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**SPRING COLLEGE ON SCIENCE AT THE NANOSCALE**  
**(24 May - 11 June 2004)**

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**Simulating Quantum Transport in Ballistic Nanowire FETS**

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*These are preliminary lecture notes, intended only for distribution to participants.*

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# Simulating Quantum Transport in Ballistic Nanowire FETs

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Network for Computational Nanotechnology

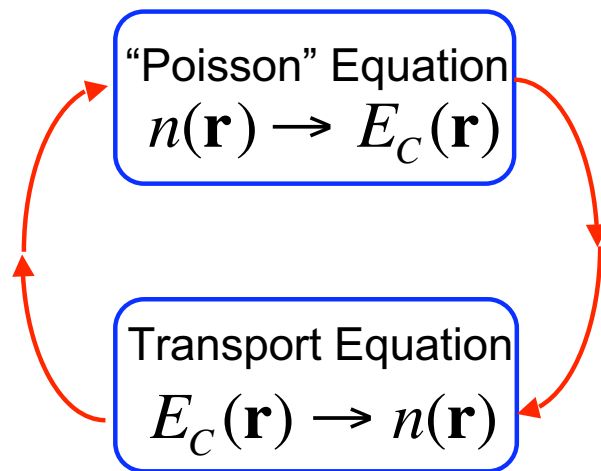
- 1) Introduction
- 2) Solving the Wave Equation
- 3) Computing  $n(x)$
- 4) Computing  $I_D$
- 5) Discussion
- 6) Summary



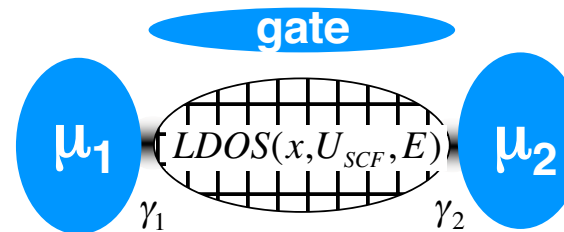
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## I) Introduction: device simulation



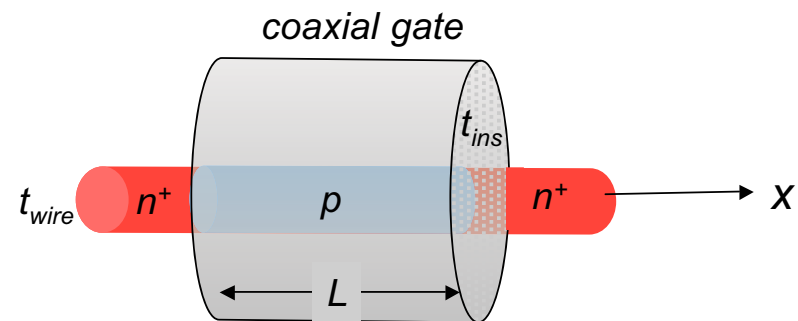
{ Boltzmann Transport Equation  
Schrödinger Equation  
NEGF  
etc.



