2, which do not necessarily involve canonical modules, arise in the following manner. The proofs are left as exercises.

Proposition 1.9 *Let R be a Noetherian domain with the condition S_2 and let A be an integral, rational extension of R. Let $\mathfrak{c} = \mathrm{Hom}_R(A,R)$ be the conductor ideal of A over R. Then $B = \mathrm{Hom}_R(\mathfrak{c},\mathfrak{c})$ is an integral, rational extension of A with the condition S_2.*

Proposition 1.10 *Let R be a Noetherian domain with the condition S_2 and let A be an integral domain containing R and R-finite. Then*

$$B = \mathrm{Hom}_A(\mathrm{Hom}_R(A,R), \mathrm{Hom}_R(A,R))$$

is an integral, rational extension of A with the condition S_2.

Remark 1.11 One important case this result applies is to Rees algebras of equimultiple ideals. If R is a Gorenstein ring and I is an equimultiple ideal and J is a complete intersection that is a reduction of I, if we set $S = R[Jt]$ and $A = R[It]$, S is a Cohen-Macaulay ring which is Gorenstein in codimension 1 (proof left to the reader) and $\mathrm{Hom}_S(\mathrm{Hom}_S(A,S),S)$ is the S_2-ification of A.

Conductors and Affine Algebras

Let A be a reduced Noetherian ring and denote by $\overline{A}$ its integral closure in the total ring of fractions K. The *conductor* of A is the ideal

$$\mathfrak{c}(A) = \mathrm{ann}\ (\overline{A}/A) = \mathrm{Hom}_A(\overline{A},A).$$

We note that

$$\overline{A} = \mathrm{Hom}_A(\mathfrak{c}(A),\mathfrak{c}(A)),$$

since $\mathfrak{c}(A)$ is an ideal of $\overline{A}$. This shows that there is an equivalence of access $\mathfrak{c}(A) \Leftrightarrow \overline{A}$.

Access to $\mathfrak{c}(A)$ being usually only found once $\overline{A}$ has been determined, one may have to appeal to what was flippantly termed in [Va98b, Chapter 6] *semi-conductors* of A: nonzero subideals of $\mathfrak{c}(A)$. Let us denote one of such nonzero ideal as C and assume that it contains regular elements.

Let us give a scenario for a description of $\overline{A}$:

Proposition 1.12 *If* height $(C) \geq 2$, *then* $\overline{A} = C^{-1}$.

Proof. First, observe that for each prime ideal $\mathfrak{p}$ of A of codimension at most one, $C \not\subset \mathfrak{p}$, and therefore $A_{\mathfrak{p}} = \overline{A}_{\mathfrak{p}}$. This implies that

$$A_{\mathfrak{p}} = \bigcap_{\mathfrak{q}} \overline{A}_{\mathfrak{q}},$$

where $\mathfrak{q}$ runs over the primes of $\overline{A}$ lying over $\mathfrak{p}$, which in turn shows that

$$\overline{A} = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}.$$

We can now prove our assertion. Since $(C^{-1})_{\mathfrak{p}} = A_{\mathfrak{p}}$, C^{-1} is contained in all these intersections and therefore is contained in $\overline{A}$. On the other hand, for any regular element $b \in \overline{A}$, from $Cb \subset A$, we get $C^{-1}b^{-1} \supset A$, that is $b \in C^{-1}$. Finally, we observe that $\overline{A}$ is generated by regular elements as an A-module. □

The Jacobian Ideal

One of the most useful of such (conductor) ideals is the Jacobian ideal of an affine algebra A. Suppose that

$$A = k[x_1, \ldots, x_n]/I,$$

where I is an unmixed ideal of codimension g. If $I = (f_1, \ldots, f_m)$, the *Jacobian ideal* $J(A)$ is the image in A of the ideal generated by the $g \times g$ minors of the Jacobian matrix

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}.$$

There are relative versions one of which we are going to make use of.

The relationship between Jacobian ideals and conductors was pointed out in [No50] and strengthened in [LS81].

Theorem 1.13 *Let A be an affine domain and let S be one of its Noether normalizations. Suppose the field of fractions of A is a separable extension of the field of fractions of S. Let $A = S[x_1, \ldots, x_n]/P$ be a presentation of A and let J_0 denote the Jacobian ideal of A relative to the variables $x_1, \ldots, x_n$. If $\overline{A}$ is the integral closure of A then $J_0 \cdot \overline{A} \subset A$. In particular*

$$A = A : J_0$$

if and only if A is integrally closed.

Theorem 1.14 *Let A be an affine domain over a field of characteristic zero. If $J(A)$ and $\mathfrak{c}(A)$ denote the Jacobian ideal and the conductor of A, then*

$$\boxed{J(A) \subset \mathfrak{c}(A).} \qquad (1)$$

Algebras with the Condition R_1 and One-Step Normalization

There are very few general 'explicit' descriptions of the integral closure of an algebra. One that comes close to meeting this requirement occurs from the computation of a very special module of syzygies.

Theorem 1.15 *Let A be a reduced affine algebra that satisfies the condition R_1 of Serre. There exist two elements f and g in the conductor ideal $\mathfrak{c}(A)$ such that*

$$L = \{(a,b) \in A \times A \mid af - bg = 0\} \rightsquigarrow \overline{A} = \{a/g \mid (a,b) \in L\} = Af^{-1} \bigcap Ag^{-1}.$$

Proof. By assumption, the conductor ideal $\mathfrak{c}(A)$ has codimension at least 2; pick $f, g \in \mathfrak{c}(A)$, both regular elements of A, generating an ideal of codimension 2 (like in the converse of Krull's principal ideal theorem). We note that the module defined by the right hand side of (2) is

$$A :_K (f,g).$$

Since f and g are contained in the conductor of A, we have

$$\overline{A} \subset A :_K (f,g).$$

To prove the reverse containment, we use that

$$\overline{A} = \bigcap_{\mathfrak{p}} \overline{A}_{\mathfrak{p}}, \quad \mathfrak{p} \text{ prime ideal of } A \text{ of codimension } 1.$$

Note that each of these localizations is a semi-local ring whose maximal ideals correspond to the prime ideals of $\overline{A}$ lying above $\mathfrak{p}$. □

Remark 1.16 If A is a reduced equidimensional affine algebra over a field of characteristic zero with the condition R_1 then the elements f, g can be chosen in the Jacobian ideal $J(A)$.

Corollary 1.17 *If A is an algebra as above, satisfying the condition R_1 of Serre, then $\overline{A}$ is the S_2-ification of A, that is, the smallest extension $A \subset B \subset K$ that satisfies the condition S_2.*

This formulation permits other representations for $\overline{A}$: If the Jacobian ideal $J(A)$ has codimension at least 2, and $S \subset A$ is a hypersurface ring over which A is integral, then

$$\overline{A} = \mathrm{Hom}_S(\mathrm{Hom}_S(A,S),S).$$

Localization and Normalization

We treat a role that localization plays in the construction of the normalization of certain rings. The setting will be that of an affine domain A over a field k with a Noether normalization

$$R = k[x_1, \ldots, x_d] \hookrightarrow A.$$

To start, consider the following recast of the Krull-Serre criterion:

Proposition 1.18 *Let A be a Noetherian domain and let f, g be a regular sequence of R. If $\overline{A}$ is the integral closure of A, then*

$$\overline{A} = \overline{A}_f \cap \overline{A}_g.$$

The assertion holds for arbitrary Noetherian domains although we will only use it for affine domains. When f and g are taken as a regular sequence in R, we still have the equality $\overline{A} = \overline{A}_f \cap \overline{A}_g$. Since $\overline{A_f} = \overline{A}_f$ (and similarly for g), there are R-submodules $C = (c_1, \ldots, c_r)$ and $D = (d_1, \ldots, d_s)$ such that $C_f = \overline{A}_f$ and $D_f = \overline{A}_g$.

Proposition 1.19 *If f, g is a regular sequence in R, setting $B = (c_1, \ldots, c_r, d_1, \ldots, d_s)$, one has a natural isomorphism*

$$\boxed{B^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(B, R), R) \cong \overline{A}.} \qquad (2)$$

Proof. Consider the inclusion $B \subset \overline{A}$. Since $\overline{A}$ satisfies the condition S_2, it will also satisfy S_2 relative to the subring R. This means that the bidual of B will be contained in $\overline{A}$, $B^{**} \hookrightarrow \overline{A}$. To prove they are equal, it will suffice to show that for each prime ideal $\mathfrak{p} \subset R$ of codimension 1, $B_{\mathfrak{p}}^{**} = \overline{A}_{\mathfrak{p}}$. But from our choices of f and g, $B_{\mathfrak{p}} = \overline{A}_{\mathfrak{p}}$. □

A special case is when f belongs to the conductor of A, since $A_f = \overline{A}_f$ already, so that we simply take $C = A$. For simplicity we denote $B = A(f, g)$, and the special case by $A(1, g)$. As an application, let us consider a reduction technique that converts the problem of finding the integral closure of a standard graded algebra into another involving finding the integral closure of a lower dimension affine domain (but not graded) and the computation of duals.

Proposition 1.20 *Suppose as above that f lies in the conductor of A and g is a form of degree 1 (in particular, $d \geq 2$)). The integral closure of A is obtained as the R-bidual of a set of generators of the integral closure of $A/(g-1)$.*

Proof. The localization A_f has a natural identification with the ring $A_g = S[T, T^{-1}]$, where S is the set of fractions in A_g of degree 0 and T is an indeterminate. Further, as it is well-known (see [Ei95, Exercise 2.17]), $S \cong A/(g-1)$. □

R_1-ification

We consider two constructions, considered in [Va91b] and [Jo98] (using [GRe84, p. 127]), respectively; our presentation is also informed by [Mat0]. Its tests provide proactive normality criteria:

Theorem 1.21 *Let A be a reduced equidimensional affine ring over a field of characteristic zero, and denote by J its Jacobian ideal. A is integrally closed in the following two cases:*

(i) $A = \mathrm{Hom}_A(J^{-1}, J^{-1})$;

(ii) $A = \mathrm{Hom}_A(\sqrt{J}, \sqrt{J})$.

Proof. Let $\mathfrak{p}$ be a prime associated to a principal ideal (f) generated by a regular element, and consider $A_{\mathfrak{p}}$. Both conditions (i) and (ii) are maintained by localizing. We will then use the symbol A for $A_{\mathfrak{p}}$.

(i) The first idealizer ring condition can be rewritten as $A = (J \cdot J^{-1})^{-1}$. This means that the ideal $J \cdot J^{-1} = A$, and therefore J is a principal ideal. By [Li69a] $J = A$, and A is a discrete valuation domain by the Jacobian criterion.

(ii) Suppose $A \neq \overline{A}$. According to Theorem 1.13, J is contained in the conductor ideal of A. Let $\mathfrak{p}$ be an associated prime of $\overline{A}/A$. Pick $x \in \overline{A} \setminus A$ such that $A : x = \mathfrak{p}$. Since $J \subset \mathfrak{p}$, we also have $\sqrt{J} \subset \mathfrak{p}$. Let

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

be an equation of integral dependence of x over A. For any $y \in \sqrt{J}$, multiplying the equation by y^n we get that $(xy)^n \in \sqrt{J}$. Since $xy \in A$, we obtain $xy \in \sqrt{J}$, and therefore $x \in A$ by the hypothesis. □

Unlike the operation defined earlier, each application of P takes place in a different ring, as the new Jacobian ideal has to be assembled from a presentation of the algebra $P(B)$.

Remark 1.22 Observe that the criterion of normality expressed in the equality

$$\mathrm{Hom}_A(J^{-1}, J^{-1}) = A,$$

does not require A to satisfy the S_2 condition of Serre (as stated originally in [Va91b]). We note that one way to configure the endomorphism ring is simply as follows. Let $x \in J$ be a regular element, and denote by $(x) :_A J$ the ideal quotient in A. Since $J^{-1} = ((x) :_R J)x^{-1}$,

$$\mathrm{Hom}_A(J^{-1}, J^{-1}) = \mathrm{Hom}_A((x) : J, (x) : J).$$

Example 1.23 Let us illustrate with one example how the two methods differ markedly in the presence of the condition R_1 of Serre. Let A be an affine domain over a field of characteristic zero, and let J be its Jacobian ideal. Suppose height $J \geq 2$ (the R_1 condition). As we have discussed,

$$\overline{A} = J^{-1} \subset \text{Hom}_A(J^{-1}, J^{-1}) \subset \overline{A}.$$

Consider now the following example. Let n be a positive integer, and set

$$A = \mathbb{R}[x,y] + (x,y)^n\mathbb{C}[x,y],$$

whose integral closure is the ring of polynomials $\mathbb{C}[x,y]$. Note that $A_x = \mathbb{C}[x,y]_x$ and $A_y = \mathbb{C}[x,y]_y$. This means that $\sqrt{J}$ is the maximal ideal

$$M = (x,y)\mathbb{R}[x,y] + (x,y)^n\mathbb{C}[x,y].$$

It is clear that

$$\text{Hom}_A(M,M) = \mathbb{R}[x,y] + (x,y)^{n-1}\mathbb{C}[x,y].$$

It will take precisely n passes of the operation to produce the integral closure. In particular, neither the dimension ($d = 2$), nor the multiplicity ($e = 1$) play any role.

Presentation of $\text{Hom}_A(J,J)$**.** The Jacobian algorithms require presentations of $\text{Hom}_A(J,J)$ as A-algebras, where J is some ideal obtained from the Jacobian ideal of the algebra A. More precisely, the passage from J_n to J_{n+1} requires a set of generators and relations for $A_{n+1} = \text{Hom}(J_n, J_n)$. In case $w_1, \ldots, w_s$ is a set of generators of A_{n+1} as an A-module, a set of generators and relations of A_{n+1} as a k-algebra can be obtained from the following observation ([Ca84]):

$$A_{n+1} = k[x_1, \ldots, x_m, W_1, \ldots, W_s]/(I, H_n, K_n),$$

where H_n are the generators for the linear syzygies of the w_i over A,

$$a_1W_1 + \cdots + a_sW_s,$$

and K_n is the set of quadratic relations

$$H_{ij} = W_iW_j - \sum_{k=1}^{s} a_{ijk}W_k,$$

that express the algebra structure of A_{n+1}.

1.2 Divisorial Extensions of an Affine Algebra

We treat here an approach of [Va0] to the analysis of the construction of the integral closure $\overline{A}$ of an affine ring A. It will apply to an examination of the complexity of some of the existing algorithms ([Jo98], [Va91b]). These algorithms put their trust blindly on the Noetherian condition, without any *a priori* numerical certificate of termination. We remedy this for any algorithm that uses a particular class of extensions termed *divisorial*. In addition, we analyze in detail an approach to the computation of $\overline{A}$ that is theoretically distinct from the current methods.

Divisorial Extensions of Gorenstein Rings

Throughout we will assume that A is a reduced affine ring and $\overline{A}$ is its integral closure.

Definition 1.24 An integral extension B of A is *divisorial* if $A \subset B \subset \overline{A}$ and B satisfies the S_2 condition of Serre. The set of divisorial extensions of A will be denoted by $\mathcal{S}_2(A)$.

Let $A = k[x_1,\ldots,x_n]/I$ be a reduced equidimensional affine algebra over a field k of characteristic zero, let $R = k[x_1,\ldots,x_d] \subset A$ be a Noether normalization and $S = k[x_1,\ldots,x_d,x_{d+1}]/(f)$ a hypersurface ring such that the extension $S \subset A$ is birational. Denote by J the Jacobian ideal of S, that is the image in S of the ideal generated by the partial derivatives of the polynomial f.

From $S \subset A \subset \overline{S} = \overline{A}$ and [No50] we have that J is contained in the conductor of $\overline{S}$. To fix the terminology, we denote the annihilator of the S-module A/S by $\mathfrak{c}(A/S)$. Note the identification $\mathfrak{c}(A/S) = \mathrm{Hom}_S(A,S)$.

We want to benefit from the fact that S is a Gorenstein ring, in particular that its divisorial ideals have a rich structure. Let us recall some of those. Denote by K the total ring of fractions of S. A finitely generated submodule L of K is said to be *divisorial* if it is faithful and the canonical mapping

$$L \mapsto \mathrm{Hom}_S(\mathrm{Hom}_S(L,S),S)$$

is an isomorphism. Since S is Gorenstein, for a proper ideal $L \subset S$ this simply means that all the primary components of L have codimension 1. We sum up some of these properties in (see [BH93] for general properties of Gorenstein rings and [Va98b, Section 6.3] for specific details on the S_2 condition of Serre):

Proposition 1.25 *Let S be a hypersurface ring as above (or more generally a Gorenstein ring). Then*

(a) *A finitely generated faithful submodule of K is divisorial if and only if it satisfies the S_2 condition of Serre.*

(b) *Let $A \subset B$ be finite birational extensions of S such that A and B have the condition S_2 of Serre. Then $A = B$ if and only if $\mathfrak{c}(A/S) = \mathfrak{c}(B/S)$.*

(c) *If A is an algebra such that $S \subset A \subset \overline{S}$, then*

$$\mathrm{Hom}_S(\mathrm{Hom}_S(A,S),S) = \mathrm{Hom}_S(\mathfrak{c}(A/S),S)$$

is the S_2-closure of A.

Remark 1.26 We will also denote $A^{-1} = \mathrm{Hom}_S(A,S)$, and the S_2-closure of an algebra A by $C(A) = (A^{-1})^{-1}$. Note that one has the equality of conductors $\mathfrak{c}(A/S) = \mathfrak{c}(C(A)/S)$. Actually these same properties of the duality theory over Gorenstein rings hold if S is S_2 and is Gorenstein in codimension at most one.

We now define our two notions and begin exploring their relationship.

Definition 1.27 Let I be an ideal containing regular elements of a Noetherian ring S. The *degree* of I is the integer

$$\deg(I) = \sum_{\text{height } \mathfrak{p}=1} \lambda((S/I)_{\mathfrak{p}}).$$

Definition 1.28 A *proper operation* for the purpose of computing the integral closure of S is a method such that whenever $S \subset A \subsetneq \overline{S}$ is given, the operation produces a divisorial extension B such that $A \subsetneq B \subset \overline{S}$.

The next result highlights the fact that most extensions used in the computation of the integral closure satisfy the descending chain condition.

Theorem 1.29 *Let A be a Gorenstein ring with a finite integral closure $\overline{A}$. Let $S_2(A)$ be the set of extensions $A \subset B \subset \overline{A}$ that satisfy the S_2 condition of Serre and denote by $\mathfrak{c}$ the conductor of $\overline{A}$ over A. There is an inclusion reversing one-one correspondence between that elements of $S_2(A)$ and a subset of divisorial ideals of S containing $\mathfrak{c}$. In particular $S_2(A)$ satisfies the descending chain condition.*

Proof. Any ascending chain of divisorial extensions

$$A \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subset \overline{A}$$

gives rise to a descending chain of divisorial ideals

$$\mathfrak{c}(A_1/A) \supset \mathfrak{c}(A_2/A) \supset \cdots \supset \mathfrak{c}(A_n/A)$$

of the same length by Proposition 1.25(b). But each of these divisorial ideals contain $\mathfrak{c}$, which gives the assertion. □

We collect these properties: Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of $\mathfrak{c}$ and let $U = \bigcup_{i=1}^{n} \mathfrak{p}_i$. There is an embedding of partially ordered sets

$$S_2(A)' \hookrightarrow \text{Ideals}(S_U/\mathfrak{c}_U),$$

where $S_2(A)'$ is $S_2(A)$ with the order reversed. Note that $S_U/\mathfrak{c}_U$ is an Artinian ring.

Of course the conductor ideal $\mathfrak{c}$ is usually not known in advance, or the ring A is not always Gorenstein. In case A is a reduced equidimensional affine algebra over a field of large characteristic we may replace it by a hypersurface subring S with integral closure $\overline{A}$. On the other hand, by [No50], the Jacobian ideal J of S is contained in the conductor of S. We get the less tight but more explicit rephrasing of Theorem 1.29:

Theorem 1.30 *Let S be a reduced hypersurface ring*

$$S = k[x_1, \ldots, x_{d+1}]/(f)$$

over a field of characteristic zero and let J be its Jacobian ideal. Then the integral closure of S can be obtained by carried out in at most $\deg(J)$ *proper operations on S.*

In characteristic zero, the relationship between the Jacobian ideal J of S and the conductor $\mathfrak{c}$ of $\overline{S}$ is difficult to express in detail. In any event the two ideals have the same associated primes of codimension one, a condition we can write as the equality of radical ideals $\sqrt{\mathfrak{c}} = \sqrt{(J^{-1})^{-1}}$.

Corollary 1.31 *Let k be a field of characteristic zero and let A be a standard graded domain over k of dimension d and multiplicity e. Let S be a hypersurface subring of A such that $S \subset A$ is finite and birational. Then the integral closure of A can be obtained after $(e-1)^2$ proper operations on S.*

Proof. Denote $S = k[x_1, \ldots, x_{d+1}]/(f)$, where f is a form of degree $e = \deg(A)$. By Euler's formula, $f \in L = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{d+1}}\right)$. Let then g, h be forms of degree $e-1$ in L forming a regular sequence in $T = k[x_1, \ldots, x_{d+1}]$. Clearly we have that $\deg(g,h)S \geq \deg(J)$. On the other hand, we have the following estimation of ordinary multiplicities

$$\begin{aligned}
(e-1)^2 = \deg(T/(g,h)) &= \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(g,h)_{\mathfrak{P}}) \deg(T/\mathfrak{P}) \\
&\geq \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(g,h)_{\mathfrak{P}}) \\
&\geq \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(f,g,h)_{\mathfrak{P}}) \\
&= \deg((g,h)S) \\
&\geq \deg(\mathfrak{c}),
\end{aligned}$$

as required. □

Note that this is a pessimistic bound, that would be already cut in half by the simple hypothesis that no minimal prime of J is monomial. One does not need the base ring to be a hypersurface ring, the Gorenstein condition will do. We illustrate with the following:

Corollary 1.32 *Let A be a reduced equidimensional Gorenstein algebra over a field of characteristic zero. Let J be the Jacobian ideal of S and let L the corresponding divisorial ideal, $L = (J^{-1})^{-1}$. If L is a radical ideal then $\overline{A}$ is the only proper divisorial extension of A.*

Proof. Let $A \subset B \subset C$ be divisorial extensions of A, and let $K \subset I$ be the conductors of the extensions C and B, respectively. We will show that $K = I$. If $\mathfrak{p}$ be a prime of codimension one that it is not associated to I, $B_{\mathfrak{p}}$ is integrally closed and $B_{\mathfrak{p}} = C_{\mathfrak{p}}$. Thus I and K have the same primary components since they both contain L. □

Let us highlight the boundedness of chains of divisorial subalgebras.

Corollary 1.33 *Let A be a reduced equidimensional standard graded algebra over a field of characteristic zero, and set $e = \deg(A)$. Then any sequence*

$$A = A_1 \subset A_2 \subset \cdots \subset A_n \subset \overline{A}$$

of finite extensions of A with the property S_2 of Serre has length at most $(e-1)^2$.

Non-Homogeneous Algebras

We will now treat affine algebras which are not homogeneous. Suppose A is a reduced equidimensional algebra over a field of characteristic zero, of dimension d. Let

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \hookrightarrow A$$

be a hypersurface ring over which A is finite and birational. The degree of the polynomial f will play the role of the multiplicity of A. Of course, we may choose f of as small degree as possible.

Our aim is to find estimates for the length of chains of algebras

$$S = A_0 \subset A_1 \subset \cdots \subset A_q = \overline{A}$$

satisfying the condition S_2, between S and its integral closure $\overline{A}$. The argument we used required the length estimates for the length of the total ring of fractions of $S/\mathfrak{c}$, where $\mathfrak{c}$ is the conductor ideal of S, ann $(\overline{A}/S)$. Actually, it only needs estimates for the length of the total ring of fractions of $(S/\mathfrak{c})_{\mathfrak{m}}$, where $\mathfrak{m}$ ranges over the maximal ideals of S.

