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Finite free resolutions

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These are preliminary lecture notes, intended only for distribution to participants

FINITE FREE RESOLUTIONS

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INTRODUCTION

With these lectures we aim to give a survey on the theory of finite free resolutions. We will treat the Buchsbaum-Eisenbud acyclicity criterion, discuss upper and lower bounds for Betti-numbers, including the Evans-Griffith syzygy theorem, and compare the graded Betti-numbers of an ideal and with those of its generic initial ideal.

1. LECTURE: BASIC CONCEPTS; ACYCLICITY CRITERIA

Throughout these lectures (R, \mathfrak{m}, k) denotes either a Noetherian local ring or a standard graded k -algebra with graded maximal ideal \mathfrak{m} . All modules considered in these lectures will be finitely generated, and will be graded if R is graded.

Let M be an R -module, m_1, \dots, m_r a minimal system of (homogeneous) generators of M . Let F_0 be a free R -module with basis e_1, \dots, e_r , and let $\varepsilon: F_0 \rightarrow M$ be surjective R -module homomorphism defined by $\varepsilon(e_i) = m_i$ for $i = 1, \dots, r$. Nakayama's lemma implies that $\text{Ker}(\varepsilon) \subset \mathfrak{m}F_0$. Since R is Noetherian, $\text{Ker}(\varepsilon)$ is finitely generated, and there is again a free R -module F_1 and an epimorphism $F_1 \rightarrow \text{Ker}(\varepsilon)$, whose kernel is a submodule of $\mathfrak{m}F_1$. Composing $F_1 \rightarrow \text{Ker}(\varepsilon)$ with the inclusion map $\text{Ker}(\varepsilon) \subset F_0$, we get a homomorphism $\varphi_1: F_1 \rightarrow F_0$ such that

$$F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

is exact and $\text{Im}(\varphi_1) \subset \mathfrak{m}F_0$. Proceeding this way one constructs an exact sequence

$$\dots \longrightarrow F_p \xrightarrow{\varphi_p} \dots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Definition 1.1. Let M be an R -module. A complex

$$\mathbb{F}: \dots \longrightarrow F_p \xrightarrow{\varphi_p} \dots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

of finitely generated free R -modules is called a *minimal free R -resolution of M* , if

- (i) $\varphi(\mathbb{F}) \subset \mathfrak{m}\mathbb{F}$;
- (ii) $H_0(\mathbb{F}) \cong M$ and $H_i(\mathbb{F}) = 0$ for $i > 0$.

A minimal free resolution always exist as we have just seen. It is called minimal since for each i the basis elements of F_i are mapped to a minimal set of generators of $\text{Ker } \varphi_{i-1}$.

Any two minimal free resolutions of M are isomorphic, that is, if \mathbb{F} and \mathbb{G} are minimal free resolutions of M , then there is an isomorphism of complexes $\mathbb{F} \cong \mathbb{G}$.

If (\mathbb{F}, φ) is a minimal free resolution of M , then

$$\text{syz}_i(M) = \text{Im}(\varphi_i)$$

is called the i th syzygy module of M .

In the graded case, by choosing in each step of the construction of the minimal free resolution a minimal system of *homogeneous* generators of $\text{Ker } \varphi_i$, one obtains a graded minimal free resolutions, that is, a minimal free resolution \mathbb{F} such that

- (iii) $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$ for all i ;
- (iv) $\varphi_i: F_i \rightarrow F_{i-1}$ is homogeneous of degree 0.

Definition 1.2. Let \mathbb{F} be a minimal free resolution of M . Then $\beta_i = \text{rank } F_i$ is called the *ith Betti-number of M* .

Remark 1.3. (a) In the graded case, $\beta_i = \sum_j \beta_{ij}$ for all i . The numbers β_{ij} are called the *graded Betti-numbers of M* ;

(b) Let \mathbb{F} be a minimal free resolution of M . Since

$$\text{Tor}_i^R(M, k) = H_i(\mathbb{F} \otimes k) = F_i \otimes k = F_i/\mathfrak{m}F_i,$$

it follows that $\beta_i = \dim_k \text{Tor}_i(M, k)$.

In the graded case, $\text{Tor}_i^R(M, k)$ is a graded k -vector space, and $\beta_{ij} = \dim_k \text{Tor}_i^R(M, k)_j$.

Let \mathbb{F} be the minimal free resolution of M . We say that M has a *finite free resolution*, if there exists an integer i such that $F_i = 0$.

Note that M has finite free resolution if one of the equivalent conditions are satisfied: there exists an integer i such that

- (a) $F_j = 0$ for all $j \geq i$;
- (b) $\text{Tor}_i(M, k) = 0$;
- (c) $\text{Tor}_j(M, k) = 0$ for all $j \geq i$.

Suppose that M has a finite free resolution. The maximal number i with $\text{Tor}_i(M, k) \neq 0$ is called the *projective dimension of M* , and denoted $\text{proj dim } M$.

If R is regular, then *all* modules have a finite free resolution. Indeed, let $\mathbf{x} = x_1, \dots, x_n$ be a regular system of parameters of R . In the graded case, R is the polynomial ring, and for \mathbf{x} we may choose the variables.

Let \mathbb{K} be the Koszul complex attached to \mathbf{x} . Then \mathbb{K} is exact, since \mathbf{x} is a regular sequence. Thus \mathbb{K} is a minimal free resolution of k , and hence

$$\text{Tor}_i(M, k) = H_i(M \otimes \mathbb{K}) \quad \text{for all } i.$$

Since $\mathbb{K}_{n+1} = 0$, we see that $\text{Tor}_{n+1}(M, k) = 0$. Hence we conclude that

$$\text{proj dim } M \leq n = \dim R$$

for all R -modules M .

In these lectures we are mostly interested in finite free resolutions. The following natural question arises:

What can be said about the Betti-numbers?

To be more specific we ask:

- (1) What can be said about the projective dimension?
- (2) Given a finite complex of free R -modules. When is it exact?
- (3) Fix certain data like the projective dimension or, in the graded case, the Hilbert-function. Are there lower or upper bounds for the Betti-numbers for such modules?

- (4) Suppose R is a polynomial ring and $I \subset R$ is a graded ideal. Given a term order. How are the Betti numbers of I and its initial ideal $\text{in}(I)$ related to each other?
- (5) What can be said about the graded Betti-numbers of a monomial ideal? In the context of (4) this question is of interest.

The answer to question (1) is classical

Theorem 1.4 (Auslander-Buchsbaum). *Suppose M has a finite free resolution. Then*

$$\text{proj dim } M + \text{depth } M = \text{depth } R.$$

In particular, $\text{proj dim } M \leq \text{depth } R$.

Proof. We proceed by induction on $c := \text{depth } R - \text{depth } M$. Suppose $c \leq 0$ and let $t = \text{depth } R$. Then there exists a sequence $\mathbf{x} = x_1, \dots, x_t$ which is regular on R and M .

Suppose that $\text{proj dim } M = p > 0$, and let \mathbb{F} be the minimal free resolution of M . Then $\bar{\mathbb{F}} = \mathbb{F}/(\mathbf{x})\mathbb{F}$ is a minimal free resolution of $M/(\mathbf{x})M$, and hence $\bar{\varphi}_p : \bar{F}_p \rightarrow \bar{F}_{p-1}$ is injective. However, since $\text{depth } \bar{F}_p = 0$, there exists $a \in \bar{F}_p$, $a \neq 0$ with $ma = 0$. Since $\bar{\varphi}_p(\bar{F}_p) \subset \mathfrak{m}\bar{F}_{p-1}$, it follows that $\bar{\varphi}_p(a) = 0$, contradiction.

Suppose now that $c > 0$. Then $\text{depth syz}_1(M) = \text{depth } M + 1$, so that

$$\text{depth } R - \text{depth syz}_1(M) = c - 1.$$

By induction hypothesis, we have $\text{proj dim syz}_1(M) + \text{depth syz}_1(M) = \text{depth } R$. Hence, since $\text{proj dim } M = \text{proj dim syz}_1(M) - 1$, the assertion follows. \square

Now we will deal with the second question. Suppose \mathbb{F} is a finite complex of free R -modules, and suppose we want to prove it is acyclic, i.e. $H_i(\mathbb{F}) = 0$ for $i > 0$. Assuming it is not acyclic, we could localize at a suitable prime ideal P such that after localization, $H_i(\mathbb{F})$ is finite length module for all $i > 0$. In this situation we can apply

Theorem 1.5 (Lemme d'acyclicité, Peskine-Szpiro [28]). *Let*

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be finite complex of free R -modules with $p \leq \text{depth } R$, and suppose that $\text{depth } H_i(\mathbb{F}) = 0$ for all $i > 0$. Then \mathbb{F} is acyclic.

Proof. We may assume that $p > 0$, and prove by induction on i that $H_{p-i}(\mathbb{F}) = 0$.

For $i = 0$, $H_p(\mathbb{F})$ is submodule of F_p of depth 0. Since $\text{depth } F_p > 0$, this submodule must be zero.

Now given i with $0 < i < p$. By induction hypothesis we have that $H_p(\mathbb{F}) = H_{p-1}(\mathbb{F}) = H_{p-i+1}(\mathbb{F}) = 0$. Hence

$$0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \cdots \rightarrow F_{p-i+1} \rightarrow \text{Im}(\varphi_{p-i+1}) \rightarrow 0$$

is exact. It follows that $\text{depth Im}(\varphi_{p-i+1}) = \text{depth } R - (i-1) \geq p - i + 1 > 1$.

Suppose that $H_{p-i}(\mathbb{F}) \neq 0$. Then $\text{depth } H_{p-i}(\mathbb{F}) = 0$, and since $\text{depth Ker}(\varphi_{p-i}) > 0$, the exact sequence

$$0 \rightarrow \text{Im}(\varphi_{p-i+1}) \rightarrow \text{Ker}(\varphi_{p-i}) \rightarrow H_{p-i}(\mathbb{F}) \rightarrow 0$$

implies that $\text{depth Im}(\varphi_{p-i+1}) = 1$, a contradiction. \square

In the proof of Theorem 1.5 it was not important that the modules F_i are free, and one could have replaced them by any other modules satisfying $\text{depth} F_i \geq i$, and would have obtained the same conclusion.

Let Q be the ring of fractions of R . An R -module M has rank r if $M \otimes Q$ is free of rank r . It is easy to see that M has rank r , if and only if $M_{\mathfrak{p}}$ is free of rank r for all $\mathfrak{p} \in \text{Ass}(R)$.

The rank is additive on short exact sequences: suppose $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of R -modules. If two of the modules U , M or N have a rank, then the third does, and $\text{rank} M = \text{rank} U + \text{rank} N$.

The additivity of rank implies

Proposition 1.6. *Suppose M has a finite free resolution \mathbb{F} . Then*

$$\text{rank} M = \sum_i (-1)^i \beta_i.$$

Corollary 1.7. *Let $I \neq 0$ be an ideal with finite free resolution. Then I contains a non-zerodivisor.*

Proof. By Proposition 1.6, I has a rank, and $\text{rank} I + \text{rank} R/I = \text{rank} R = 1$. Since $I \neq 0$ and $I \otimes Q \rightarrow R \otimes Q = Q$ is injective, it follows that $\text{rank} I = 1$. Therefore $\text{rank} R/I = 0$, and so R/I is annihilated by a non-zerodivisor. \square

Let $\varphi: M \rightarrow N$ be an R -module homomorphism. We say that φ has rank r , if $\text{Im } \varphi$ has rank r .