- 1) Compute  $U_{SCF}$
- 2) Compute LDOS  
Fill LDOS

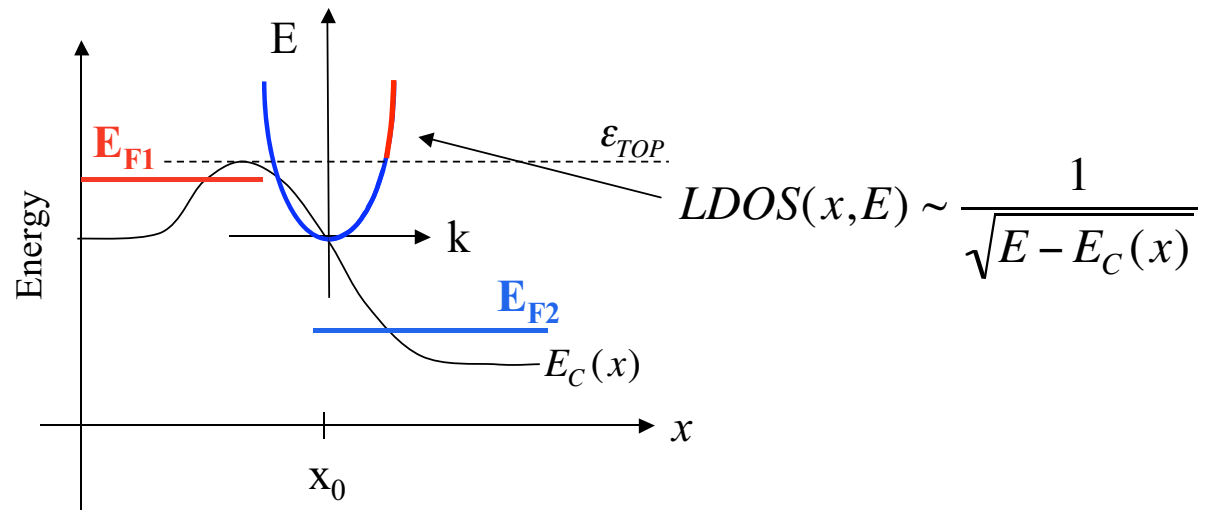
## I) Introduction: nanowire MOSFET

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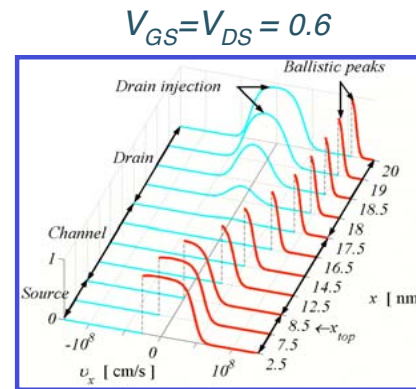
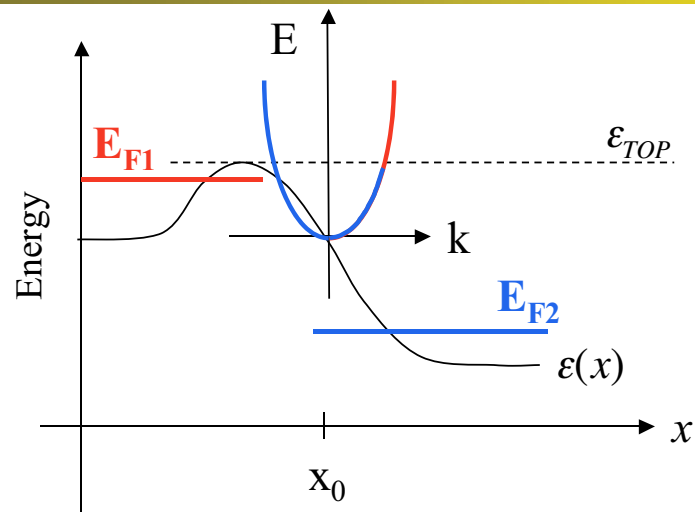


# I) Introduction: semiclassical

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# I) Introduction: semiclassical ballistic transport