In the homogeneous case, we found convenient to estimate these lengths in terms of the multiplicities of $(S/\mathfrak{c})_{\mathfrak{m}}$; we will do likewise here.

A first point to be made is the observation that we may replace k by $K \cong S/\mathfrak{m}$ and $\mathfrak{m}$ by a maximal ideal $\mathfrak{M}$ of $K \otimes_k A$ lying over it. In other words, we can replace R by a faithfully flat (local) extension R'. The conditions are all preserved in that $S' = K \otimes_k S$ is reduced, $\overline{S'} = K \otimes_k \overline{A}$, the conductor of S extends to the conductor of S', and chains of extensions with the S_2 conditions give like to likewise extensions of K-algebras. Furthermore the length of the total ring of fractions of $R/\mathfrak{c}$ is bounded by the length of the total ring of fractions of $R'/\mathfrak{c}'$.

What this all means is that we may assume that $\mathfrak{m}$ is a rational point of the hypersurface $f = 0$. We may change the coordinates so that $\mathfrak{m}$ corresponds to the actual origin.

Proposition 1.34 *Let $A = k[x_1,\ldots,x_d]$ be the ring of polynomials over the infinite field k and let f,g be polynomials in A vanishing at the origin. Suppose f,g is a regular sequence and* $\deg f = m \leq n = \deg g$. *Then the multiplicity of the local ring $(A/(f,g))_{(x_1,\ldots,x_n)}$ is at most nm^2.*

Proof. Write f as the sum of its homogeneous components,

$$f = f_m + f_{m-1} + \cdots + f_r,$$

and similarly for g,

$$g = g_n + g_{n-1} + \cdots + g_s.$$

We first discuss the route the argument will take. Suppose that g_s is not a multiple of f_r. We denote by R the localization of A at the origin, and its maximal ideal by $\mathfrak{m}$. We observe that $A/(f_r)$ is the associated graded ring of $R/(f)$, and the image of g_s is the initial form g. Thus the associated graded ring of $R/(f,g)$ is a homomorphic image of $A/(f_r,g_s)$. If f_r and g_s are relatively prime polynomials, it will follow that the multiplicity of $R/(f,g)$ will be bounded by $r\cdot s$,

$$\deg R/(f,g) \leq r\cdot s.$$

We are going to ensure that these conditions on f and g are realized for f and another element h of the ideal (f,g). After a linear, homogeneous change of variables (as k is infinite), we may assume that each non-vanishing component of f and of g has unit coefficient in the variable x_d. For that end it suffices to use the usual procedure on the product of all nonzero components of f and g. At this point we may assume that f and g are monic.

Rewrite now

$$\begin{aligned} f &= x_d^m + a_{m-1}x_d^{m-1} + \cdots + a_0, \\ g &= x_d^n + b_{n-1}x_d^{n-1} + \cdots + b_0 \end{aligned}$$

with the a_i, b_j in $k[x_1,\ldots,x_{d-1}]$. Consider now the resultant of these two polynomials with respect to x_d:

$$\operatorname{Res}(f,g) = \det \begin{bmatrix} 1 & a_{m-1} & a_{m-2} & \ldots & a_0 & & & \\ & 1 & a_{m-1} & \ldots & a_1 & a_0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & a_{m-1} & a_{m-2} & \ldots & a_0 \\ 1 & b_{n-1} & b_{n-2} & \ldots & b_0 & & & \\ & 1 & b_{n-1} & \ldots & b_1 & b_0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & b_{n-1} & b_{n-2} & \ldots & a_0 \end{bmatrix}.$$

We recall that $h = \operatorname{Res}(f,g)$ lies in the ideal (f,g). Scanning the rows of the matrix above (n rows of entries of degree at most m, m rows of entries of degree at most n), it follows that $\deg h \leq 2mn$. A closer examination of the distribution of the degrees shows that $\deg h \leq mn$ (see [Wal62, Theorem 10.9]). If h_p is the initial form of h, then clearly h_p and f_r are relatively prime since the latter is monic in x_d, while h_p lacks any term with x_d.

Assembling the estimates, one has

$$\deg R/(f,g) \leq \deg R/(f,h) \leq r \cdot p \leq m \cdot mn = nm^2,$$

as claimed. □

Corollary 1.35 *Let $S = k[x_1, \ldots, x_{d+1}]/(f)$ be a reduced hypersurface ring over a field of characteristic zero, with $\deg f = e$. Then any chain of algebras between S and its integral closure, satisfying the condition S_2, has length at most $e(e-1)^2$.*

Remark 1.36 One must be careful to use this notion of *multiplicity* in these estimates instead of the more classical one. Consider the case of the graded ring $S = k[x^2, x^{2n+1}]$. Its ordinary multiplicity at the origin is 2, while the multiplicity derived from the Noether normalization is $e = n+1$. This is the value to be used in the estimation since there are chains of length $n+1$ extensions between S and its integral closure $k[x]$. (The author thanks W. Heinzer for this observation.)

A Rees Algebra Approach to the Integral Closure

We now introduce and analyze in detail one proper operation which does not use so extensively Jacobian ideals. It is part of another methodology of finding integral closures. Thus far, our approach to passing from one extension to another involved the addition of a batch of new variables. More precisely, given say a setting as above [using the same notation]

$$S \hookrightarrow A, \quad S \neq A,$$

using a presentation of A to obtain its Jacobian ideal, one defines

$$B = \mathrm{Hom}_S(\tilde{J}, \tilde{J}),$$

where $\tilde{J}$ is a divisorial ideal constructed from J. The net result is that iteration of this construction quickly leads to too many variables. It might be advisable to carry out most of the construction in the original ring or even in S. Let us describe such approach. We will refer to it as a *modified* proper operation.

Proposition 1.37 *There is at least one modified proper operation whose iteration leads to the integral closure of S.*

Proof. We assume that we have $S \subsetneq A \subset \overline{S}$ and now we introduce a construction of a divisorial algebra B that enlarges A whenever $A \neq \overline{S}$. Denote $I = \mathfrak{c}(A/S)$. Note that $I = \mathrm{Hom}_S(A, S)$ is the canonical ideal of A, and two possibilities may occur:

(a) $I \cong A$: In this case A is quasi-Gorenstein.

(b) $I \ncong A$

(a) In this case in the method we are going to use for case (b) would come to a halt. To avoid that, we apply the Jacobian method to A to get (if $A \neq \overline{A}$, when the whole process would be halted anyway) an extension $A \neq B$. Observe $B = \operatorname{Hom}_S(\tilde{J},\tilde{J})$ is computed in S and $I = \mathfrak{c}(B)$ as well. In other words, we can discard the variables used to find J. In case again $\mathfrak{c}(B) \cong B$ we have no choice but repeat the previous step.

(b) A natural choice is:

$$B = \bigcup_{n\geq 1} \operatorname{Hom}_S(I^n, I^n).$$

In other words, if $R[It]$ is the Rees algebra of the ideal I, $X = \operatorname{Proj}(R[It])$, then $B = H^0(X, O_X)$. To show that B is properly larger than A it suffices that exhibit this at one of the associated prime ideals of I. Let $\mathfrak{p}$ be a minimal prime of I for which $S_\mathfrak{p} \neq A_\mathfrak{p}$. If $I_\mathfrak{p}$ is principal we have that $I_\mathfrak{p} = tS_\mathfrak{p}$ and therefore $S_\mathfrak{p} = A_\mathfrak{p}$, which is a contradiction. This means that the ideal $I_\mathfrak{p}$ has a minimal reduction of reduction number at least one, that is, $I_\mathfrak{p}^{r+1} = uI_\mathfrak{p}^r$ for some $r \geq 1$. This equation shows that

$$I_\mathfrak{p}^r u^{-r} \subset B_\mathfrak{p},$$

which gives the desired contradiction since $I_\mathfrak{p}^r u^{-r}$ contains $A_\mathfrak{p}$ properly as $I_\mathfrak{p}$ is not principal.

We observe that a bound for the integer r can be found as well: Let $g \in J$ be a form of degree $e-1$ which together with f form a regular sequence of $R = k[x_1, \ldots, x_{d+1}]$.

We recall ([Va98b, Chapter 9]):

Theorem 1.38 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and infinite residue field. If I is an $\mathfrak{m}$-primary ideal whose multiplicity is $e(I)$, then*

$$\boxed{\mathrm{r}(I) \leq d \cdot e(I) - 2d + 1.} \tag{3}$$

Furthermore, if $I \subset \mathfrak{m}^s$, then

$$\boxed{\mathrm{r}(I) \leq d \cdot \frac{e(I)}{s} - 2d + 1.} \tag{4}$$

Thus the reduction number of the ideal $I_\mathfrak{p}$ is bounded by the Hilbert-Samuel multiplicity $e(I_\mathfrak{p})$ of $I_\mathfrak{p}$ minus one. In a manner similar as we have argued earlier,

$$e(I_\mathfrak{p}) \leq e((g_\mathfrak{p})) \leq e(e-1).$$

By taking divisorial closures, we have

$$\mathcal{C}(B) = \mathcal{C}(\mathrm{Hom}_S(I^r, I^r))$$

for $r < e(e-1)$, as desired. □

Suppose in a sequence of computations there is no halt. The proper operation above then leads to a faster approach to the integral closure according to the following observation:

Proposition 1.39 *Let $S \subset A$ be a divisorial extension and let $I = \mathfrak{c}(A/S)$. If B is the divisorial closure of $\bigcup_{n\geq 1} \mathrm{Hom}_S(I^n, I^n)$, then*

$$\deg(\mathfrak{c}(A/S)) \geq \deg(\mathfrak{c}(B/S)) + 2.$$

In particular, for any standard graded algebra of multiplicity e, the number of terms in any chain of algebras obtained in this manner will have at most $\left\lceil \frac{(e-1)^2}{2} \right\rceil$ divisorial extensions.

Proof. Suppose that the degrees of the conductors of A and B differ by 1. This means that the two algebras agree at all localizations of S at height 1 primes, except at $R = S_{\mathfrak{p}}$, and that $\lambda((B/A)_{\mathfrak{p}}) = 1$. We are going to show that for the given choice of how B is built that is a contradiction.

We localize S, A, B, I at $\mathfrak{p}$ but keep a simpler notation $S = S_{\mathfrak{p}}$, etc. Let (u) be a minimal reduction of I, $I^{r+1} = uI^r$. We know that $r \geq 1$ since I is not a principal ideal, as $A \neq B$. We note that $A \subset Iu^{-1} \subset B$. Since B/A is a simple S-module, we have that either $A = Iu^{-1}$ or $B = Iu^{-1}$. We can readily rule out the first possibility. The other leads to the equality

$$Iu^{-1} \cdot Iu^{-1} = Iu^{-1},$$

since B is an algebra. But this implies that

$$Iu^{-1} \cdot I \subset I \subset S,$$

in other words, that $Iu^{-1} \subset \mathrm{Hom}_S(I, S) = A$. □

Remark 1.40 In case A is a non-homogeneous algebra, the cubic estimate of Corollary 1.35 must be considered.

1.3 Tracking Number of an Algebra

We now give a more conceptual explanation of the boundedness of the chains of the previous section based on another family of divisors attached to the extensions ([DV3]). Its added usefulness will include extensions to all characteristics.

Let E be a finitely generated graded module over the polynomial ring $R = k[x_1, \ldots, x_d]$. If $\dim E = d$ denote by $\det_R(E)$ the determinantal divisor of E: If E has multiplicity e,

$$\det(E) = (\wedge^e E)^{**} \cong R[-\delta].$$

Definition 1.41 The integer δ will be called the *tracking number* of E: $\delta = \mathrm{tn}(E)$.

The terminology *tracking number* (or *twist*) refers to the use of integers as locators, or tags, for modules and algebras in partially ordered sets. A forerunner of this use was made in [Va0], when divisorial ideals were employed to bound chains of algebras with the property S_2 of Serre.

Chern Coefficients

We first develop the basic properties of this notion. It may help to begin with these examples:

Example 1.42 If A is a homogeneous domain over a field k, R is a homogeneous Noether normalization and S is a hypersurface ring over which A is birational,

$$R \subset S = R[t]/(f(t)) \subset A$$

($f(t)$ is a homogeneous polynomial of degree e), we have $(\wedge^e S)^{**} = R[-\binom{e}{2}]$. As a consequence $\mathrm{tn}(A) \leq \binom{e}{2}$.

Another illustrative example, is that of the fractionary ideal $I = (x^2/y, y)$. I is positively generated and $I^{**} = R(1/y)$: $\mathrm{tn}(I) = -1$.

The following observation shows the use of tracking numbers to locate the members of certain chains of modules.

Proposition 1.43 *If $E \subset F$ are modules with the same multiplicity that satisfy the S_2 condition of Serre, then $\mathrm{tn}(E) \geq \mathrm{tn}(F)$ with equality only if $E = F$.*

Proof. Consider the exact sequence

$$0 \to E \xrightarrow{\varphi} F \longrightarrow C \to 0,$$

and we will key on the annihilator of C (the *conductor* of F into E). Localization at height 1 prime $\mathfrak{p}$ gives a free resolution of $C_{\mathfrak{p}}$ and the equality

$$\det(F) = \det(\varphi_{\mathfrak{p}}) \cdot \det(E).$$

Therefore $\det(E) = \det(F)$ if and only if C has codimension at least two, which is not possible if $C \neq 0$ since it has depth at least 1. □

Corollary 1.44 *If the modules in the (strict) chain*

$$E_0 \subset E_1 \subset \cdots \subset E_n$$

have the same multiplicity and satisfy the condition S_2 of Serre, then $n \leq \mathrm{tn}(E_0) - \mathrm{tn}(E_n)$.

Proposition 1.45 *If the complex of finitely generated graded R-modules*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is an exact sequence of free modules in each localization $R_{\mathfrak{p}}$ *at height one primes, then* $\text{tn}(B) = \text{tn}(A) + \text{tn}(C)$.

Proof. We break up the complex into simpler exact complexes:

$$0 \to \ker(\varphi) \longrightarrow A \longrightarrow A' = \text{image}(\varphi) \to 0$$

$$0 \to A' \longrightarrow \ker(\psi) \longrightarrow \ker(\psi)/A' \to 0$$

$$0 \to B' = \text{image}(\psi) \longrightarrow C \longrightarrow C/B' \to 0$$

and

$$0 \to \ker(\psi) \longrightarrow B \longrightarrow B' \to 0.$$

We note that by hypothesis, $\text{codim} \ker(\varphi) \geq 1$, $\text{codim} C/B' \geq 2$, $\text{codim} \ker(\psi)/A' \geq 2$, so that we have the equality of determinantal divisors:

$$\det(A) = \det(A') = \det(\ker(\psi)), \text{ and } \det(C) = \det(B').$$

What this all means is that we may assume the given complex is exact.

Suppose $r = \text{rank}(A)$ and $\text{rank}(C) = s$ and set $n = r + s$. Consider the pair $\wedge^r A$, $\wedge^s C$. For $v_1, \ldots, v_r \in A$, $u_1, \ldots, u_s \in C$, pick w_i in B such that $\psi(w_i) = u_i$ and consider

$$v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s \in \wedge^n B.$$

Different choices for w_i would produce elements in $\wedge^n B$ that differ from the above by terms that would contain at least $r + 1$ factors of the form

$$v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots,$$

with $v_i \in A$. Such products are torsion elements in $\wedge^n B$. This implies that modulo torsion we have a well defined pairing

$$[\wedge^r A/\text{torsion}] \otimes_R [\wedge^s C/\text{torsion}] \longrightarrow [\wedge^n B/\text{torsion}].$$

When localized at primes $\mathfrak{p}$ of codimension at most 1 the complex becomes an exact complex of projective $R_{\mathfrak{p}}$-modules and the pairing is an isomorphism. Upon taking biduals and the $\circ$ divisorial composition, we obtain the asserted isomorphism. □

Corollary 1.46 *Let*

$$0 \to A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \to 0$$

be a complex of graded R-modules and homogeneous homomorphisms which is an exact complex of free modules in codimension 1. *Then*

$$\sum_{i=1}^{n} (-1)^i \text{tn}(A_i) = 0.$$

Proposition 1.47 *Let $R = k[x_1, \ldots, x_d]$ and let*

$$0 \to A \longrightarrow B \longrightarrow C \longrightarrow D \to 0,$$

be an exact sequence of graded R-modules and homogeneous homomorphisms. If $\dim B = \dim C = d$, $\operatorname{codim} A \geq 1$ *and* $\operatorname{codim} D \geq 2$, *then* $\operatorname{tn}(B) = \operatorname{tn}(C)$.

Corollary 1.48 *If E is a graded R-module of dimension d, then*

$$\operatorname{tn}(E) = \operatorname{tn}(E/\text{mod torsion}) = \operatorname{tn}(E^{**}).$$

Let A be a homogeneous algebra defined over a field k that admits a Noether normalization $R = k[x_1, \ldots, x_d]$, then clearly $\operatorname{tn}_R(A) = \operatorname{tn}_{R'}(A')$, where K is a field extension of k, $R' = K \otimes_k R$ and $A' = K \otimes_k A$. Partly for this reason, we can always define the tracking number of an algebra by first enlarging the ground field. Having done that and chosen a Noether normalization R that is a standard graded algebra, it will follow that $\operatorname{tn}_R(A)$ is independent of R: the R-torsion submodule A_0 of A is actually an ideal of A whose definition is independent of R.

Calculation Rules

Proposition 1.49 *Let E be a finitely generated graded module over the polynomial ring $R = k[x_1, \ldots, x_d]$. If E is torsionfree over R,* $\operatorname{tn}(E) = e_1(E)$, *the first Chern number of E.*

Proof. Let

$$0 \to \oplus_j R[-\beta_{d,j}] \to \cdots \to \oplus_j R[-\beta_{1,j}] \to \oplus_j R[-\beta_{0,j}] \to E \to 0$$

be a (graded) free resolution of E. The integer

$$e_1(E) = \sum_{i,j} (-1)^i \beta_{i,j}$$

is (see [BH93, Proposition 4.1.9]) the next to the leading Hilbert coefficient of E. It is also the integer that one gets by taking the alternating product of the determinants in the free graded resolution (see Corollary 1.46). □

In general, the connection between the tracking number and the first Hilbert coefficient has to be 'adjusted' in the following manner.

Proposition 1.50 *Let E be a finitely generated graded module over $R = k[x_1, \ldots, x_d]$. If* $\dim E = d$ *and E_0 is its torsion submodule,*

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

then

$$\operatorname{tn}(E) = \operatorname{tn}(E') = e_1(E') = e_1(E) + \hat{e}_0(E_0),$$

where $\hat{e}_0(E)$ is the multiplicity of E_0 if $\dim E_0 = d - 1$, *or* 0 *otherwise.*

Proof. Denote by $H_A(t)$ the Hilbert series of an R-module A (see [BH93, Chap. 4]) and write

$$H_A(t) = \frac{h_A(t)}{(1-t)^d},$$

if $\dim A = d$. For the exact sequence defining E', we have

$$h_E(t) = h_{E'}(t) + (1-t)^r h_{E_0}(t),$$

where $r = 1$ if $\dim E_0 = d-1$, or $r \geq 2$ otherwise. Since

$$e_1(E) = h'_E(1) = h'_{E'}(1) + r(1-t)^{r-1}|_{t=1} h_{E_0}(1),$$

the assertion follows. □

Remark 1.51 This suggests a reformulation of the notion of tracking number. By using exclusively the Hilbert function, the definition could be extended to all finite modules over a graded algebra.

Corollary 1.52 *Let E and F be graded R-modules of dimension d. Then*

$$\operatorname{tn}(E \otimes_R F) = \deg(E) \cdot \operatorname{tn}(F) + \deg(F) \cdot \operatorname{tn}(E).$$

Proof. By Corollary 1.48, we may assume that E and F are torsionfree modules. Let $\mathbb{P}$ and $\mathbb{Q}$ be minimal projective resolutions of E and F, respectively. The complex $\mathbb{P} \otimes_R \mathbb{Q}$ is acyclic in codimension 1, by the assumption on E and F. We can then use Corollary 1.46,

$$\operatorname{tn}(E \otimes_R F) = \sum_{k \geq 0} (-1)^k \operatorname{tn}(\oplus_{i+j=k} \mathbb{P}_i \otimes_R \mathbb{Q}_j).$$

Expanding gives the desired formula. □

Theorem 1.53 *Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and denote by $k[\Delta]$ the corresponding Stanley-Reisner ring. If $\dim k[\Delta] = d$,*

$$\operatorname{tn}(k[\Delta]) = d f_{d-1} - f_{d-2} + f'_{d-2},$$

where f_i denotes the number of faces of dimension i, and f'_{d-2} denotes the number of maximal faces of dimension $d-2$.

Proof. Set $k[\Delta] = S/I_\Delta$, and decompose $I_\Delta = I_1 \cap I_2$, where I_1 is the intersection of the primary components of dimension d and I_2 of the remaining components. The exact sequence

$$0 \to I_1/I_\Delta \longrightarrow S/I_\Delta \longrightarrow S/I_1 \to 0,$$

gives, according to Proposition 1.50,

$$\operatorname{tn}(k[\Delta]) = e_1(k[\Delta]) + \hat{e}_0(I_1/I_\Delta).$$

From the Hilbert function of $k[\Delta]$ ([BH93, Lemma 5.1.8]), we have that $e_1 = d f_{d-1} - f_{d-2}$, while if I_1/I_Δ is a module of dimension $d-1$, its multiplicity is the number of maximal faces of dimension $d-2$. □

Theorem 1.54 *Let $S = k[x_1,\dots,x_n]$ be a ring of polynomials, and $A = S/I$ a graded algebra. For a monomial ordering $>$, denote by $I' = in_>(I)$ the initial ideal associated to I and set $B = S/I'$. Then* $\mathrm{tn}(B) \geq \mathrm{tn}(A)$.

Proof. Let J be the component of I of maximal dimension and consider the exact sequence

$$0 \to J/I \longrightarrow S/I \longrightarrow S/J \to 0.$$

$\dim J/I < \dim A$ and therefore $\mathrm{tn}(A) = \mathrm{tn}(S/J) = e_1(S/J)$. Denote by J' the corresponding initial ideal of J, and consider the sequence

$$0 \to J'/I' \longrightarrow S/I' \longrightarrow S/J' \to 0.$$

Noting that S/I and S/J have the same multiplicity, and so do S/I' and S/J' by Macaulay's theorem, $\dim J'/I' < \dim A$. This means that

$$\mathrm{tn}(S/I') = \mathrm{tn}(S/J') = e_1(S/J') + \hat{e}_0(J'/I') = e_1(S/J) + \hat{e}_0(J'/I') = \mathrm{tn}(A) + \hat{e}_0(J'/I').$$

Example 1.55 Let $A = k[x,y,z,w]/(x^3 - yzw, x^2y - zw^2)$. The Hilbert series of this (Cohen-Macaulay) algebra is

$$H_A(t) = \frac{h_A(t)}{(1-t)^2} = \frac{(1+t+t^2)^2}{(1-t)^2},$$

so that

$$\mathrm{tn}(A) = e_1(A) = h'_A(1) = 18.$$

Consider now the algebra $B = k[x,y,z,w]/J$, where J is the initial ideal of I for the Deglex order. A calculation with *Macaulay2* gives

$$J = (x^2y, x^3, xzw^2, xy^3zw, y^5zw).$$

By Macaulay's Theorem, B has the same Hilbert function as A. An examination of the components of B, gives an exact sequence

$$0 \to B_0 \longrightarrow B \longrightarrow B' \to 0,$$

where B_0 is the ideal of elements with support in codimension 1. By Corollary 1.48,

$$\mathrm{tn}(B) = \mathrm{tn}(B') = e_1(B').$$

At same time, one has the equality of h-polynomials,

$$h_B(t) = h_{B'}(t) + (1-t)h_{B_0}(t),$$

and therefore

$$e_1(B') = e_1(B) + e_0(B_0).$$

A final calculation of multiplicities gives $e_0(B_0) = 5$, and

$$\mathrm{tn}(B) = 18 + 5 = 23.$$

The example shows that $\mathrm{tn}(A)$ is independent of the Hilbert function of the algebra.