An R -module homomorphism $\varphi: F \rightarrow G$ of finite free R -modules is given by a matrix A with respect to bases of F and G . We denote by $I_t(\varphi)$ the ideal generated by all t -minors of φ , and set $I(\varphi) = I_r(\varphi)$, if r is the rank of φ . We also set $I_t(\varphi) = R$ if $t \leq 0$ and $I_t(\varphi) = 0$ if $t > \min\{\text{rank} F, \text{rank} G\}$. The definitions do not depend on the chosen bases of F and G .

For the next theorem we shall need the following facts:

Proposition 1.8. *Let $\varphi: F \rightarrow G$ be homomorphism of finite free R -modules, and \mathfrak{p} a prime ideal. Then*

- (a) $I_t(\varphi) \not\subseteq \mathfrak{p} \iff (\text{Im } \varphi)_{\mathfrak{p}}$ contains a free direct summand of $G_{\mathfrak{p}}$ of rank t ;
- (b) $I_t(\varphi) \not\subseteq \mathfrak{p}$ and $I_{t+1}(\varphi)_{\mathfrak{p}} = 0 \iff (\text{Im } \varphi)_{\mathfrak{p}}$ is a free direct summand of $G_{\mathfrak{p}}$ of rank t ;
- (c) $\text{rank } \varphi = r \iff \text{grade} I_r(\varphi) \geq 1$ and $I_{r+1}(\varphi) = 0$.

Theorem 1.9 (Buchsbaum-Eisenbud [13]). *Let*

$$\mathbb{F}: 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a finite complex of free R -modules, and let $r_i = \sum_{j \geq i} (-1)^{j-i} \text{rank} F_j$. Then the following conditions are equivalent:

- (a) \mathbb{F} is acyclic;
- (b) $\text{grade} I_{r_i}(\varphi_i) \geq i$ for $i = 1, \dots, p$;
- (c) (i) $\text{rank} F_i = \text{rank } \varphi_i + \text{rank } \varphi_{i+1}$ for $i = 1, \dots, p$;
- (ii) $\text{grade} I(\varphi_i) \geq i$ for $i = 1, \dots, p$.

Proof. (a) \Rightarrow (b): The acyclicity of \mathbb{F} and the additivity of rank imply that $r_i = \text{rank } \varphi_i$. Therefore, Proposition 1.8 implies that $\text{grade } I_{r_i}(\varphi_i) \geq 1$ for $i = 1, \dots, p$. Hence there exists a non-zero-divisor x which is contained in all the ideals $I_{r_i}(\varphi_i)$. If x is a unit, then $I_{r_i}(\varphi_i) = R$ for all i and we are done. Otherwise $x \in \mathfrak{m}$, and x is non-zero-divisor on all F_i and on $\text{Im}(\varphi_1)$. Let $\bar{}$ denote residue classes modulo x . Then $0 \rightarrow \bar{F}_p \rightarrow \bar{F}_{p-1} \rightarrow \dots \rightarrow \bar{F}_2 \rightarrow \bar{F}_1 \rightarrow 0$ is acyclic. By induction we have $\text{grade } I_{r_i}(\bar{\varphi}_i) \geq i - 1$. Hence, since $I_{r_i}(\varphi_i)\bar{} = I_{r_i}(\bar{\varphi}_i)$, we conclude that $\text{grade } I_{r_i}(\varphi_i) \geq i$ for $i = 2, \dots, p$.

(b) \Rightarrow (a): By induction on p , may assume that $0 \rightarrow F_p \rightarrow \dots \rightarrow F_1 \rightarrow 0$ is acyclic, and have to show that $H_1(\mathbb{F}) = 0$.

Set $M_i = \text{Coker}(\varphi_{i+1})$ for $i = 1, \dots, p$. We first show by descending induction that $\text{depth}(M_i)_{\mathfrak{p}} \geq \min\{i, \text{depth } R_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{Spec } R$ and $i = 1, \dots, p$.

The assertion is trivial for $i = p$, since $M_p = F_p$. Now let $i < p$ and consider the exact sequence $0 \rightarrow M_{i+1} \rightarrow F_i \rightarrow M_i \rightarrow 0$.

If $\text{depth } R_{\mathfrak{p}} \geq i + 1$, then our induction hypothesis implies that $\text{depth}(M_{i+1})_{\mathfrak{p}} \geq i + 1$, and hence $\text{depth}(M_i)_{\mathfrak{p}} \geq i$.

If $\text{depth } R_{\mathfrak{p}} \leq i$, then (b) implies that $I_{r_{i+1}}(\varphi_{i+1}) \not\subset \mathfrak{p}$, and since $\text{rank } M_{i+1} = r_{i+1}$ we have $I_t(\varphi_{i+1}) = 0$ for $t > r_{i+1}$. Thus Proposition 1.8 implies that $(M_i)_{\mathfrak{p}}$ is free, and hence $\text{depth}(M_i)_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}}$.

Now assume that $H_1(\mathbb{F}) \neq 0$, and let $\mathfrak{p} \in \text{Ass } H_1(\mathbb{F})$. If $\text{depth } R_{\mathfrak{p}} \geq 1$, then $\text{depth}(M_1)_{\mathfrak{p}} \geq 1$, and hence $\text{depth } H_1(\mathbb{F})_{\mathfrak{p}} \geq 1$, since $H_1(\mathbb{F}) = \text{Ker}(M_1 \rightarrow F_0)$. This is a contradiction.

On the other hand, if $\text{depth } R_{\mathfrak{p}} = 0$, then $I_{r_1}(\varphi_1) \not\subset \mathfrak{p}$ and

$$U := \text{Im}((\varphi_1)_{\mathfrak{p}}) = \text{Im}((M_1)_{\mathfrak{p}} \rightarrow (F_0)_{\mathfrak{p}})$$

contains a free direct summand of $(F_0)_{\mathfrak{p}}$ of rank r_1 , see 1.8. However since $(M_1)_{\mathfrak{p}}$ is a free module of rank r_1 , the surjective map $(M_1)_{\mathfrak{p}} \rightarrow U$ must be an isomorphism, i.e. $H_1(\mathbb{F})_{\mathfrak{p}} = 0$. This is again a contradiction.

(a), (b) \Rightarrow (c): Since \mathbb{F} is acyclic, the sequences $0 \rightarrow \text{Im } \varphi_{i+1} \rightarrow F_i \rightarrow \text{Im } \varphi_i \rightarrow 0$ are exact. Thus the additivity of rank implies condition (c)(i).

As noticed in (a) \Rightarrow (b), we have $r_i = \text{rank } \varphi_i$ for $i = 1, \dots, p$. Hence (b) implies (c)(ii).

(c) \Rightarrow (b): It follows from (c)(i) that $r_i = \text{rank } \varphi_i$. Hence (ii) implies (b). \square

As an application we prove

Theorem 1.10 (Hilbert-Burch). *Let I be an ideal with free resolution*

$$0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists $a \in R$ such that $I = aI_n(\varphi)$. Moreover, if $\text{grade } I \geq 2$, then $I = I_n(\varphi)$ and $\text{grade } I = 2$.

Proof. Let φ be given by the $(n+1) \times n$ matrix A with respect to the canonical bases of R^n and R^{n+1} , and let $\pi: R^{n+1} \rightarrow R$ the homomorphism which sends the canonical basis element e_i to $(-1)^i \delta_i$, where δ_i denotes the minor of A with the i th row deleted. Let B be the $(n+1) \times (n+1)$ -matrix which is obtained from A by adding the j th column of A to A as an $(n+1)$ th column. Then B has two equal columns, and hence $\det B = 0$. Expanding $\det B$ with respect to the $(n+1)$ th column we therefore get

$$0 = \sum_i a_{ij} (-1)^j \delta_i.$$

This shows that

$$(1) \quad 0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \xrightarrow{\pi} R \longrightarrow 0$$

is a complex.

Since we assume that $0 \rightarrow R^n \rightarrow R^{n+1} \rightarrow I \rightarrow 0$ is exact, Theorem 1.9 implies that $\text{grade } I_n(\varphi) \geq 2$. Therefore, since $I_1(\pi) = I_n(\varphi)$, it follows from Theorem 1.9 that complex (1) is exact. Hence $I_n(\varphi) \cong \text{Coker } \varphi \cong I$. Composing this isomorphism with the inclusion map $I \subset R$ we obtain a monomorphism $I_n(\varphi) \rightarrow R$. However since $\text{grade } I_n(\varphi) \geq 2$, we have $\text{Hom}_R(I_n(\varphi), R) = R$. Thus the monomorphism $I_n(\varphi) \rightarrow R$ is multiplication by an element $a \in R$. It follows that $I = aI_n(\varphi)$. By Corollary 1.7 the element a must be a non-zero-divisor.

Suppose now that $\text{grade } I \geq 2$. Then, since $I = aI_n(\varphi) \subset (a)$, it follows that a is unit, and $I = I_n(\varphi)$. Finally, since $\text{grade } I \leq \text{proj dim } R/I = 2$, we get $\text{grade } I = 2$. \square

2. SECOND LECTURE: LOWER BOUNDS

In our discussion on the question which are the possible Betti-numbers of a module of finite projective dimension, we will concentrate in this section on lower bounds.

The following simple result gives us a hint what kind of bounds could be expected.

Proposition 2.1. *Suppose R is regular, and let $I \subset R$ be a radical ideal of grade g . Then $\beta_i(R/I) \geq \binom{g}{i}$.*

Proof. Let \mathfrak{p} be a minimal prime ideal of I . Then $R_{\mathfrak{p}}$ is a regular local ring of dimension $\geq g$ with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Since I is a radical ideal it follows that $IR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$.

Let \mathbb{F} be a minimal free resolution of R/I . Since localization is an exact functor, $\mathbb{F}_{\mathfrak{p}}$ is a resolution of $(R/I)_{\mathfrak{p}}$. This resolution may not be minimal. Nevertheless we conclude that $\beta_i(R/I) \geq \beta_i(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \geq \binom{g}{i}$. The last inequality follows since $\mathfrak{p}R_{\mathfrak{p}}$ is generated by a regular sequence of length $\geq g$. \square

Corollary 2.2. *Let R be the polynomial ring and $I \subset R$ a monomial ideal of grade g . Then $\beta_i(R/I) \geq \binom{g}{i}$.*

Proof. Let u_1, \dots, u_m be the minimal set of monomial generators of I , say $u_i = \prod_j x_j^{a_{ij}}$, and let S be the polynomial ring in the new set of variables x_{ij} .

The monomial $v_i = \prod_{j=1}^n \prod_{k=1}^{a_{ij}} x_{jk}$ is called the polarization of u_i , and the ideal $I^p = (v_1, \dots, v_m) \subset S$ the polarization of I .