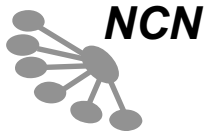


$$n(x_0) = \int [LDOS_1(E, x_0) f(E_{F1}) + LDOS_2(E, x_0) f(E_{F2})] dE$$

$$I_D = \frac{2q}{h} \int T(E) [f(E_{F1}) - f(E_{F2})] dE$$

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- 1) Introduction
- 2) Solving the Wave Equation
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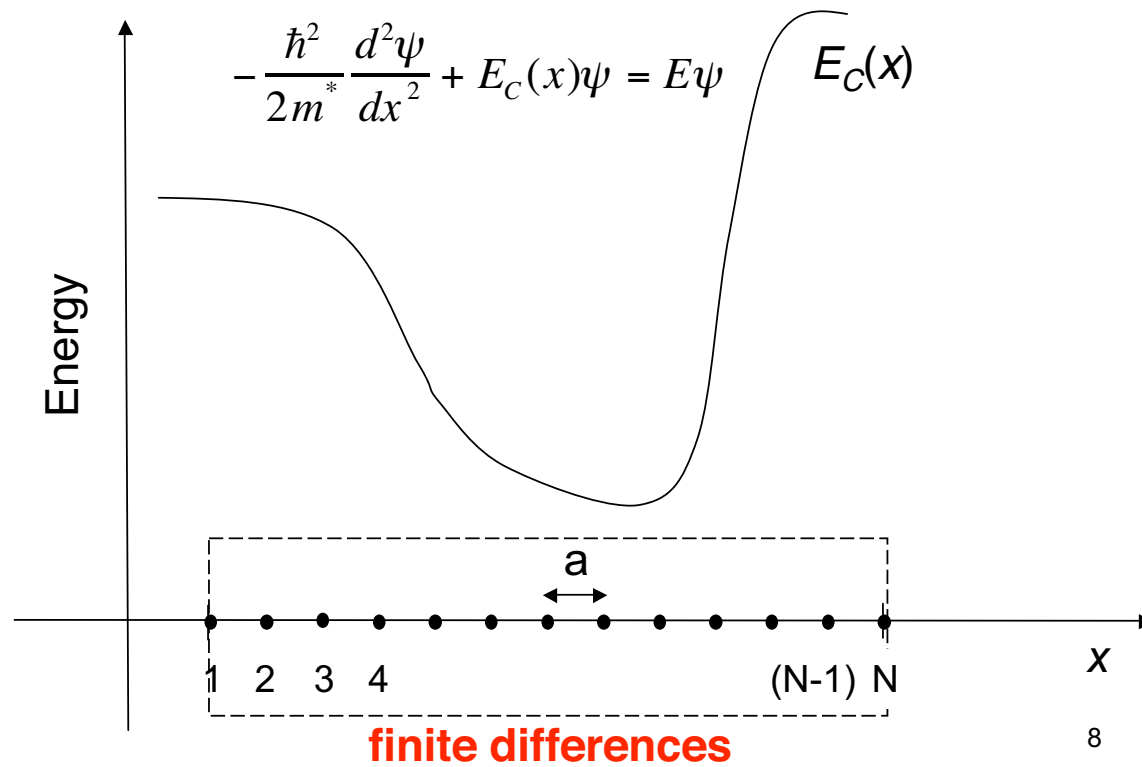
## II) Solving the Wave Equation

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$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} \left\{ \begin{array}{l} -\frac{\hbar^2}{2m^*} \frac{d^2\psi}{dx^2} + E_C(x)\psi = E\psi \\ \frac{d^2\psi}{dx^2} + k^2\psi = 0 \\ k \equiv \frac{\sqrt{2m^*[E - E_C(x)]}}{\hbar} \end{array} \right.$$



## II) Solving the Wave Equation: Bound States

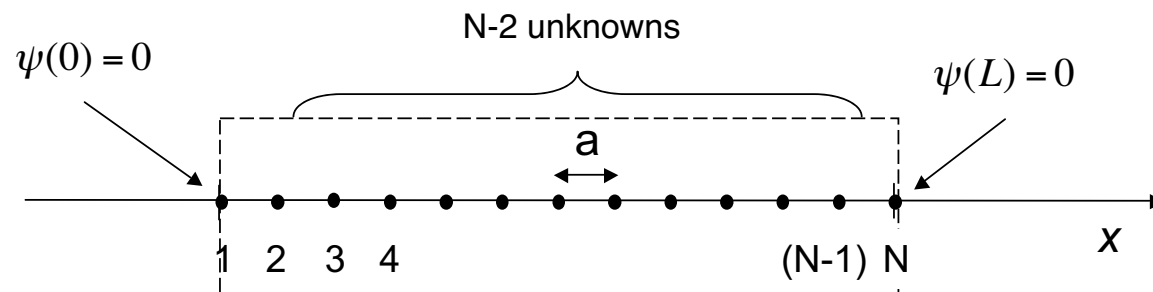


## II) Solving the Wave Equation: Discretization

$$-\frac{\hbar^2}{2m^*} \frac{d^2\psi}{dx^2} + E_C(x)\psi = E\psi \quad \left[ -t_0\psi_{i-1} + (2t_0 + E_{Ci})\psi_i - t_0\psi_{i+1} \right] = E\psi_i$$

$$i = 2, 3, \dots, N-1$$

$$\left. \frac{d^2\psi}{dx^2} \right|_i = \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{a^2} \quad t_0 \equiv \frac{\hbar^2}{2m^* a^2}$$



