SMR1563/3

School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics

(24 May - 11 June 2004)

Singular algebraic curves

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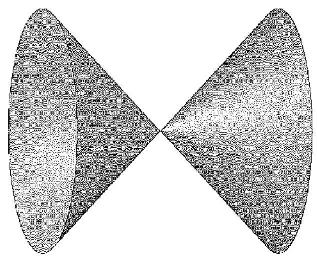
Hypersurface Singularities



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Extracts of a Preliminary Version of the book

Singular Algebraic Curves

by Gert-Martin Greuel, Christoph Lossen and Eugenii Shustin

To appear in Springer-Verlag, 2005.

Preface

In the course, I give a short introduction to some basic notions of singularity theory, in particular to hypersurface singularities. Singularity theory, mainly developed by John Milnor and Vladimir I. Arnold in the sixties and seventies, is a wonderful example where topology and algebra had substantial impact on each other. For example, the Milnor number, which is the number of vanishing cycles in the nearby smooth fibre of a hypersurface singularity, can be described algebraically as the dimension of some Artinian algebra. Indeed, the surprising interplay between geometry on the one side and algebra on the other side is one of the reasons why singularity has developed so rapidly and still is a very active area of research.

In my course I follow roughly the notes, aiming at the classification of the so-called simple or ADE singularities.

As the notes are an extract (Sections I.2.1, I.2.3 and I.2.4) of a forthcoming book of C. Lossen, E. Shustin and myself (see the end of these notes for a table of contents), they are naturally quite detailed and contain full proofs. Although the proofs sometimes refer to previous chapters, I hope that they are sufficiently self-contained, and that the notes are useful as a short tour into some of the most basic but also most exciting pieces of singularity theory.

I should like to thank the organizers of the "School on Commutative Algebra and Interactions with Algebraic Geometry and Combinatorics" in particular Guiseppe Valla and Ngo Viet Trung for creating an interesting and stimulating school, Christoph Lossen for preparing the notes, and the ICTP for its hospitality.

Gert-Martin Greuel

I. Singularity Theory

2 Hypersurface Singularities

This section is devoted to the study of isolated hypersurface singularities. We introduce basic invariants like the Milnor and Tjurina number and show that they behave semicontinuously under deformations. This will be the first important application of the finite coherence theorem proved in Section 1.

When dealing with hypersurface singularities given by a convergent power series f, $f(0) = 0$, one can either consider the (germ of the) *function* f or, alternatively, the *zero set* of f, that is, the complex space germ $V(f) = f^{-1}(0)$ at 0. With respect to these different points of view we have different equivalence relations, different notions of deformation, etc. For example, we have two equivalence relations for hypersurface singularities: *right equivalence* (referring to functions) and *contact equivalence* (referring to zero sets of functions). Although we are mainly interested in contact equivalence we treat both cases in parallel. In most cases statements about right equivalence turn out to be a special case of statements about contact equivalence.

We prove a finite determinacy theorem for isolated hypersurface singularities under right, as well as under contact, equivalence. Using finite determinacy and properties of invariants we give a complete proof of the classification of the so-called *simple* or *ADE*-singularities, which turns out to be the same for right, as well as for contact, equivalence.

2.1 Invariants of Hypersurface Singularities

We study the Milnor and Tjurina number and its behaviour under deformations.

Definition 2.1. Let $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ be a convergent power series.

(1) The ideal
$$j(f) := \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathbb{C}\{\boldsymbol{x}\}}$$
is called the *Jacobian ideal*, or the *Milnor ideal* of f, and

$$\langle f, j(f)\rangle = \Big\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big\rangle_{\mathbb{C}\{\boldsymbol{x}\}}$$

is called the *Tjurina ideal* of f.

(2) The analytic algebras

$$M_f := \mathbb{C}\{\boldsymbol{x}\}/j(f), \qquad T_f := \mathbb{C}\{\boldsymbol{x}\}/\langle f, j(f)\rangle$$

are called the *Milnor* and *Tjurina algebra* of f, respectively.

(3) The numbers

$$\mu(f) := \dim_{\mathbb{C}} M_f, \qquad \tau(f) := \dim_{\mathbb{C}} T_f$$

are called the *Milnor* and *Tjurina number* of f, respectively.

The Milnor and the Tjurina algebra and, in particular, their dimensions play an important role in the study of isolated hypersurface singularities.

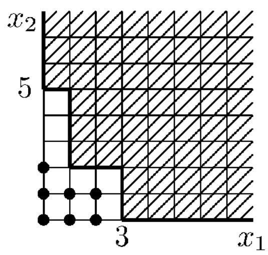
Let us consider some examples.

Example 2.1.1. (1) $f = x_1(x_1^2 + x_2^3) + x_3^2 + \ldots + x_n^2$, $n \geq 2$, is called an E_7-singularity. Since

$$j(f) = \langle 3x_1^2 + x_2^3, x_1x_2^2, x_3, \ldots, x_n\rangle$$

we see that $x_1^3, x_2^5 \in j(f)$, in particular, $f \in j(f)$.

As $\mathbb{C}\{x_1, \ldots, x_n\}/j(f) \cong \mathbb{C}\{x_1, x_2\}/\langle 3x_1^2 + x_2^3, x_1x_2^2\rangle$ we can draw the monomial diagram of $j(f)$ in the 2-plane.



The monomials belonging to the shaded region are contained in $j(f)$ and it is easy to see that none of the monomials below the shaded region belongs to $j(f)$. The only relations between these monomials are $3x_1^2 \equiv -x_2^3 \bmod j(f)$ and, hence, $3x_1^2x_2 \equiv -x_2^4 \bmod j(f)$. It follows that $1, x_1, x_1^2, x_2, x_1x_2, x_1^2x_2, x_2^2$ is a $\mathbb{C}$-basis of both M_f and T_f and, thus, $\mu(f) = \tau(f) = 7$.

(2) $f = x^5 + y^5 + x^2y^2$ has $j(f) = \langle 5x^4 + 2xy^2, 5y^4 + 2x^2y\rangle$. We can compute a $\mathbb{C}$-basis of T_f as $1, x, \ldots, x^4, xy, y, \ldots, y^4$ and a $\mathbb{C}$-basis of M_f, which has an additional monomial y^5. Hence, $10 = \tau(f) < \mu(f) = 11$.

Such computations are quite tedious by hand, but can easily be done with a computer by using a computer algebra system which allows calculations in local rings. Here is the SINGULAR code:

```
ring r=0,(x,y),ds; // a ring with a local ordering
poly f=x5+y5+x2y2;
ideal j=jacob(f);
vdim(std(j));       // the Milnor number
// -> 11

ideal fj=f,j;
vdim(std(fj));      // the Tjurina number
// -> 10
kbase(std(fj));
// -> _[1]=y4   _[2]=y3   _[3]=y2   _[4]=xy   _[5]=y
// -> _[6]=x4   _[7]=x3   _[8]=x2   _[9]=x    _[10]=1
```

Moreover, if f satisfies a certain non-degeneracy (NND) property then there is a much more handy way to compute the Milnor number. Indeed, it can be read from the Newton diagram of f (see Proposition 2.17 below).

Critical and Singular Points. Let $U \subset \mathbb{C}^n$ be an open subset, $f : U \to \mathbb{C}$ a holomorphic function and $x \in U$. We set

$$j(f) := \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \mathcal{O}(U) \subset \mathcal{O}(U)$$

and define

$$M_{f,x} := \mathcal{O}_{\mathbb{C}^n,x}/j(f)\mathcal{O}_{\mathbb{C}^n,x}\,, \qquad T_{f,x} := \mathcal{O}_{\mathbb{C}^n,x}/\langle f, j(f)\rangle \mathcal{O}_{\mathbb{C}^n,x}$$

to be the *Milnor* and *Tjurina algebra* of f at x. Furthermore, we introduce

$$\mu(f,x) := \dim_{\mathbb{C}} M_{f,x}\,, \qquad \tau(f,x) := \dim_{\mathbb{C}} T_{f,x}\,,$$

and call these numbers the *Milnor* and *Tjurina number* of f at x.

It is clear that $\mu(f,x) \neq 0$ if and only if $\frac{\partial f}{\partial x_i}(x) = 0$ for all i, and $\tau(f,x) \neq 0$ if and only if additionally $f(x) = 0$. Hence, we see that μ counts the singular points of the *function* f, while τ counts the singular points of the *zero set* of f, each with multiplicity $\mu(f,x)$, respectively $\tau(f,x)$. The following definition takes care of this difference:

Definition 2.2. Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ a holomorphic function, and $X = V(f) = f^{-1}(0)$ the hypersurface defined by f in U. We call

$$\mathrm{Crit}(f) := \mathrm{Sing}(f) := \left\{ x \in U \;\middle|\; \frac{\partial f}{\partial x_1}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$$

the set of *critical*, or *singular, points of* f and

$$\mathrm{Sing}(X) := \left\{ x \in U \;\middle|\; f(x) = \frac{\partial f}{\partial x_1}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}$$

the set of *singular points of* X.

A point $x \in U$ is called an *isolated critical point* of f, if there exists a neighbourhood V of x such that $\mathrm{Crit}(f) \cap V \setminus \{x\} = \emptyset$. It is called an *isolated singularity* of X if $x \in X$ and $\mathrm{Sing}(X) \cap V \setminus \{x\} = \emptyset$. Then we say also that the germ $(X, x) \subset (\mathbb{C}^n, x)$ is an *isolated hypersurface singularity*.

Note that the latter definition is a special case of Definition 1.33.

Lemma 2.3. *Let $f : U \to \mathbb{C}$ be holomorphic and $x \in U$, then the following are equivalent.*

(i) x is an isolated critical point of f,
(ii) $\mu(f, x) < \infty$,
(iii) x is an isolated singularity of $f^{-1}(f(x)) = V(f - f(x))$,
(iv) $\tau(f - f(x), x) < \infty$.

Proof. (i), respectively (iii) say that x is an isolated point of the fibre over 0 (if it is contained in the fibre) of the morphisms

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) : U \longrightarrow \mathbb{C}^n, \quad \left(f - f(x), \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) : U \longrightarrow \mathbb{C}^{n+1},$$

respectively. Hence, the equivalence of (i) and (ii), respectively of (iii) and (iv), is a consequence of the Hilbert-Rückert Nullstellensatz.

Since $\mu(f, x) \leq \tau(f - f(x), x)$, the implication (ii) $\Rightarrow$ (iv) is evident. Finally, (iii) $\Rightarrow$ (i) follows from the following lemma, which holds also for non-isolated singularities. $\square$

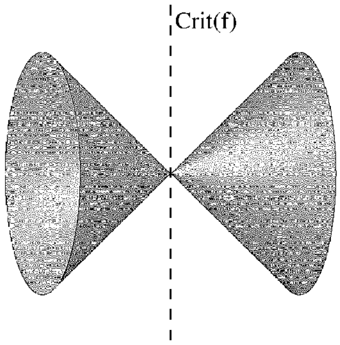
Lemma 2.4. *Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ a holomorphic function, $x \in U$ and $f(x) = 0$. Then there is a neighbourhood V of x in U such that*

$$\mathrm{Crit}(f) \cap V \subset f^{-1}(0).$$

In other words, the nearby fibres $f^{-1}(t) \cap V$, t sufficiently small, are smooth.

Proof. Consider $C = \mathrm{Crit}(f)$ with its reduced structure. As a reduced complex space, the regular points of C, $\mathrm{Reg}(C)$, are open and dense in C. Since $\frac{\partial f}{\partial x_i}$ vanishes on C for $i = 1, \dots, n$, f is locally constant on the complex manifold $\mathrm{Reg}(C)$. A sufficiently small neighbourhood V of x intersects only the connected components of $\mathrm{Reg}(C)$ having x in its closure. If $x \notin C$ the result is trivial. If $x \in C$ then $f|_{V \cap C} = 0$, since f is continuous and $f(x) = 0$. $\square$

Hence, it cannot happen that the critical set of f (the dashed line) meets $f^{-1}(0)$ as in the following picture.



Semicontinuity of Milnor and Tjurina number. In the sequel we study the behaviour of μ and τ under deformations. Loosely speaking, a deformation of a power series $f \in \mathbb{C}\{\boldsymbol{x}\}$, usually called an *unfolding*, is given by a power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ such that, setting $F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{t})$, $F_0 = f$, while a *deformation* of the hypersurface germ $f^{-1}(0)$ is given by any power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\}$ satisfying $F_0^{-1}(0) = f^{-1}(0)$. So far, unfoldings and deformations are both given by a power series F, the difference appears later when we consider isomorphism classes of deformations. However, for the moment we only consider the power series F.

Definition 2.5. A power series $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\} = \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ is called an *unfolding* of $f \in \mathbb{C}\{x_1, \dots, x_n\}$ if $F(\boldsymbol{x}, 0) = f(\boldsymbol{x})$. We use the notation

$$F_{\boldsymbol{t}}(\boldsymbol{x}) \;=\; F(\boldsymbol{x}, \boldsymbol{t})\,, \quad \boldsymbol{t} \in T\,,$$

for the family of power series $F_{\boldsymbol{t}} \in \mathbb{C}\{\boldsymbol{x}\}$ or, after choosing a representative $F\colon U \times T \to \mathbb{C}$, for the family $F_{\boldsymbol{t}}\colon U \to \mathbb{C}$ of holomorphic functions parametrized by $t \in T$, where $U \subset \mathbb{C}^n$ and $T \subset \mathbb{C}^k$ are open neighbourhoods of 0.

Theorem 2.6 (Semicontinuity of μ and τ).
Let $F \in \mathbb{C}\{\boldsymbol{x}, \boldsymbol{t}\} = \mathbb{C}\{x_1, \dots, x_n, t_1, \dots, t_k\}$ be an unfolding of $f \in \mathbb{C}\{\boldsymbol{x}\}$, $f(0) = 0$, and assume that 0 is an isolated critical point of f. Then there are neighbourhoods $U = U(0) \subset \mathbb{C}^n$, $V = V(0) \subset \mathbb{C}$, $T = T(0) \subset \mathbb{C}^k$, such that F converges on $U \times T$ and, setting $F_{\boldsymbol{t}}\colon U \to V$, $\boldsymbol{x} \mapsto F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{t})$, the following holds for each $\boldsymbol{t} \in T$:

(1) $0 \in U$ is the only critical point of $f = F_0\colon U \to V$, and $F_{\boldsymbol{t}}$ has only isolated critical points in U.
(2) For each $y \in V$,

$$\mu(f, 0) \geq \sum_{x \in \mathrm{Sing}(F_{\boldsymbol{t}}^{-1}(y))} \mu(F_{\boldsymbol{t}}, \boldsymbol{x}) \quad \textit{and}$$

$$\tau(f, 0) \geq \sum_{x \in \mathrm{Sing}(F_{\boldsymbol{t}}^{-1}(y))} \tau(F_{\boldsymbol{t}} - y, \boldsymbol{x}).$$

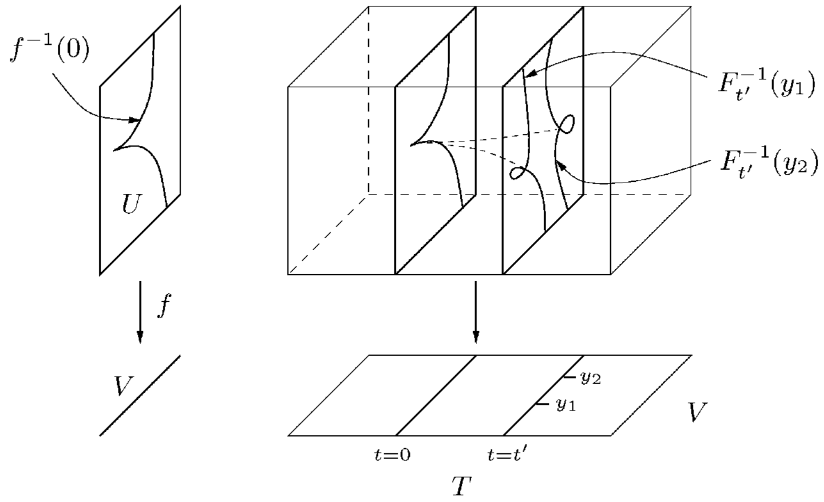


FIGURE 2.1. Deformation of an isolated hypersurface singularity

(3) Furthermore,

$$\mu(f,0) = \sum_{x\in \mathrm{Crit}(F_{\boldsymbol{t}})} \mu(F_{\boldsymbol{t}},\boldsymbol{x}).$$

Proof. (1) Choose U such that 0 is the only critical point of F_0 and consider the map

$$\Phi\colon U\times T \to \mathbb{C}^n \times T\,, \quad (\boldsymbol{x},\boldsymbol{t}) \mapsto \left(\frac{\partial F_{\boldsymbol{t}}}{\partial x_1}(\boldsymbol{x}),\dots,\frac{\partial F_{\boldsymbol{t}}}{\partial x_n}(\boldsymbol{x}),\boldsymbol{t}\right).$$

Then $\Phi^{-1}(0,0) = \mathrm{Crit}(F_0)\times\{0\} = (0,0)$ by the choice of U. Hence, by the local finiteness theorem, Φ is a finite morphism if we choose U,T to be sufficiently small. This implies that Φ has finite fibres, in particular, $\mathrm{Crit}(F_{\boldsymbol{t}})\times\{\boldsymbol{t}\} = \Phi^{-1}(0,\boldsymbol{t})$ is finite.