It is a basic fact [12, Lemma 4.2.16] that the sequence of linear forms $x_{j1} - x_{jk}$ with $j = 1, \dots, n$ and $k = 2, 3, \dots$ form a regular sequence ℓ on S/I^p , and that $(S/I^p)/\ell(S/I^p) \cong R/I$. In particular, $\beta_i(R/I) = \beta_i(S/I^p)$. Since $\text{grade } I = \text{grade } I^p$, and since I^p is a radical ideal, the conclusion follows. \square

These results indicate that the following may be true

Conjecture 2.3. *Let M be an R -module of grade g with finite free resolution. Then $\beta_i(M) \geq \binom{g}{i}$.*

In case R is regular and M is a module of finite length, this conjecture is known as the Buchsbaum-Eisenbud and Horrocks conjecture.

The conjecture is widely open. There are few cases in which the conjectured lower bound for the Betti numbers is known:

- (a) (Buchsbaum-Eisenbud, [14]) R is regular, $M = R/I$ has finite length, and the free resolution \mathbb{F} of R/I has an algebra structure;
- (b) (Huneke-Ulrich, [25]) R is regular, $M = R/I$, and I is in the linkage class of a complete intersection;
- (c) (Herzog-Hibi-Kühl, [22] and [21]) R is regular and M is componentwise linear.

The argument of Buchsbaum-Eisenbud is as follows: choose a regular sequence $\mathbf{x} = x_1, \dots, x_n$ with all $x_i \in I$. We may assume that $n > 1$. Let \mathbb{K} be the Koszul complex of \mathbf{x} with $K_1 = \bigoplus_i^n R e_i$ and $\partial(e_i) = x_i$. Then the inclusion $(\mathbf{x}) \subset I$ can be lifted to a linear map $\alpha_1: K_1 \rightarrow F_1$ such that $\varphi_1(\alpha_1(e_i)) = x_i$ for all i . Now for each integer $k \geq 1$ let $\alpha_k: K_k \rightarrow F_k$ be defined by

$$\alpha_j(e_{i_1} \wedge \dots \wedge e_{i_k}) = \alpha_1(e_{i_1}) \cdot \dots \cdot \alpha_1(e_{i_k}).$$

Then $\alpha: \mathbb{K} \rightarrow \mathbb{F}$ is an algebra and complex homomorphism, whose kernel is a graded ideal \mathfrak{a} in \mathbb{K} .

Suppose $\mathfrak{a} \neq 0$. Let $a \in \mathfrak{a}$ be a non-zero element of degree j . Then there exist $b \in K_{n-j}$ with $b \wedge a \neq 0$. Since $b \wedge a \in \mathfrak{a}$ it follows that $\mathfrak{a}_n \neq 0$. We identify K_n with R . Then the image of α_n is a cyclic submodule of \mathbb{F}_n which is isomorphic to R/\mathfrak{a}_n . Since R is a domain the annihilator of a non-zero submodule \mathbb{F}_n is zero. It follows that $R/\mathfrak{a}_n = 0$, so that $\alpha_n = 0$. The canonical epimorphism $R/(\mathbf{x}) \rightarrow R/I$ with kernel, say C , induces the exact sequence

$$\text{Ext}_R^{n-1}(C, R) \longrightarrow \text{Ext}_R^n(R/I, R) \xrightarrow{\psi} \text{Ext}_R^n(R/(\mathbf{x}), R).$$

Here the homomorphism ψ is induced by α_n , and hence is the zero map, and $\text{Ext}^{n-1}(C, R) = 0$, since C is of dimension zero. It follows that $\text{Ext}^n(R/I, R) = 0$, a contradiction.

Thus we conclude that $\mathfrak{a} = 0$. Therefore α is injective, and it follows that

$$\beta_i(R/I) = \text{rank } F_i \geq \text{rank } K_i = \binom{n}{i} \quad \text{for all } i.$$

Unfortunately, not all finite minimal free resolutions admit an algebra structure. In [4] Avramov discovered obstructions to the existence of such structures, and later Srinivasan [31] showed that despite the vanishing of the obstructions defined by Avramov, a finite minimal free resolution still may not admit an algebra structure.

Discussion of (c): In [21] componentwise linear modules are introduced: a graded R -module is called *componentwise linear* if for all j the submodule $M_{\langle j \rangle}$ generated by the j th component M_j of M has a linear resolution.

By assumption, $R = k[x_1, \dots, x_n]$ is the polynomial ring. We may assume that k is infinite. Then for a generic choice of linear forms y_1, \dots, y_n one has for $i = 1, \dots, n$ that

$$A_i = \text{Ker}(M/(y_1, \dots, y_{i-1})M \xrightarrow{y_i} M/(y_1, \dots, y_{i-1})M)$$

is a module of finite length. We set

$$\alpha_i = \ell(A_i),$$

and call $\alpha_1, \dots, \alpha_n$ the *generic annihilation numbers of M* .

It will be shown in the next section (see Corollary 3.2 and Theorem 3.5) that

$$\beta_i \leq \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j,$$

with equality if and only if M is componentwise linear.

For the proof of (c) we need the following

Lemma 2.4. *With the notation and assumptions introduced suppose that $\text{depth} M = t$, and let $\alpha_1, \dots, \alpha_n$ be the generic annihilation numbers of M . Then $\alpha_i = 0$ for $i \leq t$, and $\alpha_i \neq 0$ for $i > t$.*

Proof. Suppose $\text{depth} M > 0$. Then a generic linear form y is a non-zerodivisor. This shows that $\alpha_i = 0$ for $i \leq t$.

In order to prove that $\alpha_i \neq 0$ for $i > t$, it suffices to show: if $\text{depth} M = 0$, and y is a generic linear form, then (i) $(0 :_M y) \neq 0$, and (ii) $\text{depth} M/yM = 0$.

Statement (i) is obvious. For the proof of (ii) we consider for all i the map

$$y^{i-1}M/y^iM \rightarrow y^iM/y^{i+1}M$$

induced by multiplication by y .

Let C_i be the kernel of this map, and let $c + y^iM \in C_i$. Then $c = y^{i-1}a$ with $a \in M$ and there exists $b \in M$ such that $yc = y^i a = y^{i+1}b$. Hence $y(c - y^i b) = 0$, and so $m^n c \in y^i M$ for some n , since y is a generic linear form. This shows that C_i is a finite length module for all i .

Suppose now that $\text{depth} M/yM > 0$. We show by induction on i , that $y^{i-1}M/y^iM \rightarrow y^iM/y^{i+1}M$ is an isomorphism. In fact, for each i we have the exact sequence

$$0 \longrightarrow C_i \longrightarrow y^{i-1}M/y^iM \longrightarrow y^iM/y^{i+1}M \longrightarrow 0.$$

For $i = 1$, $\text{depth} M/yM > 0$ and $\ell C_1 < \infty$. This implies that $C_1 = 0$. Therefore $M/yM \rightarrow yM/y^2M$ is an isomorphism.

Now let $i > 0$. By induction we may assume that $y^{j-1}M/y^jM \cong y^jM/y^{j+1}M$ for all $j < i$. In particular, it follows that $M/yM \cong y^{i-1}M/y^iM$, so that $\text{depth} y^{i-1}M/y^iM > 0$. However since $\ell C_i < \infty$, the above exact sequence shows again that $C_i = 0$ and that $y^{i-1}M/y^iM \rightarrow y^iM/y^{i+1}M$ is an isomorphism.

On the other hand, since $\text{depth} M = 0$, there exists $c \in M$, $c \neq 0$, such that $yc = 0$. Let i be such that $c \in y^{i-1}M \setminus y^iM$. Then $c + y^iM \neq 0$ but $y(c + y^iM) = 0$. This is a contradiction since $C_i = 0$. \square

Now statement (c) will be a consequence of the following stronger result.

Theorem 2.5. *Let R be the polynomial ring, and M a componentwise linear R -module with $\text{proj dim} M = p$. Then $\beta_i(M) \geq \binom{p}{i}$.*

Proof. Let $t = \text{depth} M$. Then $t = n - p$, by the Auslander-Buchsbaum formula. Therefore, $\alpha_i > 0$ for $i = n - p + 1, \dots, n$, by Lemma 2.4. Thus, since M is componentwise

linear,

$$\begin{aligned}\beta_i(M) &= \sum_{j=n-p+1}^{n-i+1} \binom{n-j}{i-1} \alpha_j \geq \sum_{j=n-p+1}^{n-i+1} \binom{n-j}{i-1} \\ &= \sum_{j=i-1}^{p-1} \binom{j}{i-1} = \binom{p}{i}.\end{aligned}$$

□

In view of this result one may hope that for any R -module M of projective dimension p one has $\beta_i(M) \geq \binom{p}{i}$. However by a theorem of Bruns [10, Satz 3], if N is an i th syzygy module of a module of finite projective dimension, then N is also the i th syzygy module of an ideal generated by 3 elements. In particular, if N is the second syzygy module of a module of projective dimension p , then there exists an ideal I generated by 3 elements whose second syzygy module is N , one has $\beta_2(R/I) = 3 < \binom{p}{2}$, if $p > 3$.

The following concrete very simple example was communicated to us by Conca: let $I = (-x_1x_2 + x_3x_4, x_2^2, x_3^2) \subset R = k[x_1, x_2, x_3, x_4]$. Then R/I has the resolution

$$0 \longrightarrow R \longrightarrow R^4 \longrightarrow R^5 \longrightarrow R^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

The theorem of Bruns also tells us that the resolution of an ideal generated by 3 elements can have arbitrary high projective dimension. On the other hand, it is conjectured by Stillman that if we fix a sequence of numbers d_1, \dots, d_r , then there is a number p such that any ideal in a polynomial (over a field K) which is generated by forms of degree d_1, \dots, d_r has projective dimension $\leq p$. This conjecture is known to be true only in a few special cases.

For monomial ideals the strong lower bound for the Betti-numbers holds. More generally one has the following result ([9, Theorem 1.1])

Theorem 2.6 (Brun, Römer). *Let M be a \mathbb{Z}^n -graded module with $\text{proj dim } M = p$. Then $\beta_i(M) \geq \binom{p}{i}$.*

There is a strengthening of Conjecture 2.3 in a different direction

Conjecture 2.7. *Let M be an R -module of grade g with finite projective dimension. Then $\text{rank syz}_i(M) \geq \binom{g-1}{i-1}$.*

Of course the additivity of rank yields that Conjecture 2.7 implies Conjecture 2.3.

The best known general result concerning lower bounds for the syzygy modules is the famous

Theorem 2.8 (Evans-Griffith [19]). *Suppose that R contains a field. Let M be an R -module with $\text{proj dim } M = p$. Then $\text{rank syz}_i(M) \geq i$ for $i = 1, \dots, p-1$.*

Of course we must exclude $i = p$ in the statement of the theorem, since for example the p th syzygy module of a regular sequence of length p is only of rank 1.

Since the rank is additive we immediately obtain

Corollary 2.9. *With the assumptions of 2.8 one has*

$$\beta_i(M) \geq \begin{cases} 2i+1, & \text{for } i = 0, \dots, p-1, \\ p, & \text{for } i = p-1, \\ 1, & \text{for } i = p. \end{cases}$$

For the proof of Theorem 2.8 we follow the presentation given in [12] and in the paper [11] of Bruns. This requires some preparations: let M be an R -module, and $x \in M$. Then

$$\mathcal{O}(x) = \{\varphi(x) : \varphi \in \text{Hom}_R(M, R)\},$$

is an ideal, the so-called *order ideal of x* .