Bounding Tracking Numbers

We now describe how the technique of generic hyperplane sections leads to bounds of various kinds. We are going to assume that the algebras are defined over infinite fields.

One of the important properties of the tracking number is that it will not change under hyperplane sections as long as the dimension of the ring is at least 3. So one can answer questions about the tracking number just by studying the 2 dimensional case. The idea here is that tracking number is more or less the same material as e_1 and hence cutting by a superficial element will not change it unless the dimension is to drop below 2.

Proposition 1.56 *Let E be a finitely generated graded module of dimension d over $R = k[x_1,\ldots,x_d]$ with $d > 2$. Then for a general element h of degree one $R' = R/(h)$ is also a polynomial ring, and* $\mathrm{tn}_R(E) = \mathrm{tn}_{R'}(E')$, *where $E' = E/hE$.*

Proof. First we will prove the statement for a torsion free module E. Consider the exact sequence

$$0 \to E \longrightarrow E^{**} \longrightarrow C \to 0.$$

Note that C has codimension at least 2 since after localization at any height 1 prime E and E^{**} are equal. Now for a linear form h in R that is a superficial element for C we can tensor the above exact sequence with $R/(h)$ to get the complex

$$\mathrm{Tor}_1(C,R/(h)) \longrightarrow E/hE \longrightarrow E^{**}/hE^{**} \longrightarrow C/hC \to 0.$$

Now as an R-module C/hC has codimension at least 3, so as an $R' = R/(h)$ module it has codimension at least 2. Also as $\mathrm{Tor}_1(C,R/(h))$ has codimension at least 2 as an R-module, so it is a torsion $R/(h)$-module. Hence we have $\mathrm{tn}_{R'}(E/hE) = \mathrm{tn}_{R'}(E^{**}/hE^{**})$. But E^{**} is a torsion free $R/(h)$-module, so

$$\mathrm{tn}_{R'}(E/hE) = e_1(E^{**}/hE^{**}) = e_1(E^{**}) = \mathrm{tn}_R(E^{**}) = \mathrm{tn}_R(E).$$

To prove the statement for a general R module E, we consider the short exact sequence

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

where E_0 is the torsion submodule of E. E' is torsion free, so by the first case we know that for a general linear element h of R, $\mathrm{tn}_{R/(h)}(E'/hE') = \mathrm{tn}_R(E')$. Now if in addition we restrict ourselves to those h that are superficial for E and E_0, we can tensor the above exact sequence with $R/(h)$ and get

$$0 = \mathrm{Tor}_1(E',R/(h)) \longrightarrow E_0/hE_0 \longrightarrow E/hE \longrightarrow E'/hE' \to 0,$$

but since E_0/hE_0 is a torsion $R' = R/(h)$-module, $\mathrm{tn}_{R'}(E/hE) = \mathrm{tn}_{R'}(E'/hE') = \mathrm{tn}_R(E') = \mathrm{tn}_R(E)$. □

We shall now derive the first of our general bounds for $\mathrm{tn}(E)$ in terms of the Castelnuovo-Mumford regularity $\mathrm{reg}(E)$ of the module. For terminology and basic properties of the $\mathrm{reg}(\cdot)$ function, we shall use [Ei95, Section 20.5].

Theorem 1.57 *Let $R = k[x_1,\ldots,x_d]$ and E a generated graded R-module of dimension d. Then*
$$\mathrm{tn}(E) \leq \deg(E)\cdot \mathrm{reg}(E).$$

Proof. The assertion is clear if $d = 0$. For $d \geq 1$, if E_0 denotes the submodule of E of the elements with finite support, $\deg(E) = \deg(E/E_0)$, $\mathrm{tn}(E) = \mathrm{tn}(E/E_0)$ and $\mathrm{reg}(E/E_0) \leq \mathrm{reg}(E)$, the latter according to [Ei95, Corollary 20.19(d)]. From this reduction, the assertion is also clear if $d = 1$.

If $d \geq 3$, we use a hyperplane section h so that $\mathrm{tn}_R(E) = \mathrm{tn}_{R/(h)}(E/hE)$, according to Proposition 1.56, and $\mathrm{reg}(E/hE) \leq \mathrm{reg}(E)$, according to [Ei95, Proposition 20.20]. (Of course, $\deg(E) = \deg(E/hE)$.)

With these reductions, we may assume that $d = 2$ and that depth $E > 0$. Denote by E_0 now the torsion submodule of E and consider the exact sequence
$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0.$$
Noting that either E_0 is zero or depth $E_0 > 0$, taking local cohomology with respect to the maximal ideal $\mathfrak{m} = (x_1,x_2)$, we have the exact sequence
$$0 \to H^1_{\mathfrak{m}}(E_0) \to H^1_{\mathfrak{m}}(E) \to H^1_{\mathfrak{m}}(E') \to H^2_{\mathfrak{m}}(E_0) = 0 \to H^2_{\mathfrak{m}}(E) \to H^2_{\mathfrak{m}}(E') \to 0,$$
from which we get
$$\mathrm{reg}(E) = \max\{\mathrm{reg}(E_0), \mathrm{reg}(E')\}.$$
This provides the final reduction to $d = 2$ and E torsionfree. Let
$$0 \to \bigoplus_{j=1}^{s} R[-b_j] \longrightarrow \bigoplus_{i=1}^{r} R[-a_i] \longrightarrow E \longrightarrow 0$$
be a minimal projective resolution of E. From Corollary 1.46, we have
$$\mathrm{tn}(E) = \sum_{i=1}^{r} a_i - \sum_{j=1}^{s} b_j.$$
Reducing this complex modulo a hyperplane section h, we get a minimal free resolution for the graded module E/hE over the PID $R/(h)$. By the basic theorem for modules over such rings, after basis change, we may assume that
$$b_j = a_j + c_j, \quad c_j > 0, \quad j = 1\ldots s.$$
Noting that $\alpha = \mathrm{reg}(E) = \max\{a_i, b_j - 1 \mid i = 1\ldots r, j = 1\ldots s\} \geq 0$, and $\deg(E) = r - s$, we have
$$\begin{aligned}
\deg(E)\mathrm{reg}(E) - \mathrm{tn}(E) &= (r-s)\alpha - \sum_{i=1}^{r} a_i + \sum_{j=1}^{s}(a_j + c_j) \\
&= \sum_{i=s+1}^{r} (\alpha - a_i) + \sum_{j=1}^{s} c_j \\
&\geq 0,
\end{aligned}$$
as desired. □

Positivity of Tracking Numbers

We shall now prove our main result, the somewhat surprising fact that for a reduced homogeneous algebra A, $\mathrm{tn}(A) \geq 0$. Since such algebras already admit a general upper bound for $\mathrm{tn}(A)$ in terms of its multiplicity, together these statements are useful in the construction of integral closures by all algorithms that use intermediate extensions that satisfy the condition S_2 of Serre.

Theorem 1.58 *Let A be a reduced, non-negatively graded algebra that is finite over a standard graded Noether normalization R. Then* $\mathrm{tn}(A) \geq 0$.

Proof. Let $A = S/I$, $S = k[x_1, \ldots, x_n]$, be a graded presentation of A. From our earlier discussion, we may assume that I is height unmixed (as otherwise the lower dimensional components gives rise to the torsion part of A, which is dropped in the calculation of $\mathrm{tn}(E)$ anyway).

Let $I = P_1 \cap \cdots \cap P_r$ be the primary decomposition of I, and define the natural exact sequence

$$0 \to S/I \longrightarrow S/P_1 \times \cdots \times S/P_r \longrightarrow C \to 0,$$

from which a calculation with Hilbert coefficients gives

$$\mathrm{tn}(A) = \sum_{i=1}^{r} \mathrm{tn}(S/P_i) + \hat{e}_0(C).$$

This shows that it suffices to assume that A is a domain.

Let $\overline{A}$ denote the integral closure of A. Note that $\overline{A}$ is also a non-negatively graded algebra and that the same Noether normalization R can be used. Since $\mathrm{tn}(A) \geq \mathrm{tn}(\overline{A})$, we may assume that A is integrally closed.

Since the cases $\dim A \leq 1$ are trivial, we may assume $\dim A = d \geq 2$. The case $d = 2$ is also clear since A is then Cohen-Macaulay. Assume then $d > 2$. We are going to change the base field using rational extensions of the form $k(t)$, which do not affect the integral closure condition. (Of course we may assume that the base field is infinite.)

If h_1 and h_2 are linearly independent hyperplane sections in R, they define a regular sequence in A, since the algebra being normal satisfies the S_2 condition of Serre. Effecting a change of ring of the type $k \longrightarrow k(t)$ gives a hyperplane section $h_1 - t \cdot h_2 \in R(t)$, which is a prime element in A, according to Nagata's trick. Clearly we can choose h_1 and h_2 so that $h_1 - t \cdot h_2$ is a generic hyperplane section for the purpose of applying Proposition 1.56 to A. This completes the reduction to domains in dimension $d - 1$. □

One application is to the study of constructions of the integral closure of an affine domain (see [Va0] where details are given and the characteristic zero case is exploited via resultants).

Theorem 1.59 *Let A be a standard graded domain over a field k and let $\overline{A}$ be its integral closure. Then any chain of distinct subalgebras satisfying the condition S_2 of Serre,*

$$A \subset A_1 \subset \cdots \subset A_n = \overline{A},$$

has length at most $\binom{e}{2}$, where $e = \deg(A)$.

Proof. It will suffice, according to Corollary 1.44, to show that $0 \leq \operatorname{tn}(A) \leq \binom{e}{2}$. The non-negativity having been established in Theorem 1.58, we now prove the upper bound.

If k is a field of characteristic zero, by the theorem of the primitive element, A contains a hypersurface ring $S = R[t]/(f(t))$, where R is a ring of polynomials $R = k[z_1, \ldots, z_d]$, $\deg(z_i) = 1$, and $f(t)$ is a homogeneous polynomial in t of degree e. As $\operatorname{tn}(A) \leq \operatorname{tn}(S) = \binom{e}{2}$, the assertion would hold in this case. (We also observe that when $\dim A \geq 3$, any hyperplane section, say h, used to reduce the dimension that were employed in the proof of Theorem 1.58, could be chosen so that the image of S in $\overline{A}/h\overline{A}$ would be $S/(h)$, and therefore we would maintain the same upper bound.)

To complete the proof in other characteristics we resort to the following construction developed in Proposition 1.1. If $A = R[y_1, \ldots, y_n]$, let

$$E = \sum R y_1^{j_1} \cdots y_n^{j_n}, \quad 0 \leq j_i < r_i = [F_i : F_{i-1}]$$

be the R-module of rank e constructed there. As the rank satisfies the equality $e = \prod_{i=1}^n r_i$, e is the number of 'monomials' $y_1^{j_1} \cdots y_n^{j_n}$. Their linear independence over R is a simple verification. Note also that there are monomials of all degrees between 0 and $\sum_{i=1}^n (r_i - 1)$. Thus according to Proposition 1.49, the bound for $\operatorname{tn}(E)$ is obvious, with equality holding only when E is a hypersurface ring over R. (The precise value for $\operatorname{tn}(E)$ could be derived from Corollary 1.52.) □

1.4 Embedding Dimension of the Integral Closure

Let k be a field and let A be a reduced ring which is a finitely generated k-algebra. The integral closure $\overline{A}$ is also an affine algebra over k,

$$A = k[x_1, \ldots, x_n]/I \hookrightarrow \overline{A} = k[y_1, \ldots, y_m]/J.$$

The least n, among all such presentations of A, is the *embedding dimension* of A, $\operatorname{embdim}(A)$. We will be focused on the number of indeterminates needed to present $\overline{A}$. If A is $\mathbb{N}$-graded, $\overline{A}$ is similarly graded and there will be presentations where the y_i are homogeneous elements. The degree of a presentation of $\overline{A}$ is the maximum of the $\deg(y_i)$. We will denote by $\operatorname{embdeg}(\overline{A})$ the minimum achieved among all presentations; we call that integer the *embedding degree* of $\overline{A}$. (For the moment we shall blur the fact that a short presentation-one yielding $\operatorname{embdim}(\overline{A})$-may not correspond to the presentation giving rise to $\operatorname{embdeg}(\overline{A})$.)

These numbers are major measures of the complexity of computing $\overline{A}$. Our aim here is to give estimates for the embedding dimension of $\overline{A}$, and in the graded case to $\operatorname{embdeg}(\overline{A})$, under some restrictions. Whenever possible we will seek to control the embedding dimension of all intermediate rings which occur in the construction of $\overline{A}$. A major concern is to make use of known properties of A, such as information about its singular locus, or expected geometric properties of $\overline{A}$, in particular regarding its depth.

We introduce here a geometric approach to this problem. A direct approach would be to attempt to bound degree data on the presentation ideal J in terms of n and I. We will follow a different path, as we will assume that we possess information about A with a geometric content, such as the dimension of A and its multiplicity, and on certain instances some fine data on its singular locus. Our results will then be expressed by bounding either $\operatorname{embdim}(B)$ or $\operatorname{embdeg}(B)$ in terms of that data.

In order to explain our results, let us bring out the invariants of an affine ring A which are likely to occur in any estimate for $\operatorname{embdim}(\overline{A})$ and for $\operatorname{embdeg}(\overline{A})$. We shall assume that A is reduced and equidimensional. One source of difficulty lies in that the ring $B = \overline{A}$ may also be the integral closure of other rings. This fuzziness is at the same time a path to dealing with this problem in some cases of interest. Assume that k is an infinite field, so that after a possible change of variables, the subring generated by the images of the first d variables x's is a Noether normalization of A,

$$T = k[x_1, \ldots, x_d] \hookrightarrow A.$$

If A is reduced and equidimensional, the rank of A over T is the ordinary torsionfree rank over A as a T-module. Denote by $\deg(A)$ the least torsionfree rank of A over T for all possible Noether normalizations; this number is equal to the ordinary multiplicity $\deg(A)$ of A, when A is a graded algebra. By the theorem of the primitive element, there exists an element $u \in A$, satisfying an equation $f(u) = 0$, where $f(t)$ is a monic, irreducible polynomial of $T[t]$, $\deg f(t) = \deg(A)$. This will imply that $S = T[t]/(f(t))$ is a subring of A such that B is also the integral closure of S. This shows that the only general numerical invariants we can really use for $\overline{A}$ are its dimension and multiplicity, and to a lesser extent the Jacobian ideal of S.

There are many cases when an estimate for $\operatorname{embdim}(\overline{A})$ is easy to obtain: $\overline{A}$ is Cohen-Macaulay. The ring $\overline{A}$ is then a free module over T, of rank $\deg(A)$ having T as a summand. Therefore $\operatorname{embdim}(\overline{A}) \leq \dim A + \deg(A) - 1$. For example, if $\dim A \leq 2$ this will always hold. Another instance are the rings A generated by monomials; by the well-known theorem of Hochster ([Ho72]), $\overline{A}$ will be Cohen-Macaulay.

Our overall aim is to derive elementary functions $\beta(d,e)$ and $\delta(d,e)$, polynomial in e for fixed d, such that for any standard graded equidimensional reduced algebra A of dimension $d = \dim A$ and multiplicity $e = \deg(A)$,

$$\begin{aligned} \operatorname{embdim}(\overline{A}) &\leq \beta(\dim A, \deg(A)) \\ \operatorname{embdeg}(\overline{A}) &\leq \delta(\dim A, \deg(A)). \end{aligned}$$

The existence of such functions, albeit not in any explicit form, and therefore without the link to complexity, has been established in [DK84, Theorem 3.1] in a model theoretic formulation grounded on the explicit construction of integral closures of [Sei75] and [Sto68]. The bounds given here, beyond their effective character, seek to derive formulas for $\beta(d,e)$ and $\delta(d,e)$ that are sensitive to additional information that is known about A, such as the case when the dimension is at most 3 or the singular locus is small. To exercise this kind of control one must however work in very strict characteristics, usually zero.

Cohen-Macaulay Integral Closure

We begin by introducing some notation and terminology and make some general comments. Throughout we use standard terminology: an affine algebra A will denote a finitely generated algebra $k[a_1,\ldots,a_n]$ over a field k, and it said to be a standard graded algebra if it is affine and generated by its elements of degree 1.

Suppose A is an affine algebra, and $R = k[\mathbf{z}] = k[z_1,\ldots,z_d]$ is one of its Noether normalizations. The rank of A over R is the dimension of the K-vector space $B\otimes_R K$, where K is the field of fractions of R. This dimension may vary with the choice of the Noether normalization. When A is a standard graded algebra and the z are homogeneous of degree 1, this rank is equal to the multiplicity $\deg(A)$ of A as provided by its Hilbert function. It is thus independent of the choice of such Noether normalization. In the non-graded case, by abuse of terminology, we let $\deg A$ denote the infimum of the ranks of A over its various Noether normalizations, a terminology that is consistent in the graded case.

Theorem 1.60 *Let A be a reduced, equidimensional affine domain over a field k. Let $R = k[x_1,\ldots,x_d] \hookrightarrow A$ be a Noether normalization and suppose that* $\operatorname{rank}_R(A) = e > 1$.

(i) *If the integral closure B of A is Cohen-Macaulay then* $\operatorname{embdim}(B) \leq d+e-1$.

(ii) *If moreover A is a standard graded algebra and R is generated by forms of degree 1, then B can be generated by elements of degree at most $e-1$*

Proof. (i) If B is Cohen-Macaulay, then B is simply a free R-module so B can be generated by $\deg(B)$ module generators over R. Furthermore the embedding $R \hookrightarrow B$ gives a splitting and consequently

$$\operatorname{embdim}(B) \leq d + \deg(B) - 1.$$

(ii) We may assume that k is an infinite field. Let R be a Noether normalization as above and denote by $S = R[t]/(f)$ a hypersurface ring contained in B, where f is a form of degree e. Consider the exact sequence

$$0 \to S \longrightarrow B \longrightarrow C \to 0$$

viewed as modules over a Noether normalization $T = k[z_1,\ldots,z_d]$ of S, where the z_i are 1-forms. Since C is a Cohen-Macaulay module of dimension $d-1$, we may even assume that $\mathbf{z} = z_1, \ldots, z_{d-1}$ is a regular sequence on C. Reduction modulo $\mathbf{z}$ gives an exact sequence of graded $k[z_d]$-modules

$$0 \rightarrow \overline{S} \longrightarrow \overline{B} \longrightarrow \overline{C} \rightarrow 0,$$

where $\overline{C}$ is a torsion $k[z_d]$-module. By the fundamental theorem for free $k[z_d]$-modules and its submodules, there exists a homogeneous basis of $\overline{B}$, $\varepsilon_1, \ldots, \varepsilon_e$ such that for some integers $d_1, \ldots, d_e$, the $z_d^{d_i}\varepsilon_i$ form a basis of $\overline{S}$. This shows that $\deg \varepsilon_i \leq e-1$ since S is generated by elements of degree at most $e-1$. □

This is the optimal case but it is rarely achieved. It will always be the case when $d \leq 2$, or $\deg(A) = 2$, and by a theorem of Hochster ([Ho72]), for all monomial subrings. In many other constructions it is assured not to happen. Here is a partial explanation.

Proposition 1.61 *Let A be an integrally closed affine domain over a field of characteristic zero. If A is not Cohen-Macaulay then any finite integral extension B which is a domain is not Cohen-Macaulay either.*

This is explained by the following fact. Let $K \subset L$ be the field of fractions of A and B, respectively, and denote by N the Galois closure of L/K. The *trace* map $tr_{N/K}$ induces a A-module map

$$\frac{1}{s} tr_{N/K} : B \mapsto A,$$

where s is the relative degree of the fields $[N : K]$. It provides a splitting decomposition

$$B = A \oplus M.$$

On the other hand, we have the standard inequality of direct summands

$$\text{depth B} \leq \text{depth A}.$$

An example of such rings is built as follows. Let $S = k[x,y,z]/(f)$, say $f = x^3 + y^3 + z^3$, over a field k of characteristic zero. Let R be the Rees algebra of $(x,y,z)S$. It is easy to see that R is normal but not Cohen-Macaulay.

Dimension 3^{++}

The case of $\dim A \leq 2$ being considered above, we focus here on the conditions that are always present when $\dim A \geq 3$ and $\deg(A) \geq 3$. Because of its dependence on Jacobian ideals most of our assertions will only be valid over fields of large characteristic.

Let $A = k[x_1, \ldots, x_n]/I$ be a reduced equidimensional affine algebra over a field of characteristic zero, $n = \text{embdim}(A)$, of dimension $\dim A = d$. We call $\text{ecodim}(A) = n - d$ the *embedding codimension* of A. We denote by $\Omega_k(A)$ the module of Kähler differentials of A/k. By $J(A)$ we denote the Jacobian ideal $\text{Fitt}_d(\Omega_k(A))$ of A. Recall that $V(J(A)) = \text{Sing}(A)$ if k is perfect and A is equidimensional.

Lemma 1.62 *Let k be a field of characteristic zero, let S be a reduced and equidimensional standard graded k-algebra of dimension d and multiplicity $e > 2$ and assume that* $\mathrm{embdim}(S) \leq d+1$. *If $S \subset B$ is a finite and birational extension of rings, then the S-module B/S satisfies*

$$\deg(B/S) \leq (e-1)^2.$$

Proof. We may assume that $S \neq B$. As S satisfies S_2 and the extension $S \subset B$ is finite and birational, it follows that the S-module B/S is of pure codimension one. Thus, since S is Gorenstein,

$$\deg(B/S) = \deg(\mathrm{Ext}^1_S(B/S,S)).$$

Applying $\mathrm{Hom}_S(\cdot,S)$ to the exact sequence

$$0 \to S \longrightarrow B \longrightarrow B/S \to 0$$

yields an exact sequence

$$0 \to S/S :_S B \longrightarrow \mathrm{Ext}^1_S(B/S,S) \longrightarrow \mathrm{Ext}^1_S(B,S) \to 0.$$

Since B has the property S_2 over the Gorenstein ring S, the S-module $\mathrm{Ext}^1_S(B,S)$ has codimension at least 2. Thus

$$\deg(\mathrm{Ext}^1_S(B/S,S)) = \deg(S/S :_S B).$$

On the other hand $J(S) \subset S :_S \overline{S}$ by [No50]. Hence $J(S) \subset S :_S B$ and then, as both ideals have height one,

$$\deg(S/S :_S B) \leq \deg(S/J(S)).$$

Finally, write $S = R/(f)$ with $R = k[x_1,\ldots,x_n]$ a polynomial ring and f a form of degree e. Set $J = R\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)$. One has $f \in J$ since characteristic $k = 0$ and hence $S/J(S) = R/J$. But J is an ideal of height 2 generated by forms of degree $e-1$. Therefore

$$\deg(S/J(S)) = \deg(R/J) \leq (e-1)^2.$$

We observe the following slight improvement which will apply to the integral closure of A.