(2) The first inequality follows from (3). For the second consider the map

$$\Psi\colon U\times T \longrightarrow V\times\mathbb{C}^n\times T\,, \quad (\boldsymbol{x},\boldsymbol{t}) \mapsto \left(F_{\boldsymbol{t}}(\boldsymbol{x}),\frac{\partial F_{\boldsymbol{t}}}{\partial x_1}(\boldsymbol{x}),\dots,\frac{\partial F_{\boldsymbol{t}}}{\partial x_n}(\boldsymbol{x}),\boldsymbol{t}\right).$$

Then $\Psi^{-1}(0,0,0) = \mathrm{Sing}(f_0^{-1}(0))\times\{0\}$ and, again by the local finiteness theorem, $\mathrm{Sing}\left(F_{\boldsymbol{t}}^{-1}(y)\right)\times\{\boldsymbol{t}\} = \Psi^{-1}(y,0,\boldsymbol{t})$ is finite for U,V,T sufficiently small and $y\in V$, $\boldsymbol{t}\in T$. Moreover, the direct image sheaf $\Psi_*\mathcal{O}_{U\times T}$ is coherent on $V\times\mathbb{C}^n\times T$. The semicontinuity of fibre functions for finite maps implies that the function

$$\nu(y,\boldsymbol{t}) := \nu\left(\Psi_*\mathcal{O}_{U\times T}, (y,0,\boldsymbol{t})\right) = \sum_{(\boldsymbol{x},\boldsymbol{t})\in\Psi^{-1}(y,0,\boldsymbol{t})} \dim_{\mathbb{C}} \mathcal{O}_{U\times T,(\boldsymbol{x},\boldsymbol{t})}/\mathfrak{m}_{(y,0,\boldsymbol{t})}\mathcal{O}_{U\times T,(\boldsymbol{x},\boldsymbol{t})}$$

is upper semicontinuous. Since

$$\mathcal{O}_{U\times T,(\boldsymbol{x},\boldsymbol{t})}/\mathfrak{m}_{(y,0,\boldsymbol{t})}\mathcal{O}_{U\times T,(\boldsymbol{x},\boldsymbol{t})} \cong \mathcal{O}_{U,x}\Big/\left\langle F_{\boldsymbol{t}}-y, \frac{\partial F_{\boldsymbol{t}}}{\partial x_1},\dots,\frac{\partial F_{\boldsymbol{t}}}{\partial x_n}\right\rangle$$

we have $\nu(0,0)=\tau(f,0)$ and $\nu(y,\boldsymbol{t})=\sum_{\boldsymbol{x}\in\mathrm{Sing}(F_{\boldsymbol{t}}^{-1}(y))}\tau(F_{\boldsymbol{t}},\boldsymbol{x})$, and the result follows.

(3) We consider again the morphism Φ and have to show that the function

$$\nu(\boldsymbol{t}) := \nu\left(\Phi_*\mathcal{O}_{U\times T}, (0,\boldsymbol{t})\right) = \sum_{\boldsymbol{x}\in\mathrm{Crit}(F_{\boldsymbol{t}})} \dim_{\mathbb{C}} \mathcal{O}_{U,\boldsymbol{x}}\Big/\left\langle \frac{\partial F_{\boldsymbol{t}}}{\partial x_1},\dots,\frac{\partial F_{\boldsymbol{t}}}{\partial x_n}\right\rangle$$

is locally constant on T. Thus, by Theorems 1.75 and 1.76 we have to show that Φ is flat at $(0,0)$.

Since $\mathcal{O}_{U\times T,(0,0)} \cong \mathbb{C}\{x_1,\dots,x_n,t_1,\dots,t_k\}$ is a regular local ring, and since the $n+k$ component functions $\frac{\partial F_{\boldsymbol{t}}}{\partial x_1},\dots,\frac{\partial F_{\boldsymbol{t}}}{\partial x_n},t_1,\dots,t_k$ define a 0-dimensional, hence $(n+k)$-codimensional germ, the flatness follows from the following proposition. □

Proposition 2.7. *(1) Let $f=(f_1,\dots,f_k)\colon (X,x)\to(\mathbb{C}^k,0)$ be a holomorphic map germ and M a finitely generated $\mathcal{O}_{X,x}$-module. Then M is f-flat if and only if the sequence $f_1,\dots,f_k$ is an M-regular sequence*[1]*. In particular, f is flat if and only if $f_1,\dots,f_k$ is a regular sequence.*
(2) If (X,x) is the germ of an n-dimensional complex manifold, then $f_1,\dots,f_k$ is $\mathcal{O}_{X,x}$-regular if and only if $\dim(f^{-1}(0),x)=n-k$.

Example 2.7.1. (1) Consider the unfolding $F_t(x,y)=x^2-y^2(t+y)$ of the cusp singularity $f(x,y)=x^2-y^3$. We compute $\mathrm{Crit}(F_t)=\{(0,0),(0,-\frac{2}{3}t)\}$ and $\mathrm{Sing}(F_t^{-1}(0))=\{(0,0)\}$. Moreover, $\mu(f)=\tau(f)=2$, while for $t\neq 0$ we have $\mu(F_t,(0,0))=\tau(F_t,(0,0))=1$ and $\mu(F_t,(0,-\frac{2}{3}t))=1$.

(2) For the unfolding $F_t(x,y)=x^5+y^5+tx^2y^2$ we compute the critical locus to be $\mathrm{Crit}(F_t)=V(5x^4+2txy^2,5y^4+2tx^2y)$. The only critical point of F_0 is the origin $0=(0,0)$, and we have $\mu(F_0,0)=\tau(F_0,0)=16$. Using SINGULAR we compute that, for $t\neq 0$, F_t has a critical point at 0 with $\mu(F_t,0)=11$, $\tau(F_t,0)=10$, and five further critical points with $\mu=\tau=1$ each. This shows that $\mu(F_0,0)=\sum_{\boldsymbol{x}\in\mathrm{Crit}(F_t)}\mu(F_t,\boldsymbol{x})$ for each t as stated in Theorem 2.6.

[1] Recall that $f_1,\dots,f_k$ is an *M-regular sequence* or *M-regular* if and only if f_1 is a non-zerodivisor of M and f_i is a non-zerodivisor of $M/(f_1M+\ldots+f_{i-1}M)$ for $i=1,\dots,k$.

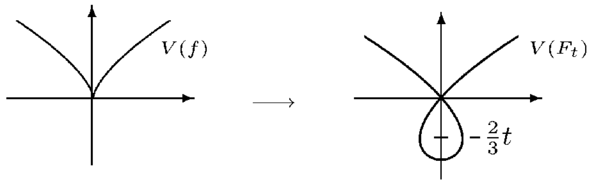


FIGURE 2.2. Deformation of a cusp singularity

But $\tau(F_0,0) = 16 > 15 = \sum_{\boldsymbol{x}\in\mathrm{Crit}(F_t)} \tau(F_t - F_t(\boldsymbol{x}), \boldsymbol{x})$, that is, even the "total" Tjurina number is not constant.

(3) The local, respectively total, Milnor number can be computed in SINGULAR by the same formulas but with a different choice of monomial ordering:

```
ring r=0,(x,y),ds;  // local ordering
poly f=x5+y5;
ideal i=f,jacob(f);
vdim(std(i));       // local Tjurina number (at 0)
// -> 16

ring R=0,(x,y),dp;  // global ordering
poly F=x5+y5+x2y2;
ideal i=F,jacob(F);
vdim(std(i));       // total (affine) Tjurina number
// -> 10
```

If the first inequality in Theorem 2.6 (2) happens to be an equality (for some $y = y(\boldsymbol{t})$, $y(0) = 0$) then the fibre $F_{\boldsymbol{t}}^{-1}(y)$ contains only one singular point:

Theorem 2.8. *Let $F \in \mathbb{C}\{\boldsymbol{x},\boldsymbol{t}\} = \mathbb{C}\{x_1,\dots,x_n,t_1,\dots,t_k\}$ be an unfolding of $f \in \langle \boldsymbol{x}\rangle^2 \subset \mathbb{C}\{\boldsymbol{x}\}$. Moreover, let $T \subset \mathbb{C}^k$, $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}$ be open neighbourhoods of the origin, and let $F_{\boldsymbol{t}}\colon U \to V$, $\boldsymbol{x} \mapsto F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x},\boldsymbol{t})$. If 0 is the only singularity of the special fibre $F_0^{-1}(0) = f^{-1}(0)$ and, for all $\boldsymbol{t} \in T$,*

$$\sum_{x\in\mathrm{Sing}(F_{\boldsymbol{t}}^{-1}(0))} \mu(F_{\boldsymbol{t}},\boldsymbol{x}) = \mu(f,0)$$

then all fibres $F_{\boldsymbol{t}}^{-1}(0)$, $\boldsymbol{t} \in T$, have a unique singular point (with Milnor number $\mu(f,0)$).

This was proven independently by Lazzeri [Laz1] and Gabrièlov [Gab1].

Right and Contact Equivalence. Now let us consider the behaviour of μ and τ under coordinate transformation and multiplication with units.

Definition 2.9. Let $f, g \in \mathbb{C}\{x_1,\dots,x_n\}$.

(1) f is called *right equivalent* to g, $f \overset{r}{\sim} g$, if there exists an automorphism φ of $\mathbb{C}\{\boldsymbol{x}\}$ such that $\varphi(f) = g$.

(2) f is called *contact equivalent* to g, $f \overset{c}{\sim} g$, if there exists an automorphism φ of $\mathbb{C}\{\boldsymbol{x}\}$ and a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ such that $f = u \cdot \varphi(g)$

If $f, g \in \mathcal{O}_{\mathbb{C}^n,x}$ then we sometimes also write $(f,x) \overset{r}{\sim} (g,x)$, respectively $(f,x) \overset{c}{\sim} (g,x)$.

Remark 2.9.1. (1) Of course, $f \overset{r}{\sim} g$ implies $f \overset{c}{\sim} g$. The converse, however, is not true (cf. Exercise 2.10, below).

(2) Any $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$ determines a biholomorphic local coordinate change $\Phi = (\Phi_1, \ldots, \Phi_n)\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ by $\Phi_i = \varphi(x_i)$, and, vice versa, any isomorphism of germs Φ determines $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$ by the same formula. We have $\varphi(g) = g \circ \Phi$ and, hence,

$$f \overset{r}{\sim} g \iff f = g \circ \Phi$$

for some biholomorphic map germ $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, that is, the diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow[\cong]{\Phi} & (\mathbb{C}^n, 0) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & (\mathbb{C}, 0) & \end{array}$$

commutes. The notion of right equivalence results from the fact that, on the level of space germs, the group of local coordinate changes acts from the right.

(3) Since f and g generate the same ideal in $\mathbb{C}\{\boldsymbol{x}\}$ if and only if there is a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ such that $f = u \cdot g$ we see that $f \overset{c}{\sim} g \Leftrightarrow \langle f \rangle = \langle \varphi(g) \rangle$ for some $\varphi \in \operatorname{Aut} \mathbb{C}\{\boldsymbol{x}\}$. Moreover, since any isomorphism of analytic algebras lifts to the power series ring by Lemma 1.12, we get

$$f \overset{c}{\sim} g \iff \mathbb{C}\{\boldsymbol{x}\}/\langle f \rangle \cong \mathbb{C}\{\boldsymbol{x}\}/\langle g \rangle \text{ as analytic } \mathbb{C}\text{-algebras.}$$

Equivalently, $f \overset{c}{\sim} g$ if and only if the complex space germs $(f^{-1}(0), 0)$ and $(g^{-1}(0), 0)$ are isomorphic.

Hence, $f \overset{r}{\sim} g$ if and only if f and g define, up to a change of coordinates in $(\mathbb{C}^n, 0)$, the same map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, while $f \overset{c}{\sim} g$ if and only if f and g have, up to coordinate change, the same zero-fibre.

Exercise 2.10. Consider the unfolding

$$f_t(x, y, z) = x^p + y^q + z^r + txyz, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Show that for all $t, t' \neq 0$, $f_t \overset{c}{\sim} f_{t'}$ but $f_t \overset{r}{\not\sim} f_{t'}$.

Lemma 2.11. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$. Then* *(i)* $f \overset{r}{\sim} g$ *implies that* $M_f \cong M_g$ *and* $T_f \cong T_g$ *as analytic algebras. In particular,* $\mu(f) = \mu(g)$ *and* $\tau(f) = \tau(g)$.

(ii) $f \overset{c}{\sim} g$ *implies that* $T_f \cong T_g$ *and hence* $\tau(f) = \tau(g)$.

Proof. (i) If $g = \varphi(f) = f \circ \Phi$, then

$$\left(\frac{\partial f \circ \Phi}{\partial x_1}(\boldsymbol{x}), \dots, \frac{\partial f \circ \Phi}{\partial x_n}(\boldsymbol{x})\right) = \left(\frac{\partial f}{\partial x_1}(\Phi(\boldsymbol{x})), \dots, \frac{\partial f}{\partial x_n}(\Phi(\boldsymbol{x}))\right) \cdot D\Phi(\boldsymbol{x}),$$

where $D\Phi$ is the Jacobian matrix of Φ, which is invertible in a neighbourhood of $\boldsymbol{x}$. It follows that $j(\varphi(f)) = \varphi(j(f))$ and $\langle \varphi(f), j(\varphi(f)) \rangle = \varphi(\langle f, j(f) \rangle)$, which proves the claim.

(ii) By the product rule we have $\langle u \cdot f, j(u \cdot f) \rangle = \langle f, j(f) \rangle$, which together with (i) implies $T_f \cong T_g$. □

In characteristic 0 it is even true that $f \overset{c}{\sim} g$ implies $\mu(f) = \mu(g)$, but this is more difficult (for an analytic proof showing that

$$\mu(f) = \dim_{\mathbb{C}} \Omega^{n-1}_{X_0} / d\Omega^{n-2}_{X_0}, \quad \text{if } n \geq 2,$$

which holds for complete intersections, cf. [Gre]). Even more, $\mu(f)$ *is a topological invariant* of $(f^{-1}(0), 0)$ (cf. [Mil] in general, respectively Section 3.4 for curves).

Example 2.11.1. (1) Consider the unfolding $F_t(x, y) = x^2 + y^2(t+y)$ with $\mathrm{Crit}(F_t) = \{(0,0), (0, -\frac{2}{3}t)\}$. The coordinate change $\varphi_t\colon x \mapsto x,\ y \mapsto y\sqrt{t+y}$, $(t \neq 0)$, satisfies $\varphi_t(x^2 + y^2) = x^2 + y^2(t+y) = F_t(x, y)$.