Suppose for example that $M = F$ is free with basis e_1, \dots, e_n , and that $x \in F$. Then $x = \sum_{i=1}^n a_i e_i$ for some $a_i \in R$. Since the linear forms $\varphi_i: F \rightarrow R$ with $\varphi_i(e_j) = \delta_{ij}$ generate $\text{Hom}_R(F, R)$, and since $\varphi_i(x) = a_i$ for $i = 1, \dots, n$, it follows that in this case

$$\mathcal{O}(x) = (a_1, \dots, a_n).$$

We have

Lemma 2.10. *Let M be an R -module, $x \in M$ and $\mathfrak{p} \in \text{Spec}(R)$. Then $x \in M$ generates a free direct summand of $M_{\mathfrak{p}}$ if and only if $\mathcal{O}(x) \not\subseteq \mathfrak{p}$.*

Proof. The order ideal $\mathcal{O}(x)$ localizes since $\text{Hom}_R(M, R)_{\mathfrak{p}}$ is naturally isomorphic to $\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})$. Thus we assume that $\mathfrak{p} = \mathfrak{m}$, and hence $\mathcal{O}(x) \not\subseteq \mathfrak{m}$ if and only if $\mathcal{O}(x) = R$. This is equivalent to say that there exists $\varphi: M \rightarrow R$ with $\varphi(x) = 1$.

Suppose $M = Rx \oplus N$, then the projection to the first summand composed with the isomorphism $Rx \rightarrow R, x \mapsto 1$, yields $\varphi: M \rightarrow R$ with $\varphi(x) = 1$. Conversely, given such φ we have $M = Rx \oplus \text{Ker } \varphi$. \square

The next result is one important step in the proof of Theorem 2.8

Theorem 2.11. *Suppose R contains a field. Let*

$$\mathbb{F}: 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a complex of finitely free R -modules such that $\text{Im}(\varphi_i) \subset \mathfrak{m}F_{i-1}$ for all i . Let $t \geq 0$ be an integer and set $r_i = \sum_{j=i}^p (-1)^{j-i} \text{rank } F_j$. Suppose that $\text{codim } I_{r_i}(\varphi_i) \geq i+t$ for all i . Then, for $j = 1, \dots, p$ and every $e \in F_j \setminus \mathfrak{m}F_j$, one has $\text{codim } \mathcal{O}(\varphi_j(e)) \geq j+t$.

Proof. Let $J = \mathcal{O}(\varphi_j(e))$. We may assume that $J \subset \mathfrak{m}$. We set $\bar{R} = R/J$ and $\bar{\mathbb{F}} = \mathbb{F} \otimes \bar{R}$. Then $\bar{\varphi}_j(\bar{e}) = 0$, and $I_{r_i}(\bar{\varphi}_i) = (I_{r_i}(\varphi_i) + J)/J$.

Assume that $\text{codim } J \leq j+t-1$. Then we obtain

$$\dim(\bar{R}/I_{r_i}(\bar{\varphi}_i)) \leq \dim(R/I_{r_i}(\varphi_i)) \leq \dim R - i - t \leq \dim \bar{R} - i + j - 1,$$

which implies that $\text{codim } I_{r_i}(\bar{\varphi}_i) \geq i - j + 1$ for all $i \geq j$.

Let

$$\mathbb{G}: 0 \longrightarrow G_{p-j+1} \xrightarrow{\psi_{p-j+1}} G_{p-j} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\psi_1} G_0 \longrightarrow 0,$$

with $G_i = \bar{F}_{i+j-1}$ and $\psi_i = \bar{\varphi}_{i+j-1}$. Then \mathbb{G} is a complex with $\text{codim } I_{r_i}(\psi_i) \geq i$ for $i = 1, \dots, p-j+1$. If we would have $\text{grade } I_{r_i}(\psi_i) \geq i$ for all i , then the Eisenbud-Buchsbaum acyclicity criterion would imply that \mathbb{G} is acyclic. In order to remedy this

defect, we choose a balanced big Cohen-Macaulay module for \bar{R} and consider the complex $\mathbb{G} \otimes M$. This is precisely the step in the proof where we need that \bar{R} contains a field, because in this case it is known that there exists a balanced big Cohen-Macaulay \bar{R} -module, that is, a Cohen-Macaulay \bar{R} -module (not necessarily finitely generated) such that every system of parameters of \bar{R} is an M -regular sequence, see [12, Corollary 8.5.3]. It follows that $\text{grade}(I_{r_i}(\psi_i), M) \geq i$ for all i . An obvious modification of the Eisenbud-Buchsbaum acyclicity criterion then implies that $\mathbb{G} \otimes M$ is acyclic.

Since $\psi_1(\bar{e}) = 0$, it follows that $(\psi_1 \otimes M)(\bar{e} \otimes M) = 0$. However, since $\mathbb{G} \otimes M$ is acyclic it follows that $\text{Ker}(\psi_1 \otimes M) = \text{Im}(\psi_2 \otimes M)$. Therefore $\bar{e} \otimes M \subset \text{Im}(\psi_2 \otimes M) \subset \mathfrak{m}(G_1 \otimes M)$.

On the other hand, since $\bar{e} \notin \mathfrak{m}G_1$, it follows that the image of $\bar{e} \otimes M$ under the canonical epimorphism $G_1 \otimes M \rightarrow (G_1 \otimes M)/\mathfrak{m}(G_1 \otimes M) = G_1/\mathfrak{m}G_1 \otimes M/\mathfrak{m}M$ is isomorphic to $M/\mathfrak{m}M \neq 0$. This implies that $\bar{e} \otimes M \not\subset \mathfrak{m}G_1 \otimes M$, a contradiction. \square

For the inductive proof of the Evans-Griffith theorem we need the following technical

Lemma 2.12. *Let M be an R -module. Then there exists a free R -module F and a homomorphism $\varphi: M \rightarrow F$ with the following property: If \mathfrak{p} is a prime ideal and $N \subset M_{\mathfrak{p}}$ is free direct $R_{\mathfrak{p}}$ -summand, then $\varphi_{\mathfrak{p}}(N)$ is a free direct summand of $F_{\mathfrak{p}}$ with $\text{rank } \varphi_{\mathfrak{p}}(N) = \text{rank } N$.*

Proof. Denote by W^* , the R -dual of an R -modules W . We choose a G a free R -module and an epimorphism $\pi: G \rightarrow M^*$, and let $h: M \rightarrow M^{**}$ be the canonical homomorphism. Then $F = G^*$ and $\varphi = \pi^* \circ h$ have the desired property. Indeed, since R is Noetherian and all modules are finitely generated, the construction of F and φ localize. Thus we may assume that $R = R_{\mathfrak{p}}$. Since N is a free direct summand of M , there exist $g_1, \dots, g_r \in N$ and $\alpha_1, \dots, \alpha_r \in M^*$ such that $\alpha_i(g_j) = \delta_{ij}$. Choose $\beta_i \in G$ with $\pi(\beta_i) = \alpha_i$ for $i = 1, \dots, r$. Then

$$\varphi(g_i)(\beta_j) = h(g_i)(\pi(\beta_j)) = h(g_i)(\alpha_j) = \alpha_j(g_i) = \delta_{ij}.$$

This proves that $\varphi(N)$ is a free direct summand of F with $\text{rank } N = \text{rank } \varphi(N)$. \square

Proof of Theorem 2.8. We prove more generally the following statement (*): let

$$\mathbb{F}: 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a complex of finitely generated free R -modules such that $\text{Im}(\varphi_i) \subset \mathfrak{m}F_{i-1}$ for all i , and set $r_i = \sum_{j=i}^p (-1)^{j-i} \text{rank } F_j$. Suppose that there exists an integer $t \geq 0$ such that $\text{codim } I_{r_i}(\varphi_i) \geq i+t$ for all i . Then $r_i \geq i+t$ for $i = 1, \dots, p-1$.

Let M be a balanced big Cohen-Macaulay module of R . Since $\text{codim } I_{r_i}(\varphi_i) \geq i$, there exists a sequence x_1, \dots, x_i in $I_{r_i}(\varphi_i)$ which is part of a system of parameters of R , and hence a regular sequence on M . This implies that $\text{grade}(\varphi_i, M) \geq i$ for all i . Thus $\mathbb{F} \otimes M$ is acyclic. In particular, $r_i \geq 1$ for $i = 1, \dots, p$.

We prove (*) by induction on p . If $p = 1$, then there is nothing to show. Suppose now that $p > 1$, and consider the complex

$$\mathbb{G}: 0 \longrightarrow G_{p-1} \xrightarrow{\psi_{p-1}} G_{p-2} \longrightarrow \dots \longrightarrow G_1 \xrightarrow{\psi_1} G_0 \longrightarrow 0,$$

with $G_i = F_{i+1}$ and $\psi_i = \varphi_{i+1}$ for $k = 1, \dots, p-1$. Let $s_i = \sum_{j=i}^{p-1} (-1)^{j-i} \text{rank } G_j$. Then $\text{codim } I_{s_i}(\psi_i) \geq i + (1+t)$ for all i . Hence by induction hypothesis, we have $r_{i+1} = s_i \geq (i+1) + t$ for $i = 1, \dots, p-1$.

Thus it remains to show that $r_1 \geq 1 + t$. We show this by induction on t . The assertion is clear if $t = 0$. Suppose now that $t > 0$. We choose $e \in F_1 \setminus \mathfrak{m}F_1$, replace F_1 by $F'_1 = F_1/Re$ and φ_2 by the induced map $\varphi'_2: F_2 \rightarrow F'_1$. Furthermore, we choose F'_0 and $\text{Coker } \varphi'_2 \rightarrow F'_0$ as described in Lemma 2.12. This yields a map $\varphi'_1: F'_1 \rightarrow F'_0$, so that we obtain the complex

$$\mathbb{F}': 0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\varphi'_2} F'_1 \xrightarrow{\varphi'_1} F'_0 \longrightarrow 0.$$

We show (i) $\text{codim } I_{r'_2}(\varphi'_2) \geq t + 1$ and (ii) $\text{codim } I_{r'_1}(\varphi'_1) \geq t$. Then \mathbb{F}' satisfies the hypotheses of $(*)$ with $t - 1$ instead of t .

It may be that $\text{Im } \varphi'_1 \not\subset \mathfrak{m}F'_0$. In this case one can split off a direct summand without affecting (i) and (ii). Applying our induction hypothesis to this cancelled complex, we obtain $r'_1 \geq t$, and hence $r_1 \geq t + 1$, as desired.