Corollary 1.63 *If moreover the algebra B satisfies S_2 then*

$$\deg(B/S) \leq e(e-2).$$

Proof. We pick up the proof above after observing that $\mathrm{Ext}^1_S(B,S)$ has codimension at least 3. Let g,h be forms of degree $e-1$ in J generating a regular sequence. Consider the induced exact sequence

$$0 \to L \longrightarrow R/(g,h) \longrightarrow \mathrm{Ext}^1_S(C,S) \longrightarrow \mathrm{Ext}^1_S(B,S) \to 0. \qquad (5)$$

We cannot have $L = 0$ as $R/(f,g)$ is Cohen-Macaulay of dimension $d-1$ and therefore $\mathrm{Ext}^1_S(B,S)$ would have codimension at most 2. Thus L must be nonzero, of dimension $d-1$, and therefore

$$\deg(S/J(S)) < \deg(R/(g,h)) - 1 \leq (e-1)^2 - 1 = e(e-2).$$

Theorem 1.64 *Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded k-algebra of dimension d and multiplicity* $e > 1$. *If* $A \subset B$ *is a finite and birational extension of graded rings with* $\mathrm{depth}_A B \geq d-1$, *then*

$$\nu_A(B) \leq (e-1)^2$$

and

$$\mathrm{embdim}(B) \quad \leq \quad (e-1)^2 + d + 1. \tag{6}$$

Proof. There exists a homogeneous subalgebra S of A so that $\mathrm{embdim}(S) \leq d+1$ and the extension $S \subset A$ is finite and birational. Notice that $\deg(S) = \deg(A) = e$. The S-module B/S is Cohen-Macaulay. Therefore $\nu_S(B/S) \leq \deg(B/S)$ and the assertions follow by Lemma 1.62. □

Corollary 1.65 *Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded k-algebra of dimension* 3 *and multiplicity e. The integral closure* $B = \overline{A}$ *satisfies*

$$\mathrm{embdim}(B) \leq (e-1)^2 + 4.$$

Remark 1.66 *We will make two observations about this quadratic estimate. First, the assertion is not strictly module theoretic, the algebra structure of B really matters, not depending merely on the fact that B is a reflexive module over the Noether normalization. For example, if* $R = k[x,y,z]$ *and* $\wp$ *is one of the Macaulay's prime ideals with a large number of generators, there are exact sequences of the form*

$$0 \to R \longrightarrow E \longrightarrow \wp \to 0,$$

with E a reflexive module. Thus $\deg(E) = 2$ *but* $\nu(E) \gg 0$. *According to the result mentioned, E could not be the underlying module of an integral domain finite over R.*

The other point is that even the quadratic bound may be too large and perhaps a bound of type $C\deg(A)^{3/2}$ *suffices. In examples it is all we have been able to achieve.*

Remark 1.67 *Let R be a ring of polynomials over a field k of characteristic* $\neq 2$ *and let* $u \in R$ *be a polynomial which is not a square. Let* $B = R[T]/(T^2 - u) = R[t]$. *For any ideal I of R,* $A = R + It$ *is an affine domain whose integral closure is B. Note the significance of the* S_2 *condition of Serre for A: It will limit its embedding dimension to* $\dim A + 1$, *otherwise it will be determined by the number of generators of I if* $\dim A > 1$.

Remark 1.68 *If in the setting of Theorem 1.64,* $\operatorname{depth}_A B \leq d-2$ *and* B *satisfies* S_2 *(as an* A*-module or, equivalently, as a ring), then* $\nu_A(B) \geq e+3$*. Indeed, for a homogeneous Noether normalization* T *of* A*,* $\operatorname{depth}_T B \leq d-2$ *and the* T*-module* B *satisfies* S_2*. Thus any first syzygy module of* B *over* T *is a nonfree third syzygy of finite projective dimension, thus has rank at least 3 by the Syzygy Theorem ([EG81]). It follows that* $\nu_A(B) \geq \nu_T(B) \geq \operatorname{rank}_S B + 3 = e+3$.

Non-Homogeneous Algebras

We now formulate a version of the estimates of the two previous sections valid for general affine algebras. We will deal with the non-homogeneous version of Lemma 1.62 and Theorem 1.64. To this end we use a technique employed in [Va0].

Suppose A is a reduced, equidimensional, affine algebra over a field k of characteristic zero. We denote by $R = k[\mathbf{z}] = k[z_1, \ldots, z_d]$ one of its Noether normalizations such that the rank e of A over R is minimal. As earlier, we set $S = R[z_{d+1}]/(f)$ to be a hypersurface ring $R \subset S \subset A$ over which A is rational, so that $\deg f = e$. With abuse of terminology we shall refer to e as the 'multiplicity' of A.

Lemma 1.69 *Let* k *be a field of characteristic zero, let* S *be a reduced and equidimensional* k*-algebra of dimension* d *and multiplicity* $e > 1$ *and assume that* $\operatorname{embdim}(S) \leq d+1$*. If* $S \subset B$ *is a finite and birational extension of rings, then for each maximal ideal* $\mathfrak{P}$ *of* S*, the* $S_{\mathfrak{P}}$*-module* $(B/S)_{\mathfrak{P}}$ *satisfies*

$$\deg(B/S)_{\mathfrak{P}} \quad \leq \quad e(e-1)^2. \tag{7}$$

Proof. Let $S = k[z_1, \ldots, z_d, z_{d+1}]/(f)$, $\deg f = e$. We may assume that k is algebraically closed and $\mathfrak{P}$ is defined by the origin. Since f is square free and the characteristic of k is zero, after a possible linear change of variables we may assume that f and one of its partial derivatives, g, form a regular sequence. We now appeal to Proposition 1.34. □

Theorem 1.70 *Let* k *be a field of characteristic zero and let* A *be a reduced and equidimensional* k*-algebra of dimension* d *and multiplicity* $e > 1$*. If* $A \subset B$ *is a finite and birational extension of graded rings and for each maximal ideal* $\mathfrak{P}$ *of* A*,* $\operatorname{depth}_{A_{\mathfrak{P}}} B_{\mathfrak{P}} \geq d-1$*, then*

$$\nu_A(B) \leq e(e-1)^2 + d + 1$$

and

$$\operatorname{embdim}(B) \quad \leq \quad e(e-1)^2 + 2d + 1. \tag{8}$$

Proof. The proof is nearly the same as that of Theorem 1.64 once we have established the local estimates for the number of generators of B/A. We complete by appealing to Swan's Theorem. □

We obtain a general bound in the special case of $\dim A = 3$:

Corollary 1.71 *Let k be a field of characteristic zero and let A be a reduced and equidimensional k-algebra of dimension 3 and multiplicity e. The integral closure $B = \overline{A}$ satisfies*

$$\mathrm{embdim}(B) \leq \begin{cases} (e-1)^2+4 & \text{if } A \text{ is homogeneous} \\ e(e-1)^2+7 & \text{if } A \text{ is non homogeneous.} \end{cases}$$

Small Singularities

Let A be a reduced equidimensional affine algebra over a perfect field k, let J be its Jacobian ideal and B its integral closure. We treat the case of $\dim A = d \geq 4$ and assume that the singular locus of A is suitably small.

Suppose that height $J \geq 2$, a condition equivalent with $A_{\wp} = B_{\wp}$ for each prime ideal of A of height 1. This means that B is the S_2-ification of A, and one may describe B as the outcome of a single operation. According to Theorem 1.15,

$$B = A : J.$$

The issue is to estimate how many generators this process requires. One of our results, for $d = 4$, will do precisely this. For higher dimensions we shall require a condition of the order R_{d-3} (see further for more precise statements). Before we can get to this we need to examine various 'approximations' of B by some more tractable subrings much in the manner that the hypersurface subring S was used in Section 3. They will be aimed at converting information about the degrees of the generators of a 'thick' portion of the Jacobian ideal J in terms of the multiplicity of A.

Proposition 1.72 *Let k be an infinite perfect field, let A be a finitely generated k-algebra of dimension d, and let $\mathfrak{q}$ be a prime ideal of A. If* $\mathrm{ecodim}(A_{\mathfrak{q}}) \leq g$ *for some $g \geq 1$, then there exists a k-subalgebra $S = k[x_1, \ldots, x_{d+g}] \subset A$ so that the extension $S \subset A$ is finite and*

$$S_{\mathfrak{q} \cap A} = A_{\mathfrak{q} \cap A} = A_{\mathfrak{q}}.$$

If in addition A is standard graded then $x_1, \ldots, x_{d+g}$ can be chosen to be linear forms, and if A is reduced and equidimensional then the extension $S \subset A$ can be chosen to be birational.

Proof. Write $A = k[y_1, \ldots, y_n]$, where $y_1, \ldots, y_n$ are chosen to be linear forms if A is standard graded. Let $x_1, \ldots, x_{d+g}$ be sufficiently general k-linear combinations of $y_1, \ldots, y_n$. Write $S = k[x_1, \ldots, x_{d+g}]$ and $\mathfrak{p} = \mathfrak{q} \cap S$. Obviously A is finite over S. If A is reduced and equidimensional then the extension is birational since $g \geq 1$.

To show that the inclusion $S \subset A$ induces an isomorphism $S_{\mathfrak{p}} = A_{\mathfrak{q}}$, write $k(\mathfrak{q})$ for the residue field of $\mathfrak{q}$ and consider the exact sequence

$$0 \to \mathbb{D} \longrightarrow R = k(\mathfrak{q}) \otimes_k A \longrightarrow k(\mathfrak{q}) \to 0 \qquad (9)$$

induced by the multiplication map $A\otimes_k A\to A$. Notice that $\mathbb{D}=\sum_{i=1}^{n} R(\overline{y_i}\otimes 1-1\otimes y_i)$ is a maximal ideal of R. Since

$$\nu(\mathbb{D}_{\mathbb{D}})=\nu(\Omega_k(A_{\mathfrak{q}}))=\operatorname{embdim}(A_{\mathfrak{q}})+\dim A/\mathfrak{q}\le d+g,$$

one has $\mathbb{D}_{\mathbb{D}}=\sum_{i=1}^{d+g} R_{\mathbb{D}}(\overline{x_i}\otimes 1-1\otimes x_i)$. On the other hand as $\dim R=d<d+g$, $\mathbb{D}=\sqrt{\sum_{i=1}^{d+g} R(\overline{x_i}\otimes 1-1\otimes x_i)}$; to see this recall that every ideal in a d-dimensional ring containing an infinite field k is generated up to radical by $d+1$ general k-linear combinations of its generators. It follows that $\mathbb{D}=\sum_{i=1}^{d+g} R(\overline{x_i}\otimes 1-1\otimes x_i)$. Thus by (9), $k(\mathfrak{q})\otimes_S A\cong k(\mathfrak{q})$. Comparing numbers of generators over the ring $S_{\mathfrak{p}}$ we conclude that $\nu_{S_{\mathfrak{p}}}(A_{\mathfrak{q}})=1$, hence $S_{\mathfrak{p}}=A_{\mathfrak{q}}$. □

Corollary 1.73 *Let k be an infinite perfect field and let A be a reduced and equidimensional standard graded k-algebra of dimension d. Write $\mathcal{A}$ for the set of all homogeneous subalgebras $S=k[x_1,\ldots,x_{d+1}]$ of A so that the extension $S\subset A$ is finite and birational, and set $J=\sum_{S\in\mathcal{A}} A\cdot J(S)$. Then $\sqrt{J(A)}=\sqrt{J}$.*

Proof. For a fixed prime ideal $\mathfrak{q}$ of A we apply Proposition 1.72 with $g=1$. The proposition shows that $A_{\mathfrak{q}}$ is regular if and only if $S_{\mathfrak{q}\cap S}$ is regular for some $S\in\mathcal{A}$. Since S is again equidimensional one has $J(A)\subset\mathfrak{q}$ if and only if $J(S)\subset\mathfrak{q}\cap S$ if and only if $A\cdot J(S)\subset\mathfrak{q}$ for some $S\in\mathcal{A}$. □

Corollary 1.74 *Let k be an infinite perfect field and let A be a reduced and equidimensional standard graded k-algebra with $e=\deg(A)$ and $c=\operatorname{codim}(\operatorname{Sing}(A))$. Then the conductor of A contains an ideal of height c generated by forms of degree $e-1$.*

Proof. We claim that the ideal J of Corollary 1.73 has the desired properties. First, by the corollary, height $J=c$. Now let $S\in\mathcal{A}$. Since $S=k[x_1,\ldots,x_{d+1}]$ is reduced and equidimensional of dimension d we can write $S=k[X_1,\ldots,X_{d+1}]/(f)$ where f is a form. As the homogeneous extension $S\subset A$ is finite and birational one has $\deg(S)=\deg(A)$, thus $\deg f=e$. Therefore $J(S)$ is generated by forms of degree $e-1$ and hence J has the same property. Finally by [No50], $J(S)$ is contained in the conductor $\mathfrak{c}(S)$ of S. But $\mathfrak{c}(S)\subset\mathfrak{c}(A)$ since the extension $S\subset A$ is finite and birational. Thus indeed $J\subset\mathfrak{c}(A)$. □

Theorem 1.75 *Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded algebra of dimension $d\ge 4$ and multiplicity e. Let $A\subset B$ be a finite and birational extension of graded rings and let t be a positive integer with $2\le t\le\min\{d-2,\operatorname{depth}_A B\}$. If A satisfies R_{d-t-1} then*

$$\nu_A(B)\ \le\ (e(e-1))^{2^{d-t-1}}-(2e(e-1))^{2^{d-t-2}}+2$$

$$\operatorname{embdim}(B)\ \le\ (e(e-1))^{2^{d-t-1}}-(2e(e-1))^{2^{d-t-2}}+t+3.$$

Proof. It suffices to show that $\nu_S(B/A) \leq (e(e-1))^{2^{d-t-1}} - (2e(e-1))^{2^{d-t-2}} + 1$ for some homogeneous k-subalgebra $S = k[x_1, \ldots, x_{t+2}]$ of A.

Let $\mathbf{x} = x_1, \ldots, x_{t-1}$ be a sequence of general linear forms in A. Write $B' = B/B(\mathbf{x})$ and A' for the image of A in B'. Since $\dim_A(B/A) \leq t$ one has $\dim_{A'} B'/A' \leq \dim_A B'/A' \leq 1$. Furthermore B' satisfies R_{d-t-1} ([Fl77]), and is reduced and equidimensional of dimension $d-t+1 \geq 3$. Thus the standard graded k-algebra A' has the same properties. Moreover $\deg(A') = \deg_{A'}(B') = \deg_A(B) = \deg(A) = e$. By the graded Nakayama lemma, as $B'/A' \subset \overline{A'}/A'$, it suffices to show that for every A'-submodule C of $\overline{A'}/A'$ and some homogeneous k-subalgebra S' of A' with $\operatorname{embdim}(S') \leq 3$,

$$\nu_{S'}(C) \leq (\deg(A')(\deg(A')-1))^{2^{\dim A'-2}} - (2\deg(A')(\deg(A')-1)^{2^{\dim A'-3}} + 2.$$

Changing notation we prove the following: Let A be a reduced and equidimensional standard graded k-algebra of dimension $d \geq 3$ and multiplicity e satisfying R_{d-2}. There exists a homogeneous k-subalgebra S of A with $\operatorname{embdim}(S) \leq 3$ so that for every A-submodule of $B/A = \overline{A}/A$

$$\nu_S(C) \leq (e(e-1))^{2^{d-2}} - (2e(e-1))^{2^{d-3}} + 1. \tag{10}$$

Write $\mathfrak{c}$ for the conductor $\operatorname{ann}_A(B/A)$ of A. By Corollary 1.74, $\mathfrak{c}$ contains forms of degree $e-1$ that generate an ideal J of height $d-1$. Let x be a general k-linear combination of these forms. Write $B' = B/Bx$ and let S', A' be the images of S, A in B'. As height $J \geq d-1$, it follows that B' satisfies R_{d-3} ([Fl77, Theorem 4.7]), and B' is reduced and equidimensional of dimension $d-1 \geq 2$. Since $\dim_{A'} B'/A' \leq 1$, A' then has the same properties. Notice that $\deg(A') = e(e-1)$ because x is a B-regular form of degree $e-1$. On the other hand, as $x \in \mathfrak{c} = \operatorname{ann}_A(B/A)$ we have $B/A = B'/A'$ and therefore $C \subset B/A = B'/A' \subset \overline{A'}/A'$.

We are now ready to prove (10) by induction on $d \geq 3$. If $d = 3$ let x_1, x_2, x_3 be general linear forms in A, and write $S = k[x_1, x_2, x_3]$ and S' for the image of S in A'. The extension $S' \subset A'$ is finite and birational, S' is a Cohen-Macaulay ring of dimension 2, $\operatorname{embdim}(S') \leq 3 = \dim S' + 1$, and $\deg(S') = \deg(A')$. Thus the S-module B'/S' is Cohen-Macaulay of dimension 1 and has multiplicity at most $(\deg(A')-1)^2$ by Lemma 1.62. Let $D \subset B'/S'$ be the preimage of $C \subset B'/A'$ under the natural projection from B'/S' to B'/A'. As an S'-module of B'/S', D is also Cohen-Macaulay of dimension 1. Therefore

$$\begin{aligned}\nu_S(C) = \nu_{S'}(C) \leq \nu_{S'}(D) &\leq \deg(D) \leq \deg(B'/S') \leq (\deg(A')-1)^2 \\ &= (e(e-1)-1)^2 = (e(e-1))^2 - 2e(e-1) + 1\end{aligned}$$

establishing (10) for $d = 3$.

If $d \geq 4$ then by the induction hypothesis there exists a homogeneous k-subalgebra S' of A' with $\operatorname{embdim} S' \leq 3$ so that

$$\begin{aligned}\nu_{S'}(C) &\leq (\deg(A')(\deg(A')-1))^{2^{\dim A'-2}} - (2\deg(A')(\deg(A')-1))^{2^{\dim A'-1}} + 1 \\ &= (e(e-1)(e(e-1)-1))^{2^{d-3}} - (2e(e-1)(e(e-1)-1))^{2^{d-4}} \\ &\leq (e(e-1))^{2^{d-2}} - (2e(e-1))^{2^{d-3}} + 1.\end{aligned}$$

To see the last inequality one may assume that $e \geq 2$; set $a = e(e-1) \geq 2$ and $\alpha = 2^{d-4} \geq 1$, and use the fact that $(a-1)^{\alpha} \leq a^{\alpha} - a^{\alpha-1}$ to deduce $(a(a-1))^{2\alpha} - (2a(a-1))^{\alpha} \leq a^{4\alpha} - (2a)^{2\alpha}$. Finally, let S be any homogeneous k-subalgebra of A with $\mathrm{embdim}(S) \leq 3$ mapping onto S' and notice that $\nu_S(C) = \nu_{S'}(C)$. □

Corollary 1.76 *Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded algebra of dimension $d \geq 4$ and multiplicity e. If A satisfies R_{d-3} then for $B = \overline{A}$*

$$\mathrm{embdim}(B) \leq (e(e-1))^{2^{d-3}} - (2e(e-1))^{2^{d-4}} + 5.$$

In particular if $d = 4$ and A satisfies R_1 then

$$\mathrm{embdim}(B) \leq (e(e-1))^2 - 2e(e-1) + 5.$$

Bounds on Degrees

In this section we give degree bounds for the generators of the integral closure of a standard graded integral domains over a field of characteristic zero.

Our first bound has a conjectural basis, as it depends on the validity of the conjecture of [EG84]:

Conjecture 1.77 (Eisenbud-Goto) *Let A be a homogeneous integral domain over an integrally closed field of characteristic zero. Then the Castelnuovo-Mumford regularity of A is bounded by*

$$\mathrm{reg}(A) \leq \deg(A) - \mathrm{codim}(A) + 1.$$

Theorem 1.78 *Let k be a field of characteristic zero and let A be a standard graded domain over k of dimension d and multiplicity e. Write $B = \overline{A}$ and suppose that A satisfies the condition R_1. If (1.77) holds in dimension $\leq d-1$ then*

$$\mathrm{embdeg}(B) \leq (e-1)^2. \tag{11}$$

Proof. Consider the exact sequence

$$0 \to A \longrightarrow B \longrightarrow D \to 0,$$

and we will seek to estimate the degrees of the generators of D. We set $A = R/\mathfrak{p}$ where R is a ring of polynomials. By Corollary 1.74 and Bertini's theorem ([Fl77, Theorem 4.7]), there exists a form f of degree $e-1$ in the conductor of A such that B/fB is an integral domain. Tensoring the exact sequence above by $A/(f)$, we obtain the exact sequence

$$0 \to D[-(e-1)] \longrightarrow R/(\mathfrak{p}, f) \longrightarrow B/fB \longrightarrow D \to 0.$$

We write

$$D[-(e-1)] = \mathfrak{P}/(\mathfrak{p}, f) \subset R/(\mathfrak{p}, f),$$

and observe that $R/\mathfrak{P}$ is a subring of B/fB and therefore it is an integral domain of dimension $d-1$, and multiplicity $e(e-1)$, since $\dim D \leq d-2$.

When we apply (1.77) to the standard graded domain $R/\mathfrak{P}$, we obtain that $\mathfrak{P}$ is generated by elements of degree $\leq \deg R/\mathfrak{P} = e(e-1)$. This means that $D[-(e-1)]$ is also generated by elements of this degree, and therefore D can be generated by elements of degree at most $e(e-1)-(e-1) = (e-1)^2$. □

The bound predicted by (1.77) have been established in very low dimensions (≤ 3), often with additional restrictions. On the other hand, for isolated singularities there is the bound of Mumford [BM91, Theorem 3.12(b)], that implies that B is generated in degrees at most $d(e-2)+2$.

Proposition 1.79 *Let k be an infinite perfect field, let $A = k[y_1, \ldots, y_n]$ be an equidimensional k-algebra of dimension d, and let s be a positive integer. Let $x_1, \ldots, x_{d+s}$ be general k-linear combinations of $y_1, \ldots, y_n$ and write $S = k[x_1, \ldots, x_{d+s}] \subset A$. If* $\operatorname{ecodim}(A_\mathfrak{q}) \leq \dim A_\mathfrak{q}$ *for every prime $\mathfrak{q}$ of A with* $\dim A_\mathfrak{q} \leq s-1$, *then $S_\mathfrak{p} = A_\mathfrak{p}$ for every prime $\mathfrak{p}$ of S with* $\dim S_\mathfrak{p} \leq s-1$.

Proof. Consider the exact sequence

$$0 \to \mathbb{D} \longrightarrow R = A \otimes_k A \stackrel{\text{mult}}{\longrightarrow} A \to 0. \tag{12}$$

The ring R is equidimensional and $\mathbb{D} = \sum_{i=1}^n R(y_i \otimes 1 - 1 \otimes y_i)$ is an ideal of height d. Furthermore by our assumption, $\nu(\mathbb{D}_Q) \leq \dim R_Q$ for every $Q \in V(\mathbb{D})$ with $\dim R_Q \leq d+s-1$, i.e., $\mathbb{D}$ satisfies the condition G_{d+s}. But then

$$\text{height } \left(\sum_{i=1}^{d+s} R(x_i \otimes 1 - 1 \otimes x_i)\right) : \mathbb{D} \geq d+s. \tag{13}$$

Now let $\mathfrak{p}$ be a prime of S with $\dim S_\mathfrak{p} \leq s-1$. As $\dim A_\mathfrak{p} \otimes_k A_\mathfrak{p} \leq \dim A_\mathfrak{p} + \dim A \leq d+s-1$, (13) implies that

$$(A_\mathfrak{p} \otimes_k A_\mathfrak{p})\mathbb{D} = \sum_{i=1}^{d+s} (A_\mathfrak{p} \otimes_k A_\mathfrak{p})(x_i \otimes 1 - 1 \otimes x_i).$$

Thus $A_\mathfrak{p} \otimes_{S_\mathfrak{p}} A_\mathfrak{p} \cong A_\mathfrak{p}$ by (12). Since $A_\mathfrak{p}$ is a finite $S_\mathfrak{p}$-module, comparing numbers of generators then yields $\nu_{S_\mathfrak{p}}(A_\mathfrak{p}) = 1$, hence $S_\mathfrak{p} = A_\mathfrak{p}$. □

Theorem 1.80 *Let k be a perfect field, let A be a reduced and equidimensional standard graded k-algebra of dimension d and multiplicity $e \geq 2$, and let $A \subset B$ be a finite and birational extension of graded rings. If A satisfies R_1 and* $\operatorname{depth}_A B \geq d-1$, *then the A-module B is generated in degrees at most* $3e-4$.