Hence, $(F_t, 0) \overset{r}{\sim} (x^2 + y^2, 0)$ for $t \neq 0$. Thus, we have $\tau(x^2 + y^3, 0) = 2$, but for $t \neq 0$ we have $\tau(F_t, 0) = 1$, $\tau(F_t, (0, -\frac{2}{3}t)) = 1$. Hence $(F_t, 0)$ and $(F_t, (0, -\frac{2}{3}t))$ are not contact equivalent to $(f, 0)$.

(2) Consider the unfolding $F_t(x, y) = x^2 + y^2 + txy = x(x+ty) + y^2$. The coordinate change $\varphi\colon x \mapsto x - \frac{1}{2}ty,\ y \mapsto y$ satisfies $\varphi(F_t) = x^2 + y^2(1 - \frac{1}{4}t^2)$, which is right equivalent to $x^2 + y^2$ for $t \neq \pm 2$. In particular, $(F_t, 0) \overset{r}{\sim} (F_0, 0)$ for all sufficiently small $t \neq 0$.

(3) The Milnor number is not an invariant of the contact class in positive characteristic: $f = x^p + y^{p+1}$ has $\mu(f) = \infty$, but $\mu((1+x)f) < \infty$ in $K[[x, y]]$ where K is a field of characteristic p.

Quasihomogeneous singularities. The class of those isolated hypersurface singularities, for which the Milnor and Tjurina number coincide, attains a particular importance. Of course, an isolated hypersurface singularity $(X, x) \subset (\mathbb{C}^n, x)$ belongs to this class if and only if $f \in j(f)$ for any (hence, by the chain rule, all) local equation(s) $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$. In the following, we give an alternative description of this class:

Definition 2.12. $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}[x_1 \dots, x_n]$ is called *weighted homogeneous* or *quasihomogeneous* (*QH*) of *type* $(\boldsymbol{w}; d) = (w_1, \dots, w_n; d)$ if w_i, d are positive integers satisfying

$$\boldsymbol{w}\text{-deg}(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle = w_1\alpha_1 + \ldots + w_n\alpha_n = d$$

for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $a_{\boldsymbol{\alpha}} \neq 0$. The numbers w_i are called the *weights* and d the *weighted degree* or the $\boldsymbol{w}$-*degree* of f.

Note that this property is not invariant under coordinate changes (if the w_i are not all the same then it is not even invariant under linear coordinate changes).

In the above Example 2.1.1 (1), f is QH of type $(6, 4, 9; 18)$, while in Example 2.1.1 (2), f is not QH, not even after a change of coordinates.

Remark 2.12.1. A quasihomogeneous polynomial f of type $(\boldsymbol{w}; d)$ obviously satisfies the relations[2]

$$d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} \quad \text{in } \mathbb{C}[\boldsymbol{x}],$$

$$f(t^{w_1}x_1, \dots, t^{w_n}x_n) = t^d \cdot f(x_1, \dots, x_n) \quad \text{in } \mathbb{C}[\boldsymbol{x}, t].$$

The first relation implies that f is contained in $j(f)$, hence, $\mu(f) = \tau(f)$.

The second relation implies that the hypersurface $V(f) \subset \mathbb{C}^n$ is invariant under the $\mathbb{C}^*$-*action* $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$, $(\lambda, \boldsymbol{x}) \mapsto \lambda \circ \boldsymbol{x} := (\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n)$. In particular, the complex hypersurface $V(f) \subset \mathbb{C}^n$ is contractible.

Moreover, $\text{Sing}(f)$ and $\text{Crit}(f)$ are also invariant under $\mathbb{C}^a st$ and, hence, the union of $\mathbb{C}^*$-orbits. It follows that if $V(f)$ has an isolated singularity at 0 then 0 is the only singular point of $V(f)$. Furthermore, $\boldsymbol{x} \mapsto \lambda \circ \boldsymbol{x}$ maps $V(f - t)$ isomorphically onto $V(f - \lambda^d t)$. Since $f \in j(f)$, $\text{Sing}(f)$ and $\text{Sing}(V(f))$ coincide in this situation.

Definition 2.13. An isolated hypersurface singularity $(X, x) \subset (\mathbb{C}^n, x)$, is called *quasihomogeneous* (*QH*) if there exists a quasihomogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \dots, x_n]$ such that $\mathcal{O}_{X,x} \cong \mathbb{C}\{\boldsymbol{x}\}/\langle f \rangle$.

Lemma 2.14. *Let $f \in \mathbb{C}[\boldsymbol{x}]$ be quasihomogeneous and $g \in \mathbb{C}\{\boldsymbol{x}\}$ arbitrary. Then $f \overset{c}{\sim} g$ if and only if $f \overset{r}{\sim} g$.*

Proof. Let f be weighted homogeneous of type $(w_1, \dots, w_n; d)$. If $f \overset{c}{\sim} g$ then there exists a unit $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ and an automorphism $\varphi \in \text{Aut}\,\mathbb{C}\{\boldsymbol{x}\}$ such that $u \cdot f = \varphi(g)$. Choose a d-th root $u^{1/d} \in \mathbb{C}\{\boldsymbol{x}\}$. The automorphism

$$\psi \colon \mathbb{C}\{\boldsymbol{x}\} \longrightarrow \mathbb{C}\{\boldsymbol{x}\}, \quad x_i \mapsto u^{w_i/d} \cdot x_i$$

yields $\psi\big(f(\boldsymbol{x})\big) = f\big(u^{w_1/d}x_1, \dots, u^{w_n/d}x_n\big) = u \cdot f(\boldsymbol{x})$ by Remark 2.12.1, implying the result. □

[2] The first relation generalizes *Euler's formula* for homogeneous $f \in \mathbb{C}[x_0, \dots, x_n]$: $x_0\frac{\partial f}{\partial x_0} + \ldots + x_n\frac{\partial f}{\partial x_n} = \deg(f) \cdot f$.

It is clear that for QH isolated hypersurface singularities the Milnor and Tjurina number coincide (since $f \in j(f)$). It is a remarkable theorem of K. Saito [Sai] that for an isolated singularity the converse does also hold. Let $(X,x) \subset (\mathbb{C}^n, x)$ be an isolated hypersurface singularity and let $f \in \mathbb{C}\{x_1,\dots,x_n\}$ be any local equation for (X,x), then

$$(X,x) \text{ quasihomogeneous} \iff \mu(f) = \tau(f)\,.$$

Since $\mu(f)$ and $\tau(f)$ are computable, the latter equivalence gives an effective characterization of isolated quasihomogeneous hypersurface singularities.

Newton Non-Degenerate and Semiquasihomogeneous Singularities. As mentioned before, for certain classes of singularities there is a much more handy way to compute the Milnor number. It can be read from the Newton diagram of an appropriate defining power series:

Definition 2.15. Let $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1,\dots,x_n\}$, $a_{\mathbf{0}} = 0$. Then the convex hull of the support of f in $\mathbb{R}^n$,

$$\Delta(f) := \operatorname{conv}\bigl\{\boldsymbol{\alpha} \in \mathbb{N}^n \bigm| a_{\boldsymbol{\alpha}} \neq 0\bigr\}\,,$$

is called the *Newton polytope of* f. We introduce $K(f) := \operatorname{conv}(\{\mathbf{0}\} \cup \Delta(f))$, and denote by $K_0(f)$ the closure of the set $K(f) \setminus \Delta(f)$. Then

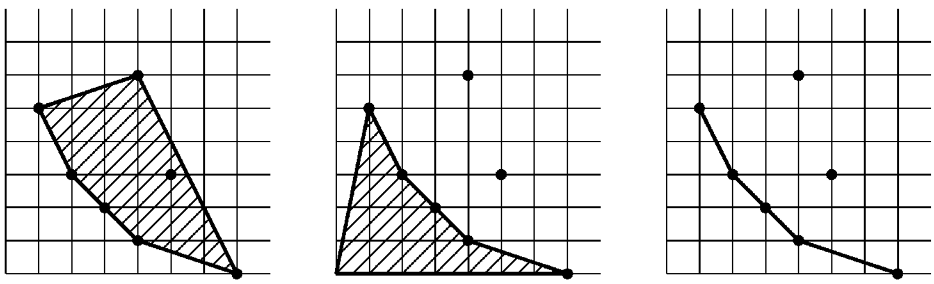
$$\Gamma(f,0) := K_0(f) \cap \Delta(f)$$

is called *Newton diagram of* f *at* the origin. Moreover, we introduce for a face $\sigma \subset \Gamma(f,0)$ the *truncation*

$$f^\sigma := \sum_{\boldsymbol{\alpha}\in\sigma} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{i}\in\sigma\cap\mathbb{N}^n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}\,,$$

that is, the sum of the monomials in f corresponding to the integral points in σ.

Example 2.15.1. Let $f = x \cdot (y^5 + xy^3 + x^2y^2 - x^2y^4 + x^3y - 10x^4y + x^6)$.



$\Delta(f)$ $K_0(f)$ $\Gamma(f,0)$

In particular, the Newton diagram at 0 has three one-dimensional faces, of slopes $-2, -1, -\frac{1}{3}$.

Definition 2.16. A power series $f = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in \mathbb{C}\{\boldsymbol{x}\}$ with $a_{\mathbf{0}} = 0$ is called *Newton non-degenerate (NND)* at 0 (or 0 is called a Newton non-degenerate (singular) point of f) if the Newton diagram $\Gamma(f,0)$ is bounded (that is, meets all coordinate axes) and if, for all faces $\sigma \subset \Gamma(f,0)$, the hypersurface $\{f^\sigma = 0\}$ has no singular point in the torus $(\mathbb{C}^*)^n$.

In the above example, we have 3 truncations on one-dimensional faces σ of $\Gamma(f,0)$, $f^\sigma = y^5 + xy^3$, $xy^3 + x^2y^2 + x^3y$ and $x^3y + x^6$, respectively. None of the corresponding hypersurfaces $\{f^\sigma = 0\}$ is singular in $(\mathbb{C}^*)^2$, hence, f is Newton non-degenerate (the 0-dimensional faces are monomials and define obviously non-degenerate truncations).

On the other hand, $f + x^2y^2$ is Newton degenerate, since its truncation at the face with slope -1, $xy^3 + 2x^2y^2 + x^3y = xy(x+y)^2$, is singular along the line $\{x + y = 0\}$.

Proposition 2.17. *Let $f \in \mathbb{C}\{x_1, \ldots x_n\}$ be Newton non-degenerate. Then the Milnor number of f satisfies*

$$\mu(f) = n!\,\mathrm{Vol}_n\big(K_0(f)\big) + \sum_{i=1}^{n} (-1)^{n-i}(n-i)! \cdot \mathrm{Vol}_{n-i}\big(K_0(f) \cap H_{n-i}\big)\,,$$

where H_i denotes the union of all i-dimensional coordinate planes, and where Vol_i *denotes the i-dimensional Euclidean volume.*

Proof. Cf. [Kou, Thm. I(ii)]) □

In the above Example 2.15.1, we compute

$$\mu(f) = 2 \cdot \frac{19}{2} - 11 + 1 = 9\,.$$

Note that, in general, the mixed sum of volumes on the right hand side (the so-called *Newton number of f*) gives a lower bound for $\mu(f)$ (cf. [Kou]).

Another important class of singularities is given by the class of semiquasihomogeneous singularities, which are characterized by means of the Newton diagram, too:

Definition 2.18. A power series $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is called *semiquasihomogeneous (SQH)* at 0 (or, 0 is called a semiquasihomogeneous point of f) if there is a face $\sigma \subset \Gamma(f,0)$ of dimension $n-1$ (called the *main*, or *principal, face*) such that the truncation f^σ has no critical points in $\mathbb{C}^n \setminus \{0\}$. f^σ is called the *main part*, or *principal part*, of f.

Note that f^σ is a quasihomogeneous polynomial, hence, it is contained in the ideal generated by its partials. It follows that f^σ has no critical point in $\mathbb{C}^n \setminus \{0\}$ iff the hypersurface $\{f^\sigma = 0\} \subset \mathbb{C}^n$ has an isolated singularity at 0. In other words, due to Lemma 2.3, f is SQH iff we can write

$$f = f_0 + g\,, \quad \mu(f_0) < \infty$$

with $f_0 = f^\sigma$ a quasihomogeneous polynomial of *type* $(\boldsymbol{w}; d)$ and all monomials of g being of $\boldsymbol{w}$-degree at least $d+1$.

We should like to point out that we do not require that the Newton diagram $\Gamma(f,0)$ meets all coordinate axes (as for NND singularities). For instance, $xy + y^3 + x^2y^2 \in \mathbb{C}\{x,y\}$ is SQH with main part $xy + y^3$ (which is $\boldsymbol{w} = (2,1)$-weighted homogeneous of weighted degree 3); but it is not Newton non-degenerate, since the Newton diagram does not meet the x-axis.

Anyhow, in some cases we want to assume that $\Gamma(f,0)$ meets all coordinate axes. Then we speak about *convenient* power series. Actually, the results of the next section show that each SQH power series is right equivalent to a convenient one.

Note that each convenient SQH power series $f \in \mathbb{C}\{x,y\}$ is NND, while for higher dimensions this is not true. For instance $f = (x+y)^2 + xz + z^2$ is SQH (with $f = f_0$) and convenient, but Newton degenerate (the truncation $(x+y)^2$ at one of the one-dimensional faces has singular points in $(\mathbb{C}^*)^3$).

Corollary 2.19. *Let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be SQH with principal part f_0. Then f has an isolated singularity at 0 and $\mu(f) = \mu(f_0)$.*

Proof. Let $f_0 \in \mathbb{C}[\boldsymbol{x}]$ be quasihomogeneous of type $(\boldsymbol{w}; d)$ and write

$$f = f_0 + \sum_{i \geq 1} f_i$$

with f_i quasihomogeneous of type $(\boldsymbol{w}; d+i)$. Clearly, f is singular at 0 iff f_0 is singular at 0. Consider for $t \in \mathbb{C}$ the unfolding

$$F_t(\boldsymbol{x}) := f_0(\boldsymbol{x}) + \sum_{i \geq 1} t^i f_i(\boldsymbol{x})\,,$$

which satisfies $F_0 = f_0$ and $F_1 = f_1$. Theorem 2.6 (1) implies that, for $t_0 \neq 0$ sufficiently small, F_{t_0} has an isolated critical point at 0. Since, for every $t \in \mathbb{C}^*$,

$$F_t(x_1, \ldots, x_n) = \frac{1}{t^d} \cdot f(t^{w_1}x_1, \ldots, t^{w_n}x_n)\,,$$

the $\mathbb{C}^*$-action $x \mapsto (t^{w_1}x_1, \ldots, t^{w_n}x_n)$ maps

$$\operatorname{Crit}(F_t) \cap \left\{ \boldsymbol{x} \;\middle|\; \forall\, i:\ |x_i| < \frac{\varepsilon}{|t|^{w_i}} \right\} \xrightarrow{\cong} \operatorname{Crit}(f) \cap \left\{ \boldsymbol{x} \;\middle|\; \forall\, i:\ |x_i| < \varepsilon \right\}.$$

Hence, we can find some $\varepsilon > 0$, independent of t, such that, for all $|t| \leq 1$, $\operatorname{Crit}(F_t : B_\varepsilon(0) \to \mathbb{C}) = \{0\}$. Finally, the statement follows from Theorem 2.6 (3). □

Again, the SQH and NND property are both not preserved under analytic coordinate changes, for instance, $x^2 - y^3 \in \mathbb{C}\{x, y\}$ is SQH and NND, but $(x+y)^2 - y^3 \in \mathbb{C}\{x, y\}$ is neither SQH nor NND. Anyhow, we can make the following definition:

Definition 2.20. An isolated hypersurface singularity $(X, x) \subset (\mathbb{C}^n, x)$, is called *Newton non-degenerate* (respectively *semiquasihomogeneous*), if there exists a NND (respectively SQH) power series $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ such that $\mathcal{O}_{X,x} \cong \mathbb{C}\{\boldsymbol{x}\}/\langle f \rangle$.