Proof of (i): let \mathfrak{p} be a prime ideal with $\text{codim } \mathfrak{p} \leq t$. Then $I_{r_i}(\varphi_i) \not\subset \mathfrak{p}$, so that $\mathbb{F} \otimes R_{\mathfrak{p}}$ is split acyclic. In particular, $(F_1)_{\mathfrak{p}} \cong (\text{Im } \varphi_2)_{\mathfrak{p}} \oplus (\text{Coker } \varphi_2)_{\mathfrak{p}}$ with $\text{rank}(\text{Im } \varphi_2)_{\mathfrak{p}} = r_2$ and $\text{rank}(\text{Coker } \varphi_2)_{\mathfrak{p}} = r_1$. Moreover, by Theorem 2.11 we have $\text{codim } \mathcal{O}(\varphi_1(e)) \geq t + 1$. Therefore Lemma 2.10 implies that $\varphi_1(e)$ generates a non-zero free summand of $(F_0)_{\mathfrak{p}}$. Consequently, the image \bar{e} of e under the residue class map $F_1 \rightarrow \text{Coker } \varphi_2$ generates a non-zero free direct summand of $(\text{Coker } \varphi_2)_{\mathfrak{p}}$. Hence $(\text{Coker } \varphi'_2)_{\mathfrak{p}} \cong (\text{Coker } \varphi_2)_{\mathfrak{p}}/R_{\mathfrak{p}}\bar{e}$ is free of rank $r'_1 = r_1 - 1$, and the exact sequence

$$0 \longrightarrow (\text{Im } \varphi'_2)_{\mathfrak{p}} \longrightarrow (F'_1)_{\mathfrak{p}} \longrightarrow (\text{Coker } \varphi'_2)_{\mathfrak{p}} \longrightarrow 0$$

splits. In particular, $(\text{Im } \varphi'_2)_{\mathfrak{p}}$ is free direct summand of $(F'_1)_{\mathfrak{p}}$ of rank r_2 . Thus Proposition 1.8 implies that $I_{r'_2}(\varphi'_2) \not\subset \mathfrak{p}$, as desired.

Proof of (ii): We choose \mathfrak{p} as before. We have already seen that $(\text{Coker } \varphi'_2)_{\mathfrak{p}}$ is free of rank r'_1 . As φ'_1 is constructed as described in Lemma 2.12, $(\text{Coker } \varphi'_2)_{\mathfrak{p}}$ is mapped isomorphically onto a free direct summand of F'_0 . This implies that $I_{r'_1}(\varphi'_1) \not\subset \mathfrak{p}$, and shows $\text{codim } I_{r'_1}(\varphi'_1) \geq 1 + t$, which is even more than required. \square

3. LECTURE: UPPER BOUNDS

In the remaining sections, unless otherwise stated, $R = k[x_1, \dots, x_n]$ is the polynomial ring, and M is a finitely generated graded R -module. As indicated in Section 1 we want to relate the Betti-numbers $\beta_i(M)$ of M to the generic annihilator numbers $\alpha_i(M)$ of M .

Let $y = y_1, \dots, y_n$ be generic linear forms. Then

$$A_j = ((y_1, \dots, y_{j-1})M :_M y_j) / (y_1, \dots, y_{j-1})M$$

is a module of finite length. We set

$$\alpha_i(M) = \ell(A_i).$$

We denote by $H_i(j; M)$ the Koszul homology $H_i(y_1, \dots, y_j; M)$ of the partial sequence y_1, \dots, y_j , and set $h_i(j; M) = \dim_K H_i(j; M)$. If there is no danger of confusion, we simply write β_i , α_i , $H_i(j)$ and $h_i(j)$ for $\beta_i(M)$, $\alpha_i(M)$, $H_i(j; M)$ and $h_i(j; M)$ respectively.

Attached with y there are long exact sequences

$$\begin{aligned} \cdots &\longrightarrow H_i(j-1) \xrightarrow{\varphi_{i,j-1}} H_i(j-1) \longrightarrow H_i(j) \longrightarrow H_{i-1}(j-1) \\ \cdots &\longrightarrow H_0(j-1) \xrightarrow{\varphi_{0,j-1}} H_0(j-1) \longrightarrow H_0(j) \longrightarrow 0. \end{aligned}$$

Here $\varphi_{i,j-1}: H_i(j-1) \rightarrow H_i(j-1)$ is the map given by multiplication with $\pm y_j$. Note that A_j is the Kernel of the map $\varphi_{0,j-1}$. We conclude

- (*) $h_1(j) = h_1(j-1) + \alpha_j - \dim_K \text{Im } \varphi_{1,j-1}$ for $i = 1$;
- (**) $h_i(j) = h_i(j-1) + h_{i-1}(j-1) - \dim_K \text{Im } \varphi_{i,j-1} - \dim_K \text{Im } \varphi_{i-1,j-1}$ for $i > 1$.

With the notation introduced we now have:

Proposition 3.1. *Given integers $1 \leq i \leq j$ we define the set*

$$C_{i,j} = \{(a,b) \in \mathbb{N}^2 : 1 \leq b \leq j-1 \text{ and } \max(i-j+b, 1) \leq a \leq i\}.$$

Then we have

- (a) $h_i(j) \leq \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_k$ for all $i \geq 1$ and $j \geq 1$;
- (b) For given $i \geq 1$ and $j \geq 1$ the following conditions are equivalent:
 - (i) $h_i(j) = \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_k$;
 - (ii) $\varphi_{ab} = 0$ for all $(a,b) \in C_{i,j}$;
 - (iii) $\mathfrak{m}H_a(b) = 0$ for all $(a,b) \in C_{i,j}$.

Proof. By induction on j and using equations (*) and (**) one proves that

$$h_i(j) = \sum_{k=1}^{j-i+1} \binom{j-k}{i-1} \alpha_{jk} - \sum_{(a,b) \in C_{i,j}} \binom{j-b}{i-a} \dim_K \text{Im } \varphi_{a,b}$$

Then (a) and the equivalence of (i) and (ii) in (b) follow immediately. For the equivalence of (ii) and (iii) we notice that a generic linear form annihilates $H_a(b)$ if and only if $\mathfrak{m}H_a(b) = 0$. \square

By taking $j = n$ we obtain the following upper bound

Corollary 3.2. $\beta_i \leq \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$ for all $i \geq 1$.

When this upper bound is reached is described in the next corollary in terms of vanishing of Koszul homology

Corollary 3.3. (a) For a given integer i the following conditions are equivalent:

- (i) $\beta_i = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$,
- (ii) $\mathfrak{m}H_a(b) = 0$ for all $(a,b) \in C_{i,n}$;

(b) The following conditions are equivalent:

- (i) $\beta_i = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j$ for all $i \geq 1$,
- (ii) $\mathfrak{m}H_a(b) = 0$ for all b and for all $a \geq 1$.

We now want to discuss when condition (b)(ii) is satisfied. We first note that it implies that y_1, \dots, y_n is a proper sequence in the sense of [23].

Definition 3.4. Let R be an arbitrary commutative ring, and M and R -module. A sequence y_1, \dots, y_r of elements of R is called a *proper M -sequence*, if $y_{j+1}H_i(j; M) = 0$ for all $i \geq 1$ and $j = 0, \dots, r-1$.

In [26] Kühl proved the following remarkable fact: The sequence y_1, \dots, y_r is a proper M -sequence if and only if

$$y_{j+1}H_1(j; M) = 0 \quad \text{for } j = 0, \dots, r-1.$$

Now we have

Theorem 3.5 (Conca-Herzog-Hibi). *Let $I \subset R$ be a graded ideal, and let $y = y_1, \dots, y_n$ be a sequence of generic linear forms. The following conditions are equivalent:*

(a) R/I has maximal Betti numbers, i.e.

$$\beta_i(R/I) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j(R/I) \quad \text{for all } i \geq 1;$$

(b) y is a proper R/I -sequence;

(c) I is componentwise linear.

Proof. Let z be a generic linear form. Then $zH_i(p) = 0$ if and only if $\mathfrak{m}H_i(p) = 0$. Thus the equivalence of (a) and (b) follows from 3.3 (b). The equivalence of (b) and (c) can be found in [16, Theorem 4.5]. \square

Another important method to obtain upper bounds for resolutions is to compare the resolution of an ideal I with the resolution of its initial ideal $\text{in}(I)$ with respect to some term order $<$ on R . The basic fact is the following

Theorem 3.6. *Let $I \subset R$ be a graded ideal. Then for any term order $<$ one has*

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\text{in}_<(I)) \quad \text{for all } i, j.$$

Proof. Let \tilde{R} be the $k[t]$ -algebra $R[t]$, where t is an indeterminate of degree 0. By [17, Theorem 15.17] there exists a graded ideal $\tilde{I} \subset \tilde{R}$ such that the $k[t]$ -algebra \tilde{R}/\tilde{I} is a free $k[t]$ -module (and thus flat over $k[t]$), and such that

$$(2) \quad (\tilde{R}/\tilde{I})/t(\tilde{R}/\tilde{I}) \cong R/\text{in}_<(I),$$

and

$$(3) \quad (\tilde{R}/\tilde{I})_t \cong (R/I) \otimes_k k[t, t^{-1}],$$

as graded k -algebras. The ideal $\tilde{I} \subset \tilde{R}$ is constructed by means of a weight function.

Let \mathbb{F} be the minimal graded free \tilde{S} -resolution of \tilde{R}/\tilde{I} . Then (2) implies that $\mathbb{F}/t\mathbb{F}$ is a graded minimal free R -resolution of R/I , so that $\beta_{ij}(\tilde{R}/\tilde{I}) = \beta_{ij}(R/\text{in}_<(I))$ for all i and j , and (3) implies that the localized complex \mathbb{F}_t is a graded (not necessarily minimal) free $R \otimes_K K[t, t^{-1}]$ resolution of $(R/I) \otimes_K K[t, t^{-1}]$. Thus, $\beta_{ij}(R/I) = \beta_{ij}((R/I) \otimes_K K[t, t^{-1}]) \leq \beta_{ij}(\tilde{R}/\tilde{I})$, as desired. \square

Let M be a finitely generated graded S -module. The *regularity* of M is defined to be the number $\text{reg}(M) = \max\{j - i : \beta_{ij}(M) \neq 0\}$. As an immediate consequence of 3.6 we have

Corollary 3.7. *Let $I \subset R$ be a graded ideal. Then for any term order $<$ one has:*

- (a) $\text{projdim } R/I \leq \text{projdim } R/\text{in}_{<}(I)$.
- (b) $\text{depth } R/I \geq \text{depth } R/\text{in}_{<}(I)$.
- (c) *If $R/\text{in}_{<}(I)$ is Cohen-Macaulay (Gorenstein), then so is S/I .*
- (d) $\text{reg } R/I \leq \text{reg } R/\text{in}_{<}(I)$.

We shall see in the next section that all inequalities of 3.7 become equalities, if $\text{in}_{<}(I)$ is replaced by the generic initial ideal $\text{Gin}(I)$ with respect to the reverse lexicographic order.

We fix a term order $<$ satisfying $x_1 > x_2 > \dots > x_n$. Let $I \subset R$ be an ideal. The *generic initial ideal* $\text{Gin}(I)$ with respect to this term order is defined as follows: let $GL(n)$ denote the general linear group with coefficients in k . Any $\varphi = (a_{ij}) \in GL(n)$ induces an automorphism of the graded k -algebra R , again denoted by φ , namely

$$\varphi(f(x_1, \dots, x_n)) = f\left(\sum_{i=1}^n a_{i1}x_i, \dots, \sum_{i=1}^n a_{in}x_i\right) \quad \text{for all } f \in R.$$

For the proof of the following result we refer to [17, Theorem 15.18]

Theorem 3.8 (Galligo, Bayer and Stillman). *Let $I \subset R$ be a graded ideal. Then there is a nonempty Zariski open set $U \subseteq GL(n)$ such that $\text{in}(\varphi(I))$ does not depend on $\varphi \in U$. Moreover, U meets non trivially the Borel subgroup of $GL(n)$ consisting of all upper triangular invertible matrices.*

For $\varphi \in U$ the monomial ideal $\text{in}(\varphi(I))$ is called the *generic initial ideal of I* , and will be denoted $\text{Gin}(I)$.