Proof. We may assume that k is infinite and then by Proposition 1.79 we may reduce to the case where $A = k[x_1, \ldots, x_{d+2}]$ with the x_i linear forms. Take $x_1, \ldots, x_{d+2}$ to be general, map the polynomial ring $R = k[X_1, \ldots, X_{d+2}]$ onto A by sending X_i

to x_i, and write $S = k[x_1, \ldots, x_{d+1}] \subset A$. One has $S = k[X_1, \ldots, X_{d+1}]/(f)$ for some form f of degree e. Since S is reduced there exists a form $g \in k[X_1, \ldots, X_{d+1}]$ of degree $e-1$ so that f, g are a regular sequence on $k[X_1, \ldots, X_{d+1}]$ and the image of g in S lies in $J(S)$, hence in the conductor of S ([No50]). Write $k[x_1, \ldots, x_d, x_{d+2}] = k[X_1, \ldots, X_d, X_{d+2}]/(h)$ where h is a form of degree e. As h is monic in X_{d+2}, it follows that f, g, h is an R-regular sequence.

Consider the ring $A' = R/(f, h)$ and its homogeneous Noether normalization

$$S' = k[X_1, \ldots, X_{d-1}, g] \hookrightarrow A'.$$

Since the socle of the complete intersection $A'/(x_1, \ldots, x_{d-1}, g)$ is concentrated in degree $\deg f + \deg h + \deg g - 3 = 3e - 4$, we conclude that the S'-module A' is generated in degrees at most $3e-4$. But A' maps onto A and $gB \subset A$. Thus as modules over the polynomial ring $T = k[X_1, \ldots, X_{d-1}]$, A/S is generated in degrees at most $3e-4$ and B/S is finite.

Now consider the exact sequence of graded T-modules,

$$0 \to M = A/S \longrightarrow N = B/S \longrightarrow B/A \to 0.$$

Here N is a maximal Cohen-Macaulay module, hence free, whereas $\dim N/M = \dim B/A \leq d-2 < \dim T$. If the degree of a homogeneous basis element of N exceeds $3e-4$, then N/M has a nontrivial free summand, which is impossible. Thus the T-module N is generated in degrees at most $3e-4$, and then the same holds for the A-module B. □

1.5 Homological Degrees of Biduals

Let A be an affine domain over a field and let E be a finitely generated torsionfree A–module. The construction of bidual, $E^{**} = \mathrm{Hom}_A(\mathrm{Hom}_A(E, A), A)$, occurs often in our treatment of integral closure and a comparison between the number of generators of E and of E^{**} is required, at least in the case when A is a Gorenstein ring. This is not always possible but we are going to show how the theory of homological degrees developed in [Va98a] may help.

Proposition 1.81 *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d, and let E be a torsionfree R–module and let F be its bidual. If $\dim F/E = 2$, then*

$$\mathrm{hdeg}(F) \leq \begin{cases} \mathrm{hdeg}(E) & \text{if } d \geq 6, \\ 2 \cdot \mathrm{hdeg}(E) & \text{if } d = 4, 5. \end{cases}$$

Proof. Set $C = F/E$ and note that in general $\dim C \leq d-2$. We note also that the case $\dim C \leq 1$ will easily be gleaned in the proof. Consider the exact sequence

$$0 \to E \longrightarrow F \longrightarrow C \to 0,$$

and apply the functor $\mathrm{Hom}_R(\cdot, R)$. We observe that by local duality $\mathrm{Ext}^i_R(C, R) = 0$ for $i \leq d-3$, $\mathrm{Ext}^i_R(F, R) = 0$ for $i \geq d-1$, that $\mathrm{Ext}^{d-2}_R(F, R)$ has finite length because F satisfies the condition S_2 of Serre, and in a similar manner $\mathrm{Ext}_R^d(E,R) = 0$ and $\mathrm{Ext}_R^{d-1}(E,R)$ has finite length because E is torsionfree. With this the long exact sequence of cohomology breaks down into the shorter exact sequences:

$$\mathrm{Ext}_R^i(F,R) = \mathrm{Ext}_R^i(E,R), \quad i \leq d-4,$$

$$0 \to \mathrm{Ext}_R^{d-3}(F,R) \to \mathrm{Ext}_R^{d-3}(E,R) \to \mathrm{Ext}_R^{d-2}(C,R) \to \mathrm{Ext}_R^{d-2}(F,R) \to \mathrm{Ext}_R^{d-2}(E,R) \to \mathrm{Ext}_R^{d-1}(C,R) \to 0,$$

$$\mathrm{Ext}_R^{d-1}(E,R) = \mathrm{Ext}_R^d(C,R).$$

We break down the mid sequence into short exact sequences

$$0 \to \mathrm{Ext}_R^{d-3}(F,R) \to \mathrm{Ext}_R^{d-3}(E,R) \to L_0 \to 0$$

$$0 \to L_0 \to \mathrm{Ext}_R^{d-2}(C,R) \to L_1 \to 0$$

$$0 \to L_1 \to \mathrm{Ext}_R^{d-2}(F,R) \to L_2 \to 0$$

$$0 \to L_2 \to \mathrm{Ext}_R^{d-2}(E,R) \to \mathrm{Ext}_R^{d-1}(C,R) \to 0.$$

We recall the expression for $\mathrm{hdeg}(E)$

$$\begin{aligned}\mathrm{hdeg}(E) &= \deg(E) + \\ &\sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}(\mathrm{Ext}_R^i(E,R)),\end{aligned}$$

and identify similar values for the expression for $\mathrm{hdeg}(F)$. Note that for $i \leq d-4$, they are the same. We are going to argue that for the higher dimensions the contributions of $\mathrm{Ext}_R^i(E,R)$ are at least as high as those of $\mathrm{Ext}_R^i(F,R))$. We have to concern ourselves with $\mathrm{Ext}_R^{d-3}(F,R)$ and $\mathrm{Ext}_R^{d-2}(F,R)$.

The key module to examine is $H = \mathrm{Ext}_R^{d-2}(C,R)$. Since $\dim C = 2$, its bottom cohomology module is a module of dimension 2 of depth 2 as well: $H = \mathrm{Hom}_{R/I}(C,R/I)$, I being a complete intersection of codimension $d-2$ contained in the annihilator of C. In particular, L_0 is a nonzero module of positive depth and dimension 2, since $\mathrm{Ext}_R^{d-3}(F,R)$ has dimension at most 1, as remarked at the outset; it is the full submodule of $\mathrm{Ext}_R^{d-3}(E,R)$ of dimension ≤ 1.

Since $\mathrm{Ext}_R^{d-2}(C,R)$ has depth 2, we have that $\mathrm{Ext}_R^{d-1}(L_0,R) = \mathrm{Ext}_R^d(L_1,R)$, a module of the same length as L_1. On the other hand, the cohomology of the sequence defining L_0 gives the exact sequences

$$\mathrm{Ext}_R^{d-2}(L_0,R) = \mathrm{Ext}_R^{d-2}(\mathrm{Ext}_R^{d-3}(E,R),R)$$

$$0 \to \mathrm{Ext}_R^{d-1}(L_0,R) \to \mathrm{Ext}_R^{d-1}(\mathrm{Ext}_R^{d-3}(E,R),R) \to \mathrm{Ext}_R^{d-1}(\mathrm{Ext}_R^{d-3}(F,R),R) \to 0$$

$$\mathrm{Ext}_R^d(\mathrm{Ext}_R^{d-3}(E,R),R) = \mathrm{Ext}_R^d(\mathrm{Ext}_R^{d-3}(F,R),R),$$

that establishes the equality

$$\mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(E,R)) = \mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(F,R)) + \mathrm{hdeg}(L_0). \qquad (14)$$

As for $\mathrm{Ext}_R^{d-2}(F,R)$, the modules L_1 and L_2 have finite length and

$$\mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(F,R)) = \lambda(L_1) + \lambda(L_2) \leq \mathrm{hdeg}(L_0) + \mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(E,R)). \tag{15}$$

The corresponding terms in the formula for $\mathrm{hdeg}(E)$ and $\mathrm{hdeg}(F)$ are respectively

$$\binom{d-1}{d-4} \cdot \mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(E,R)) + \binom{d-1}{d-3} \cdot \mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(E,R)) \tag{16}$$

$$\binom{d-1}{d-4} \cdot \mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(F,R)) + \binom{d-1}{d-3} \cdot \mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(F,R)). \tag{17}$$

Replacing $\mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(E,R))$ from (14) by $\mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(F,R)) + \mathrm{hdeg}(L_0)$, and moving the term with $\mathrm{hdeg}(L_0)$ over we get

$$\binom{d-1}{d-4}\mathrm{hdeg}(\mathrm{Ext}_R^{d-3}(F,R)) + \binom{d-1}{d-3}\mathrm{hdeg}(\mathrm{Ext}_R^{d-2}(E,R)) + \binom{d-1}{d-4}\mathrm{hdeg}(L_0). \tag{18}$$

For $d \geq 6$, $\binom{d-1}{d-4} \geq \binom{d-1}{d-3}$, which shows that taking (15) into account the expression (18) exceeds (17), as desired.

In case of $d = 4, 5$, taking twice $\binom{d-1}{d-4}$ will serve the stated purpose. □

Remark 1.82 Since for any module F, $\nu(F) \leq \mathrm{hdeg}(F)$, the result bounds $\nu(F)$ in terms of $\mathrm{hdeg}(E)$. The coefficient of 2 in case of $d = 4, 5$ probably can be refined further.

Corollary 1.83 *Let R be a Gorenstein local ring of dimension* 4. *For any finitely generated torsionfree R–module E,* $\mathrm{hdeg}(E^{**}) \leq 2 \cdot \mathrm{hdeg}(E)$.

Proof. Note that the hypothesis on C is always satisfied when $d = 4$. □

2 Normalization of Ideals and Modules

An issue is how to introduce numerical measures, with similar properties to those we employed in the case of algebras, which are appropriate to the normalization of ideals.

2.1 Normalization of Ideals

This section examines several auxiliary constructions and devices to examine the integral closure of ideals, and to study the properties and applications to normalization of ideals.

A non direct construction of the integral closure of an ideal can be sketched as follows. Let R be a normal domain and let I be an ideal. $\overline{I}$ is the degree 1 component of the integral closure of the Rees algebra of I:

$$I \leadsto \overline{R[It]} = R + \boxed{\overline{I}t} + \overline{I^2}t^2 + \cdots \leadsto \overline{I}$$

This begs the issue since the construction of $\overline{R[It]}$, for arbitrary ideals, may verge on the impossibility. It takes place in a much larger setting (that of a presentation $R[It] = R[T_1, \ldots, T_n]/L$). By a *direct* construction $I \leadsto \overline{I}$ we mean an algorithm whose steps take place entirely in R. These are lacking in the literature. We provide in full details a simple situation, that of monomial ideals of finite colength.

The significant difference between the construction of the integral closure of an affine algebra A and that of $\overline{I}$ lies in the ready existence of *conductors*: Given A by generators and relations (at least in characteristic zero) the Jacobian ideal J of A has the property

$$J \cdot \overline{A} \subset A,$$

in other words, $\overline{A} \subset A : J$. This fact lies at the root of all current algorithms to build $\overline{A}$. There is no known corresponding *annihilator* for $\overline{I}/I$. In several cases, one can cheat by borrowing part of the Jacobian ideal of $R[It]$ by proceeding as follows. Let (this will be part of the cheat) $R[It] = R[T_1, \ldots, T_n]/L$ be a presentation of the Rees algebra of I. The Jacobian ideal is a graded ideal

$$J = J_0 + J_1 t + J_2 t^2 + \cdots,$$

with the components obtained by taking selected minors of the Jacobian matrix. This means that to obtain some of the generators of J_i we do not need to consider all the generators of L. Since J annihilates $\overline{R[It]}/R[It]$, we have that for each i

$$J_i \cdot \overline{I} \subset I^{i+1},$$

and therefore

$$\overline{I} \subset \bigcap_{i \geq 0} I^{i+1} : J_i.$$

Of course when using subideals $J_i' \subset J_i$, or further when only a few J_i' are used, the comparison gets overstated.

Despite these obstacles, in a number of important cases, one is able to understand relatively well the process of integral closure and normalization of ideals. These include monomial ideals and ideals of finite colength in regular local rings. Even here the full panoply of techniques of commutative algebra must be brought to play.

Definition 2.1 Let R be a quasi-unmixed normal domain and let I be an ideal.

(i) The *normalization index* of I is the smallest integer $s = s(I)$ such that

$$\overline{I^{n+1}} = I \cdot \overline{I^n} \quad n \geq s.$$

(ii) The *generation index* of I is the smallest integer $s_0 = s_0(I)$ such that

$$\sum_{n \geq 0} \overline{I^n} t^n = R[\overline{I}t, \ldots, \overline{I^{s_0}} t^{s_0}].$$

For example, if $R = k[x_1, \ldots, x_d]$ and $I = (x_1^d, \ldots, x_d^d)$, then $I_1 = \overline{I} = (x_1, \ldots, x_d)^d$. It follows that $s_0(I) = 1$, while $s(I) = \mathrm{r}_I(I_1) = d - 1$.

The notion of special fiber can also be useful in treating *normalization indices*. Let $(R, \mathfrak{m})$ be a local, normal domain and let I be an ideal such that its Rees algebra $A = R[It]$ has a finite integral closure $B = \sum_{n \geq 0} \overline{I^n} t^n$. We defined in the previous section, the notions of indices of normalization:

$$\begin{aligned} s(I) &= \inf\{n \mid B = \sum_{k \leq n} A\overline{I^k} t^k\} \\ s_0(I) &= \inf\{n \mid B = A[\overline{I}t, \ldots, \overline{I^k} t^k].\} \end{aligned}$$

This means that $s(I)$ measures the 'degree' of B as an A-module, while $s_0(I)$ measures the 'degree' of B as an A-algebra. If we set

$$F = \mathcal{F}(B) = B/(\mathfrak{m}, It)B = \sum_{n \geq 0} F_n,$$

we get an Artinian local ring and use Nakayama Lemma to derive simple relations between $s(I)$ and $s_0(I)$.

Proposition 2.2 *For an ideal I as above,*

$$\begin{aligned} s(I) &= \sup\{n \mid F_n \neq 0\}. \\ s_0(I) &= \inf\{n \mid F = F_0[F_1, \ldots, F_n]\}. \end{aligned}$$

Furthermore, if the index of nilpotency of F_i is r_i, then

$$s(I) \leq \sum_{i \leq s_0(I)} (r_i - 1).$$

Although these integers are well defined–since $\overline{R[It]}$ is finite over $R[It]$–it is not clear, even in case R is a regular local ring, which invariants of R and of I have a bearing on the determination of $s(I)$. An affirmative case is that of a monomial ideal I of a ring of polynomials in d indeterminates over a field–when $s \leq d - 1$ (according to Theorem 2.7).

Equimultiple Ideals

For primary ideals and some other equimultiple ideals there are relations between the two indices of normalization.

Proposition 2.3 *Let $(R,\mathfrak{m})$ be an integrally closed, local Cohen-Macaulay domain such that the maximal ideal $\mathfrak{m}$ is normal. Let I be $\mathfrak{m}$-primary ideal of indices of normalization $s(I)$ and $s_0(I)$. Then*

$$s(I) \leq e(I)((s_0(I)+1)^d - 1) - s_0(I)(2d-1) + 1),$$

where $e(I)$ is the multiplicity of I.

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Set $S = R[It]$ and denote by B its integral closure. We form the special fiber of B,

$$F = B/(\mathfrak{m}, It)B = \sum_{n\geq 0} F_n,$$

and following Proposition 2.2, we estimate $s(I)$ (the Castelnuovo-Mumford regularity of F) of the algebra F in terms of the indices of nilpotency of the components F_n, for $n \leq s_0(I)$.

Let $J = (z_1,\ldots,z_d)$ be a minimal reduction of I. For each component $I_n = \overline{I^n}$ of B, we collect the following data:

$$\begin{aligned} J_n &= (z_1^n,\ldots,z_d^n), \text{ a minimal reduction of } I_n \\ e(I_n) &= e(I)n^d, \text{ the multiplicity of } I_n \\ r_n = r_{J_n}(I_n) &\leq \frac{e(I_n)}{n}d - 2d + 1, \text{ a bound on the reduction number of } I_n. \end{aligned}$$

The last assertion follows from Theorem 1.38, once it is observed that $I_n \subset \overline{\mathfrak{m}^n} = \mathfrak{m}^n$, by the normality of $\mathfrak{m}$.

We are now ready to estimate the index of nilpotency of the component F_n. With the notation above, we have $I_n^{r_n+1} = J_n I_n^{r_n}$. When this relation is read in the special fiber ring F, it means that $r_n + 1 \geq$ index of nilpotency of F_n.

Following Proposition 2.2, we have

$$s(I) \leq 1 + \sum_{n=1}^{s_0(I)} r_n = 1 + \sum_{n=1}^{s_0(I)} e(I)dn^{d-1} - s_0(I)(2d-1) + 1,$$

which we approximate with an elementary integral to get the assertion. □

We can do considerably better when R is a ring of polynomials over a field of characteristic zero.

Reduction Number of Good Filtrations

We have, on several occasions, mentioned the construction of Rees algebras associated to filtrations different from adic ones. An important case is the integral closure filtration associated to an ideal. Let R be a quasi unmixed integral domain and let I be a nonzero ideal. Denote by

$$A = \sum_{n\geq 0} \overline{I^n}t^n,$$

the Rees algebra attached to $\{\overline{I^n}\}$. Noting that while A may not be a standard graded algebra, it is finitely generated over $R[It]$. We may thus apply the theory of Castelnuovo-Mumford regularity to A in order to deduce results about the reduction number of the filtration. We state one of these extensions.

We assume that $(R,\mathfrak{m})$ is a local ring and set $\ell(A) = \dim A/\mathfrak{m}A$ for the analytic spread of A (which is equal to the analytic spread of I). If J is a minimal reduction of I, setting $B = R[Jt]$, we can apply the previous results to the pair (B,A):

Theorem 2.4 *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring and let $\{I_n, \neq 0, I_0 = R\}$ be a multiplicative filtration such that the Rees algebra $A = \sum_{n\geq 0} I_n t^n$ is Cohen-Macaulay and finite over $R[I_1t]$. Suppose that* height $I_1 \geq 0$ *and let J be a minimal reduction of I_1. Then*

$$I_{n+1} = JI_n = I_1I_n, \quad n \geq \ell(I_1) - 1,$$

and in particular, A is generated over $R[It]$ by forms of degrees at most $\ell - 1 = \ell(I_1) - 1$,

$$\sum_{n\geq 0} I_n t^n = R[It, \ldots, I_{\ell-1}t^{\ell-1}].$$

Theorem 2.5 *Let $R = k[x_1, \ldots, x_d]$, where k is a field of characteristic zero and let I be a homogeneous ideal that is $(x_1, \ldots, x_d)$-primary. If $s(I)$ and $s_0(I)$ are the indices of normalization of I, then*

$$s(I) \leq (e(I) - 1)s_0(I) + 1,$$

where $e(I)$ is the multiplicity of I.

Proof. We begin by localizing R at the maximal homogeneous ideal and picking a minimal reduction J of I. We denote the associated graded ring of the filtration of integral closures $\{I_n = \overline{I^n}\}$ by G,

$$G = \sum_{n\geq 0} I_n/I_{n+1}.$$

In this affine ring we can take for a Noether normalization a ring $A = k[z_1, \ldots, z_d]$, where the z_i's are the images in G_1 of a minimal set of generators of J.

There are two basic algebraic facts about the algebra G. First, its multiplicity as a graded A-module is the same as that of the associated graded ring of I, that is,

$e(I)$. Second, since the Rees algebra of the integral closure filtration is a normal domain, so is the extended Rees algebra

$$C = \sum_{n\in\mathbb{Z}} I^n t^n,$$

and consequently the algebra $G = C/(t^{-1})$ will satisfy the condition S_1 of Serre. This means that as a module over A, C is torsionfree.

We now apply the theory of Cayley-Hamilton equations to the elements of the components of G (see [Va98b, Chapter 9]): For $u \in G_n$, we have an equation of integrality over A

$$u^r + a_1 u^{r-1} + \cdots + a_r = 0,$$

where a_i are homogeneous forms of A, in particular $a_i \in A_{ni}$, and $r \leq e(G) = e(I)$. Since k has characteristic zero, using the argument of [Va98b, Proposition 9.3.5], we obtain an equality

$$G_n^r = A_n G_n^{r-1}.$$

At the level of the filtration, this equality means that

$$I_n^r \subset J^n I_n^{r-1} + I_{nr+1},$$

which we weaken by

$$I_n^r \subset I \cdot I_{nr-1} + \mathfrak{m} I_{nr},$$

where we used

Proposition 2.6 *Let $R = k[x_1, \ldots, x_n]$ be a ring of polynomials over a field of characteristic zero. For any homogeneous ideal I and any positive integer r,*

$$\overline{I^r} \subset (x_1, \ldots, x_n)\overline{I^{r-1}}.$$

Finally, in the special fiber ring $F(B)$, this equation shows that the indices of nilpotency of the components F_n are bounded by $e(I)$, as desired. Now we apply Proposition 2.2 (and delocalize back to the original homogeneous ideals). □

Normalization of Monomial Ideals

In the case of monomial ideals the picture is very clear, according to:

Theorem 2.7 *Let R be a ring of polynomials over a field k, $R = k[x_1, \ldots, x_d]$, and let I be a monomial ideal. Then*

$$\overline{I^n} = I\overline{I^{n-1}}, \quad n \geq d. \tag{19}$$

Proof. Let $\overline{R}$ be the integral closure of the Rees algebra of the ideal I,

$$\overline{R} = \sum_{n\geq 0} \overline{I^n} t^n.$$

To prove the asserted equality, since the ideals are homogeneous, it is enough to localize at the maximal homogeneous ideal M of R. The ring $\overline{R}$ is Cohen-Macaulay by Hochster's theorem ([BH93, Theorem 6.3.5]).

We first assume that k is an infinite field, and apply Theorem 2.4 by letting J be a minimal reduction of I_1. Then

$$I_{n+1} = JI_n = I_1 I_n, \quad n \geq \ell(I_1) - 1,$$

and in particular, A is generated over $R[It]$ by forms of degrees at most $\ell - 1 = \ell(I_1) - 1$,

$$\sum_{n \geq 0} I_n t^n = R[It, \ldots, I_{\ell-1} t^{\ell-1}].$$

The extension to all fields is now immediate: The equality (19) of monomial ideals is equivalent of an equality of products of its monomial generators. Since these are independent of the characteristic or any field extension, being completely determined by the convex hull of the corresponding exponent vectors, the assertion is clear. □

The following normality criterion was first shown in [RRV2]:

Theorem 2.8 (Reid-Roberts-Vitulli) *Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials over a field k, and let I be a monomial ideal. If I^i is integrally closed for $i < d$ then I is normal.*

Corollary 2.9 *Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials over a field k, and let I be a monomial ideal. Then for $n \geq d - 1$, the ideal $\overline{I^n}$ is normal.*

Remark 2.10 Let R be a regular ring of dimension d and let I be one of its ideals. If $d = 2$, from Zariski's theory ([ZS60]),

$$\overline{I^n} = (\overline{I})^n, \quad \forall n \geq 1.$$

Moreover, by [LT81] it will follow that the integral closure of $R[It]$ is Cohen-Macaulay. For $d = 3$, no general bound exists.