2.3 Algebraic Group Actions

The classification with respect to right, respectively contact, equivalence may be considered in terms of algebraic group actions.

Definition 2.21. The group $\mathcal{R} := \mathrm{Aut}(\mathbb{C}\{\boldsymbol{x}\})$ of automorphisms of the analytic algebra $\mathbb{C}\{\boldsymbol{x}\}$ is called the *right group*. The *contact group* is the semidirect product $\mathcal{K} := \mathbb{C}\{\boldsymbol{x}\}^* \ltimes \mathcal{R}$ of $\mathcal{R}$ with the group of units of $\mathbb{C}\{\boldsymbol{x}\}$, where the product in $\mathcal{K}$ is defined by

$$(u', \varphi')(u, \varphi) = (u'\varphi'(u), \varphi'\varphi).$$

These groups act on $\mathbb{C}\{\boldsymbol{x}\}$ by

$$\begin{array}{rcl} \mathcal{R} \times \mathbb{C}\{\boldsymbol{x}\} & \longrightarrow & \mathbb{C}\{\boldsymbol{x}\}\,, \\ (\varphi, f) & \longmapsto & \varphi(f)\,, \end{array} \qquad \begin{array}{rcl} \mathcal{K} \times \mathbb{C}\{\boldsymbol{x}\} & \longrightarrow & \mathbb{C}\{\boldsymbol{x}\}\,, \\ ((u, \varphi), f) & \longmapsto & u \cdot \varphi(f). \end{array}$$

We have

$$f \overset{\mathrm{r}}{\sim} g \iff f \in \mathcal{R} \cdot g\,, \qquad f \overset{\mathrm{c}}{\sim} g \iff f \in \mathcal{K} \cdot g,$$

where $\mathcal{R} \cdot g$ (respectively $\mathcal{K} \cdot g$) denotes the orbit of g under $\mathcal{R}$ (respectively $\mathcal{K}$), that is, the image of $\mathcal{R} \times \{f\}$, respectively $\mathcal{K} \times \{f\}$, in $\mathbb{C}\{\boldsymbol{x}\}$ under the maps defined above.

Neither $\mathcal{R}$ nor $\mathcal{K}$ are algebraic groups or Lie groups, since they are infinite dimensional. Therefore we pass to the k-jets of these groups

$$\mathcal{R}^{(k)} := \big\{ \mathrm{jet}(\varphi, k) \bigm| \varphi \in \mathcal{R} \big\}\,,\ \ \mathcal{K}^{(k)} := \big\{ \big(\mathrm{jet}(u, k), \mathrm{jet}(\varphi, k)\big) \bigm| (u, \varphi) \in \mathcal{K} \big\},$$

where $\mathrm{jet}(\varphi, k)(x_i) = \mathrm{jet}(\varphi(x_i), k)$ is the truncation of the power series of the component functions of φ.

As we shall show below, $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are algebraic groups acting algebraically on the jet space $J^{(k)}$, which is a finite dimensional complex vector space. The action is given by

$$\varphi \cdot f = \mathrm{jet}(\varphi(f), k)\,, \qquad (u, \varphi) \cdot f = \mathrm{jet}(u \cdot \varphi(f), k)\,,$$

for $\varphi \in \mathcal{R}^{(k)}$, $(u, \varphi) \in \mathcal{K}^{(k)}$. Hence, we can apply the theory of algebraic groups to the action of $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$. If k is bigger or equal to the determinacy of g, then $g \overset{r}{\sim} f$ (respectively $g \overset{c}{\sim} f$) if and only if $g \in \mathcal{R}^{(k)} f$ (respectively $g \in \mathcal{K}^{(k)} f$). Hence, the orbits of these algebraic groups are in one-to-one correspondence with the equivalence classes.

Before we make use of this point of view, we recall some basic facts about algebraic group actions. For a detailed study we refer to [Bor, Spr, Kra].

Definition 2.22. (1) An (affine) *algebraic group* G (over an algebraically closed field K) is a reduced (affine) algebraic variety over K, which is also a group such that the group operations are morphisms of varieties. That is, there exists an element $e \in G$ (the unit element) and morphisms of varieties over K

$$\begin{aligned} G \times G &\longrightarrow G\,, \quad (g,h) \mapsto g \cdot h \quad \text{(the multiplication)},\\ G &\longrightarrow G\,, \quad g \mapsto g^{-1} \quad \text{(the inverse)} \end{aligned}$$

satisfying the usual group axioms.

(2) A *morphism of algebraic groups* is a group homomorphism, which is also a morphism of algebraic varieties over K.

Example 2.22.1. (i) $GL(n,K)$ and $SL(n,K)$ are affine algebraic groups.

(ii) For any field K, the additive group $(K,+)$ and the multiplicative group $(K^*,\cdot)$ of K are affine algebraic groups.

(iii) The groups $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are algebraic groups for any $k \geq 1$. This can be seen as follows: an element φ of $\mathcal{R}^{(k)}$ is uniquely determined by

$$\varphi^{(i)} := \varphi(x_i) = \sum_{j=1}^{n} a_j^{(i)} x_j + \sum_{|\alpha|=2}^{k} a_\alpha^{(i)} \boldsymbol{x}^\alpha\,, \quad i = 1, \dots, n,$$

such that $\det\big(a_j^{(i)}\big) \neq 0$. Hence, $\mathcal{R}^{(k)}$ is an open subset of a finite dimensional K-vectorspace (with coordinates the coefficients $a_j^{(i)}$ and $a_\alpha^{(i)}$). It is affine, since it is the complement of a hypersurface.

The elements of the contact group $\mathcal{K}^{(k)}$ are given by pairs (u,φ), $\varphi \in \mathcal{R}^{(k)}$, $u = u_0 + \sum_{|\alpha|=1}^{k} u_\alpha \boldsymbol{x}^\alpha$ with $u_0 \neq 0$, hence $\mathcal{K}^{(k)}$ is also open in some finite dimensional vectorspace and an affine variety.

The group operations are morphisms of affine varieties, since the component functions are rational functions. Indeed the coefficients of $\varphi \cdot \psi$ are polynomials in the coefficients of φ, ψ, while the coefficients of φ^{-1} are determined by solving linear equations and involve $\det\big(a_j^{(i)}\big)$ (respectively $\det\big(a_j^{(i)}\big)$ and u_0) in the denominator.

Proposition 2.23. *Every algebraic group G is a smooth variety.*

Proof. Since G is a reduced variety, it contains smooth points by Corollary 1.85. For any $g \in G$ the translation $h \mapsto hg$ is an automorphism of G and in this way G acts transitively on G. Hence, a smooth point can be moved to any other point of G by some automorphism of G. □

Definition 2.24. (1) An (*algebraic*) *action* of G on an algebraic variety X is given by a morphism of varieties

$$G \times X \longrightarrow X, \quad (g, x) \mapsto g \cdot x,$$

satisfying $ex = x$ and $(gh)x = g(hx)$ for all $g, h \in G$, $x \in X$.

(2) The *orbit* of $x \in X$ under the action of G on X is the subset

$$Gx := G \cdot x := \{g \cdot x \in X \mid g \in G\} \subset X,$$

that is, the image of $G \times \{x\}$ in X under the orbit map $G \times X \to X$.

(3) G acts *transitively* on X if $Gx = X$ for some (and then for any) $x \in X$.

(4) The *stabilizer* of $x \in X$ is the subgroup $G_x := \{g \in G \mid gx = x\}$ of G, that is, the preimage of x under the induced map $G \times \{x\} \to X$.

In this sense $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ act on $J^{(k)}$. Note that the somehow unexpected multiplication on $\mathcal{K}^{(k)}$ as a semidirect product (and not just as direct product) was introduced in order to guarantee $(gh)x = g(hx)$ (check this!).

For the classification of singularities we need the following important properties of orbits.

Theorem 2.25. *Let G be an affine algebraic group acting on an algebraic variety X, and $x \in X$ an arbitrary point. Then*

(1) Gx is open in its (Zariski-) closure $\overline{Gx}$.
(2) Gx is a smooth subvariety of X.
(3) $\overline{Gx} \setminus Gx$ is a union of orbits of smaller dimension.
(4) G_x is a closed subvariety of G.
(5) If G is connected, then $\dim(Gx) = \dim(G) - \dim(G_x)$.

Proof. (1) By Theorem 2.36, below, Gx contains an open dense subset of $\overline{Gx}$; in particular, it contains interior points of $\overline{Gx}$. For any $g \in G$, $g \cdot \overline{Gx}$ is closed and contains Gx. Hence, $\overline{Gx} \subset g \cdot \overline{Gx}$. Replacing g with g^{-1} and then multiplying with g we also obtain $g \cdot \overline{Gx} \subset \overline{Gx}$. It follows that $\overline{Gx} = g \cdot \overline{Gx}$, that is, $\overline{Gx}$ is stable under the action of G.

Now, consider the induced action of G on $\overline{Gx}$. Since G acts transitively on Gx and Gx contains an interior point of its closure, every point of Gx is an interior point of $\overline{Gx}$, that is, Gx is open in its closure.

(2) Gx with its reduced structure contains a smooth point and, hence, it is smooth everywhere by homogeneity (cf. the proof of Proposition 2.33).

(3) $\overline{Gx} \setminus Gx$ is closed, of dimension strictly smaller than $\dim Gx$ and G-stable, hence a union of orbits.

(4) follows, since G_x is the fibre of a morphism (we consider only closed points).

(5) Consider the map $f\colon G \times \{x\} \to \overline{Gx}$ induced by $G \times \{x\} \to X$. Then f is dominant and $G_x = f^{-1}(x)$. Since G is connected, $G \times \{x\}$ and $\overline{Gx}$ are both

irreducible. Since for $y = gx \in Gx$ we have $Gx = Gy$ and $G_y = gG_xg^{-1}$, the statement is independent of the choice of $y \in Gx$. Hence, the result follows from (2) of the following theorem. □

We recall that a morphism $f\colon X \to Y$ of algebraic varieties is called *dominant* if for any open dense set $U \subset Y$, $f^{-1}(U)$ is dense in X. When we study the (non-empty) fibres $f^{-1}(y)$ of any morphism $f\colon X \to Y$ we may replace Y by $\overline{f(X)}$, that is, we may assume that $f(X)$ is dense in Y. If X and Y are irreducible, then f is dominant iff $\overline{f(X)} = Y$.

The following theorem concerning the dimension of the fibres of a morphism of algebraic varieties has many applications.

Theorem 2.26. *Let $f\colon X \to Y$ be a dominant morphism of irreducible varieties, $W \subset Y$ an irreducible, closed subvariety and Z an irreducible component of $f^{-1}(W)$. Put $r = \dim X - \dim Y$.*

(1) If Z dominates W then $\dim Z \geq \dim W + r$. In particular, for $y \in f(X)$, any irreducible component of $f^{-1}(y)$ has dimension $\geq r$.
(2) There is an open dense subset $U \subset Y$ (depending only on f) such that $U \subset f(X)$ and $\dim Z = \dim W + r$ or $Z \cap f^{-1}(U) = \emptyset$. In particular, for $y \in U$, any irreducible component of $f^{-1}(y)$ has dimension equal to r.
(3) The open set U in (2) may be chosen such that $f\colon f^{-1}(U) \to U$ factors as follows

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\ \pi\ } & U \times \mathbb{A}^r \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

with π finite and pr_1 denoting the projection onto the first factor.

Proof. See [Mum1, Chap. I, §8] and [Spr, Thm. 4.1.6]. □

Observe that the theorem implies that for $f\colon X \to Y$ dominant there is an open dense subset U of Y such that $U \subset f(X) \subset Y$.

Recall that a morphism $f\colon X \to Y$ of algebraic varieties with algebraic structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ is *finite* if there exists a covering of Y by open, affine varieties U_i such that for each i, $f^{-1}(U_i)$ is affine and such that $\mathcal{O}_X\left(f^{-1}(U_i)\right)$ is a finitely generated $\mathcal{O}_Y(U_i)$-module.

If $f\colon X \to Y$ is finite, then the following holds

(1) f is a closed map,
(2) for each $y \in Y$ the fibre $f^{-1}(y)$ is a finite set,
(3) for *every* open affine set $U \subset Y$, $f^{-1}(U)$ is affine and $\mathcal{O}_X\left(f^{-1}(U)\right)$ is a finitely generated $\mathcal{O}_Y(U)$-module.
(4) If X and Y are affine, then f is surjective if and only if the induced map of coordinate rings $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective.

(5) Moreover, if $f\colon X \to Y$ is a dominant morphism of irreducible varieties and $y \in f(X)$ such that $f^{-1}(y)$ is a finite set, then there exists an open, affine neighbourhood U of y in Y such that $f^{-1}(U)$ is affine and $f\colon f^{-1}(U) \to U$ is finite.

For proofs see [Mum1, Chap. I, §7], [Spr, Chap. 4.2] and [Har4, Chap. II, Exe. 3.4–3.7].

Now, let $f\colon X \to Y$ be a morphism of algebraic varieties over $\mathbb{C}$, and let $f^{\text{an}}\colon X^{\text{an}} \to Y^{\text{an}}$ be the induced morphism of complex spaces. It follows that f finite implies that f^{an} is finite. The converse, however, is not true (cf. [Har4, Chap. II, Exe. 3.5(c)]).

Let $f\colon X \to Y$ be a morphism of algebraic varieties, $x \in X$ a point and $y = f(x)$. Then the induced map of local rings $f^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a K-linear map $\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2 \to \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ of the cotangent spaces and, hence, its dual is a K-linear map on the level on tangent spaces

$$T_x f\colon T_x X \longrightarrow T_y Y$$

where $T_x X = \text{Hom}_K(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, K)$ is the Zariski tangent space of X at x.

Observe that the cotangent and, hence, the tangent spaces coincide, independently of whether we consider X as an algebraic variety or as a complex space. Hence, if $f^{\text{an}}\colon X^{\text{an}} \to Y^{\text{an}}$ is the induced map of complex spaces, then the induced map $(f^{\text{an}})^{\#}\colon \mathcal{O}_{Y^{\text{an}},y} \to \mathcal{O}_{X^{\text{an}},x}$ induces the same map as $f^{\#}$ on the cotangent spaces and, hence, on the Zariski tangent spaces.

Proposition 2.27. *Let $f\colon X \to Y$ be a dominant morphism of irreducible complex algebraic varieties. Then there is an open dense subset $V \subset X$ such that for each $x \in V$ the map $T_x f\colon T_x X \longrightarrow T_{f(x)} Y$ is surjective.*

Proof. By Theorem 2.36 there is an open dense subset $U \subset Y$ such that the restriction $f\colon f^{-1}(U) \to U$ is surjective.

By deleting the (closed) set $A := f(\text{Sing}(f^{-1}(U)) \cup \text{Sing}(U) \subset U$ and considering $f\colon f^{-1}(U \setminus A) \to U \setminus A$, we obtain a map f between complex manifolds. The tangent map of f is just given by the (transpose of the) Jacobian matrix of f with respect to local analytic coordinates, which is surjective on the complement of the vanishing locus of all maximal minors. □

Another corollary of Theorem 2.36 is the theorem of Chevalley. For this recall that a subset Y of a topological space X is called *constructible* if it is a finite union of locally closed subsets of X. We leave it as an exercise to show that a constructible set Y contains an open dense subset of $\overline{Y}$. Moreover, the system of constructible subsets is closed under the Boolean operations of taking finite unions, intersections and differences.

If X is an algebraic variety (with Zariski topology) and $Y \subset X$ is constructible, then $Y = \bigcup_{i=1}^{s} L_i$ with L_i locally closed, and we can define the *dimension* of Y as the maximum of $\dim L_i$, $i = 1, \ldots, s$.