A monomial ideal I is called *strongly stable*, if $x_i(u/x_j) \in I$ for all monomials $u \in I$, all x_j that divides u , and all $i < j$. The ideal I is called *stable*, if $x_i(u/x_{m(u)}) \in I$ for all monomials $u \in I$, and all $i < m(u)$. Here $m(u) = \max\{i: x_i \text{ divides } u\}$.

For a monomial ideal I it is customary to denote the unique minimal set of monomial generators by $G(I)$. It is easy to see that I is strongly stable if $x_i(u/x_j) \in I$ for all monomials $u \in G(I)$, all x_j that divides u , and all $i < j$. A similar statement holds for stable ideals.

Stable monomial ideals were introduced by Eliahou and Kervaire [18] who also gave an explicit resolution of such ideals. Such ideals are important because of the following result [17, Theorem 15.23]

Theorem 3.9. *Suppose that $\text{char } k = 0$, and let $I \subset R$ be a graded ideal. Then the generic initial ideal $\text{Gin}(I)$ of I with respect to the reverse lexicographical order is strongly stable.*

Also in positive characteristic $\text{Gin}(I)$ has a nice (but much more complicated) combinatorial structure.

The Koszul homology of a stable monomial ideal I can be easily computed. We let $\varepsilon: R \rightarrow R/I$ be the canonical epimorphism, and set $u' = u/x_{m(u)}$ for all $u \in G(I)$.

Theorem 3.10. *Let $I \subset R$ be a stable ideal. For all $j = 1, \dots, n$ and $i > 0$, the Koszul homology $H_i(x_j, \dots, x_n)$ is annihilated by $\mathfrak{m} = (x_1, \dots, x_n)$. In other words, all these homology modules are k -vector spaces. A basis of $H_i(x_j, \dots, x_n)$ is given by the homology*

classes of the cycles

$$\varepsilon(u')e_\sigma \wedge e_{m(u)}, \quad u \in G(I), \quad |\sigma| = i-1, \quad j \leq \min(\sigma), \quad \max(\sigma) < m(u).$$

Proof. We proceed by induction on $n-j$. For $j=n$, we only have to consider $H_1(x_n)$ which is obviously minimally generated by the homology classes of the elements $\varepsilon(u')e_n$ with $u \in G(I)$ such that $m(u) = n$. Since by the definition of stable ideals $x_i u' \in I$ for all i , we see that $H_1(x_n)$ is a k -vector space.

Now assume that $j < n$, and that the assertion is proved for $j+1$. Then $x_j H_i(x_{j+1}, \dots, x_n) = 0$ for all $i > 0$, so that the long exact sequence

$$\begin{aligned} \cdots \quad & \xrightarrow{x_j} H_i(x_{j+1}, \dots, x_n) \longrightarrow H_i(x_j, \dots, x_n) \longrightarrow H_{i-1}(x_{j+1}, \dots, x_n) \\ & \xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_n) \longrightarrow H_{i-1}(x_j, \dots, x_n) \longrightarrow \cdots \end{aligned}$$

splits into the exact sequences

$$(4) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j, \dots, x_n) \longrightarrow R_j/I_j \xrightarrow{x_j} R_j/I_j$$

and

$$(5) \quad 0 \longrightarrow H_i(x_{j+1}, \dots, x_n) \longrightarrow H_i(x_j, \dots, x_n) \longrightarrow H_{i-1}(x_{j+1}, \dots, x_n) \longrightarrow 0.$$

for $i > 0$. Here R_j is the polynomial ring $k[x_1, \dots, x_j]$, I_j the ideal in R_j generated by the monomials $u \in G(I)$ which are not divisible by any x_i with $i > j$, in other words, $I_j = I \cap S_j$.

In sequence (4), $\text{Ker } x_j$ is minimally generated by the residues of the monomials u' with $u \in G(I)$ and $m(u) = j$. Note that the sets $\{u \in G(I) : m(u) = j\}$ and $\{u \in G(I_j) : m(u) = j\}$ are equal, and that I_j is a stable ideal in R_j . Therefore $\text{Ker } x_j$ is a k -vector space.

We now consider the short exact sequence

$$(6) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j, \dots, x_n) \longrightarrow \text{Ker } x_j \longrightarrow 0.$$

It is clear that the elements $\varepsilon(u')e_j$, $u' \in G(I)$, $m(u) = j$ are cycles in $K_1(x_j, \dots, x_n)$ such that $\delta([\varepsilon(u')e_j]) = u' + I_j$. Therefore, by (6) and our induction hypothesis, it follows that the set $\mathcal{S} = \{[\varepsilon(u')e_i] : u \in G(I), m(u) = i \geq j\}$ generates $H_1(x_j, \dots, x_n)$. Since I is a stable ideal we see that $x_j[\varepsilon(u')e_i] = 0$ for all $j = 1, \dots, n$ and all $[\varepsilon(u')e_i] \in \mathcal{S}$. In other words, $H_1(x_j, \dots, x_n)$ is a k -vector space. Finally, since the number of elements of \mathcal{S} equals $\dim_k H_1(x_{j+1}, \dots, x_n) + \dim \text{Ker } x_j$, we conclude that \mathcal{S} is a basis of $H_1(x_j, \dots, x_n)$.

In order to prove our assertion for $i > 1$ we consider the exact sequences (5). By induction hypothesis the homology module $H_{i-1}(x_{j+1}, \dots, x_n)$ is a k -vector space with basis

$$[\varepsilon(u')e_\sigma \wedge e_{m(u)}], \quad u \in G(I), \quad |\sigma| = i-2, \quad j+1 \leq \min(\sigma), \quad \max(\sigma) < m(u).$$

Given such a homology class, consider the element $\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}$. It is clear that this element is a cycle in $K_i(x_j, \dots, x_n)$, and that

$$\delta([\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}]) = \pm[\varepsilon(u')e_\sigma \wedge e_{m(u)}].$$

Thus from the exact sequence (5) and our induction hypothesis it follows that the homology classes of the cycles described in the theorem generate $H_i(x_j, \dots, x_n)$. Again the stability of the ideal I implies that m annihilates all these homology classes, so that

$H_i(x_j, \dots, x_n)$ is a K -vector space. Finally, just as for $i = 1$, a dimension argument shows that these homology classes form a basis of $H_i(x_j, \dots, x_n)$. \square

Let I be a monomial ideal. We denote by $G(I)_j$ the set of monomial generators of I of degree j . The following result of Eliahou and Kervaire [18] follows immediately from 3.10.

Corollary 3.11. *Let $I \subset SR$ be a stable ideal. Then*

- (a) $\beta_{ii+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i}$;
- (b) $\text{proj dim } R/I = \max\{m(u) : u \in G(I)\}$;
- (c) $\text{reg}(I) = \max\{\text{deg}(u) : u \in G(I)\}$.

If we consider upper bounds for the Betti-numbers of an ideal we have to fix a class \mathcal{C} of ideals and to ask if there is an upper bound for the Betti-numbers of the ideals within this class. We have already seen that the class of ideals whose residue class ring has a given sequence of annihilator numbers has such an upper bound.

Here we now consider the class \mathcal{C} of ideals with given Hilbert function. Within this class there is a distinguished ideal. In fact, let $>$ be the lexicographical monomial order induced by $x_1 > x_2 > \dots > x_n$. Recall that a monomial ideal $I \subset R$ is called a *lexsegment ideal*, if for each monomial $u \in I$, all monomials $v > u$ belong to I as well.

Let $B \subset R_d$ be a set of monomials. Then B is called a *lexsegment* if with each $u \in B$ we have $v \in B$ for all $v > u$ in the lexicographical order. A lexsegment ideal is an ideal which is spanned in each degree by a lexsegment set of monomials.

We denote by $\text{Shad}(B)$ the *shadow* of B , i.e. the set of monomials

$$\{x_1, \dots, x_n\}B = \{x_i u : u \in B, i = 1, \dots, n\}.$$

The set B is called a (*strongly*) *stable* set of monomials if the ideal generated by B is (strongly) stable.

We let $m_i(B)$ the number of elements $u \in B$ with $m(u) = i$, and set $m_{\leq i}(B) = \sum_{j=1}^i m_j(B)$. Then we have

Lemma 3.12. *Let $B \subset M_d$ be a stable set of monomials. Then*

- (a) $m_i(\text{Shad}(B)) = m_{\leq i}(B)$;
- (b) $|\text{Shad}(B)| = \sum_{i=1}^n m_{\leq i}(B)$.

Proof. (b) is of course a consequence of (a). For the proof of (a) we note that the map

$$\varphi: \{u \in B : m(u) \leq i\} \rightarrow \{u \in \text{Shad}(B) : m(u) = i\}, \quad u \mapsto ux_i$$

is a bijection. In fact, φ is clearly injective. To see that φ is surjective, we let $v \in \text{Shad}(B)$ with $m(v) = i$. Since $v \in \text{Shad}(B)$, there exists $w \in B$ with $v = x_j w$ for some $j \leq i$. It follows that $m(w) \leq i$. If $j = i$, then we are done. Otherwise, $j < i$ and $m(w) = i$. Hence, since B is stable it follows that $u = (x_j/x_i)w \in B$. The assertion follows, since $v = ux_i$. \square

The following result is crucial.

Theorem 3.13 (Bayer [5]). *Let $L \subset R_d$ be a lexsegment, and $B \subset R_d$ be a stable set of monomials with $|L| \leq |B|$. Then $m_{\leq i}(L) \leq m_{\leq i}(B)$ for $i = 1, \dots, n$.*

We denote by B^{lex} the unique lexsegment set of monomials with $|B^{lex}| = |B|$. Now Lemma 3.12 and Theorem 3.13 imply

Corollary 3.14. *Let $B \subset R_d$ be a stable set of monomials, then $|\text{Shad}(B^{lex})| \leq |\text{Shad}(B)|$.*

Using all this we now get

Theorem 3.15. *Let $I \subset R$ be a graded ideal. Then there exists a unique lexsegment ideal in R , denoted I^{lex} , such that R/I and R/I^{lex} have the same Hilbert function.*

Proof. Let $<$ be any monomial order. It is easy to see that S/I and $S/\text{in}_{<}(I)$ have the same Hilbert function. Hence we may replace I by $\text{in}_{<}(I)$, and thus may assume that I is a monomial ideal. Then for any field L the Hilbert function of $L[x_1, \dots, x_n]/(G(I))$ does not depend on L . Thus we may replace k by L if necessary, and thus may as well assume that $\text{char } k = 0$. Then by Theorem 3.9 the generic initial ideal $\text{Gin}(I)$ of I with respect to the reverse lexicographical order is strongly stable.

For each d let I_d be spanned by the set of monomials N_d . Then N_d is a strongly stable set of monomials. Let I_d^{lex} be the subspace of R_d spanned by N_d^{lex} . We set $I^{lex} = \bigoplus_{d \geq 0} I_d^{lex}$, and show that I^{lex} is an ideal, in other words, that $\{x_1, \dots, x_n\} I_d^{lex} \subset I_{d+1}^{lex}$ for all d .