## II) Solving the Wave Equation

---

$$\left[ -t_0\psi_{i-1} + (2t_0 + E_{Ci})\psi_i - t_0\psi_{i+1} \right] = E\psi_i \quad (i = 2, 3 \dots N-1)$$

$$\mathbf{H}\psi = E\psi$$

$N-2 \times N-2$      $(N-2) \times 1$

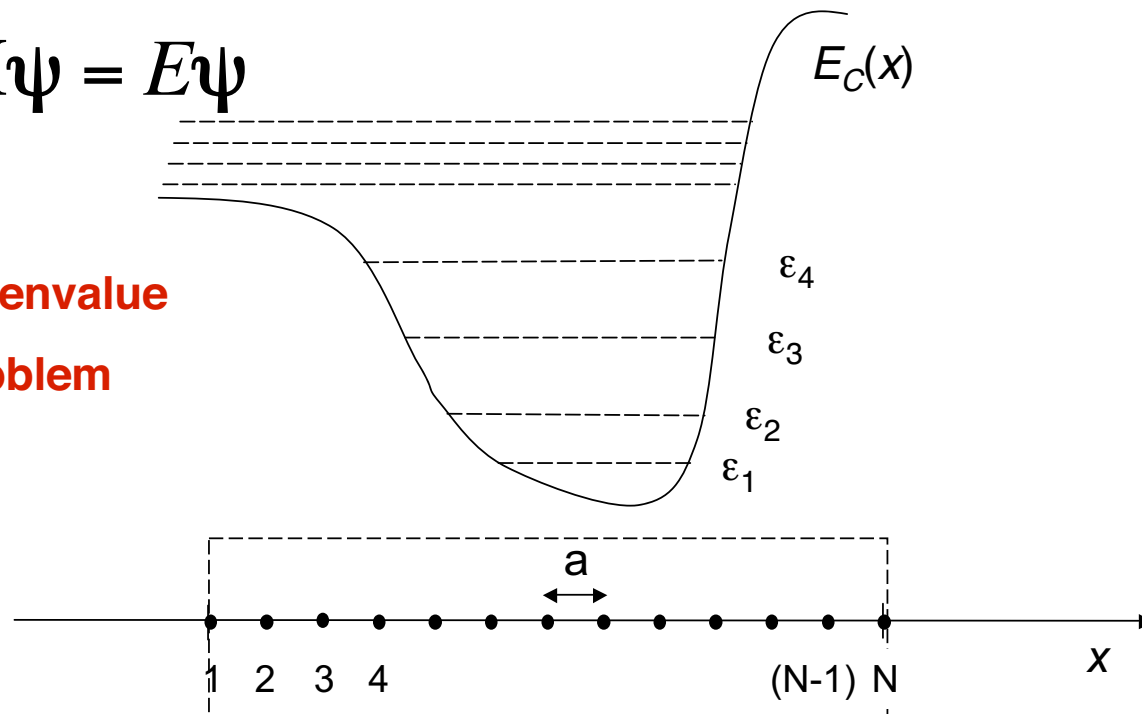
$$\begin{bmatrix} & & & \\ & & & \\ -t_0 & & & \\ & (2t_0 + E_{Ci}) & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \end{bmatrix} = E \begin{bmatrix} \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \\ \psi \end{bmatrix}$$