Theorem 2.28 (Chevalley). *Let $f\colon X \to Y$ be any morphism of algebraic varieties. Then the image of any constructible set is constructible. In particular, $f(X)$ contains an open dense subset of $\overline{f(X)}$.*

Proof. It is clear that the general case follows if we show that $f(X)$ is constructible. Since X is a finite union of irreducible varieties, we may assume that X is irreducible. Moreover, replacing Y by $\overline{f(X)}$ we may assume that Y is irreducible and that f is dominant.

We prove the theorem now by induction on $\dim Y$, the case $\dim Y = 0$ being trivial. Let the open set $U \subset Y$ be as in Theorem 2.36, then $Y \setminus U$ is closed of strictly smaller dimension. By induction hypothesis, $f(f^{-1}(Y \setminus U))$ is constructible in $Y \setminus U$ and hence in Y. Then $f(X) = U \cup f(f^{-1}(Y \setminus U))$ is constructible. □

We return to the action of $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ on $J^{(k)} = \mathbb{C}\{x_1, \dots, x_n\}/\mathfrak{m}^{k+1}$, the affine space of k-jets. Note that $\mathcal{R}^{(k)}$ and $\mathcal{K}^{(k)}$ are both connected as they are complements of hypersurfaces in some $\mathbb{C}^N$.

Proposition 2.29. *Let G be either $\mathcal{R}^{(k)}$, or $\mathcal{K}^{(k)}$, and for $f \in J^{(k)}$ let Gf be the orbit of f under the action of G on $J^{(k)}$. We denote by $T_f(Gf)$ the tangent space to Gf at f, considered as a linear subspace of $J^{(k)}$. Then, for $k \geq 1$,*

$$\begin{aligned} T_f(\mathcal{R}^{(k)} f) &= \big(\mathfrak{m} \cdot j(f) + \mathfrak{m}^{k+1}\big)/\mathfrak{m}^{k+1}, \\ T_f(\mathcal{K}^{(k)} f) &= \big(\mathfrak{m} \cdot j(f) + \langle f \rangle + \mathfrak{m}^{k+1}\big)/\mathfrak{m}^{k+1}. \end{aligned}$$

Proof. Note that the orbit map and translation by $g \in G$ induce a commutative diagram

$$\begin{array}{ccc} T_e G & \longrightarrow & T_f(Gf) \\ \cong\downarrow & & \downarrow\cong \\ T_g G & \longrightarrow & T_{gf}(Gf) \end{array} .$$

Since the orbit map $G \times \{f\} \to Gf$ satisfies the assumptions of Proposition 2.37, $T_g G \to T_{gf}(Gf)$ and, hence, $T_e G \to T_f(Gf)$ are surjective. Hence, the tangent space to the orbit at f is the image of the tangent map at $e \in G$ of the map $\mathcal{R}^{(k)} \to J^{(k)}$, $\Phi \mapsto f \circ \Phi$, respectively $\mathcal{K}^{(k)} \to J^{(k)}$, $(u, \Phi) \mapsto u \cdot (f \circ \Phi)$.

In the following, we treat only the contact group (the statement for the right group follows with $u \equiv 1$): consider a curve $t \mapsto (u_t, \Phi_t) \in \mathcal{K}^{(k)}$ such that $u_0 = 1$, $\Phi_0 = \mathrm{id}$, that is,

$$\begin{aligned} \Phi(\boldsymbol{x}, t) = \boldsymbol{x} + \varepsilon(\boldsymbol{x}, t) &\colon\ (\mathbb{C}^n \times \mathbb{C}, (0,0)) \longrightarrow (\mathbb{C}^n, 0) \\ u(\boldsymbol{x}, t) = 1 + \delta(\boldsymbol{x}, t) &\colon\ (\mathbb{C}^n \times \mathbb{C}, (0,0)) \longrightarrow \mathbb{C}, \end{aligned}$$

with $\varepsilon(\boldsymbol{x}, t) = \varepsilon^1(\boldsymbol{x})\, t + \varepsilon^2(\boldsymbol{x})\, t^2 + \dots$, $\varepsilon^i = (\varepsilon^i_1, \dots, \varepsilon^i_n)$ such that $\varepsilon^i_j \in \mathfrak{m}$, and $\delta(\boldsymbol{x}, t) = \delta_1(\boldsymbol{x})\, t + \delta_2(\boldsymbol{x})\, t^2 + \dots$, $\delta_i \in \mathbb{C}\{\boldsymbol{x}\}$. The image of the tangent map are all vectors of the form

$$\frac{\partial}{\partial t}\Big((1+\delta(\boldsymbol{x},t))\cdot f(\boldsymbol{x}+\varepsilon(\boldsymbol{x},t))\Big)\Big|_{t=0} \mod \mathfrak{m}^{k+1}$$
$$= \delta_1(\boldsymbol{x})\cdot f(\boldsymbol{x}) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\boldsymbol{x})\cdot \varepsilon_j^1(\boldsymbol{x}) \mod \mathfrak{m}^{k+1},$$

which proves the claim. □

Of course, instead of the analytic proof using curves, we could have used the interpretation of the Zariski tangent space T_xX as morphisms $T_\varepsilon \to X$, where $T_\varepsilon = \operatorname{Spec}\big(\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle\big)$.

In view of Proposition 2.39 we call $\mathfrak{m}\cdot j(f)$, respectively $\mathfrak{m}\cdot j(f)+\langle f\rangle$ the *tangent space at* f *to the orbit* of f under the right action $\mathcal{R}\times\mathbb{C}\{\boldsymbol{x}\}\to\mathbb{C}\{\boldsymbol{x}\}$, respectively the contact action $\mathcal{K}\times\mathbb{C}\{\boldsymbol{x}\}\to\mathbb{C}\{\boldsymbol{x}\}$.

Corollary 2.30. *For $f\in\mathbb{C}\{x_1,\dots,x_n\}$, $f(0)=0$, the following are equivalent.*

(1) f has an isolated critical point.
(2) f is right finitely-determined.
(3) f is contact finitely-determined.

Proof. (1) ⇒ (2). By Corollary 2.25, f is $\mu(f)+1$-determined. On the other hand, $\mu(f)<\infty$ due to Lemma 2.3. Since the implication (2) ⇒ (3) is trivial, we are left with (3) ⇒ (1). Let f be contact k-determined and $g\in\mathfrak{m}^{k+1}$. Then $f_t = f+tg\in\mathcal{K}^{(k+1)}f \mod \mathfrak{m}^{k+2}$ and, hence,

$$g = \frac{\partial f_t}{\partial t}\Big|_{t=0} \in \mathfrak{m}\cdot j(f)+\langle f\rangle \mod \mathfrak{m}^{k+2},$$

by Proposition 2.39. By Nakayama's lemma $\mathfrak{m}^{k+1}\subset\mathfrak{m}\cdot j(f)+\langle f\rangle$, the latter being contained in $j(f)+\langle f\rangle$. Hence, $\tau(f)<\infty$ and f has an isolated critical point by Lemma 2.3. □

Lemma 2.31. *Let $f\in\mathfrak{m}^2\subset\mathbb{C}\{x_1,\dots,x_n\}$ be an isolated singularity. Let k satisfy $\mathfrak{m}^{k+1}\subset\mathfrak{m}\cdot j(f)$, respectively $\mathfrak{m}^{k+1}\subset\mathfrak{m}\cdot j(f)+\langle f\rangle$, and call*

$$\begin{aligned} r\text{-}\operatorname{codim}(f) &:= \textit{codimension of } \mathcal{R}^{(k)}f \textit{ in } J^{(k)}\,, \ \textit{respectively}\\ c\text{-}\operatorname{codim}(f) &:= \textit{codimension of } \mathcal{K}^{(k)}f \textit{ in } J^{(k)} \end{aligned}$$

the codimension of the orbit *of f in $J^{(k)}$ under the action of $\mathcal{R}^{(k)}$, respectively $\mathcal{K}^{(k)}$. Then*

$$r\text{-}\operatorname{codim}(f) = \mu(f)+n\,, \qquad c\text{-}\operatorname{codim}(f) = \tau(f)+n.$$

Proof. In view of Proposition 2.39 and the definition of $\mu(f)$ and $\tau(f)$, one has to show that

$$\dim_{\mathbb{C}}\frac{j(f)}{\mathfrak{m}\cdot j(f)} = \dim_{\mathbb{C}}\frac{j(f)+\langle f\rangle}{\mathfrak{m}\cdot j(f)+\langle f\rangle} = n\,. \tag{2.3.1}$$

Both linear spaces in question are generated by $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$, and it is sufficient to prove that none of these derivatives belongs to the ideal $\mathfrak{m} \cdot j(f) + \langle f \rangle$. Arguing to the contrary, assume, for example, that $\frac{\partial f}{\partial x_1} \in \mathfrak{m} \cdot j(f) + \langle f \rangle$, which immediately implies

$$\frac{\partial f}{\partial x_1} = \sum_{i=2}^{n} \alpha_i(\boldsymbol{x}) \frac{\partial f}{\partial x_i} + \beta(\boldsymbol{x}) f, \quad \alpha_2, \dots, \alpha_n \in \mathfrak{m},\ \beta \in \mathbb{C}\{\boldsymbol{x}\}\,.$$

The system of differential equations

$$\frac{dx_i}{dx_1} = -\alpha_i(x_1, \dots, x_n), \quad x_i(0) = y_i \in \mathbb{C}, \quad i = 2, \dots, n\,,$$

has an analytic solution

$$x_i = \varphi_i(x_1, y_2, \dots, y_n), \quad i = 2, \dots, n\,,$$

convergent in a neighbourhood of zero. The latter formulae define an isomorphism $\mathbb{C}\{x_1, x_2, \dots, x_n\} \cong \mathbb{C}\{x_1, y_2, \dots, y_n\}$ which takes $f(x_1, \dots, x_n)$ to

$$\widetilde{f}(x_1, y_2, \dots, y_n) = f(x_1, \varphi_2(x_1, y_2, \dots, y_n), \dots, \varphi_n(x_1, y_2, \dots, y_n))$$

such that

$$\frac{\partial \widetilde{f}}{\partial x_1} = \left(\frac{\partial f}{\partial x_1} - \sum_{i=2}^{n} \alpha_i(\boldsymbol{x}) \frac{\partial f}{\partial x_i} \right)_{\substack{x_i = \varphi_i(x_1, y_2, \dots, y_n) \\ i = 2, \dots, n}} = \widetilde{\beta}(x_1, y_2, \dots, y_n) \cdot \widetilde{f}\,.$$

However, the latter relation means that $\widetilde{f}$ does not depend on x_1, and hence $\widetilde{f}$, $\partial \widetilde{f} / \partial x_i$, $i = 1, \dots, n$, vanish along the line $(t, 0, \dots, 0)$ contradicting the assumption on the isolated singularity at the origin. □

2.4 Classification of Simple Singularities

We want to classify singularities having no "moduli" up to contact equivalence. No moduli means that, in a sufficiently high jet space, there exists a neighbourhood of f, which meets only finitely many orbits of the contact group. A singularity having no moduli is also simply called 0-*modal*, while k-*modal* means, loosely speaking, that any small neighbourhood of f meets k- (and no higher) dimensional families of orbits.

The same notion makes sense for right equivalence and, indeed, they were introduced by Arnol'd for right equivalence in a series of papers and had a great influence on singularity theory.

Here we treat simultaneously right and contact equivalence, since it means almost no additional work.

We recall that the space of k-jets $J^{(k)} = \mathbb{C}\{x_1, \dots, x_n\}/\mathfrak{m}^{k+1}$ is a finite dimensional complex vector space with a natural topology: for a power series

$f = \sum_{|\boldsymbol{\nu}|=0}^{\infty} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in \mathbb{C}\{\boldsymbol{x}\}$, we identify $f^{(k)} = \mathrm{jet}(f,k) \in J^{(k)}$ with the truncated power series $f^{(k)} = \sum_{|\boldsymbol{\nu}|=0}^{k} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$. Then an open neighbourhood of $f^{(k)}$ in $J^{(k)}$ consists of all truncated power series $\sum_{|\boldsymbol{\nu}|=0}^{k} b_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$ such that $b_{\boldsymbol{\nu}}$ is contained in some open neighbourhood of $a_{\boldsymbol{\nu}}$ in $\mathbb{C}$, for all $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| \leq k$.

Consider the projections

$$\mathbb{C}\{\boldsymbol{x}\} \longrightarrow J^{(k)}, \quad k \geq 0 .$$

The preimages of open sets in $J^{(k)}$ generate a topology on $\mathbb{C}\{\boldsymbol{x}\}$, the coarsest topology such that all projections are continuous. Hence, a neighbourhood of f in $\mathbb{C}\{\boldsymbol{x}\}$ consists of all those $g \in \mathbb{C}\{\boldsymbol{x}\}$ for which the coefficients up to some degree k are in a neighbourhood of the coefficients of f but with no restrictions on the coefficients of higher order terms. The neighbourhood becomes smaller if the coefficients up to order k get closer to the coefficients of f and if k gets bigger.

Definition 2.32. Consider the action of the right group $\mathcal{R}$, respectively of the contact group $\mathcal{K}$, on $\mathbb{C}\{\boldsymbol{x}\}$. Call $f \in \mathbb{C}\{\boldsymbol{x}\}$ *right simple*, respectively *contact simple*, if there exists a neighbourhood U of f in $\mathbb{C}\{\boldsymbol{x}\}$ such that U intersects only finitely many orbits of $\mathcal{R}$, respectively of $\mathcal{K}$.

This means that there exists some k and a neighbourhood U_k of $f^{(k)}$ in $J^{(k)}$ such that the set of all g with $g^{(k)} \in U_k$ decomposes into only finitely many right classes, respectively contact classes. It is clear that right simple implies contact simple. However, as the classification will show, the converse is also true.

We show now that for an isolated singularity f a sufficiently high jet is not only sufficient for f but also for all g in a neighbourhood of f.

Proposition 2.33. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ have an isolated singularity. Then there exists a neighbourhood U of f in $\mathbb{C}\{\boldsymbol{x}\}$ such that each $g \in U$ is right $(\mu+1)$-determined, respectively contact $(\tau+1)$-determined.*

Proof. We consider contact equivalence, the proof for right equivalence is analogous. Let $\tau = \tau(f)$, $k = \tau + 1$, and consider

$$f^{(k)} = \sum_{|\boldsymbol{\nu}|=0}^{k} a_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in J^{(k)} .$$

Then $f \overset{c}{\sim} f^{(k)}$, and any element $h = \sum_{|\boldsymbol{\nu}|=0}^{k} b_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \in J^{(k)}$ can be written as

$$h(\boldsymbol{x}) = f^{(k)}(\boldsymbol{x}) + \sum_{|\boldsymbol{\nu}|=0}^{k} t_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}} \qquad (2.4.1)$$

with $t_{\boldsymbol{\nu}} = b_{\boldsymbol{\nu}} - a_{\boldsymbol{\nu}}$. Considering $t_{\boldsymbol{\nu}}$, $|\boldsymbol{\nu}| \leq k$, as variables, then (2.4.1) defines an unfolding of $f^{(k)}$, and the semicontinuity theorem 2.6 says that there is a neighbourhood $U_k \subset J^{(k)}$ of $f^{(k)}$ such that $\tau(h) \leq \tau(f^{(k)}) = \tau(f)$ for each $h \in \mathfrak{m} \cap U_k$. Hence, h is k-determined and, therefore, also every $g \in \mathbb{C}\{\boldsymbol{x}\}$ with $g^{(k)} \in U_k$. If $U \subset \mathbb{C}\{\boldsymbol{x}\}$ is the preimage of U_k under $\mathbb{C}\{\boldsymbol{x}\} \twoheadrightarrow J^{(k)}$ then this says that every $g \in U \cap \mathfrak{m}$ is contact $(\tau+1)$-determined. □

Remark 2.33.1. We had to use μ, respectively τ, as a bound for the determinacy, since the determinacy itself is not semicontinuous. For example the singularity E_7 is 4-determined (cf. Example 2.26.1) and deforms into A_6, which is 6-determined. This will follow from the classification in below.