Primary Ideals in Regular Rings

We discuss the role of Briançon-Skoda type theorems in determining some relationships between the coefficients $e_0(I)$ and $e_1(I)$ of the Hilbert polynomial of an ideal. We consider here the case of a normal local ring $(R, \mathfrak{m})$ of dimension d and of an $\mathfrak{m}$-primary ideal I. Set $A = R[It]$ and $B = \overline{R[It]}$; we assume that B is a finite A-module. From the exact sequence

$$0 \to \overline{I^n}/I^n \longrightarrow R/I^n \longrightarrow R/\overline{I^n} \to 0, \tag{20}$$

we obtain as above the relationship

$$\overline{e}_1(I) = e_1(I) + \hat{e}_0(I),$$

where $\hat{e}_0(I)$ is the multiplicity of the module of components $\overline{I^n}/I^n$, if $\dim L = d$; otherwise it is set to zero.

Theorem 2.11 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of infinite residue field. Suppose the Briançon-Skoda number of R is $c(R)$. Then for any $\mathfrak{m}$-primary ideal I,*

$$\bar{e}_1(I) \leq c(R) \cdot e_0(I).$$

In particular, $e_1(I) \leq c(R) \cdot e_0(I)$.

Proof. We recall the definition of $c = c(I)$: For any ideal L of R

$$\overline{L^{n+c}} \subset L^n, \quad \forall n.$$

To apply this notion to our setting, let J be a minimal reduction of I. Assume $\overline{I^{n+c}} \subset J^n$ for all n. To estimate the multiplicity of the module of components $\overline{I^{n+c}}/J^{n+c}$-which is the same as that of the module of components $\overline{I^n}/J^n$-note that $\overline{I^{n+c}} \subset J^n$, and that J^n admits a filtration

$$J^n \supset J^{n+1} \supset \cdots \supset J^{n+c},$$

whose factors all have multiplicity $e_0(J)$. More precisely, for each positive integer k,

$$\lambda(J^{n+k-1}/J^{n+k}) = e_0(J)\binom{n+k-1+d-1}{d-1} = \frac{e_0(J)}{(d-1)!}n^{d-1} + \text{lower terms}.$$

As a consequence we obtain

$$\bar{e}_1(I) \leq e_1(J) + c(R) \cdot e_0(J) = c(R) \cdot e_0(I),$$

since $e_1(J) = 0$. The other inequality, $e_1(I) \leq c(R) \cdot e_0(I)$, follows from (20). □

Corollary 2.12 *Let $(R, \mathfrak{m})$ be a Japanese regular local ring of dimension d. Then for any $\mathfrak{m}$-primary ideal I,*

$$\bar{e}_1(I) \leq (d-1)e_0(I), \quad e_1(I) \leq (d-1)e_0(I).$$

Proof. In this case, the classical Briançon-Skoda theorem asserts that $c(R) = d-1$. □

Normalization of Rees Algebras

The computation (and of its control) of the integral closure of a standard graded algebra over a field benefits greatly from Noether normalizations and of the structures built upon them. If $A = R[It]$ is the Rees algebra of the ideal I of an integral domain R, it does not allow for many such constructions. We would still like to develop some tracking of the complexity of the task required to build $\overline{A}$ (assumed A-finite) through sequences of extensions

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_n = \overline{A}$$

where A_{i+1} is obtained from an specific procedure P applied to A_i. (In [Va0], with the A_i satisfying the conditionS_2, the chains were called divisorial.) At a minimum, we would want to bound the length of such chains. We are going to show how Theorem 2.11 can be used to do just that for a class of Rees algebras.

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local of dimension d, integrally closed, of Briançon-Skoda number $c(R)$, and let I be an $\mathfrak{m}$-primary ideal of multiplicity $e_0(I)$. Let A and B be distinct algebras satisfying the S_2 condition of Serre, and such that

$$R[It] \subset A \subset B \subset \overline{R[It]}.$$

For any algebra D such as these, we set $\lambda(R/D_n)$ for its Hilbert function; for $n \gg 0$, one has the Hilbert polynomial

$$\lambda(R/D_n) = e_0(D)\binom{n+d-1}{d} - e_1(D)\binom{n+d-2}{d-1} + \text{lower terms}.$$

The Hilbert coefficients satisfy $e_0(D) = e_0(I)$, and according to Theorem 2.11, $0 \leq e_1(D) \leq c(R)e_0$.

Theorem 2.13 *For any two algebras A and B as above,*

$$c(R)e_0(I) > e_1(B) > e_1(A) \geq 0,$$

in particular any chain of such algebras has length bounded by $c(R)e_0(I)$.

Proof. Set $C = B/A$. Since A has Krull dimension $d+1$ and satisfies S_2, it follows easily that C is an A-module of Krull dimension d. From the exact sequence,

$$0 \to C_n \longrightarrow R/A_n \longrightarrow R/B_n \to 0,$$

one gets that the multiplicity $e_0(C)$ of C is $e_1(B) - e_1(A)$. As $e_0(C) > 0$, we have all the assertions. □

Corollary 2.14 *If $(R,\mathfrak{m})$ is a regular local ring of dimension d and I is an $\mathfrak{m}$-primary ideal, then $(d-1)e_0(I)$ bounds the lengths of the divisorial chains between $R[It]$ and $\overline{R[It]}$.*

Remark 2.15 In sections 3 and 4 of Chapter 1, several bounds for the lengths of divisorial chains of algebras were developed in terms of the multiplicity of the algebra. In the case of a Rees algebra $R[It]$, where $(R,\mathfrak{m})$ is a local ring, the relevant multiplicity would be $\deg(R[It]_{\mathfrak{P}})$ where $\mathfrak{P} = (\mathfrak{m}, R[It]_+)$. This number may however be considerably larger that multiplicities associated to the ideal I in R. For example, if $R = k[x_1,\ldots,x_d]$ is a ring of polynomials over the field k, and $I = (x_1,\ldots,x_d)$, then $\deg(R[It] = d$.

Question 2.16 There are two interlocked issues here:

(i) The overall bounds developed above obviously extend to certain classes of singular rings (e.g. non-regular local rings which are F-regular), but one should expects the existence of bounds, particularly for $\overline{e}_1(I)$ to be more prevalent (and not necessarily in the shape above). Even more challenging is the problem of handling higher dimensional ideals (with equimultiple ideals not representing a great challenge, hopefully).

(ii) Given the existence for bounds, there remains the problem of finding methods and approaches to build the sequences of extensions.

2.2 Integral Closure of Modules

Rees algebras of modules exhibit a significant new set of problems compared to ordinary Rees algebras. In this section we study the notion of integrality on modules and of algebras built on them. In addition to the general methods of Section 1 based on Jacobian criteria, we make use of the properties specific to modules and ways to convert the problems to the setting of ideals via deformation theory or through the intervention of Fitting ideals.

Let R be a Noetherian ring with total ring of fractions K and let E be a finitely generated R-module. We say that E has a *rank* if $K \otimes_R E \cong K^r$, in which case r is said to be the rank of E. By $S(E)$ we denote the symmetric algebra of E. As for the *Rees algebra* of E, as discussed in the Introduction, in an embarrassment of riches there are several possible definitions. The path chosen here is that which most resembles the classical case of the blowup algebra of an embedding. For a module E and a mapping $f : E \longrightarrow R^r$, we take this to mean the subalgebra of $S(R^r)$ generated by the forms in $f(E)$. In other words, $f(E)$ generates a multiplicative filtration of $S(R^r)$, and $\mathcal{R}_f(E)$ is the associated graded algebra. In this definition, one may as well assume that f is an embedding. For modules of rank r we have $f : E \to R^r$, the details of the embedding get coded as well in the structure of $\mathcal{R}_f(E)$.

For such modules we may actually adopt the following definition of their Rees algebras: Let R be a Noetherian ring and E a finitely generated R-module having a rank. The *Rees algebra* $\mathcal{R}(E)$ of E is $S(E)$ modulo its R-torsion submodule. If $S(E) = \mathcal{R}(E)$ then $\mathcal{R}(E)$ is said to be of *linear type*. By abuse of terminology, E is said to be of *of linear type*.

There are several algebras that arise in this fashion. First, assume $\mathbb{A}$ is a finite dimensional algebra over the field k. $\mathbb{A}$ may be a Lie algebra or have another special structure. The *commuting variety* is

$$\mathrm{C}(\mathbb{A}) = \{(u,v) \in \mathbb{A} \times \mathbb{A} \mid uv = vu\}.$$

The defining equations for such algebraic variety are obtained by picking a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{A}$ over k, and collecting the quadratic polynomials $f_i = f_i(\mathbf{X}, \mathbf{Y})$ that occur as coefficients of the generic commutation relation

$$\sum_{i=1}^{n} f_i e_i = [\sum_{i=1}^{n} X_i e_i, \sum_{i=1}^{n} Y_i e_i].$$

We can write these polynomials as

$$[f_1,\ldots,f_n] = [Y_1,\cdots,Y_n]\cdot\varphi,$$

where φ is a matrix of linear forms in the variables X_i. It follows that the affine ring of $C(\mathbb{A})$ is the symmetric algebra $S_{k[\mathbf{X}]}(\text{coker }\varphi)$. The generic component of this algebra is the Rees algebra $\mathcal{R}(\text{coker }\varphi)$. In several cases of interest (see [Va94b, Chapter 9]) there remains the issue of deciding when these algebras coincide: If $\mathbb{A}$ is a simple Lie algebra over a field of characteristic zero, it is known that coker (φ) is a torsionfree $k[\mathbf{X}]$-module of projective dimension two, and that

$$S_{k[\mathbf{X}]}(\text{coker }\varphi)_{\text{red}} = \mathcal{R}(\text{coker }\varphi).$$

Another major class of examples are the Rees algebras directly attached to affine algebras. Suppose A is an algebra essentially of finite type over the field k, and let $\Omega = \Omega_k(A)$ be its module of Kähler differentials. The Rees algebra of A is $\mathcal{R}(\Omega)$. It is the carrier of considerable information about the tangential variety of spec (A) and of its Gauss image (see [SSU2]). In this case, rarely ω is of linear type, and the construction of $\mathcal{R}(\Omega)$ gives a mechanism to extract "nonlinear relations" from the module Ω.

A third setting is provided by conormal modules, that is if $R = S/I$, the Rees algebra of I/I^2, under natural conditions such as I is a prime ideal or is generically a complete intersection, provides the means to study features of the associated graded ring of I though the examination of the natural surjection

$$\text{gr}_I(S) \to \mathcal{R}_{R/I}(I/I^2) \to 0.$$

It is worthwhile pointing out a significant difference between the Rees algebra of an ideal I and the Rees algebra of a module E of rank $r \geq 2$. While a Veronese subring $\mathcal{R}(I)^{(e)}$ of $\mathcal{R}(I)$ is a Rees algebra, the Veronese subring $\mathcal{R}(E)^{(e)}$, for $e \geq 2$, is *not* a Rees algebra.

Dimensions of Rees Algebras and of its Fibers

There are several measures of size attached to a Rees algebra $\mathcal{R}(E)$, all derived from ordinary Rees algebras. We briefly survey them.

Proposition 2.17 *Let R be a Noetherian ring of dimension d and E a finitely generated R-module having a rank r. Then*

$$\dim \mathcal{R}(E) = d + r = d + \text{height } \mathcal{R}(E)_+.$$

Proof. We may assume that E is torsionfree, in which case E can be embedded into a free module $G = R^r$. Now $\mathcal{R}(E)$ is a subalgebra of the polynomial ring $S = \mathcal{R}(G) = R[t_1,\ldots,t_r]$. As in the case of ideals, the minimal primes of $\mathcal{R}(E)$ are exactly of the form $\mathfrak{P} = \mathfrak{p}S \cap \mathcal{R}(E)$, where $\mathfrak{p}$ ranges over all minimal primes of R. Write $\bar{R} = R/\mathfrak{p}$ and $\bar{E}$ for the image of E in $\bar{R}\otimes_R G$. Since $\mathcal{R}(E)/\mathfrak{P} \cong \mathcal{R}_{\bar{R}}(\bar{E})$, we may replace R and E by $\bar{R}$ and $\bar{E}$ to assume that R is a domain. But then the assertions follow from the dimension formula for graded domains ([Va94b, 1.2.2]). □

A special but significant class of Rees algebras are those of *linear type*. They are the algebras of the form $\mathcal{R}(E) \cong S_R(E)$. We have already discussed this notion earlier in the case of ideals. It is hard to test this condition from a presentation of the module alone:

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

Nevertheless the following criterion is helpful ([Va94b, Theorem 1.3.5]).

Proposition 2.18 *Let R be an unmixed integral domain and E a torsionfree R-module, with a presentation as above. If $\mathcal{R}(E)$ is of linear type then*

$$\text{height } I_t(\varphi) \geq \text{rank}(\varphi) - t + 2, \quad 1 \leq t \leq \text{rank}(\varphi).$$

Reductions and Integral Closure of a Module

In analogy with the case of ideals, one introduces a key notion.

Definition 2.19 Let $U \subset E$ be a submodule. We say that U is a *reduction* of E or, equivalently, E is *integral* over U if $\mathcal{R}(E)$ is integral over the R-subalgebra generated by U.

Alternatively, the integrality condition is expressed by the equations $\mathcal{R}(E)_{s+1} = U \cdot \mathcal{R}(E)_s$, $s \gg 0$. The least integer $s \geq 0$ for which this equality holds is called the *reduction number* of E *with respect to* U and denoted by $r_U(E)$. For any reduction U of E the module E/U is torsion, hence U has the same rank as E. This follows from the fact that a module of linear type such as a free module admits no proper reductions.

Let E be a submodule of R^r. The *integral closure* of E in R^r is the largest submodule $\overline{E} \subset R^r$ having E as a reduction. (A broader definition exists, but we will limit ourselves to this case.)

If R is a local ring with residue field k then the *special fiber* of $\mathcal{R}(E)$ is the ring $\mathcal{F}(E) = k \otimes_R \mathcal{R}(E)$; its Krull dimension is called the *analytic spread* of E and is denoted by $\ell(E)$.

Now assume in addition that k is infinite. A reduction of E is said to be *minimal* if it is minimal with respect to inclusion. For any reduction U of E one has $\nu(U) \geq \ell(E)$ ($\nu(\cdot)$ denotes the minimal number of generators function), and equality holds if and only if U is minimal. Minimal reductions arise from the following construction: The algebra $\mathcal{F}(E)$ is a standard graded algebra of dimension $\ell = \ell(E)$ over the infinite field k. Thus it admits a Noether normalization $k[y_1, \ldots, y_\ell]$ generated by linear forms; lift these linear forms to elements $x_1, \ldots, x_\ell$ in $\mathcal{R}(E)_1 = E$, and denote by U the submodule generated by $x_1, \ldots, x_\ell$. By Nakayama's Lemma, for all large r we have $\mathcal{R}(E)_{r+1} = U \cdot \mathcal{R}(E)_r$, making U a minimal reduction of E.

Having established the existence of minimal reductions, we can define the *reduction number* $r(E)$ of E to be the minimum of $r_U(E)$, where U ranges over all minimal reductions of E.

Proposition 2.20 *Let R be a Noetherian local ring of dimension $d \geq 1$ and let E be a finitely generated R-module having rank r. Then*

$$\boxed{r \leq \ell(E) \leq d+r-1.} \tag{21}$$

Proof. We may assume that the residue field of R is infinite. Let $\mathfrak{m}$ be the maximal ideal of R and U any minimal reduction of E. Now $r = \text{rank}\, E = \text{rank}\, U \leq \nu(U) = \ell(E)$. On the other hand, by the proof of Proposition 2.17, $\mathfrak{m}\mathcal{R}(E)$ is not contained in any minimal prime of $\mathcal{R}(E)$. Therefore $\ell(E) = \dim \mathcal{F}(E) \leq \dim \mathcal{R}(E) - 1 = d+r-1$, where the last equality holds by Proposition 2.17. □

In some cases one can fix completely the analytic spread.

Proposition 2.21 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$, with an infinite residue field, and let E be a finitely generated R-module having rank r. Suppose $E \hookrightarrow R^r$ and $0 \neq \lambda(R^r/E) < \infty$. Then $\ell(E) = d+r-1$.*

Proof. Let $R^s \xrightarrow{\varphi} R^r$ be a homomorphism with $s = \ell(E)$ and such that the image F of φ is a minimal reduction of E. For each prime ideal $\mathfrak{p} \neq \mathfrak{m}$, $F_{\mathfrak{p}}$ is a reduction of $E_{\mathfrak{p}} = R^r_{\mathfrak{p}}$, and therefore $F_{\mathfrak{p}} = R^r_{\mathfrak{p}}$. Thus the cokernel of φ is a (nonzero) module of finite length. By the Eagon-Northcott theorem, the height d of the ideal of maximal minors of φ must satisfy $d \leq s-r+1$. Together with the estimate of Proposition 2.20, we obtain the desired equality. □

Buchsbaum-Rim Multiplicity

Let R be a Noetherian local ring of dimension d. The Buchsbaum-Rim multiplicity ([BR65]) arises in the context of an embedding

$$0 \to E \xrightarrow{\varphi} R^r \longrightarrow C(\varphi) \to 0,$$

where $C(\varphi)$ has finite length.

Denote by

$$\varphi : R^m \longrightarrow R^r$$

a matrix with image $\varphi \cong E$ and grade coker $\varphi \geq 2$. There is a homomorphism

$$S(\varphi) : S(R^m) \longrightarrow S(R^r)$$

of symmetric algebras, whose image is $\mathcal{R}(E)$, and whose cokernel we denote by $C(\varphi)$,

$$0 \to \mathcal{R}(E) \longrightarrow S(R^r) \longrightarrow C(\varphi) \to 0. \tag{22}$$

This exact sequence (with a different notation) is studied in [BR65] in great detail. Of significance for us is the fact that $C(\varphi)$, with the grading induced by the homogeneous homomorphism $S(\varphi)$, has components of finite length which for $n \gg 0$ satisfy

$$\lambda(C(\varphi)_n) = \frac{\mathrm{br}(E)}{(d+r-1)!}n^{d+r-1} + \text{lower degree terms}.$$

Here $\mathrm{br}(E)$ is a non vanishing positive integer (see Theorem 2.22) called the *Buchsbaum-Rim multiplicity* of φ or of E. (In fact, for one only needs to assume that coker φ is a module of finite length.)

It would be useful to have practical algorithms to find these multiplicities. We only make the comment that there are raw comparative estimates of these multiplicities. Let $\mathfrak{a}$ be any $\mathfrak{m}$-primary ideal contained in the annihilator of coker φ, for instance $\mathfrak{a} = I_r(\varphi)$. Then

$$\begin{aligned}\lambda(C(\varphi)_n) &\leq \lambda(S_n(R^r)/\mathfrak{a}^n S_n(R^r)) \\ &= \binom{r+n-1}{r-1}\lambda(R/\mathfrak{a}^n),\end{aligned}$$

since $\mathfrak{a}^n$ annihilates $C(\varphi)_n$. Thus

$$\mathrm{br}(E) \leq \binom{d+r-1}{r-1}\cdot e(\mathfrak{a})$$

where $e(\mathfrak{a})$ is the Hilbert-Samuel multiplicity of the ideal $\mathfrak{a}$.

One should expect $\mathrm{br}(E)$ to be, in most cases, not larger than the multiplicity of $I_r(\varphi)$. To illustrate the reason, let $(R,\mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ and consider the module $E = \mathfrak{m} \oplus \mathfrak{m}$ with its natural embedding into R^2. The ideal $I = I_2(\varphi) = \mathfrak{m}^2$, which has multiplicity 2^d. The Buchsbaum-Rim's polynomial is

$$\lambda(S_n(R^2)/E^n) = \lambda(S_n(R/I^n)) = (n+1)\binom{n+d-1}{d} = \frac{d+1}{(d+1)!}n^{d+1} + \text{lower terms},$$

so that $\mathrm{br}(E) = d+1$.

Theorem 2.22 *If $C(\varphi) \neq 0$ then for $n \gg 0$, $\lambda(S_n(R^r)/E^n)$ is a polynomial in n of degree $d+r-1$.*

Corollary 2.23 *Let R be a quasi-unmixed local ring of dimension $d > 0$ and let $F \subset E \subsetneq R^r$ be R-modules with $\lambda(R^r/F) < \infty$. Then F is a reduction of E if and only if* $\mathrm{br}(F) = \mathrm{br}(E)$.

Reduction Numbers of Modules

We discuss some tools to estimate reductions of modules. It makes use of the multiplicities of the special fibers. We begin by quoting some results from [SUV3].

Special Fibers and Buchsbaum-Rim Multiplicities

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and E an ideal module of rank r that is a vector bundle (that is, E is free on the punctured spectrum of R) but it is not R-free. There is one case when we have an explicit formula for the Buchsbaum-Rim multiplicity $\mathrm{br}(E)$. It arises as follows. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let

$$E \hookrightarrow R^r$$

be a vector bundle. Assuming that R has an infinite residue field, let F be a minimal reduction of E. According to Proposition 2.21, F is minimally generated by $d + r - 1$ elements and it is also an ideal module. In the terminology of [BR65], F is a parameter module. We shall refer to such modules as *complete intersection modules*.

Theorem 2.24 *Under the conditions above, let* $\varphi : R^{d+r-1} \longrightarrow R^r$ *be a mapping whose image is* U*. Then*

$$\mathrm{br}(E) = \mathrm{br}(U) = \lambda(R^r/U) = \lambda(R/\mathrm{det}_0(U)),$$

where $\mathrm{det}_0(U)$ *is the ideal generated by the* r *by* r *minors of* φ*.*

We quote some results from [SUV3]:

Proposition 2.25 *If* R *is a Cohen-Macaulay local ring with infinite residue field then* $\mathrm{br}_0(E) \leq \mathrm{br}(E)$.

Theorem 2.26 *Let* R *be a Noetherian local ring and* E *an ideal module that is a vector bundle, but not free. Then the special fiber* $\mathcal{F}(E)$ *has multiplicity at most* $\mathrm{br}_0(E)$.

Corollary 2.27 *Let* R *and* E *be as in* Theorem 2.26 *and suppose further that the residue field of* R *has characteristic zero. If* $\mathcal{F}(E)$ *satisfies the condition* S_1 *and is equidimensional then* $\mathrm{r}(E) \leq \mathrm{br}_0(E) - 1$*. If in addition* R *is Cohen-Macaulay with infinite residue field then* $r(E) \leq \mathrm{br}(E) - 1$.

The hypothesis on the condition S_1 always holds if R is a positively graded domain over a field and E is graded, generated by elements of the same degree. In this case $\mathcal{F}(E)$ embeds into $\mathcal{R}(E)$.

Theorem 2.28 *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d \geq 1$ *and let* $E \subsetneq R^r$ *be a submodule such that* $\lambda(R^r/E) < \infty$ *and* $\nu(E) \leq d + r - 1$*. For all* $n \geq 0$

$$\lambda(S_n(R^r)/E^n) = \mathrm{br}(E) \cdot \binom{n + d + r - 2}{d + r - 1}.$$

Reduction Number of a Module

Let us first clarify, in terms of the coefficients of the Buchsbaum-Rim polynomial, when a module E is a complete intersection.