## II) Solving the Wave Equation

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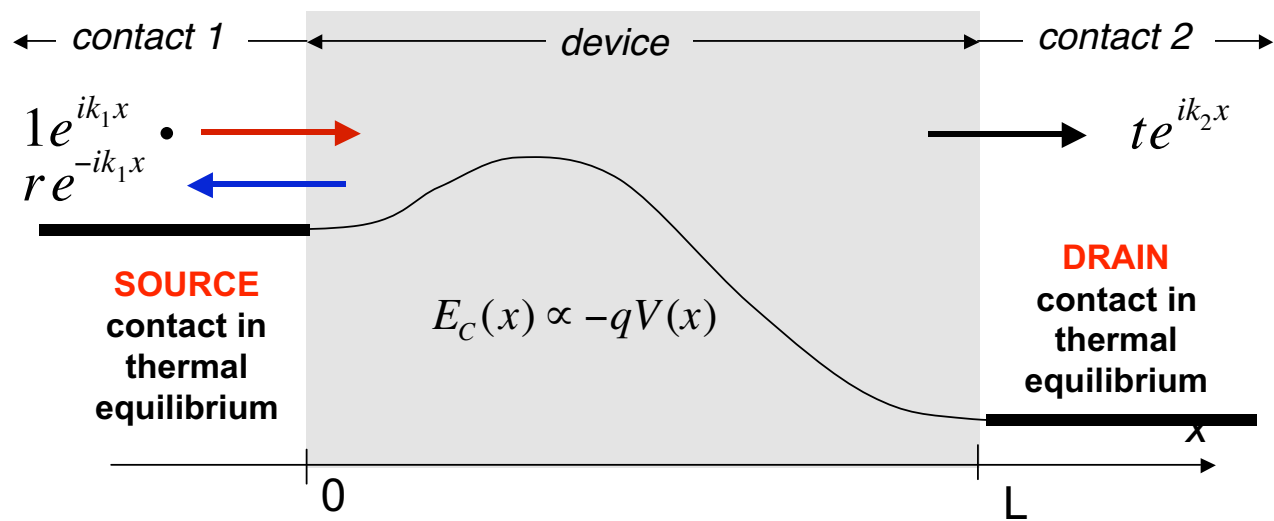
$$H\psi = E\psi$$

eigenvalue  
problem



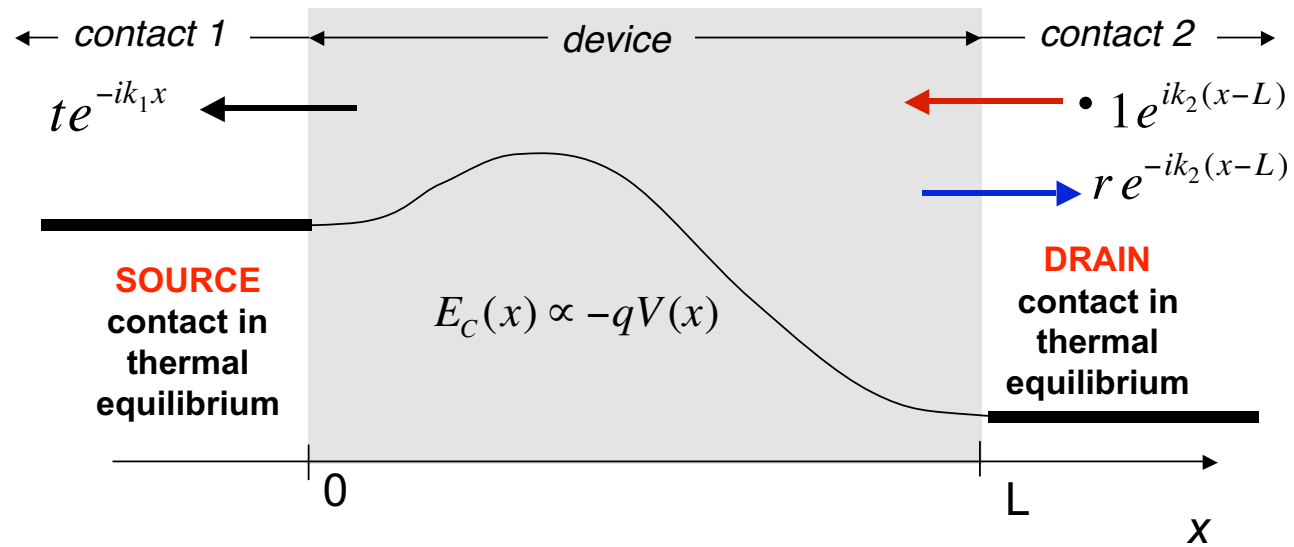
## II) Solving the Wave Equation: open systems

source injection: "scattering states"