Corollary 2.34. *Let $f \in \mathfrak{m}$ have an isolated singularity, and suppose that $k \geq \mu(f)+1$, respectively $k \geq \tau(f)+1$. Then f is right simple, respectively contact simple, if and only if there is a neighbourhood of $f^{(k)}$ in $J^{(k)}$, which meets only finitely many $\mathcal{R}^{(k)}$-orbits, respectively $\mathcal{K}^{(k)}$-orbits.*

Proof. The necessity is clear, the sufficiency is an immediate consequence of Proposition 2.43. □

Now let us start with the classification. The aim is to show that the right simple as well as the contact simple singularities $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ are exactly the so-called *ADE-singularities*:

$$
\begin{array}{lll}
A_k\colon & x_1^{k+1} + x_2^2 + \ldots + x_n^2\,, & k \geq 1\,,\\
D_k\colon & x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \ldots + x_n^2\,, & k \geq 4\,,\\
E_6\colon & x_1^3 + x_2^4 + x_3^2 + \ldots + x_n^2\,, & \\
E_7\colon & x_1(x_1^2 + x_2^3) + x_3^2 + \ldots + x_n^2\,, & \\
E_8\colon & x_1^3 + x_2^5 + x_3^2 + \ldots + x_n^2\,. &
\end{array}
$$

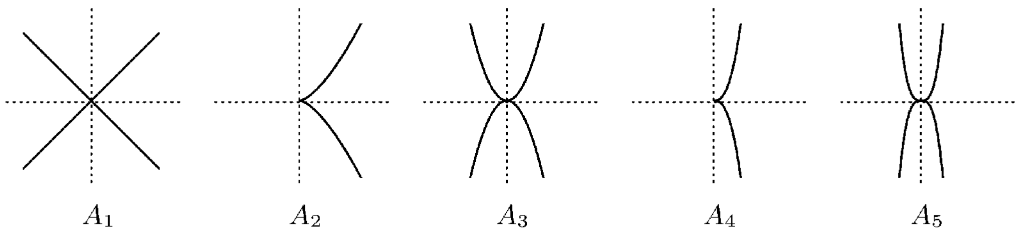
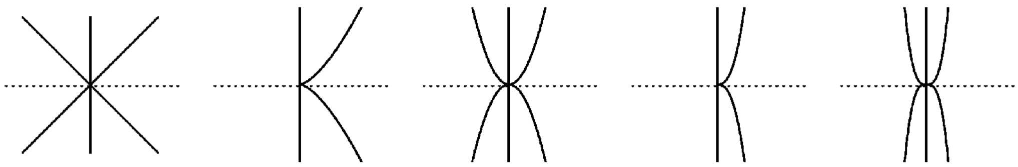


FIGURE 2.3. Real pictures of one-dimensional A_k-singularities

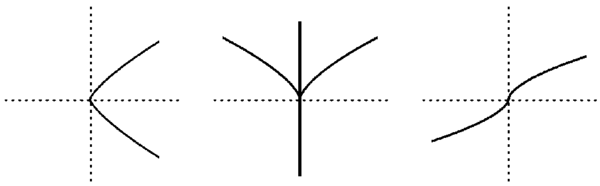
Note that A_0 is usually not included in the list of simple singularities, since it is non-singular. It is however simple in the sense of Definition 2.42, since in a neighbourhood of A_0 in $\mathbb{C}\{\boldsymbol{x}\}$ there are only smooth germs or units. A_1-singularities are also called *(ordinary) nodes*, and A_2-singularities *(ordinary) cusps*.

Classification of Smooth Germs.



D_4 D_5 D_6 D_7 D_8

FIGURE 2.4. Real pictures of one-dimensional D_k-singularities



E_6 E_7 E_8

FIGURE 2.5. Real pictures of one-dimensional E_6-, E_7-, E_8-singularities

Lemma 2.35. *For $f \in \mathfrak{m} \subset \mathbb{C}\{\boldsymbol{x}\}$ the following are equivalent.*

(1) $\mu(f) = 0$,
(2) $\tau(f) = 0$,
(3) f *is non-singular,*
(4) $f \overset{r}{\sim} f^{(1)}$,
(5) $f \overset{r}{\sim} x_1$.

Proof. $\mu(f) = 0 \Leftrightarrow \tau(f) = 0 \Leftrightarrow \frac{\partial f}{\partial x_i}(0) \neq 0$ for some i $\Leftrightarrow f$ is non-singular. The remaining equivalences follow from the implicit function theorem. □

Classification of Non-Degenerate Singularities. Let $U \subset \mathbb{C}^n$ be open, and let $f\colon U \to \mathbb{C}$ be a holomorphic function. Then we denote by

$$H(f) := \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)_{i,j=1,\dots,n} \in \mathrm{Mat}(x \times n, K[\boldsymbol{x}])$$

the *Hessian (matrix)* of f.

Definition 2.36. A critical point p of f is called a *non-degenerate*, or *Morse singularity* if $\operatorname{rank} H(f)(p) = n$. The number $\operatorname{crk}(f,p) := n - \operatorname{rank} H(f)(p)$ is called the *corank* of f at p. We write $\operatorname{crk}(f)$ instead of $\operatorname{crk}(f,0)$.

The notion of non-degenerate critical points is independent of the choice of local analytic coordinates. Namely, if $\phi\colon (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ is biholomorphic, then

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\big(f\circ\phi(\boldsymbol{x})\big) = \frac{\partial}{\partial x_i}\Bigg(\sum_\nu \frac{\partial f}{\partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial \phi_\nu}{\partial x_j}(\boldsymbol{x})\Bigg)$$
$$= \sum_{\mu,\nu}\frac{\partial^2 f}{\partial x_\mu \partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial\phi_\mu}{\partial x_i}(\boldsymbol{x})\cdot\frac{\partial\phi_\nu}{\partial x_j}(\boldsymbol{x}) + \sum_\nu\frac{\partial f}{\partial x_\nu}\big(\phi(\boldsymbol{x})\big)\cdot\frac{\partial^2\phi_\nu}{\partial x_i\partial x_j}(\boldsymbol{x}).$$

Since p is a critical point of f and $\phi(p)=p$ we have $\frac{\partial f}{\partial x_\nu}\big(\phi(p)\big)=0$, hence

$$H(f\circ\phi)(p) = J(\phi)(p)^t\cdot H(f)(p)\cdot J(\phi)(p), \tag{2.4.2}$$

where $J(\phi)$ is the Jacobian matrix of ϕ, which has rank n.

Similarly, we show that if p is a singular point of the hypersurface $f^{-1}(0)$, that is, $\frac{\partial f}{\partial x_i}(p) = f(p) = 0$, then $\operatorname{rank} H(f)(p) = \operatorname{rank} H(uf)(p)$ for any unit u.

Hence, $\operatorname{crk}(f,p)$ is an invariant of the right equivalence class of f at a critical point and an invariant of the contact class at a singular point of $f^{-1}(0)$. However, if p is non-singular, then $\operatorname{rank} H(f)(p)$ may depend on the choice of coordinates.

Note that for a critical point p, $\operatorname{rank} H(f)(p)$ depends only on the 2-jet of f.

Theorem 2.37 (Morse lemma). *For $f\in\mathfrak{m}^2\subset\mathbb{C}\{x_1\dots,x_n\}$ the following are equivalent.*

(1) $\operatorname{crk}(f,0)=0$*, that is,* 0 *is a non-degenerate singularity of* f*,*
(2) $\mu(f)=1$*,*
(3) $\tau(f)=1$*,*
(4) $f\overset{r}{\sim}f^{(2)}$ *and* $f^{(2)}$ *is non-degenerate,*
(5) $f\overset{r}{\sim}x_1^2+\ldots+x_n^2$*,*
(6) $f\overset{c}{\sim}x_1^2+\ldots+x_n^2$*.*

Proof. The apparently simple proof makes use of the finite determinacy theorem. Since $f\in\mathfrak{m}^2$, we can write

$$f(x) = \sum_{1\le i,j\le n} h_{i,j}(\boldsymbol{x})x_ix_j\,,\quad h_{i,j}\in\mathbb{C}\{\boldsymbol{x}\}\,,$$

with $\big(h_{i,j}(0)\big)=\frac{1}{2}\cdot H(f)(0)$ where $H(f)(0)$ is the Hessian of f at 0.

(1) $\Rightarrow$ (2). Since $h_{i,j}(0)=h_{j,i}(0)$, we have

$$\frac{\partial f}{\partial x_\nu} = \sum_{i,j}\frac{\partial h_{i,j}}{\partial x_\nu}x_ix_j + \sum_j h_{\nu,j}x_j + \sum_i h_{i,\nu}x_i \equiv 2\cdot\sum_{j=1}^n h_{\nu,j}(0)x_j \bmod \mathfrak{m}^2.$$

Since $H(f)(0)$ is invertible by assumption, we get

$$\Big\langle\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\Big\rangle = \big\langle x_1,\dots,x_n\big\rangle \bmod \mathfrak{m}^2.$$

Nakayama's lemma implies $j(f) = \mathfrak{m}$ and, hence, $\mu(f) = 1$.

(2) $\Rightarrow$ (3) is obvious, since $\tau \le \mu$ and $\tau = 0$ can only happen if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

(3) $\Rightarrow$ (2), (4). If $\tau = 1$ then $\mathfrak{m} = \langle f, j(f) \rangle$ and, hence, by Nakayama's lemma, $\mathfrak{m} = j(f)$, since $f \in \mathfrak{m}^2$. Then $\mu(f) = 1$, and by Corollary 2.25 f is right 2-determined, whence (4).

(4) $\Rightarrow$ (5). By the theory of quadratic forms over $\mathbb{C}$ there is a non-singular matrix T such that

$$T^t \cdot \frac{1}{2} H(f)(0) \cdot T = \mathbf{1}_n ,$$

where $\mathbf{1}_n$ is the $n \times n$ unit matrix. The linear coordinate change $\boldsymbol{x} \mapsto T \cdot \boldsymbol{x}$ provides, for $f = f^{(2)}$,

$$f(T \cdot \boldsymbol{x}) = \boldsymbol{x} \cdot T^t \cdot \frac{1}{2} H(f)(0) \cdot T \cdot \boldsymbol{x}^t = x_1^2 + \ldots + x_n^2 .$$

The implication (5) $\Rightarrow$ (6) is trivial. Finally, (6) implies $\tau(f) = 1$ and, hence, (5) as shown above. The implication (5) $\Rightarrow$ (1) is again obvious. □

The Morse lemma gives a complete classification of non-degenerate singularities in a satisfying form: they are classified by an invariant, the Milnor, respectively the Tjurina, number and, moreover, we have a very simple normal form.

In general, we cannot hope for such a simple answer. There might not be a finite set of complete invariants (that is, completely determining the singularity), and there might not be just one normal form but a whole family of normal forms. However, as we shall see, the simple singularities have a similar nice classification.

Splitting Lemma and Classification of Corank 1 Singularities. The following theorem, called *generalized Morse lemma* or *splitting lemma*, allows to reduce the classification to corank n or, equivalently, to germs in $\mathfrak{m}^3$.

Theorem 2.38 (Splitting lemma). *If* $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ *has* $\operatorname{rank} H(f)(0) = k$, *then*

$$f \overset{r}{\sim} x_1^2 + \ldots + x_k^2 + g(x_{k+1}, \ldots, x_n)$$

with $g \in \mathfrak{m}^3$. *Moreover,* g *is uniquely determined up to right equivalence.*

Proof. As the Hessian of f at 0 has rank k, the 2-jet of f can be transformed into $x_1^2 + \ldots + x_k^2$ by a linear change of coordinates (cf. the proof of Theorem 2.47). Hence, we can assume that

$$f(\boldsymbol{x}) = x_1^2 + \ldots + x_k^2 + f_3(x_{k+1}, \ldots, x_n) + \sum_{i=1}^{k} x_i \cdot g_i(x_1, \ldots, x_n) ,$$

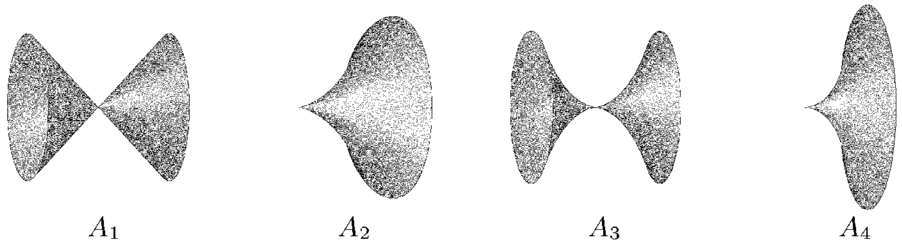


FIGURE 2.6. Real pictures of two-dimensional A_k-singularities

with $g_i \in \mathfrak{m}^2$, $f_3 \in \mathfrak{m}^3$. The coordinate change $x_i \mapsto x_i - \frac{1}{2}g_i$ for $i = 1, \ldots, k$, and $x_i \mapsto x_i$ for $i > k$, yields

$$f(\boldsymbol{x}) = x_1^2 + \ldots + x_k^2 + f_3(x_{k+1}, \ldots, x_n) + f_4(x_{k+1}, \ldots, x_n) + \sum_{i=1}^{k} x_i \cdot h_i(\boldsymbol{x}),$$

with $h_i \in \mathfrak{m}^3$, $f_4 \in \mathfrak{m}^4$. Continuing with h_i instead of g_i in the same manner, the last sum will be of arbitrary high order, hence 0 in the limit.

In case f has an isolated singularity, the result follows from the finite determinacy theorem 2.24. In general, we get at least a formal coordinate change such that $g(x_{k+1}, \ldots, x_n)$ in the theorem is a formal power series. We omit the proof of convergence.

To prove the uniqueness of g, let $\boldsymbol{x}' = (x_{k+1}, \ldots, x_n)$ and assume

$$f_0(\boldsymbol{x}) := x_1^2 + \ldots + x_k^2 + g_0(\boldsymbol{x}') \overset{\mathrm{r}}{\sim} x_1^2 + \ldots + x_k^2 + g_1(\boldsymbol{x}') =: f_1(\boldsymbol{x}).$$

Then, by Theorem 2.29, we obtain isomorphisms of $\mathbb{C}\{t\}$-algebras,

$$\mathbb{C}\{\boldsymbol{x}'\}\Big/\Big\langle \frac{\partial g_0}{\partial x_{k+1}}, \ldots, \frac{\partial g_0}{\partial x_n} \Big\rangle \cong M_{f_0} \cong M_{f_1} \cong \mathbb{C}\{\boldsymbol{x}'\}\Big/\Big\langle \frac{\partial g_1}{\partial x_{k+1}}, \ldots, \frac{\partial g_1}{\partial x_n} \Big\rangle,$$

t acting on M_{f_0}, respectively on M_{f_1}, via multiplication with f_0, respectively with f_1. It follows that M_{g_0} and M_{g_1} are isomorphic as $\mathbb{C}\{t\}$-algebras. Hence, $g_0 \overset{\mathrm{r}}{\sim} g_1$, again by Theorem 2.29. □

We use the splitting lemma to classify the singularities of corank ≤ 1.

Theorem 2.39. *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\}$ and $k \geq 1$, then the following are equivalent:*

(a) $\operatorname{crk}(f) \leq 1$ *and* $\mu(f) = k$,
(b) $f \overset{\mathrm{r}}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$, *that is, f is of type A_k,*
(c) $f \overset{\mathrm{c}}{\sim} x_1^{k+1} + x_2^2 + \ldots + x_n^2$.

Moreover, f is of type A_1 if and only if $\operatorname{crk}(f) = 0$, *and f is of type A_k for some $k \geq 2$ if and only if* $\operatorname{crk}(f) = 1$.