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let E be an ideal module which is a vector bundle. Let $E \hookrightarrow R^r$ be the natural embedding and consider the Buchsbaum-Rim polynomial

$$\lambda(S_n(R^r)/E^n) = \mathrm{br}(E)\binom{n+d+r-2}{d+r-1} - b_1\binom{n+d+r-3}{d+r-2} + \text{lower terms},$$

with $\mathrm{br}(E) = \lambda(R^r/U)$, where U is a minimal reduction of E. From Theorem 2.28, we have that

$$\lambda(S_n(R^r)/U^n) = \lambda(R^r/U)\cdot\binom{n+d+r-2}{d+r-1},$$

and therefore

$$\lambda(E^n/U^n) = b_1\binom{n+d+r-3}{d+r-2} + \text{lower terms}.$$

In this expression, it follows that $b_1 \geq 0$. Actually one can explain the case when b_1 vanishes in the following manner. Consider the embedding of Rees algebras

$$0 \to \mathcal{R}(U) \longrightarrow \mathcal{R}(E) \longrightarrow \bigoplus_{n\geq 1} E^n/U^n \to 0.$$

This is an exact sequence of finitely generated modules over $\mathcal{R}(U)$, and since $\mathcal{R}(U)$ is Cohen-Macaulay of dimension $d+r$, the module on the right must have Krull dimension precisely $d+r-1$ unless it vanishes. This means that $b_1 = 0$ only if $U = E$. This is the module extension of the result for ideals. It would be interesting to have also $b_2 \geq 0$, as it occurs in the case of ideals.

We now turn to the question of the reduction number of the module E as above. We seek submodules U, minimally generated by $\ell(E) = d+r-1$ elements such that for some 'small' integer s, $\mathcal{R}(E)_{s+1} = U\cdot\mathcal{R}(E)_s$. For that, we consider the Hilbert function of the special fiber $\mathcal{F}(E)$ in order to apply [Va98b, Theorem 9.3.2].

Theorem 2.29 *Let R be a Cohen-Macaulay local ring of dimension $d > 1$, with infinite residue field, and let $E \subset R^r$ be a submodule such that $\lambda(R^r/E) < \infty$. The reduction number of E satisfies*

$$\mathrm{r}(E) \;\leq\; (d+r-1)\cdot\mathrm{br}(E) - 2(d+r-1) + 1 = \ell(E)\cdot\mathrm{br}(E) - 2\cdot\ell(E) + 1.$$

2.3 Extended Degree for the Buchsbaum-Rim Multiplicity

Let R be a Noetherian local ring of dimension d and let $E \subset R^r$ be a module of rank r. Suppose that the ideal $\det_0(E)$ has codimension c. The function

$$n \mapsto \deg(S_n(R^r)/E^n)$$

is the Hilbert function for the *extended degree* associated to the Buchsbaum-Rim multiplicity. If $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the associated prime ideals of R^r/E of maximal dimension, by the associativity formula for multiplicities,

$$\deg(S_n(R^r)/E^n) = \sum_{i=1}^{s} \lambda((S_n(R^r)/E^n)_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i).$$

For $n \gg 0$, this is a polynomial of degree $r + \max\{\dim R_{\mathfrak{p}_i}\} - 1$. When all these primes have the same codimension, say c,

$$\deg(S_n(R^r)/E^n) = \sum_{j=0}^{c+r-1} (-1)^j b_j(E) \binom{n+c+r-j-2}{c+r-j-1},$$

where the $b_j(E)$ are assembled from the Buchsbaum-Rim multiplicities of the modules $E_{\mathfrak{p}_i}$

$$b_j(E) = \sum_{i=1}^{s} b_j(E_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i).$$

Proposition 2.30 *Let R be a Cohen-Macaulay local ring and let $E \subset R^r$ be an equimultiple module of codimension c, and let F be a minimal reduction of E. Then*

$$b_0(E) = \deg(R^r/F).$$

Proof. According to the expression above for $b_0(E)$, we have

$$\begin{aligned} b_0(E) &= \sum_{i=1}^{s} b_0(E_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i) \\ &= \sum_{i=1}^{s} \lambda((R^r/F)_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i) = \deg(R^r/F), \end{aligned}$$

by the rule for computation of multiplicities. □

Let us make a slight extension to subfiltrations of the integral closure filtration $\overline{E^n}$. We are going to assume that E is equimultiple and that the algebra $C = \sum_{n\geq 0} \overline{E^n}$ is Noetherian, that is it is finite over $\mathcal{R}(F)$, for the minimal reduction F of E. We can define the extended degree function $\deg(S_n(R^r)/\overline{E^n})$. It has similar properties to those of $\deg(S_n(R^r)/E^n)$; the coefficients of the polynomial (for $n \gg 0$) will be denoted by $\overline{b}_j(E)$.

Theorem 2.31 *Let $(R, \mathfrak{m})$ be a normal Cohen-Macaulay local of dimension d and let $E \subset R^r$ be an equimultiple module of codimension c such that $C = \sum_{n\geq 0} \overline{E^n}$ is Noetherian. Let F be a minimal reduction of E and let A and B be distinct graded subalgebras*

$$\mathcal{R}(F) \subset A = \sum_{n\geq 0} A_n \subset B = \sum_{n\geq 0} B_n \subset C.$$

If A and B satisfy the condition S_2 of Serre, then $b_0(A) = b_0(B)$ and $b_1(A) < b_1(B)$. In particular, $b_1(A) < \overline{b}_1(E)$ for all proper subalgebras of C, and all chains of graded subalgebras of C satisfying the condition S_2 have length at most $\overline{b}_1(E)$.

Proof. Consider the exact sequence of $\mathcal{R}(F)$-modules

$$0 \to A \longrightarrow B \longrightarrow D \to 0.$$

Since A satisfies S_2, it follows that D has Krull dimension $d+r-1$. We first deal with the case $c = d$. Let us examine $\deg(D_n)$ using the exact sequence

$$0 \to B_n/A_n \longrightarrow S_n(R^r)/A_n \longrightarrow S_n(R^r)/B_n \to 0,$$

in which deg is just ordinary length. Since $\dim D = d+r-1$, $\deg(D_n)$ is just the ordinary Hilbert function and therefore it is a polynomial of degree $d+r-2$ for $n \gg 0$, of leading coefficient $e_0(D)$, the ordinary multiplicity of D. On the other hand, the Buchsbaum-Rim extended degree functions $\deg(S_n(R^r)/A_n)$ and $\deg(S_n(R^r)/B_n)$ are polynomials of degree $d+r-1$ for $n \gg 0$, so their leading coefficients must be equal $b_0(A) = b_0(B)$, and $b_1(B) = b_1(A) + e_0(D)$. Therefore $b_1(B) > b_1(A)$ since $e_0(D) > 0$. (Observe that comparing individually A and B, we get that $b_1(B) > 0$.)

Assume now $c < d$. Since the special fiber $\mathcal{R}(F) \otimes R/\mathfrak{m}$ has Krull dimension $c+r-1 < d+r-1$, the module $D \otimes R/\mathfrak{m}$ has Krull dimension $< d+r-1$ as well. This means that the associated primes of D of dimension $d+r-1$ are of the form $\mathfrak{P} = \mathfrak{p} + P \subset \mathcal{R}(F)$, where $\mathfrak{p} \neq \mathfrak{m}$ and P is a homogeneous subideal of $\mathcal{R}(F)_+$. Write $T = \mathcal{R}(F)/\mathfrak{P} = R/\mathfrak{p} + Q_0$. By [Va94b, Lemma 1.2.2], $\dim T = \dim R/\mathfrak{p} +$ height (Q_0). As a localization of an equimultiple module is still equimultiple, it follows that the minimal primes we must consider are still among the associated primes $\mathfrak{p}_i$ of R^r/F. This means that $\dim R/\mathfrak{p} = d-c$ and

$$\text{height }(Q_0) = \dim T - \dim R/\mathfrak{p} = d+r-1-(d-c) = c+r-1.$$

As a consequence, $\lambda((D_n)_\mathfrak{p})$ is a polynomial of degree exactly $c+r-2$ for $n \gg 0$.

Meanwhile the functions $\deg(S_n(R^r)/A_n)$ and $\deg(S_n(R^r)/B_n)$ are both polynomials of degree $c+r-1$ for $n \gg 0$, which allows us again to obtain the equalities $b_0(A) = b_0(B)$ and $b_1(B) = b_1(A) + b_0(D)$, and our assertions follow since $b_0(D) > 0$. □

2.4 Normality of Algebras of Linear Type

In this section we examine the normality of certain classes of modules. More precisely, we want to develop effective criteria of normality for algebras of linear type. In particular, given a normal domain R and a torsionfree module E with a free resolution,

$$\cdots \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} F_0 \longrightarrow E \to 0,$$

we study the role of the matrices of syzygies in the normality of the Rees algebra of E. This goal being, in general, not too realistic, we will restrict ourselves to those algebras which are of linear type, that is, in which $\mathcal{R}(E) = S_R(E)$. Our general reference for this section is [BV3].

An underlying motivation is the following open problem. Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field k of characteristic zero. If $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ over k, we consider as in (21) the *commuting variety* $\mathbf{V} = C(\mathfrak{g})$ of $\mathfrak{g}$ (see [Va94b, Chapter 9] for details). This is an irreducible variety and an important question is whether it is *normal*. (There is also a more pointed question on whether $\mathbf{V}$ is a rational singularity.)

From the perspective of Rees algebras, there is a torsionfree module E over the ring of polynomials $R = k[x_1, \ldots, x_n]$ such that $\mathcal{R}(E)$ is the affine ring of $\mathbf{V}$. The irreducibility of $\mathbf{V}$ means that

$$\mathcal{R}(E) \cong S(E)_{\text{red}}.$$

The structure of E carries considerable information about $\mathfrak{g}$. E has projective resolution

$$0 \to R^{\ell} \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^n \longrightarrow E \to 0,$$

in which ℓ is the rank of $\mathfrak{g}$.

The mapping φ is given as a skew-symmetric matrix of linear forms, obtained as a Jacobian matrix of the equations defining commutation in $\mathfrak{g}$. Then ψ is also a Jacobian matrix obtained as follows. Let $p_1, \ldots, p_\ell$ be the fundamental invariants of the adjoint action of the corresponding Lie group G. The mapping ψ is the Jacobian matrix of these polynomials. A fundamental fact about the ideal $I = I_\ell(\psi)$ is that it has codimension 3, and its radical has one or two components, according to the root length of $\mathfrak{g}$. Elementary calculations show that I is a prime ideal in the cases of the algebras A_2 and A_3.

This wealth of information makes an appealing case for a program to examine normality in Rees algebras of modules. In this section we develop some techniques targeted at simpler classes of modules.

The main result of this section (Theorem 2.42) characterizes normality in terms of the ideal $I_c(\psi)S(E)$ and of the completeness of the first s symmetric powers of E, where $c = \text{rank}\psi$, and $s = \text{rank}F_0 - \text{rank}E$. It requires that R be a regular domain. Some of the other characterizations, although valid only for restricted classes of modules, do not assume regularity in R. Furthermore, they are more accessible for computation.

- Complete intersection modules
- Complete intersection algebras
- Almost complete intersection algebras
- Algebras of linear type

Complete Intersection Modules

Let R be a Cohen-Macaulay ring and let E be a complete intersection module. Suppose that E is given by a mapping

$$\varphi : R^{r+c-1} \longrightarrow R^r.$$

This means that $E = \text{image } \varphi$ has rank r, and that the ideal $I = \det_0(E)$ has codimension $c \geq 2$.

Our purpose is to describe, entirely in terms of the ideal I, when E is integrally closed. It turns out that more is achieved.

Theorem 2.32 *Let R be a Cohen-Macaulay integrally closed domain and let E be a complete intersection module. The following conditions are equivalent:*

(a) *E is integrally closed.*

(b) *E is normal.*

(c) $\det_0(E)$ *is an integrally closed generic complete intersection.*

Moreover if $\det_0(E)$ *is a prime ideal, then the conditions above hold.*

Example 2.33 Let $\mathfrak{p}$ be a perfect ideal of codimension 2 generated by maximal minors of the $n \times (n+1)$ matrix φ. According to the assertion above, the columns of φ generate a normal submodule of R^n. In contrast, $\mathfrak{p}$ is often a non-normal ideal.

Let R be a Cohen-Macaulay normal domain and let E be a torsionfree R-module. The criteria above can be used to test the completeness of other modules that are not complete intersection modules. Let E be an ideal module of projective dimension $c-1 \geq 1$. It will follow that the associated primes of the cokernel of the embedding $E \hookrightarrow E^{**} = R^r$ have codimension at most c. This means that in comparing E to $\overline{E}$,

$$\overline{E}/E \hookrightarrow R^r/E,$$

to compare E to $\overline{E}$ we may consider only those localizations at the associated primes of R^r/E.

We will assume one additional requirement on E, that it satisfies the condition F_1 (or, G_∞) on the local number of generators,

$$\forall \mathfrak{p} \in \text{Spec }(R), \mathfrak{p} \neq (0), \quad \nu(E_\mathfrak{p}) \leq r + \dim R_\mathfrak{p} - 1.$$

This condition is always present when the module is of linear type. If the codimension of R^r/E is c, localizing at a prime ideal associated to R^r/E, $E_\mathfrak{p}$ is a complete intersection module. As a consequence of these observations we obtain:

Corollary 2.34 *Let E be an ideal module for which E^{**}/E is equidimensional. If E satisfies the condition F_1 then E is integrally closed if and only if* $\det_0(E)$ *is an integrally closed generic complete intersection.*

Complete Intersection Algebras

Let R be an integrally closed Cohen-Macaulay domain and let E be a finitely generated torsionfree R-module. It is clear that if the Rees algebra $\mathcal{R}(E)$ of E is a complete intersection then E has a projective resolution

$$0 \rightarrow R^m \xrightarrow{\varphi} R^n \longrightarrow E \rightarrow 0, \tag{23}$$

and it is of linear type. According to [Va94b, Theorem 3.1.6], for $S(E)$ to be a complete intersection we must have

$$\text{grade } I_t(\varphi) \geq \text{rank}(\varphi) - t + 1 = m - t + 1, \quad 1 \leq t \leq m.$$

Moreover, since $S(E) = \mathcal{R}(E)$ is an integral domain, this requirement will hold modulo any nonzero element of R, so it will be strengthened to

$$\text{grade } I_t(\varphi) \geq \text{rank}(\varphi) - t + 2 = m - t + 2, \quad 1 \leq t \leq m.$$

It can be rephrased in terms of the local number of generators as

$$\nu(E_{\mathfrak{p}}) \leq n - m + \text{height } \mathfrak{p} - 1, \quad \mathfrak{p} \neq 0.$$

For these modules, the graded components of the Koszul complex of the forms of $A = R[T_1, \ldots, T_n]$ (see [Av81]),

$$[f_1, \ldots, f_m] = [T_1, \ldots, T_n] \cdot \varphi,$$

give R-projective resolutions of the symmetric powers of E:

$$0 \rightarrow \wedge^s R^m \rightarrow \wedge^{s-1} R^m \otimes R^n \rightarrow \cdots \rightarrow R^m \otimes S_{s-1}(R^n) \rightarrow S_s(R^n) \rightarrow S_s(E) \rightarrow 0.$$

In particular we have:

Proposition 2.35 *Let R be a Cohen-Macaulay normal domain and let E be a finitely generated torsionfree R-module such that the Rees algebra $\mathcal{R}(E)$ is a complete intersection defined by m equations. The non-normal R-locus of $\mathcal{R}(E)$ has codimension at most $m+1$.*

Proof. It suffices to consider the following observation. For any torsionfree R-module G, contained in a free module F, the embedding

$$\overline{G}/G \hookrightarrow F/G$$

shows that if G has projective dimension $\leq r$, the associated primes of $\overline{G}/G$ have codimension at most $r+1$. In the case of the symmetric powers $S_s(E)$ of E, the projective dimensions are bounded by m, from the comments above. $\square$

The next aim is to discuss the normality of the algebra $\mathcal{R}(E)$ versus the completeness of the module E and of a few other symmetric powers.

General Complete Intersections

When $S(E)$ is defined by more than one hypersurface,

$$S(E) = R[T_1,\ldots,T_n]/(f_1,\ldots,f_m), \quad [f_1,\ldots,f_m] = [T_1,\ldots,T_n]\cdot\varphi,$$

it may be more convenient to use an algorithmic formulation.

Theorem 2.36 *Let E be a module as above. $S(E)$ is normal if and only if the following conditions hold:*

(i) *For every prime ideal $\mathfrak{p} \supset I_t(\varphi)$ of height $m-t+2$, $R_\mathfrak{p}$ is a regular local ring.*

(ii) *The modules $S_s(E)$ are complete, for $s = 1\ldots m$.*

Proof. Let us assume that $S(E)$ is normal, and establish (i). Let $\mathfrak{p}$ be a prime ideal as in (i). Since height $I_{t-1}(\varphi) \geq m-t+3$, localizing at $\mathfrak{p}$, we obtain a presentation of the module in the form (we set still $R = R_\mathfrak{p}$)

$$0 \to R^{m-t+1} \xrightarrow{\varphi'} R^{n-t+1} \longrightarrow E \to 0,$$

where the entries of φ' lie in $\mathfrak{p}$. Changing notation, this means that we can assume that all entries of φ lie in the maximal ideal $\mathfrak{p}$. Setting $A = R[T_1,\ldots,T_n]$, and $P = \mathfrak{p}A$,

$$S(E)_P = (A/(f_1,\ldots,f_m))_P,$$

and therefore if S_P is a DVR, A_P must be a regular local ring, and therefore R will also be a regular local ring.

For the converse, let P be a prime ideal of the ring of polynomials A, of height $m+1$, containing the forms f_i's, and set $\mathfrak{p} = P\cap R$. The normality of $S(E)$ means that for all such P, $S(E)_P$ is a DVR. The claim is that failure of this to hold is controlled by either (i) or (ii).

We may localize at $\mathfrak{p}$. Suppose that height $\mathfrak{p} = m+1$. In this case, there exists $z \in \mathfrak{p}$ so that $z, f_1,\ldots,f_m$ is a regular sequence in $\mathfrak{p}A$. If $x_1,\ldots,x_{m+1}$ is a regular system of parameters of the local ring R, from a representation

$$[z, f_1,\ldots,f_m] = [x_1,\ldots,x_{m+1}]\cdot\psi,$$

we obtain the socle equality

$$(z, f_1,\ldots,f_m) : \mathfrak{p}A = (z, f_1,\ldots,f_m, \det\psi).$$

The image u of $z^{-1}\det\psi$, provides us with a nonzero form of degree m, in the field of fractions of $S(E)$. If $S(E)_P$ is not a DVR,

$$\mathrm{Hom}_{S(E)}(PS(E)_P, PS(E)_P) = (PS(E)_P)^{-1},$$

and, given that $u \in (PS(E)_P)^{-1}$, we obtained a fresh element in the integral closure of $S_m(E)$.

On the other hand, if height $\mathfrak{p} \leq m$, we may assume that for some $1 < t \leq m$, $I_t(\varphi) \subset \mathfrak{p}$, but $I_{t-1}(\varphi) \not\subset \mathfrak{p}$. This implies that height $\mathfrak{p} = m-t+2$. We can localize at $\mathfrak{p}$, and argue as in the previous case. □

Almost Complete Intersection Algebras

An *almost complete intersection* Rees algebra arises from a module E with a projective resolution

$$0 \to R \xrightarrow{\psi} R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

For the module to be of linear type, $S(E) = \mathcal{R}(E)$, the roles of the determinantal ideals $I_t(\varphi)$ and of $I = I_1(\psi)$ are less well behaved than in the case of complete intersections. They are nevertheless determined by any minimal resolution of E, the $I_t(\varphi)$ giving the Fitting ideals of E, while $I_1(\psi)$ is the annihilator of $\mathrm{Ext}^2_R(E,R)$. Let us give a summary of some of the known results, according to [Va94b, Section 3.4]:

Theorem 2.37 *Let R be a Cohen-Macaulay integral domain and let E be a module with a resolution as above.The following hold:*

(a) *If* $S(E) = \mathcal{R}(E)$ *then* height $I_1(\psi)$ *is odd.*

(b) *If I is a strongly Cohen-Macaulay ideal of codimension* 3, *satisfying the condition* $\mathcal{F}_1$, *and* E^* *is a third syzygy module, then* $S(E)$ *is a Cohen-Macaulay integral domain.*

This shows the kind of requirement that must be present when one wants to construct integrally closed Rees algebras in this class.

Example 2.38 Let $R = k[x_1,\ldots,x_d]$ be a ring of polynomials. In [Ve73], for each $d \geq 4$, it is described an indecomposable vector bundle on the punctured spectrum of R, of rank $d-2$. Its module E of global sections has a resolution

$$0 \to R \xrightarrow{\psi} R^d \xrightarrow{\varphi} R^{2d-3} \longrightarrow E \to 0,$$

with φ having linear forms as entries, and $\psi(1) = [x_1,\ldots,x_d]$. If d is odd, according to [SUV93, Corollary 3.10], E is of linear type and normal.

The analysis of the normality of the two previous classes of Rees algebras was made simpler because they were naturally Cohen-Macaulay. This is not the case any longer with almost complete intersections, requiring that the S_2 condition be imposed in some fashion. We pick one closely related to normality.

Theorem 2.39 *Let R be a regular integral domain and let E be a torsionfree module whose second Betti number is* 1. *Suppose E is of linear type. If the ideal* $I_1(\psi)\mathcal{R}(E)$ *is principal at all localizations of* $\mathcal{R}(E)$ *of depth* 1 *then* $\mathcal{R}(E)$ *satisfies the condition* S_2 *of Serre.*

Proof. The condition on $L = I_1(\psi)\mathcal{R}(E)$ means that for any localization S_P, $S = \mathcal{R}(E)$, with depth $S_P = 1$, the ideal L_P is principal. There are several global ways to recast this, such as $(L \cdot L^{-1})^{-1} = S$.

We may assume that R is a local ring, and that the resolution of E is minimal. Pick in $A = R[T_1, \ldots, T_n]$ a prime ideal P for which depth $S_P = 1$. We must show that $\dim S_P = 1$. As in the other cases, we may assume that $P \cap R$ is the maximal ideal of R.

We derive now a presentation of the ideal $J = J(\varphi) = (f_1, \ldots, f_m)A$, modulo J:

$$0 \to K \to (A/J)^m = S^m \longrightarrow J/J^2 \to 0,$$

and analyze the element of S^m induced by $v = \psi(1)$. This is a nonzero 'vector' whose entries in S_P^m generate L_P, which by assumption is a principal ideal. This means that $v = \alpha v_0$, for some nonzero $\alpha \in S_P$, where v_0 is an unimodular element S_P^m. Since $v \in K$, this means that the image u of v_0 in $(J/J^2)_P$, is a torsion element of the module. Two cases arise. If $u = 0$, $(J/J^2)_P$ is a free S_P since it has also rank $m-1$, and therefore J_P is a complete intersection in the regular local ring A_P, according to [Va67]. This implies that the Cohen-Macaulay local ring S_P has dimension 1. On the other hand, if $u \neq 0$, u is a torsion element of $(J/J^2)_P$ which is also a minimal generator of the module. We thus have that in the exact sequence

$$0 \to \text{torsion } (J/J^2)_P \to (J/J^2)_P \to C \to 0,$$

C is a torsionfree S_P-module, of rank $m-1$, generated by $m-1$ elements. We thus have that the ideal J_P of the regular local ring A_P has the property

$$J_P/J_P^2 \cong (A_P/J_P)^{m-1} \oplus (\text{torsion}).$$

According to [Va67] again, J_P is a complete intersection since it has codimension $m-1$. This shows that S_P must have dimension 1. □

In studying the normality in an algebra $S = S(E)$ (E torsionfree) of linear type, the advantage of S_2 holding is extremely useful. To recall briefly some technical facts from [Va94b, p. 138]. For a prime ideal $\mathfrak{p} \subset R$ there is an associated prime ideal in $S(E)$ defined by

$$T(\mathfrak{p}) = \ker(S_R(E) \longrightarrow S_{R/\mathfrak{p}}(E/\mathfrak{p}E)_\mathfrak{p}).$$

In our case, $T(\mathfrak{p})$ is the contraction $\mathfrak{p}S(E)_\mathfrak{p} \cap S(E)$, so height $T(\mathfrak{p}) =$ height $\mathfrak{p}S(E)$.