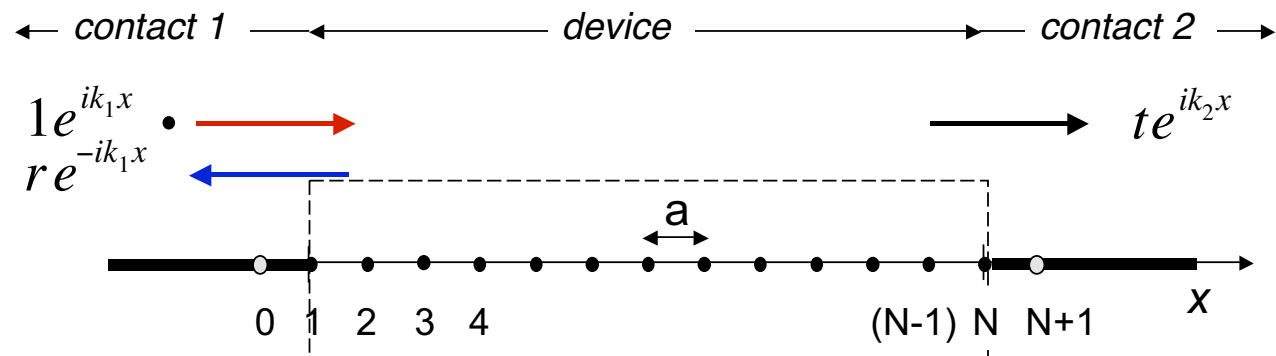


## II) Solving the Wave Equation

drain injection: "scattering states"



## II) Solving the Wave Equation: A 1D Device



$$-\frac{\hbar^2}{2m^*} \frac{d^2\psi}{dx^2} + E_C(x)\psi = E\psi$$

$$\hat{H}\psi = E\psi$$

$$(E - \hat{H})\psi = 0$$

## II) Solving the Wave Equation

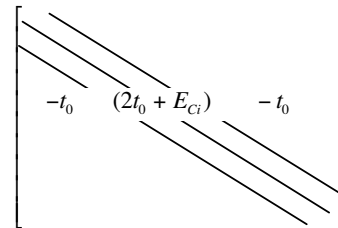
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$$E\psi_i - [-t_0\psi_{i-1} + (2t_0 + E_{Ci})\psi_i - t_0\psi_{i+1}] = 0 \quad (i = 1, 2 \dots N)$$

$$(EI - H)\psi = 0$$

$N \times N$

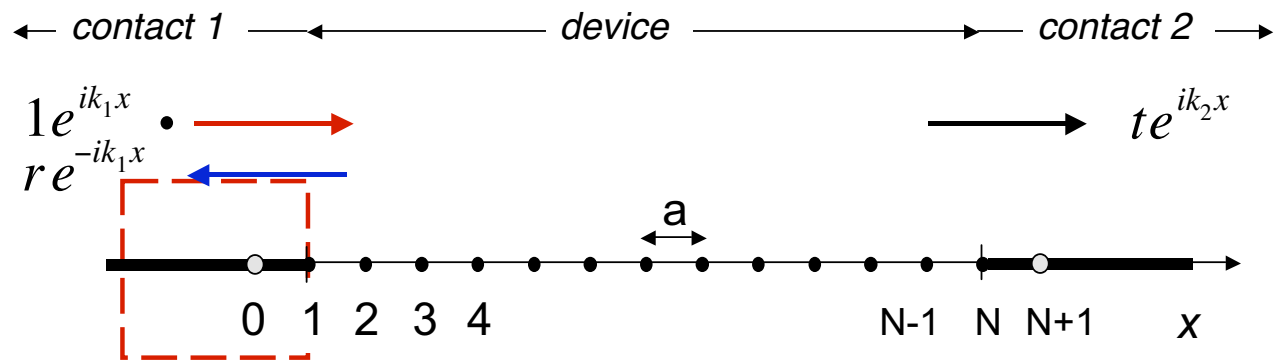
$N \times 1$



but there is a **problem** at the first node ( $i = 1$ ) and the last node, ( $i = N$ ) because they couple to nodes outside the device.