Proof. The implications $(b) \Rightarrow (c) \Rightarrow (a)$ are obvious. Hence, it is only left to prove $(a) \Rightarrow (b)$. By the splitting lemma, we may assume that

$$f = g(x_1) + x_2^2 + \ldots + x_n^2 = u \cdot x_1^{k+1} + x_2^2 + \ldots + x_n^2$$

with $u \in \mathbb{C}\{x_1\}$ a unit and $k \geq 1$, since $\mathrm{crk}(f) \leq 1$. The coordinate change $x_1' = \sqrt[k+1]{u} \cdot x_1$, $x_i' = x_i$ for $i \geq 2$ transforms f into A_k. □

Corollary 2.40. *A_k-singularities are right (and, hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_ℓ with $\ell \leq k$.*

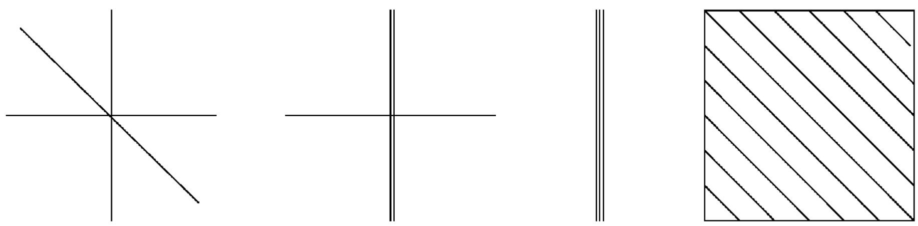
Proof. Since $\mathrm{crk}(f)$ is semicontinuous on $\mathfrak{m}^2$, a neighbourhood of A_k contains only A_ℓ-singularities. Since $\mu(f)$ is semicontinuous on $\mathfrak{m}^2$, too, we obtain $\ell = \mu(A_\ell) \leq \mu(A_k) = k$. □

On the Classification of Corank 2 Singularities. If $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\boldsymbol{x}\}$ has corank 2 then the splitting lemma implies that $f \overset{r}{\sim} g(x_1, x_2) + x_3^2 + \ldots + x_n^2$ with a uniquely determined $g \in \mathfrak{m}^3$. Hence, we may assume $f \in \mathbb{C}\{x, y\}$ and $f \in \mathfrak{m}^3$.

Proposition 2.41. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x, y\}$. Then there exists a linear automorphism $\varphi \in \mathbb{C}\{x, y\}$ such that $g = f^{(3)}$, the 3-jet of $\varphi(f)$, is of one of the following forms*

(1) $xy(x + y)$ or, equivalently, g factors into 3 different linear factors,
(2) $x^2 y$ or, equivalently, g factors into 2 different linear factors,
(3) x^3 or, equivalently, g has a unique linear factor (of multiplicity 3),
(4) 0.

We may draw the zero-sets:



(1) 3 different lines (2) a line and a double line (3) a triple line (4) a plane

Proof. Let $g = f^{(3)} = ax^3 + bx^2 y + cxy^2 + dy^3 \neq 0$. After a linear change of coordinates we may assume $a \neq 0$. Dehomogenizing g by setting $y = 1$, we get a univariate polynomial of degree 3, which decomposes into linear factors. Homogenizing the factors, we see that g factorizes into 3 homogeneous factors of degree 1, either 3 simple factors or a double factor and a simple factor or a triple factor. This corresponds to the cases (1) – (3).

To obtain the exact normal forms in (1) – (3) we may first assume $a = 1$ (replacing x by $\frac{1}{\sqrt[3]{a}}x$). Then g factors as

$$g = (x - \lambda_1 y) \cdot (x - \lambda_2 y) \cdot (x - \lambda_3 y)\,.$$

Having a triple factor would mean $\lambda_1 = \lambda_2 = \lambda_3$, and, replacing $x - \lambda_1 y$ by x, we end up with the normal form (3). One double plus one simple factor can be transformed similarly to the normal form in (2).

Three different factors can always be transformed to $xy(x - \lambda y)$ with $\lambda \neq 0$. Replacing $-\lambda y$ by y, we get $\alpha xy(x+y)$, $\alpha \neq 0$. Finally, replacing x by $\alpha^{-\frac{1}{3}}x$ and y by $\alpha^{-\frac{1}{3}}y$ yields $xy(x+y)$. □

Remark 2.41.1. If $f \in \mathbb{C}\{\boldsymbol{x}\}$ then we can always write

$$f = \sum_{i \geq d} f_i\,, \quad f_d \neq 0\,,$$

where f_i are homogeneous polynomials of degree i. The lowest non-vanishing term f_d is called the *tangent cone* of f, where $d = \operatorname{ord}(f)$ is the order of f. If f is contact equivalent to g with $u \cdot \varphi(f) = g$, $u \in \mathbb{C}\{\boldsymbol{x}\}^*$ and $\varphi \in \operatorname{Aut}\mathbb{C}\{\boldsymbol{x}\}$, then $\operatorname{ord}(f) = \operatorname{ord}(g) = d$ and

$$u^{(0)} \cdot \varphi^{(1)}(f_d) = g_d\,,$$

where $u^{(0)} = u(0)$ is the 0-jet of u and $\varphi^{(1)}$ the 1-jet of φ. In particular we have $f_d \overset{c}{\sim} g_d$, and Lemma 2.14 implies $f_d \overset{r}{\sim} g_d$.

In other words, if f is contact equivalent to g then the tangent cones are right equivalent by some linear change of coordinates, that is, they are in the same $GL(n, \mathbb{C})$-orbit acting on $\mathfrak{m}^d/\mathfrak{m}^{d+1}$.

Remark 2.41.2. During the following classification we shall make several times use of the so-called *Tschirnhaus transformation*: let A be a ring and

$$f = \alpha_d x^d + \alpha_{d-1}x^{d-1} + \ldots + \alpha_0 \in A[x]$$

a polynomial of degree d with coefficients in A. Assume that the quotient $\beta := \alpha_{d-1}/(d\alpha_d)$ exists in A. Then, substituting x by $x - \beta$ yields a polynomial of degree d with no term of degree $d-1$. In other words, the isomorphism $\varphi\colon A[x] \to A[y]$, $\varphi(x) = y - \beta$, maps f to

$$\varphi(f) = \alpha_d y^d + \beta_{d-2}y^{d-2} + \ldots + \beta_0 \in A[y]$$

for some $\beta_i \in A$.

Let us now analyse the four cases of Proposition 2.51, starting with the cases (1) and (2).

Theorem 2.42. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x,y\}$ and $k \geq 4$. Then the following are equivalent*

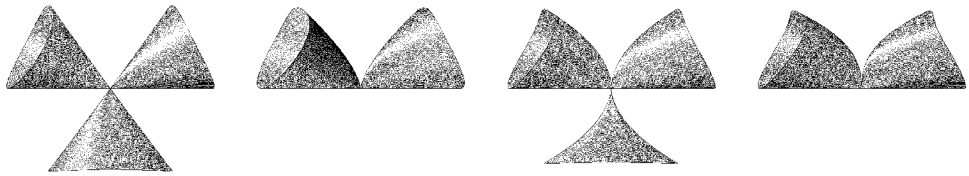

D_4 D_5 D_6 D_7

FIGURE 2.7. Real pictures of two-dimensional D_k-singularities

(a) $f^{(3)}$ *factors into at least two different factors and* $\mu(f) = k$,
(b) $f \stackrel{r}{\sim} x(y^2 + x^{k-2})$, *that is,* f *is of type* D_k,
(c) $f \stackrel{c}{\sim} x(y^2 + x^{k-2})$.

Moreover, $f^{(3)}$ *factors into three different factors if and only if* f *is of type* D_4.

The proof will also show that D_k is $(k-1)$-determined.

Proof. The implications (b) $\Rightarrow$ (c),(a) being trivial, and (c) $\Rightarrow$ (b) being implied by Lemma 2.14, we can restrict ourselves on proving (a) $\Rightarrow$ (b).

Assume that $f^{(3)}$ factors into three different factors. Then, due to Proposition 2.51 (1), $f^{(3)} \stackrel{r}{\sim} g := xy(x+y)$. But now it is easy to see that $\mathfrak{m}^4 \subset \langle \mathfrak{m}^2 \cdot j(g)$, hence g is right 3-determined due to the finite determinacy theorem. In particular, $g \stackrel{r}{\sim} f$.

If $f^{(3)}$ factors into exactly two different factors then, due to Proposition 2.51 (2), we can assume $f^{(3)} = x^2y$. Note that $f - f^{(3)} \neq 0$ (otherwise $\mu(f) = \infty$). Hence, we can define $m := \operatorname{ord}(f - f^{(3)})$ and consider the m-jet of f,

$$f^{(m)} = x^2y + \alpha y^m + \beta xy^{m-1} + x^2 \cdot h(x,y) \tag{2.4.3}$$

with $\alpha, \beta \in \mathbb{C}$, $h \in \mathfrak{m}^{m-2}$, $m \geq 4$. Applying the Tschirnhaus transformations $x = x - \frac{1}{2}\beta \cdot y^{m-2}$, $y = y - h(x,y)$ turns $f^{(m)}$ into

$$f^{(m)}(x,y) = x^2y + \alpha y^m . \tag{2.4.4}$$

Case A. If $\alpha = 0$ consider $f^{(m+1)}$, which has the form (2.4.3), hence can be transformed to (2.4.4) with m replaced by $m+1$ and, if still $\alpha = 0$, we continue. This procedure stops, since $\alpha = 0$ implies that

$$\begin{aligned}\mu(f) &\geq \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\big(j(f) + \mathfrak{m}^{m-1}\big) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\big(j(f^{(m)}) + \mathfrak{m}^{m-1}\big) \\ &= \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle x^2, xy, y^{m-1}\rangle = m\,.\end{aligned}$$

Case B. If $\alpha \neq 0$, then, replacing y by $\alpha^{-1/m}y$ and x by $\alpha^{2/m}x$, we obtain

$$f^{(m)}(x,y) = x^2y + y^m\,,$$

which is m-determined by Theorem 2.24. In particular, $f \stackrel{r}{\sim} y(x^2 + y^{m-1})$, which is a D_{m+1}-singularity. □

Corollary 2.43. *D_k-singularities are right (and, hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_ℓ for $\ell < k$ or D_k for $\ell \leq k$.*

Proof. For any $g \in \mathfrak{m}^2$ in a neighbourhood of D_k we have either $\operatorname{crk}(g) \leq 1$, which implies $g \overset{r}{\sim} A_\ell$ and $\ell \leq k$ by Theorem 2.49, respectively the semicontinuity theorem 2.6 (for the strict inequality we refer to Exercise 2.54, below), or we have $\operatorname{crk}(g) = 2$. In the latter case $g^{(3)}$ must factor into 2 or 3 different linear forms for g close to f, since this is an open property (by continuity of the roots of a polynomial, cf. the proof of Proposition 2.51). Hence, $g \overset{r}{\sim} D_\ell$ for some $\ell \leq k$. □

Exercise 2.44. Show that for $k \geq 4$ there exists a neighbourhood of D_k in $\mathfrak{m}^2$, which does not contain A_k-singularities.

Remark 2.44.1. Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x, y\}$ and $g = f^{(3)}$. Then g factors into

- three different linear factors if and only if the ring $\mathbb{C}\{x, y\}/j(g)$ has dimension 0,
- two different linear factors if and only if $\mathbb{C}\{x, y\}/j(g)$ has dimension 1, and the ring $\mathbb{C}\{x, y\}/\langle \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2} \rangle$ has dimension 0,
- one (triple) linear factor if and only if $\mathbb{C}\{x, y\}/j(g)$ has dimension 1, and the ring $\mathbb{C}\{x, y\}/\langle \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2} \rangle$ has dimension 1.

This can be seen by considering the singular locus of g, respectively the singular locus of the singular locus, and it gives in fact an effective characterization of the D_k-singularities by using standard bases in local rings (as implemented in SINGULAR).

Theorem 2.45. *Let $f \in \mathfrak{m}^3 \subset \mathbb{C}\{x, y\}$. Then the following are equivalent.*

(a) $f^{(3)}$ has a unique linear factor (of multiplicity 3) and $\mu(f) \leq 8$,
(b) $f^{(3)} \overset{r}{\sim} x^3$ and if $f^{(3)} = x^3$ then $f \notin \langle x, y^2 \rangle^3 = \langle x^3, x^2y^2, xy^4, y^6 \rangle$.
(c) $f \overset{r}{\sim} g$ with $g \in \{x^3 + y^4, x^3 + xy^3, x^3 + y^5\}$, i.e., f is of type E_6, E_7 or E_8.
(d) $f \overset{c}{\sim} g$ with $g \in \{x^3 + y^4, x^3 + xy^3, x^3 + y^5\}$.

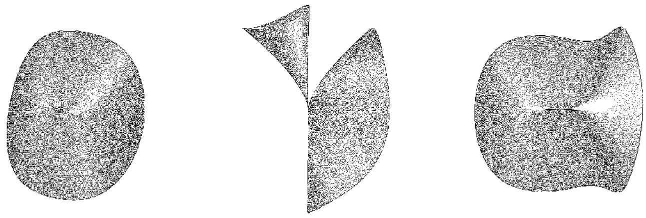
Moreover, $\mu(E_k) = k$ for $k = 6, 7, 8$.

Proof. Let's prove the implication (b) ⇒ (c). The 4-jet $f^{(4)}$ can be written as

$$f^{(4)}(x, y) = x^3 + \alpha y^4 + \beta xy^3 + x^2 \cdot h(x, y)$$

with $\alpha, \beta \in \mathbb{C}$, $h \in \mathfrak{m}^2$. After substituting $x = x - \frac{1}{3}h$, we may assume

$$f^{(4)}(x, y) = x^3 + \alpha y^4 + \beta xy^3 . \tag{2.4.5}$$



E_6 E_7 E_8

FIGURE 2.8. Real pictures of two-dimensional E_k-singularities

Case E_6: $\alpha \neq 0$ in (2.4.5). Applying a Tschirnhaus transformation (with respect to y), we obtain

$$f^{(4)}(x,y) = x^3 + y^4 + x^2 \cdot h\,, \quad h \in \mathfrak{m}^2\,,$$

and by applying another Tschirnhaus transformation (with respect to x) we obtain $f^{(4)} = x^3 + y^4$, which is 4-determined due to the finite determinacy theorem. Hence, $f \overset{r}{\sim} f^{(4)}$.

Case E_7: $\alpha = 0$, $\beta \neq 0$ in (2.4.5). Replacing y by $\beta^{-1/3}y$, we obtain the 4-jet $f^{(4)} = x^3 + xy^3$, which is 4-determined by Example 2.26.1, hence $f \overset{r}{\sim} f^{(4)}$.

Case E_8: $\alpha = 0$, $\beta = 0$ in (2.4.6). Then $f^{(4)} = x^3$, and we consider the 5-jet of f.

$$f^{(5)}(x,y) = x^3 + \alpha y^5 + \beta x y^4 + x^2 \cdot h(x,y)\,, \quad h \in \mathfrak{m}^3.$$

Replacing x by $x - \frac{1}{3}h(x,y)$ we obtain

$$f^{(5)} = x^3 + \alpha y^5 + \beta x y^4. \tag{2.4.6}$$

If $\alpha \neq 0$ then, replacing y by $\alpha^{-1/5}y$ and renaming β, we obtain

$$f^{(5)}(x,y) = x^3 + y^5 + \beta x y^4\,.$$

Applying a Tschirnhaus transformation (with respect to y) gives

$$f^{(5)}(x,y) = x^3 + y^5 + x^2 \cdot h(x,y)\,, \quad h \in \mathfrak{m}^3\,,$$

and, again replacing x by $x - \frac{1}{3}h$ yields $f^{(5)} = x^3 + y^5$, which is 5-determined due to the finite determinacy theorem.

If $\alpha = 0$ in (2.4.6) then $f^{(5)} = x^3 + \beta x y^4$ and, hence,

$$f \in \langle x^3, xy^4 \rangle_{\mathbb{C}} + \mathfrak{m}^6 \subset \langle x, y^2 \rangle^3.$$

This proves (c).