By Corollary 3.14 one has $|\text{Shad}(N_d^{lex})| \leq |\text{Shad}(N_d)| \leq |N_{d+1}| = |N_{d+1}^{lex}|$. On the other hand, since $\text{Shad}(N_d^{lex})$ and N_{d+1}^{lex} are both lexsegments, this inequality implies $\text{Shad}(N_d^{lex}) \subset N_{d+1}^{lex}$, as desired.

It is obvious from the construction that R/I and R/I^{lex} have the same Hilbert function. \square

Bigatti [8] and Hulett [24] proved independently the following theorem if the base field k of R is of characteristic 0. A proof in arbitrary characteristic was later given by Pardue [27] using a suitable polarization argument.

Theorem 3.16 (Bigatti, Hulett, Pardue). *Let $I \subset R$ be a graded ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{lex}) \quad \text{for all } i \text{ and } j.$$

In particular, among all ideals with a given Hilbert function, the unique lexsegment ideal with this given Hilbert function has the largest Betti-numbers.

Proof. We outline the proof in case $\text{char}(k) = 0$. By Theorem 3.6 we have $\beta_{ij}(I) \leq \beta_{ij}(\text{Gin}(I))$ for all i and j , where $\text{Gin}(I)$ denotes the generic initial ideal of I with respect to the reverse lexicographical order. Since we assume that $\text{char}(k) = 0$, it follows from Theorem 3.9 that $\text{Gin}(I)$ is a strongly stable ideal. We may therefore assume that I itself is a strongly stable monomial ideal. Then $\beta_{ij}(I) = \sum_{u \in G(I)_j} \binom{m(u)_i - 1}{i}$, by Corollary 3.11(a). A similar formula holds for I^{lex} , since I^{lex} is also strongly stable. These formulas for the Betti-numbers can be rewritten in terms of the numbers $m_{\leq i}(I)$ $m_{\leq i}(I^{lex})$. Then using Bayer's theorem 3.13 according to which $m_{\leq i}(I) \leq m_{\leq i}(I^{lex})$ for all i , one obtains the desired inequalities. \square

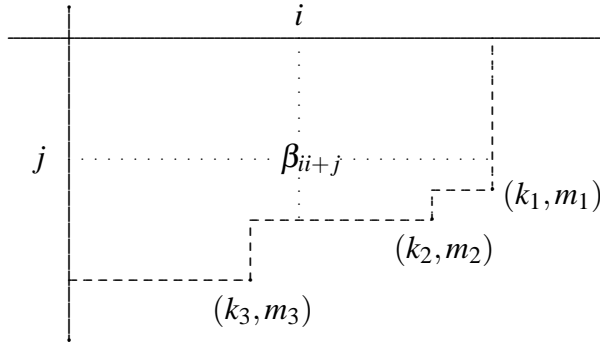
4. LECTURE: STABILITY

In this section we assume that $R = k[x_1, \dots, x_n]$ is the polynomial ring over an infinite field K , and $I \subset R$ is a graded ideal. In the previous section we have seen that for any term

order one has $\beta_{ij}(I) \leq \beta_{ij}(\text{in}(I))$. One may ask on what conditions on the term order or on the ideal one obtains more precise information in this comparison. A classical result in this direction is the theorem of Bayer-Stillman [7] which asserts that $\text{reg}(I) = \text{reg}(\text{Gin}(I))$. Here and throughout this section $\text{Gin}(I)$ denotes the generic initial ideal with respect to the reverse lexicographical order.

A remarkable extension of the Bayer-Stillman theorem, which we want to discuss next, was proved by Bayer, Charalambous and Popescu in [6]. Let M be a finitely generated graded S -module. A Betti number $\beta_{kk+m} \neq 0$ of M is called *extremal* if $\beta_{ii+j} = 0$ for all $(i, j) \neq (k, m)$ with $i \geq k$ and $j \geq m$.

The following picture displays the Betti diagram of a graded free resolution in the form of a MACAULAY output. The entry with coordinates (i, j) is the Betti number β_{ii+j} . In our picture the outside corners of the dashed line give the positions of the extremal Betti numbers.



Let M be a finitely generated graded R -module, and let $y = y_1, \dots, y_n$ be generic linear forms. As in Section 3 we set $H_i(j) = H_i(y_1, \dots, y_j; M)$, and

$$A_j = (y_1, \dots, y_{j-1})M :_M y_j / (y_1, \dots, y_{j-1})M.$$

All $H_i(j)$ as well as all A_j are R -modules of finite length and since M is a graded S -module, all $H_i(j)$ and all A_j are naturally graded, and there are graded isomorphisms $H_i(n)_j \cong \text{Tor}_i(k, M)_j$ for all i and j .

Let N be an Artinian graded module. We set

$$s(N) = \begin{cases} \max\{s : N_s \neq 0\} & \text{if } N \neq 0, \\ -\infty & \text{if } N = 0. \end{cases}$$

Now we introduce the following numbers attached to M and the sequence $y = y_1, \dots, y_n$. We set

$$r_j = \max\{s(H_i(j)) - i : i \geq 1\} \quad \text{and} \quad s_j = s(A_j) \quad \text{for } j = 1, \dots, n,$$

and put $r_0 = 0$. Observe that $\text{reg}(M) = \max\{r_n, s(M/\mathfrak{m}M)\}$.

We quote the following technical result from [2].

Theorem 4.1. *With the hypotheses and notation introduced we have*

- (a) $r_j = \max\{s_1, \dots, s_j\}$ for $j = 1, \dots, n$. In particular, $r_1 \leq r_2 \leq \dots \leq r_n$.
- (b) Let $\mathcal{J} = \{j_1, \dots, j_l\}$, $1 \leq j_1 < j_2 < \dots < j_l \leq n$, be the set of elements $j \in [n]$ such that $r_j - r_{j-1} \neq 0$. Then for all t with $1 \leq t \leq l$ and all j with $j_t \leq j$ we have
 - (i) $H_i(j)_{i+s} = 0$ for $s > r_{j_{t-1}}$ and $i > j - j_t + 1$;

$$(ii) H_{j-j_t+1}(j)_{j-j_t+1+r_{j_t}} \cong (A_{j_t-1})_{r_{j_t}}.$$

This result yields a characterization of the extremal Betti-numbers in terms of Koszul homology

Corollary 4.2. *Let the numbers j_t be defined as in 4.1, and set $k_t = n - j_t + 1$ and $m_t = r_{j_t}$. Then the graded Betti number β_{i+j} of M is extremal if and only if*

$$(i, j) \in \{(k_t, m_t) : t = 1, \dots, l\}.$$

Moreover, $\beta_{k_t, k_t+m_t} = \dim_K(A_{j_t})_{s_{j_t}}$ for $t = 1, \dots, l$.

Let $I \subset R$ be a graded ideal. We want to compare the graded Betti-numbers of R/I and $R/\text{Gin}(I)$. Choosing generic coordinates we may assume that $\text{in}(I) = \text{Gin}(I)$, and that x_n, x_{n-1}, \dots, x_1 is a generic sequence for R/I . For the reverse lexicographical order induced by $x_1 > x_2 > \dots > x_n$ one has

$$\text{in}((x_j, \dots, x_n) + I) = (x_j, \dots, x_n) + \text{in}(I)$$

and

$$\text{in}((x_j, \dots, x_n) + I) : x_{j-1} = ((x_j, \dots, x_n) + \text{in}(I)) : x_{j-1}.$$

It follows that

$$((x_j, \dots, x_n) + I) : x_{j-1} / ((x_j, \dots, x_n) + I)$$

and

$$((x_i, \dots, x_n) + \text{in}(I)) : x_{i-1} / ((x_i, \dots, x_n) + \text{in}(I))$$

have the same Hilbert function.

Let A_j be the module defined before in case that $M = R/I$. Then

$$A_j = ((x_j, \dots, x_n) + I) : x_{j-1} / ((x_j, \dots, x_n) + I).$$

We set

$$A_j^* = ((x_i, \dots, x_n) + \text{in}(I)) : x_{i-1} / ((x_i, \dots, x_n) + \text{in}(I)),$$

and $\alpha_j = \ell(A_j)$, $\alpha_j^* = \ell(A_j^*)$, $s_j = s(A_j)$ and $s_j^* = s(A_j^*)$ for $j = 1, \dots, n$.

The preceding considerations now yield

Lemma 4.3. *The modules A_j and A_j^* have the same Hilbert functions. In particular, $\alpha_j = \alpha_j^*$ and $s_j = s_j^*$ for $j = 1, \dots, n$.*

Combining this result with Theorem 4.1 and Corollary 4.2 we obtain

Theorem 4.4 (Bayer-Charalambous-S.Popescu). *Let $I \subset R$ be a graded ideal, and let $\text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. Then for any two integers $i, j \in \mathbb{N}$ one has*

- (a) *the ij th Betti number of R/I is extremal if and only if the ij th Betti number of $R/\text{Gin}(I)$ is extremal;*
- (b) *the corresponding extremal Betti numbers of R/I and $R/\text{Gin}(I)$ are equal.*

This theorem implies in particular

Corollary 4.5. *Let $I \subset R$ be a graded ideal, $\text{Gin}(I)$ the generic initial ideal of I with respect to the reverse lexicographic order. Then*

- (a) (Bayer-Stillman) $\text{reg}(I) = \text{reg}(\text{Gin}(I))$;
- (b) $\text{projdim} R/I = \text{projdim} R/\text{Gin}(I)$;
- (c) R/I is Cohen-Macaulay, if and only if $R/\text{Gin}(I)$ is Cohen-Macaulay.

Under which circumstances do all the graded Betti-numbers of I and $\text{Gin}(I)$ agree? The answer is given by

Theorem 4.6 (Aramova-Herzog-Hibi). *Suppose $\text{char} k = 0$, and let $I \subset R$ be a graded ideal. The following conditions are equivalent:*

- (a) $\beta_{i,i+j}(I) = \beta_{i,i+j}(\text{Gin}(I))$ for all i and j ;
- (b) I is componentwise linear.

For the proof of this theorem we need some preparation. We write $I_{\langle j \rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to I . Moreover, we write $I_{\geq d}$ for the ideal generated by all homogeneous polynomials of I whose degree is greater than or equal to d .

The Betti-numbers of $\text{Gin}(I)$ have the following properties

Proposition 4.7. *Let $I \subset R$ be a graded ideal generated in degree d . Then we have:*

- (a) if $\beta_{i,i+j}(\text{Gin}(I)) \neq 0$, then $\beta_{i',i'+j}(\text{Gin}(I)) \neq 0$ for all $i' < i$;
- (b) if $\beta_{0,j}(\text{Gin}(I)) \neq 0$, then $\beta_{0,j'}(\text{Gin}(I)) \neq 0$ for all $d \leq j' < j$.

Proof. Since the generic initial ideal is strongly stable, statement (a) follows from Corollary 3.11(a).