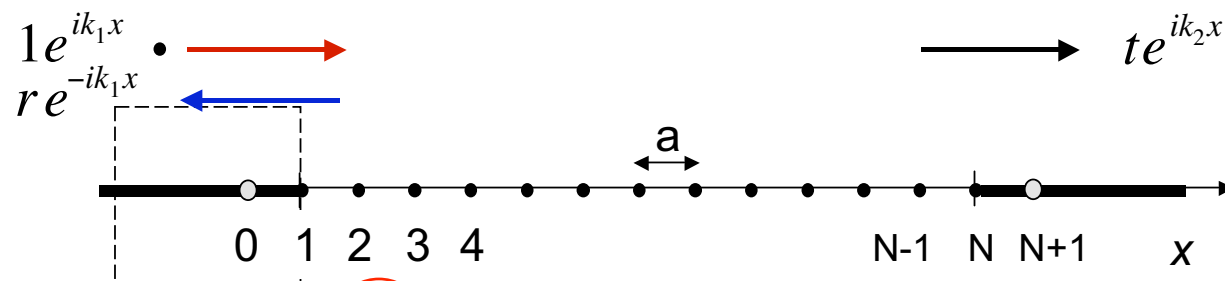


## II) Solving the Wave Equation: Open boundary conditions



$$E\psi_1 - [-t_0\psi_0 + (2t_0 + E_{C1})\psi_1 - t_0\psi_2] = 0 \quad (i = 1)$$

## II) Solving the Wave equationn: Treatment of $i = 1$



$$E\psi_1 - [-t_0\psi_0 + (2t_0 + E_{C1})\psi_1 - t_0\psi_2] = 0 \quad (i = 1)$$

$$\psi(x) = 1e^{ik_1x} + re^{-ik_1x} \quad x \leq 0$$

$$\psi(x=0) = \psi_1 = 1 + r$$

$$\psi(x=-a) = \psi_0 = e^{-ik_1a} + re^{ik_1a}$$

## II) Solving the Wave Equation: **Open boundary conditions**

---

$$E\psi_1 - [-t_0\psi_0 + (2t_0 + E_{C1})\psi_1 - t_0\psi_2] = 0$$

$$\psi_0 = \psi_1 e^{ik_1 a} - (e^{ik_1 a} - e^{-ik_1 a})$$

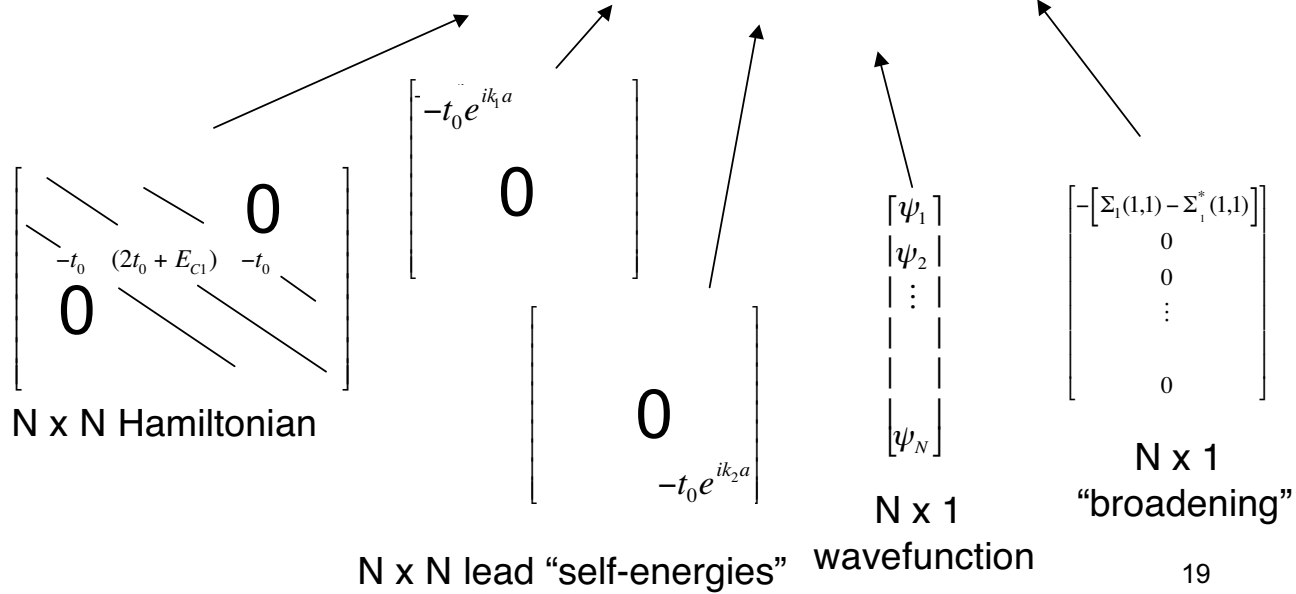
to account for the open boundary condition at  $x = 0$ , we