By Exercise 2.57 it follows that $\mu(f) > 8$ if $f \in \langle x, y^2 \rangle^3$. Since $\mu(E_k) = k$ for $k = 6, 7, 8$, we get the equivalence of (a) and (b) and the implication (c) $\Rightarrow$ (a). Finally, since E_6, E_7, E_8 are quasihomogeneous, (c) and (d) are equivalent, by Lemma 2.14. $\square$

Corollary 2.46. *E_6, E_7, E_8 are right (hence, contact) simple. More precisely, there is a neighbourhood of f in $\mathfrak{m}^2$, which meets only orbits of singularities of type A_k or D_k or E_k for k at most 8.*

Proof. Let $g \in \mathfrak{m}^2$ be in a (sufficiently small) neighbourhood of f. Then either $\operatorname{crk}(g) \leq 1$, or $\operatorname{crk}(g) = 2$.

If $\operatorname{crk}(g) \leq 1$ then $g \overset{r}{\sim} A_k$ for some k by 2.49. If $\operatorname{crk}(g) = 2$ and $g^{(3)}$ factors into three or two factors, then $g \overset{r}{\sim} D_k$ for some k by 2.52. If $g^{(3)} \overset{r}{\sim} x^3$, then g is right equivalent to E_6, E_7 or E_8 since the condition $f \notin \langle x, y^2 \rangle^3$ is open. □

Exercise 2.47. Show that $\mu(f) > 8$ if $f \in \langle x, y^2 \rangle^3$.

HINT: Choose a generic element from $\langle x, y^2 \rangle^3$ and use the semicontinuity of μ.

Remark 2.47.1. We have shown that the singularities of type A_k $(k \geq 1)$, D_k $(k \geq 4)$, and E_6, E_7, E_8 are right simple (and, hence, contact simple). Moreover, we have also shown that if $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ is not contact equivalent to one of the ADE classes, then either

(1) $\operatorname{crk}(f) \geq 3$, or

(2) $\operatorname{crk}(f) = 2$, $f \overset{r}{\sim} g(x_1, x_2) + x_3^2 + \ldots + x_n^2$ with
 (a) $g \in \mathfrak{m}^4$, or
 (b) $g \in \langle x_1, x_2^2 \rangle^3$.

We still have to show that all singularities belonging to one of these latter classes are, indeed, not contact simple. In particular, if f has a non-isolated singularity, then it must belong to class (1) or (2). An alternative way to prove that non-isolated singularities are not simple is given in the exercises below.

Theorem 2.48. *If $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ belongs to one of the classes (1),(2) above, then f is not contact simple and hence not right simple.*

Proof. (1) We may assume $f \in \mathbb{C}\{x_1, x_2, x_3\}$ and $f \in \mathfrak{m}^3 \setminus \mathfrak{m}^4$. The tangent cone $f^{(3)}$ of f is in $\mathfrak{m}^3/\mathfrak{m}^4$, which is a 10-dimensional vector space. If $f \overset{c}{\sim} g$, then $f^{(3)}$ and $g^{(3)}$ are in the same $GL(3, \mathbb{C})$-orbit. Since $\dim GL(3, \mathbb{C}) = 9$, this orbit has dimension ≤ 9 by Theorem 2.35. Since the orbits are locally closed by 2.35, and since a finite union of at most 9-dimensional locally closed subvarieties is a constructible set of dimension ≤ 9, a neighbourhood of $f^{(3)}$ in $\mathfrak{m}^3/\mathfrak{m}^4$ must meet infinitely many $GL(3, \mathbb{C})$-orbits. Hence, any neighbourhood of f in $\mathfrak{m}^3$ must meet infinitely many $\mathcal{K}$-orbits, that is, f is not contact simple.

(2) We may assume $f \in \mathbb{C}\{x, y\}$. The argument for (a) is the same as in (1) except that we consider $f^{(4)}$ in the 5-dimensional vector space $\mathfrak{m}^4/\mathfrak{m}^5$ and the action of $GL(2, \mathbb{C})$, which has dimension 4.

In case (b) it is not sufficient to consider the tangent cone. Instead we use the *weighted* tangent cone: first notice that an arbitrary element f can be written as

$$f(x,y)=\sum_{d\geq 6} f_d(x,y)\,,\quad f_d(x,y)=\sum_{2i+j=d}\alpha_{i,j}x^iy^j\,,$$

that is, f_d is weighted homogeneous of type $(2,1;d)$. The weighted tangent cone f_6 has the form

$$f_6(x,y)=\alpha x^3+\beta x^2y^2+\gamma xy^4+\delta y^6.$$

Applying the coordinate change φ given by

$$\begin{aligned}\varphi(x)&=a_1x+b_1y+c_1x^2+d_1xy+e_1y^2+\dots\,,\\ \varphi(y)&=a_2x+b_2y+c_2x^2+d_2xy+e_2y^2+\dots\,,\end{aligned}$$

we see that $\varphi(f)\in\langle x,y^2\rangle^3$ forces $b_1=0$. Then the weighted order of $\varphi(x)$ is at least 2, while the weighted order of $\varphi(y)$ is at least 1. This implies that, for all $d\geq 6$, $\varphi(f_d)$ has weighted order at least d. Therefore, only f_6 is mapped to the space of weighted 6-jets of $\langle x,y^2\rangle^3$. However, the weighted 6-jet of $\varphi(f_6)$ involves only the coefficients a_1, e_1 and b_2 of φ as a simple calculation shows. Therefore, the orbit of f_6 under the right group intersects the space of weighted 6-jets of $\langle x,y^2\rangle^3$ in a locally closed variety of dimension at most 3. Since f_6 is quasihomogeneous, the right orbit coincides with the contact orbit. As the space of weighted 6-jets of $\langle x,y^2\rangle^3$ is 4-dimensional, generated by x^3, x^2y^2, xy^4 and y^6, it must intersect infinitely many contact orbits of elements of $\langle x,y^2\rangle^3$. Hence, f is not contact simple. □

Exercise 2.49. (1) Let $f\in\mathfrak{m}^2\subset\mathbb{C}\{\boldsymbol{x}\}$ have an isolated singularity, and let $g\in\mathbb{C}\{\boldsymbol{x}\}$ satisfy $g\notin\mathfrak{m}\cdot j(f)$, respectively $g\notin\mathfrak{m}\cdot j(f)+\langle f\rangle$.
Show that $f\overset{r}{\sim}f+tg$, respectively $f\overset{c}{\sim}f+tg$, for only finitely many $t\in\mathbb{C}$.

(2) Use this to show that if f has a non-isolated singularity, then, for each $k>0$, there is some $g_k\in\mathfrak{m}^k\setminus(\mathfrak{m}\cdot j(f)+\langle f\rangle+\mathfrak{m}^{k+1})$ such that $f+tg_k\overset{c}{\nsim}f$ for arbitrary small t. Hence, f is not contact simple and, therefore, also not right simple.

We now give explicit examples of non-simple singularities belonging to the classes (1) and (2) (a,b).

Example 2.49.1. (1) Consider the family of surface singularities given by

$$E=y^2z-4x^3+g_2xz^2+g_3z^3$$

of corank 3. This equation $E=0$ defines the cone over an *elliptic curve*, defined by $E=0$ in $\mathbb{P}^2$, in *Weierstraß normal form.* The *J-invariant* of this equation is

$$J=\frac{g_2^3}{g_2^3-27g_3^2}\,.$$

The number J varies continuously in $\mathbb{C}$ if the coefficients g_2, g_3 vary, and two isomorphic elliptic curves in Weierstraß form have the same J-invariant (cf. [BrK, Sil]). Therefore the family $E = E(g_2, g_3)$ meets infinitely many right (and, hence, contact) orbits.

Another normal form is the *Hesse normal form* of an elliptic curve,

$$x^3 + y^3 + z^3 + \lambda xyz = 0\,.$$

(2) Given 4 lines in $\mathbb{C}^2$ through 0, defined by $a_i x + b_i y = 0$, then

$$f = \prod_{i=1}^{4}(a_i x + b_i y) \ \in \mathfrak{m}^4\,,$$

defines the union of these lines. Similar to the J-invariant for elliptic curves, there is an invariant of 4 lines (equivalently, 4 points in $\mathbb{P}^1$), the *cross-ratio*

$$r = \frac{(a_1b_3 - a_3b_1)\cdot(a_2b_4 - a_4b_2)}{(a_1b_4 - a_4b_1)\cdot(a_2b_3 - a_3b_2)}\,.$$

A direct computation shows that r is an invariant under linear coordinate changes. Since this is quite tideous to do by hand, we provide the SINGULAR code for checking this.

```
ring R = (0,A,B,C,D,a1,a2,a3,a4,b1,b2,b3,b4),(x,y),dp;
ideal i= Ax+By, Cx+Dy;          // the coordinate transformation
ideal i1 = subst(i,x,a1,y,b1);
ideal i2 = subst(i,x,a2,y,b2);
ideal i3 = subst(i,x,a3,y,b3);
ideal i4 = subst(i,x,a4,y,b4);

poly r1 = (a1b3-a3b1)*(a2b4-a4b2);
poly r2 = (a1b4-a4b1)*(a2b3-a3b2);
// cross-ratio = r1/r2

poly s1 = (i1[1]*i3[2]-i3[1]*i1[2])*(i2[1]*i4[2]-i4[1]*i2[2]);
poly s2 = (i1[1]*i4[2]-i4[1]*i1[2])*(i2[1]*i3[2]-i3[1]*i2[2]);
// cross-ratio of transformed lines = s1/s2

// The difference of the cross-ratios:
r1/r2-s1/s2;
// -> 0
```

(3) Consider 3 parabolas which are tangent to each other,

$$f(x,y) = (x - t_1y^2)\cdot(x - t_2y^2)\cdot(x - t_3y^2) \in \langle x, y^2\rangle^3\,.$$

Two such polynomials for different (t_1, t_2, t_3) are, in general, not contact equivalent. We show this for the family

$$f_t(x,y) = x(x-y^2)(x-ty^2)\,.$$

As in the proof of Theorem 2.58 we make a coordinate change φ and then consider the weighted 6-jet of $\varphi(f_t) - f_s$. The relation between t and s can be computed explicitly by eliminating the coefficients of the coordinate change, but for this the use of a computer is necessary. Here is the SINGULAR code.

```
ring r = 0,(a,b,c,d,e,f,g,h,i,j,s,t,x,y),dp;
poly ft = x*(x-y2)*(x-sy2);
poly fs = x*(x-y2)*(x-ty2);
ideal i = maxideal(1);
i[13] = ax+by+cx2+dxy+ey2;     // phi(x)
i[14] = fx+gy+hx2+ixy+jy2;     // phi(y)
map phi = r,i;
poly dd = phi(ft)-fs;
intvec w;
w[13],w[14]=2,1;               // weights for the variables
coef(jet(dd,3,w),xy);          // weighted 3-jet (must be 0)
// -> _[1,1]=y3
// -> _[2,1]=b3                // hence, we must have b=0
dd=subst(dd,b,0);              // set b=0

// Now consider the weighted 6-jet:
matrix C = coef(jet(dd,6,w),xy);
ideal cc=C[2,1..ncols(C)];     // note: cc=0 iff the weighted
                               // 6-jets of phi(ft), fs coincide
cc;
// -> cc[1]=eg4s-e2g2s-e2g2+e3
// -> cc[2]=ag4s-2aeg2s-2aeg2+3ae2-t
// -> cc[3]=-a2g2s-a2g2+3a2e+t+1
// -> cc[4]=a3-1

// We eliminate a,e,g in cc to get the relation between t and s:
eliminate(cc,aeg);
// -> _[1]=s6t4-s4t6-2s6t3-3s5t4+3s4t5+2s3t6+s6t2+6s5t3
// ->      -6s3t5-s2t6-3s5t2-5s4t3+5s3t4+3s2t5+3s4t+5s3t2-5s2t3
// ->      -3st4-s4-6s3t+6st3+t4+2s3+3s2t-3st2-2t3-s2+t2

// Hence, for fixed t there are at most 6 values of s such that
// ft and fs are contact equivalent
```

Algorithmic Classification of ADE-Singularities. The proof of the classification of the simple singularities is actually effective and provides a concrete algorithm for deciding if a given polynomial $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1,\dots,x_n\}$, $n > 1$, is simple or not, and if it is simple to determine the type of f.

STEP 1. Compute $\mu := \mu(f)$. If $\mu = \infty$ then f has a non-isolated singularity and, hence, is not simple.

The Milnor number can be computed as follows: compute a standard basis $sj(f)$ of $j(f)$ with respect to a local monomial ordering and let $L(j(f))$ be the ideal generated by the leading monomials of the generators of $sj(f)$. Then $\mu = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/L(f)$, which can be determined combinatorially (cf. [GrP]).

STEP 2. Assume $\mu < \infty$. Let $f^{(2)}$ be the 2-jet of f and compute

$$r := \operatorname{rank}\Big(\frac{\partial^2 f^{(2)}}{\partial x_i \partial x_j}(0)\Big).$$

Then $n-r = \operatorname{crk}(f)$ and so if $n-r \geq 3$, then f is not simple. On the other hand if $n-r \leq 1$, then $f \overset{r}{\sim} A_\mu$. If $n-r = 2$ goto Step 3.

STEP 3. Assume $n-r = 2$. Note that, in order to decide whether f is of type D or E, we need only to consider the 3-jet of $f^{(3)}$. That is, by a linear change of coordinates we get

$$f^{(3)} = x_3^2 + \ldots + x_n^2 + f_3(x_1, x_2) + \sum_{i=3}^{n} x_i g_i(\boldsymbol{x}), \quad f_3 \in \mathfrak{m}^3,\ g_i \in \mathfrak{m}^2.$$

The coordinate change $x_i \mapsto x_i - \frac{1}{2} g_i$, $i = 3, \ldots, n$ transforms $f^{(3)}$ into

$$g(x_1, x_2) + x_3^2 + \ldots + x_n^2 + h(\boldsymbol{x}), \quad g \in \mathfrak{m}^3,\ h \in \mathfrak{m}^4.$$

Assume $g \neq 0$. If g factors over $\mathbb{C}$ into two or three different factors, then $f \overset{r}{\sim} D_\mu$. If g has only one factor and $\mu \in \{6,7,8\}$, then $f \overset{r}{\sim} E_\mu$. If $g = 0$ or $\mu \notin \{6,7,8\}$, then f is not simple (and necessarily $\mu > 8$).

The splitting lemma uses linear algebra to adjust the 2-jet of f and then applies Tschirnhaus transformations in order to adjust higher and higher order terms. In order to check the number of factors of g one can apply for example the method discussed in Remark 2.54.1.

Let us treat an example with SINGULAR, using some procedures from the library `classify.lib`.

```
LIB "classify.lib";
ring R = 0,(x,y,z,t),ds;
poly f = x4+3x3y+3x2y2+xy3+y4+4y3t+6y2t2+4yt3+t4+x3+z2+zt;
ideal j = jacob(f);

vdim(std(j));                      // the Milnor number
// -> 6
corank(f);                         // the corank
// -> 2
poly g = morsesplit(f);
g;                                 // the residual part
// -> x3+x4+3x3y+3x2y2+xy3+y4+16y6
poly h = jet(g,3);                 // the 3-jet

ideal jh = jacob(h);
nvars(R) - dim(std(jh));           // codim of Sing(h)
// -> 1
```

Hence, $\dim \mathbb{C}\{x,y\}/j(g^{(3)}) = 1$ and $f \overset{r}{\sim} E_6$.

SINGULAR is also able to classify many other classes of singularities. Some of them can be identified by computing invariants without applying the splitting lemma. The procedure quickclass uses this method. Arnol'd's original method [AGV] is implemented in the procedure classify.

```
poly nf = quickclass(f);
// -> Singularity R-equivalent to :  E[6k]=E[6]
// -> normal form : z2+t2+x3+xy3+y4
nf;
// -> z2+t2+x3+xy3+y4
```

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Contents of the book "Singular Algebraic Curves"
(joint monograph with C. Lossen and E. Shustin)