Let g_1, \dots, g_m be the generators of I of degree d . Suppose that $\beta_{0,j-1}(\text{Gin}(I)) = 0$. Then consider the ideal $I_{\geq j-2}$. Since $\text{Gin}(I_{\geq j-2}) = \text{Gin}(I)_{\geq j-2}$, we may assume that $\beta_{0,d+1}(\text{Gin}(I)) = 0$. We have to show that $\text{Gin}(I)$ is generated in degree d . It follows from $\beta_{0,d+1}(\text{Gin}(I)) = 0$ that all S -polynomials of degree $d+1$ reduce to zero with respect to $\{g_1, \dots, g_m\}$. Since $(\text{in}(g_1), \dots, \text{in}(g_m))$ is a strongly stable ideal, its first syzygy module is generated in degree $d+1$, the fact that the S -polynomials of this degree reduce to zero, implies that $\{g_1, \dots, g_m\}$ is a Gröbner basis of I . From this it follows that $\text{Gin}(I)$ is generated in degree d . \square

We shall also need

Lemma 4.8. *Let I and J be graded ideals of R generated in degree d with the same graded Betti numbers. Then $I_{\geq d+1}$ and $J_{\geq d+1}$ have the same graded Betti numbers.*

Proof. The exact sequence

$$0 \longrightarrow I_{\geq d+1} \longrightarrow I \longrightarrow k(-d)^{\beta_{0,d}} \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Tor}_{i+1}(k, I_{\geq d+1})_{(i+1)+(j-1)} &\rightarrow \text{Tor}_{i+1}(k, I)_{(i+1)+(j-1)} \rightarrow \text{Tor}_{i+1}(k, k)^{\beta_{0,d}}_{(i+1)+j-(d+1)} \\ &\rightarrow \text{Tor}_i(k, I_{\geq d+1})_{i+j} \rightarrow \text{Tor}_i(k, I)_{i+j} \rightarrow \text{Tor}_i(k, k)^{\beta_{0,d}}_{i+j-d} \rightarrow \cdots \end{aligned}$$

It then follows that $\beta_{i,i+j}(I_{\geq d+1}) = \beta_{i,i+j}(I)$ for all i and for all $j \neq d, d+1$. Also, $\beta_{i,i+j}(I_{\geq d+1}) = 0$ if $j \leq d$. Now, if $j = d+1$, then the above long exact sequence becomes

$$0 \rightarrow \mathrm{Tor}_{i+1}(k, I)_{i+1+d} \rightarrow \mathrm{Tor}_{i+1}(k, k)_{i+1}^{\beta_{0,d}} \rightarrow \mathrm{Tor}_i(k, I_{\geq d+1})_{i+d+1} \rightarrow \mathrm{Tor}_i(k, I)_{i+d+1} \rightarrow 0.$$

Hence, $\beta_{i,i+d+1}(I_{\geq d+1}) = \beta_{i,i+d+1}(I) + \binom{n}{i+1} \beta_{0,d}(I) - \beta_{i+1,i+1+d}(I)$.

The same formulae are valid for $\beta_{i,i+j}(J)$. This completes the proof. \square

We are now in the position to give a proof of Theorem 4.6.

Proof of 4.6. First, suppose that I is componentwise linear. The following formula for the graded Betti numbers of a componentwise linear ideal I is known [21]:

$$\beta_{i,i+j}(I) = \beta_i(I_{\langle j \rangle}) - \beta_i(\mathfrak{m}I_{\langle j-1 \rangle}).$$

Here \mathfrak{m} is the irrelevant maximal ideal (x_1, \dots, x_n) of R . Since a strongly stable ideal is componentwise linear and since $\mathrm{Gin}(I)$ is strongly stable, the same formula is valid for $\mathrm{Gin}(I)$. Therefore, it suffices to prove that $\beta_i(I_{\langle j \rangle}) = \beta_i(\mathrm{Gin}(I)_{\langle j \rangle})$ and $\beta_i(\mathfrak{m}I_{\langle j-1 \rangle}) = \beta_i(\mathfrak{m}\mathrm{Gin}(I)_{\langle j-1 \rangle})$.

Since $I_{\langle j \rangle}$ has a linear resolution, it follows from the Bayer–Stillman theorem (Corollary 4.5), that $\mathrm{Gin}(I_{\langle j \rangle}) = \mathrm{Gin}(I)_{\langle j \rangle}$. Since $I_{\langle j \rangle}$ and $\mathrm{Gin}(I_{\langle j \rangle})$ have the same Hilbert function, and since the Betti numbers of a module with linear resolution are determined by its Hilbert function, the first equality follows. To prove the second one, we note that $\mathfrak{m}I_{\langle j-1 \rangle}$ has again a linear resolution and that, by the same reason as before, $\mathfrak{m}\mathrm{Gin}(I)_{\langle j-1 \rangle} = \mathrm{Gin}(\mathfrak{m}I_{\langle j-1 \rangle})$.

Second, suppose that I and $\mathrm{Gin}(I)$ have the same graded Betti numbers. Let $\max(I)$ (resp. $\min(I)$) denote the maximal (resp. minimal) degree of a homogeneous generator of I . To show that I is componentwise linear, we work with induction on $r = \max(I) - \min(I)$. Set $d = \min(I)$.

Let $r = 0$. Since I and $\mathrm{Gin}(I)$ have the same graded Betti numbers, it follows that $\mathrm{Gin}(I)$ is generated in degree d . Since $\mathrm{Gin}(I)$ is a strongly stable ideal, we have that $\mathrm{Gin}(I)$ has a linear resolution, hence I has a linear resolution.

Now, suppose that $r > 0$. Since $\mathrm{Gin}(I_{\geq d+1}) = \mathrm{Gin}(I)_{\geq d+1}$, our induction hypothesis and Lemma 4.8 imply that $I_{\geq d+1}$ is componentwise linear. Thus, it suffices to prove that $I_{\langle d \rangle}$ has a linear resolution. Suppose this is not the case. Then, by the Bayer–Stillman theorem, $\mathrm{Gin}(I_{\langle d \rangle})$ has regularity $> d$. Moreover, since $\mathrm{Gin}(I_{\langle d \rangle})$ is strongly stable, its regularity equals $\max(\mathrm{Gin}(I_{\langle d \rangle}))$. It follows from Theorem 4.7 that $\mathrm{Gin}(I_{\langle d \rangle})$ has a generator of degree $d+1$. Now,

$$\begin{aligned} \beta_{0,d+1}(I) &= \dim I_{d+1} - \dim(\mathfrak{m}I_{\langle d \rangle})_{d+1} \\ &= \dim I_{d+1} - \dim(I_{\langle d \rangle})_{d+1}, \end{aligned}$$

and

$$\begin{aligned} \beta_{0,d+1}(\mathrm{Gin}(I)) &= \dim \mathrm{Gin}(I)_{d+1} - \dim(\mathfrak{m}\mathrm{Gin}(I)_{\langle d \rangle})_{d+1} \\ &= \dim \mathrm{Gin}(I)_{d+1} - \dim(\mathfrak{m}\mathrm{Gin}(I_{\langle d \rangle}))_{d+1} \\ &> \dim \mathrm{Gin}(I)_{d+1} - \dim \mathrm{Gin}(I_{\langle d \rangle})_{d+1}, \end{aligned}$$

because $(\mathfrak{m} \operatorname{Gin}(I_{\langle d \rangle}))_{d+1}$ is properly contained in $\operatorname{Gin}(I_{\langle d \rangle})_{d+1}$. Hence

$$\beta_{0,d+1}(\operatorname{Gin}(I)) > \beta_{0,d+1}(I),$$

a contradiction. This completes our proof. \square

We note that Theorem 4.6 and Theorem 4.7 (b) are not valid in positive characteristic. Indeed, if characteristic $p > 0$, then $I = (x^p, y^p)$ provides a counterexample.

In the proof of Theorem 4.6 (a) \Rightarrow (b) we only used that $\beta_{0j}(I) = \beta_{0j}(\operatorname{Gin}(I))$ for all j . Thus this condition implies that $\beta_i(I) = \beta_i(\operatorname{Gin}(I))$ for all i . This was first noted by Conca in [16, Theorem 1.2]. We now generalize this observation and first show

Theorem 4.9 (Conca-Herzog-Hibi). *Let M be a graded R -module. Suppose $\beta_i(M) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j(M)$ for some i . Then*

$$\beta_k(M) = \sum_{j=1}^{n-k+1} \binom{n-j}{k-1} \alpha_j(M) \quad \text{for all } k \geq i$$

Proof. It is enough to prove the statement for $k = i + 1$. Let $y = y_1, \dots, y_n$ be a sequence of generic linear forms and denote by $H_a(b)$ the associated Koszul homology $H_a(b; M)$. By Proposition 3.1(b) we have to show that $\mathfrak{m}H_a(b) = 0$ for all $(a, b) \in C_{i,n}$ implies that $\mathfrak{m}H_a(b) = 0$ for all $(a, b) \in C_{i+1,n}$. But

$$C_{i+1,n} \setminus C_{i,n} = \{(i+1, b) : b \leq n-1\}.$$

Thus it suffices to show: if $\mathfrak{m}H_i(b) = 0$ for all b , then $\mathfrak{m}H_{i+1}(b) = 0$ for all b .

We use the theorem of Kühl quoted before Theorem 3.5. The theorem implies: if $\mathfrak{m}H_1(b; M) = 0$ for all b , then $\mathfrak{m}H_i(b; M) = 0$ for all b and all $i \geq 1$.

Now assume that we have $H_i(b; M) = 0$ for given $i > 1$ and all b . Then $H_i(b; M) \cong H_1(b; \operatorname{syz}_{i-1}(M))$ and $H_{i+1}(b; M) \cong H_2(b; \operatorname{syz}_{i-1}(M))$. Assuming that $\mathfrak{m}H_i(b; M) = 0$ for all b implies that $\mathfrak{m}H_1(b; \operatorname{syz}_{i-1}(M)) = 0$ for all b . Then the theorem of Kühl implies that $0 = \mathfrak{m}H_2(b; \operatorname{syz}_{i-1}(M)) = \mathfrak{m}H_{i+1}(b; M)$, as desired. \square

Corollary 4.10. *Assume $\operatorname{char}(k) = 0$, and let $I \subset R$ be a graded ideal. Suppose that $\beta_i(I) = \beta_i(\operatorname{Gin}(I))$ for some i . Then*

$$\beta_k(I) = \beta_k(\operatorname{Gin}(I)) \quad \text{for all } k \geq i.$$

Proof. Since we assume $\operatorname{char}(k) = 0$ the ideal $\operatorname{Gin}(I)$ is strongly stable and hence componentwise linear. It follows from 3.5 that

$$\beta_{i+1}(R/\operatorname{Gin}(I)) = \sum_{j=1}^{n-i+2} \binom{n-j}{i} \alpha_j(R/\operatorname{Gin}(I))$$

By Lemma 4.3 and our assumption this implies that

$$\beta_{i+1}(R/I) = \sum_{j=1}^{n-i+2} \binom{n-j}{i} \alpha_j(R/I)$$

Now we apply Theorem 4.9 and again Lemma 4.3 to conclude that

$$\begin{aligned}
 \beta_k(I) = \beta_{k+1}(R/I) &= \sum_{j=1}^{n-k+2} \binom{n-j}{k} \alpha_j(R/I) \\
 &= \sum_{j=1}^{n-k+2} \binom{n-j}{k} \alpha_j(R/\text{Gin}(I)) \\
 &= \beta_{k+1}(R/\text{Gin}(I)) = \beta_k(\text{Gin}(I))
 \end{aligned}$$

for $k = i, \dots, n-1$. □

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