The prime ideals we are interested in are those of height 1. If R is equidimensional, we see that

$$\text{height } T(\mathfrak{p}) = 1 \text{ if and only if } \nu(E_\mathfrak{p}) = \text{height } \mathfrak{p} + \text{rank}(E) - 1.$$

Let us quote [Va94b, Proposition 5.6.2]:

Proposition 2.40 *Let R be a universally catenarian Noetherian ring and let E be a finitely generated module such that $S(E)$ is a domain. Then the set*

$$\{T(\mathfrak{p}) \mid \text{height } \mathfrak{p} \geq 2 \textit{ and } \text{height } T(\mathfrak{p}) = 1\}$$

is finite. More precisely, for any presentation $R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0$*, this set is in bijection with*

$$\{\mathfrak{p} \subset R \mid E_{\mathfrak{p}} \textit{ not free}, \ \mathfrak{p} \in \mathrm{Min}(R/I_t(\varphi)) \textit{ and } \mathrm{height}\ \mathfrak{p} = \mathrm{rank}(\varphi) - t + 2\},$$

where $1 \leq t \leq \mathrm{rank}(\varphi)$.

Theorem 2.41 *Let R be a regular integral domain and let E be a torsionfree module with a free resolution*

$$0 \to R \xrightarrow{\psi} R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

Suppose E is of linear type. E is normal if and only if the following conditions hold:

(i) *The ideal* $I_1(\psi)S(E)$ *is principal at all localizations of* $S(E)$ *of depth* 1.

(ii) *The modules* $S_s(E)$ *are complete, for* $s = 1 \ldots m-1$.

Proof. We only have to show that (i) and (ii) imply that E is normal. In view of Theorem 2.39, it suffices to verify the condition R_1 of Serre.

Let $P \subset A$ be a prime ideal such that its image in $S = A/J$ has height 1. We may assume that $P \cap R$ is the maximal ideal of R and that E has projective dimension 2, that is $I_1(\psi) \subset \mathfrak{m}$ as otherwise we could apply Theorem 2.36.

From Proposition 2.40, and the paragraph preceding it, $\dim R = \mathrm{rank}(\varphi) - t + 2$ on the one hand and $\dim R = n - r + 1$ on the other. Thus, $t = 1$ and $m = \dim R$.From Proposition 2.40, and the paragraph preceding it, $\dim R = \mathrm{rank}(\varphi) - t + 2$ on the one hand and $\dim R = n - r + 1$ on the other. Thus, $t = 1$ and $m = \dim R$. Pick $0 \neq a \in \mathfrak{m}$ and consider the ideal $I = (a, f_1, \ldots, f_m) \subset A$. With $P = \mathfrak{m}A$, set $L = I : P$. For a set $x_1, \ldots, x_m$ of minimal generators of $\mathfrak{m}$, we have

$$[a, f_1, \ldots, f_m] = [x_1, \ldots, x_m] \cdot B(\Phi), \tag{24}$$

where $B(\Phi)$ is a $m \times (m+1)$ matrix whose first column has entries in R, and the other columns are linear forms in the T_i's. We denote by L_0 the ideal of A generated by the minors of order m that fix the first column of $B(\Phi)$. These are all forms of degree $m-1$, and $L_0 \subset L$. When we localize at P however, $L_0A_P = LA_P$, since by condition (i) and the proof of Theorem 2.39, J_P is a complete intersection and the assertion follows from a classical result of Northcott. This means that the image C of $a^{-1}L_0$ in the field of fractions of S is not contained in S and has the property that $C \cdot PS_P \subset S_P$, giving rise to two possible outcomes:

$$C \cdot PS_P = \begin{cases} PS_P \\ S_P \end{cases} \tag{25}$$

In the first case, C would consist of elements in the integral closure of S_P but it not contained in S_P. This cannot occur since by condition (ii) all the symmetric powers of E, up to order $m-1$, are complete. This means that the second possibility occurs, and S_P is a DVR. □

Effective Criteria

We are going to make a series of observations leading to the proof of a generalization of Theorem 2.41 to all algebras of linear type over regular domains. The arguments show great similarity for the following technical reason. If E is a torsion-free R-module of linear type, a prominent role is played by the prime ideals of $S(E)$ of codimension one, which we denoted $T(\mathfrak{p})$. If $\mathfrak{p}$ is the unique maximal ideal-as a reduction will lead to-height $T(\mathfrak{p}) = 1$ if and only if $\dim R + \mathrm{rank}(E) = \nu(E) + 1$, a condition equivalent to (when E has finite projective dimension) the second Betti number of E is 1.

Theorem 2.42 *Let R be a regular integral domain and let E be a torsionfree module of rank r, with a free presentation*

$$R^p \xrightarrow{\psi} R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

Suppose E is of linear type. $\mathcal{R}(E)$ is normal if and only if the following conditions hold:

(i) *The ideal $I_c(\psi)S(E)$, $c = m + r - n$, is principal at all localizations of $S(E)$ of depth 1.*

(ii) *The modules $S_s(E)$ are complete, for $s = 1 \dots n - r$.*

Proof. We note that $n - r$ is the height of the defining ideal $J(\varphi)$ of S, while $c = m + r - n$ is the rank of the second syzygy module.

The necessity of these conditions follows *ipso literis* from the discussion of Theorem 2.41, except for a clarification in the role of condition (i). We consider the complex induced by tensoring the tail of the presentation by S,

$$S^p \xrightarrow{\overline{\psi}} S^m \longrightarrow J/J^2 \to 0.$$

Note that $\overline{\psi}$ has rank c, while J/J^2 has rank $n - r$. This means that the kernel of the natural surjection $C = \mathrm{coker}\,(\overline{\psi}) \to J/J^2$ is a torsion S-module.

Lemma 2.43 *Let E be a module as above, set $A = R[T_1, \dots, T_n]$ and $J = (f_1, \dots, f_m) = [T_1, \dots, T_n] \cdot \varphi$ the defining ideal of $S(E) = A/J$. Let $P \supset J$ be a prime ideal of A. If $I_c(\overline{\psi})S_P$ is a principal ideal then J_P is a complete intersection.*

Proof. According to [Va98b, Proposition 2.4.5], since the Fitting ideal $I_c(\overline{\psi})_P$ is principal, the module C_P decomposes as

$$C_P \cong S_P^{n-r} \oplus (\text{torsion}).$$

This means that we have a surjection

$$S_P^{n-r} \to (J_P/J_P^2)/(\text{torsion})$$

of torsionfree S_P-modules of the same rank. Therefore

$$J_P/J_P^2 \cong S_P^{n-r} \oplus (\text{torsion}).$$

At this point, we invoke [Va67] to conclude that J_P is a complete intersection. □

The rest of the proof of Theorem 2.42 would proceed as in the proof of Theorem 2.41. Choosing $P = \mathfrak{m}A$ so that S_P has dimension 1, J_P is a complete intersection of codimension $n - r = \dim R - 1 = d - 1$. As in setting up the equation (24), we pick $0 \neq a \in \mathfrak{m}$, pick a minimal set $\{x_1, \ldots, x_d\}$ of generators for $\mathfrak{m}$, and define the matrix $B(\Phi)$

$$[a, f_1, \ldots, f_m] = [x_1, \ldots, x_d] \cdot B(\Phi). \tag{26}$$

Note that when localizing at P the ideals $(a, f_1, \ldots, f_m)_P$ and $(x_1, \ldots, x_d)_P$ are generated by regular sequences and we can use the same argument employed in the proof of Theorem 2.41. □

2.5 Normalization of Modules

Let R be a normal domain and consider a module with an embedding $E \hookrightarrow R^r$ ($r = \operatorname{rank}(E)$). In this section we will show how the issues of normalization of ideals and modules resemble one another. We will denote by E_n the components of the Rees algebra $\mathcal{R}(E)$.

The *normalization* problem for a module of E is understood here in several senses. A strong one as either the description of the integral closure of $\mathcal{R}(E)$ or its construction. In more general terms, as the study of the relationships (expressed in the comparison between their invariants) between the two algebras $\mathcal{R}(E)$ and $\overline{\mathcal{R}(E)}$.

The normalization problem for modules over two-dimensional regular local rings is well clarified in ([Ko95]; see also [KK97]):

Theorem 2.44 *Let R be a two-dimensional regular local ring and let E be a finitely generated torsionfree R-module. Then* $\overline{\mathcal{R}(E)} = \mathcal{R}(\overline{E})$*, and it is a Cohen-Macaulay algebra.*

In higher dimensions, or for nonregular two-dimensional normal domains, the picture is much more diffuse. To exhibit some of this diversity, first we make several observations on how the transiting between Rees algebras of ideals and of modules can take place besides the technique of Bourbaki sequences. Let $E \subset R^r$ be a submodule of a free module, and denote by $\mathcal{R}(E) \subset \mathcal{R}(R^r) = S$ the corresponding Rees algebras. We can consider the ideal (E) of S generated by the 1-forms in E. In other words, (E) is the ideal of relations of $S/(E) = S_R(R^r/E)$. In particular some properties of (E) can be examined through the general tools developed for symmetric algebras. Thus for example, to examine the codimension of (E) we may replace E by one of its reductions F, and focus on $S_R(R^r/F)$.

Our aim in this section being normalization, we start by providing a source of complete modules derived from ideals.

Proposition 2.45 *Let M be a graded ideal of the polynomial ring $S = R[x_1,\ldots,x_d]$, $M = \oplus_{n\geq 0} M_n$. If M is an integrally closed S-ideal then each component M_n is an integrally closed R-module.*

Proof. M_n is a R-module of the module S_n freely R-generated by the monomials T_α in the x_i of degree n. Denote by $\mathcal{R}(M_n) \subset \mathcal{R}(S_n)$ the corresponding Rees algebras. Let $u \in S_n$ be integral over M_n; there is an equation in $\mathcal{R}(S_n)$ of the form

$$u^m + a_1 u^{m-1} + \cdots + a_m = 0, \quad a_i \in M_n^i.$$

Map this equation using the natural homomorphism $\mathcal{R}(S_n) \to S$, that sends the variable T_α into the corresponding monomial of S. The equation converts into an equation of integrality of $u \in S$ over the ideal M. Since $M = \overline{M}$, $u \in M_n$. □

Proposition 2.46 *Let $S = \mathcal{R}(R^r) = R[x_1,\ldots,x_r]$ and denote by (E) the ideal of S generated by the forms in E. For any positive integer n,*

$$\overline{E_n} = \overline{(E^n)}_n.$$

Proof. It suffices to verify the equality of these two integrally closed modules at the valuations of R. At some valuation V, $VS = V[z_1,\ldots,z_r]$ and VE is generated by the forms $a_1z_1,\ldots,a_rz_r$, with $a_i \in V$. We must show that the ideal (VE) is normal. We may assume that a_1 divides all a_i, $a_i = a_1b_i$, so that the ideal $(VE) = a_1(z_1,b_2z_2,\ldots,b_rz_r)$, and it will be normal if and only if it is the case for $(z_1,b_2z_2,\ldots,b_rz_r)$. Obviously we can drop the indeterminate z_1 and iterate. □

We shall now apply the Briançon-Skoda theory to the ideal (E) and to its powers. To illustrate, suppose that R is a regular domain of dimension d so that the localizations of S are regular of dimension at most $d+r$; its Briançon-Skoda number is therefore $c(S) \leq d+r-1$. Since the analytic spread of E is $d+r-1$, the Briançon-Skoda number of E is $c \leq \ell(E) - 1 = d+r-2$.

Let $E \hookrightarrow R^r$ be a module of finite colength and let F be a minimal reduction of E (we may assume that R has infinite residue field). F is a complete intersection module, generated by $d+r-1$ elements ($\dim R = d$). From the definition of $c = c(E)$, we have

$$\overline{(E^{n+c})} \subset (F^n), \quad n \geq 1,$$

which in degree $n+c$ by the Proposition above can be written

$$\overline{E^{n+c}} \subset (F^n)_{n+c} = F^n S_c, \quad n \geq 1.$$

Consider now the corresponding filtrations

$$0 \to \overline{E^{n+c}}/F^{n+c} \longrightarrow S_{n+c}/F^{n+c} \longrightarrow S_{n+c}/\overline{E^{n+c}} \to 0. \qquad (27)$$

As in the case of ideals,

$$\lambda(\overline{E^{n+c}}/F^{n+c}) = \lambda(S_{n+c}/F^{n+c}) - \lambda(S_{n+c}/\overline{E^{n+c}}) - \leq \lambda(S_{n+c}/F^{n+c}) - \lambda(S_{n+c}/F^nS_c),$$

since $\overline{E^{n+c}} \subset F^n S_c$, by the definition of $c(E)$.

We now identify these numerical functions. The simplest to do is $\lambda(S_{n+c}/F^{n+c})$. This, according to Theorem 2.28, is given by the Buchsbaum-Rim polynomial of F,

$$\lambda(S_{n+c}/F^{n+c}) = \text{br}(F)\cdot\binom{n+c+d+r-2}{d+r-1}.$$

Here $\text{br}(F) = \text{br}(E) = \lambda(R^r/F)$ is the Buchsbaum-Rim multiplicity of E. More generally, we use the following notation for the Buchsbaum-Rim polynomials of E and of the integral closure of $\mathcal{R}(E)$:

$$\begin{aligned} P(n) &= \lambda(S_n/E^n) = \text{br}(E)\binom{n+d+r-2}{d+r-1} - \text{br}_1(E)\binom{n+d+r-3}{d+r-2} + \text{lower terms} \\ \overline{P}(n) &= \lambda(S_n/\overline{E^n}) = \overline{\text{br}}(E)\binom{n+d+r-2}{d+r-1} - \overline{\text{br}}_1(E)\binom{n+d+r-3}{d+r-2} + \text{lower terms} \end{aligned}$$

For any graded subalgebra $\mathcal{R}(E) \subset A \subset \overline{\mathcal{R}(E)}$, one can define $\lambda(S_n/A_n)$, and associate with it the Buchsbaum-Rim polynomial. In such case, the coefficients will be denoted $\text{br}(A)$, $\text{br}_1(A)$, and so on.

Theorem 2.47 *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay normal local domain of infinite residue field. Let E be a module of rank $r \geq 2$ and S as above such that the Rees algebra $\mathcal{R}(E)$ has finite integral closure. If the Briançon-Skoda number of S is $c(S)$, then*

$$\overline{\text{br}}_1(E) \leq \text{br}(E)\cdot\binom{r+c-1}{c-1}.$$

In addition,

$$\text{br}_1(E) \leq \text{br}(E)\cdot\binom{r+c-1}{c-1}.$$

Proof. According to the definition of $c = c(S)$ and the discussion above, $\overline{E^{n+c}} \subset F^n S_c$, for $n > 0$. We set up the exact sequences

$$0 \to \overline{E^{n+c}}/F^{n+c} \longrightarrow S_{n+c}/F^{n+c} \longrightarrow S_{n+c}/\overline{E^{n+c}} \to 0.$$

For $n \gg 0$, we use it to make comparisons between the coefficients of the polynomials. First note the embedding $\overline{E^{n+c}}/F^{n+c} \hookrightarrow F^n S_c/F^{n+c}$, which means that the Hilbert polynomial $p(n)$ of the $\mathcal{R}(F)$-module

$$C = \bigoplus_{n\geq 0} F^n S_c/F^{n+c}$$

dominates that of $\bigoplus_{n\geq 1} \overline{E^{n+c}}/F^{n+c}$, so that we have

$$p(n+c) \geq P(n+c) - \overline{P}(n+c), \quad n \gg 0.$$

Since C is annihilated by a power of $\mathfrak{m}$, its Krull dimension is at most $\dim \mathcal{R}(F) - 1 = d+r-1$, and consequently the polynomial $p(n)$ has degree at most $d+r-2$. Comparing leading coefficients for $P(n+c)$ and $\overline{P}(n+c)$, this implies that $\overline{\text{br}}(E) = \text{br}(E)$. One also has the inequality

$$0 \leq \overline{\text{br}}_1(E) \leq e_0(C),$$

where $e_0(C)$ is set to the multiplicity of C if $\dim C = d+r-1$, or to 0 otherwise. We are going to see that $\dim C = d+r-1$ and calculate its multiplicity. First note that C is defined by the exact sequence

$$0 \to \mathcal{R}(F) \longrightarrow R \oplus F \oplus \cdots \oplus F^{c-1} \oplus \mathcal{R}(F)S_c \longrightarrow C \to 0.$$

Since $\mathcal{R}(F)$ is Cohen-Macaulay, it follows that C has dimension precisely $\dim \mathcal{R}(F) - 1$.

The determination of $e_0(C)$ is more delicate but we offer an estimation that looks natural. We begin by filtering C itself: From

$$F^{n+c} \subset F^{n+c-1}S_1 \subset \cdots \subset F^n S_c$$

we define the $\mathcal{R}(F)$-modules

$$D_i = \bigoplus_{n \geq 0} F^n S_i / F^{n+1} S_{i-1}, \quad i = 1, \ldots, c,$$

that give the factors of a filtration of C.

Lemma 2.48 $\deg(D_i) \leq \text{br}(E) \cdot \binom{r+i-2}{i-1}$.

Proof. We set $S_i = S_1 \cdot S_{i-1}$, $s = \lambda(S_1/F) = \text{br}(E)$ and write

$$S_1 = (F, a_1, \ldots, a_s),$$

where the submodules of S_1,

$$F_j = (F, a_1, \ldots, a_j), \quad F_0 = F, \quad j \leq s,$$

define a composition series for S_1/F, that is $\lambda(F_{j+1}/F_j) = 1$. We use these s modules for a final filtering of the D_i obtaining factors of the form

$$\bigoplus_{n \geq 0} F^n \cdot F_{j+1} S_{i-1} / F^n \cdot F_j S_{i-1}.$$

Note that this module is annihilated by the maximal ideal of R and is generated as an $\mathcal{R}(F)$-module by $a_{j+1} \cdot m_\alpha$, where the m_α are the monomials of degree $i-1$ of the polynomial ring $S = R[x_1, \ldots, x_r]$. As a consequence it is generated by $\binom{r+i-2}{i-1}$ elements over the special fiber of $\mathcal{R}(F)$. Since F is of linear type, this ring is a ring of polynomials and the multiplicity of the module is at most $\binom{r+i-2}{i-1}$. Adding these contributions we get the asserted bound. □

To complete the proof of Theorem 2.47, collecting the partial multiplicities gives

$$e_0(C) \leq \text{br}(E) \sum_{i=1}^{c} \binom{r+i-2}{i-1} = \text{br}(E) \cdot \binom{r+c-1}{c-1},$$

establishing the desired formula. □

Corollary 2.49 *If R is a regular local ring of dimension d and E is a submodule of R^r, of finite colength so that $\mathcal{R}(E)$ has finite integral closure, then for any distinct graded subalgebras A and B,*

$$\mathcal{R}(E) \subset A \subset B \subset \overline{\mathcal{R}(E)},$$

satisfying the S_2-condition of Serre,

$$0 \leq \text{br}_1(A) < \text{br}_1(B) \leq \text{br}(E) \cdot \binom{2r+d-3}{r+d-3}.$$

In particular, any chain of such subalgebras has length at most $\text{br}(E) \cdot \binom{2r+d-3}{r+d-3}$.

Proof. As in the proof above, the module $C = B/A$ has dimension $d+r-1$, and we would obtain $b_1(A) = \text{br}_1(B) + e_0(C)$. □

Question 2.50 As it was the case of ideals, an issue is whether the length of such chains can be predicted *a priori* for more general rings.

Cohen-Macaulay Algebras

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local domain of dimension d and let E be a torsionfree R-module of rank r. Denote by A the Rees algebra of E, $\mathcal{R}(E)$, and let B be a graded, finite A-subalgebra of $\overline{A}$. The main result of this section is that if B is Cohen-Macaulay then its reduction number (in a sense to be made explicit) is at most $d-1$. In particular B is generated by its components of degree at most $d-1$.

We fix an embedding $E \hookrightarrow R^r$ and identify $\mathcal{R}(E)$ with the subalgebra of $S = R[T_1, \ldots, T_r]$ generated by the 1-forms in E. The integral closure of $\mathcal{R}(E)$ and the subalgebra B will be written

$$A = \mathcal{R}(E) \subset B = \sum_{n \geq 0} E_n \subset \overline{A} = \sum_{n \geq 0} \overline{E^n}.$$

We can refer to B as a algebra of the multiplicative filtration $\{E_n\}$ of R-submodules of S.

Theorem 2.51 *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of infinite residue field and let $\{E_n, \neq 0, E_0 = R\}$ be a multiplicative filtration such that the Rees algebra $B = \sum_{n\geq 0} E_n$ is Cohen-Macaulay and finite over A. Let G be a minimal reduction of E. Then*

$$E_{n+1} = GE_n = EE_n, \quad n \geq d-1,$$

and in particular, B is generated over A by forms of degrees at most $d-1$,

$$\sum_{n\geq 0} E_n = R[E_1, \ldots, E_{d-1}].$$

For $r = 1$, that is, the case of ideals is proved in Theorem rredintclos. That result was based on the characterization of the Cohen-Macaulayness of the Rees algebra of an ideal I in terms of its associated graded and of its reduction number. It turns out that Theorem 2.51 is a direct consequence of the ideal case and of the technique of Bourbaki sequences.

Proof. We assume that $r \geq 2$. In the algebra A, the prime ideal $A_+ = EA$ has codimension r and therefore, by the going-up theorem, height $(EB) = r$ or more precise grade $(EB) = r$ since B is Cohen-Macaulay by assumption.

Now we carry out a generic extension of $R \to R'$ to construct a prime ideal of $B \otimes_R R'$ generated by a regular sequence $r-1$ elements of $R' \otimes E$. For example, if $e_1, \ldots, e_n$ is a set of R-generators of E and $\mathbf{X}$ is a $n \times (r-1)$ matrix of distinct indeterminates over R, taking the $r-1$ linear combinations

$$F = (f_1, \ldots, f_{r-1}) = (e_1, \ldots, e_n) \cdot \mathbf{X}$$

and setting $R' = R[\mathbf{X}]_{\mathfrak{m}[\mathbf{X}]}$, gives rise to the prime ideal $(F) \subset B$ with the asserted properties. In particular, this construction gives rise to the exact sequence of R-modules (after we replace, innocuously, R by R')

$$0 \to F \longrightarrow E_1 \longrightarrow I_1 \to 0,$$

where I_1 is torsionfree R-module of rank 1, which we identity to an ideal I.

Consider the R-algebra homomorphism

$$B = \sum_{n\geq 0} E_n \longrightarrow \sum_{n\geq 0} E_n/FE_{n-1} \longrightarrow \sum_{n\geq 0} I_n = C,$$

where $E_n/FE_{n-1} = I_n$ is also torsionfree of rank 1. Observe that C is finite over the Rees algebra of I and thus C is contained in $\overline{R[It]}$. Since (F) is a complete intersection ideal of the Cohen-Macaulay ring B, C is Cohen-Macaulay. We may thus apply Theorem 2.4 to obtain a minimal reduction J for C of reduction number at most $d-1$. Lifting J to E_1 and adding F, gives rise to a reduction L for B generated by at most $d+r-1$ elements, having reduction number at most $d-1$. It will easily have the property that $E_{n+1} = LE_n = EE_n$, for $n \geq d-1$. Note that if G is any minimal reduction of E, we can change A by $\mathcal{R}(G)$ and obtain $E_{n+1} = GE_n$, $n \geq d-1$, as well. □

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