1) add  $-t_0 e^{ik_1 a}$  to the diagonal

2) add  $t_0 (e^{ik_1 a} - e^{-ik_1 a})$  to the first column of the RHS

## II) Solving the Wave Equation

$$[E\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]\psi = \mathbf{S} = i\gamma$$



## II) Solving the wave Equation: [The solution](#)

---

$$[E\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]\psi = \mathbf{S}$$

(not an eigenvalue problem - energy continuous)

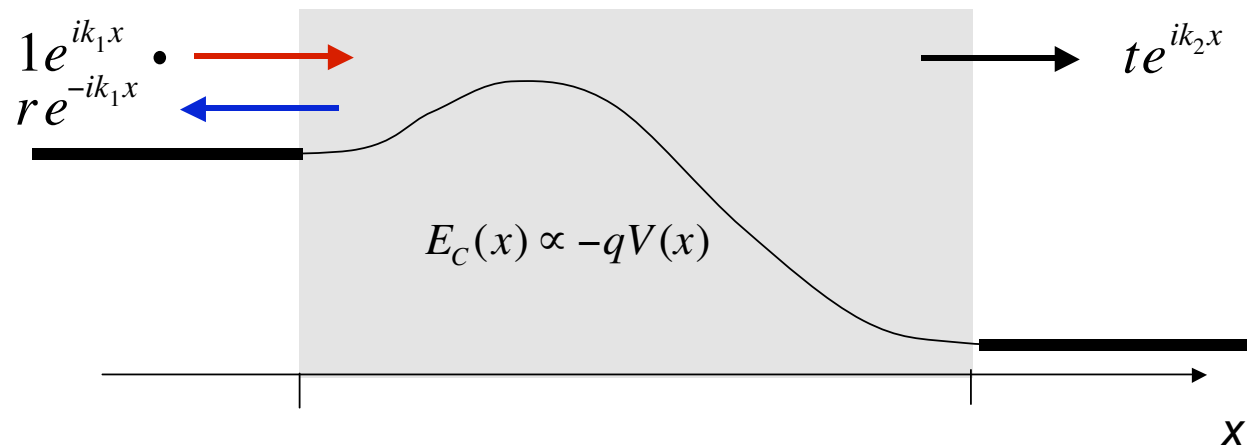
***formal solution:***

$$\psi = \mathbf{G}^R \mathbf{S}$$

$$\mathbf{G}^R(E) = [E\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]^{-1}$$

(N x N retarded Green's function)

## II) Solving the Wave Equation: The self-energies and $k$



$$\Sigma_1 = \begin{bmatrix} -t_0 e^{ik_1 a} \\ 0 \end{bmatrix}$$

$$E - E_{C0} = \hbar^2 k^2 / 2m^*$$

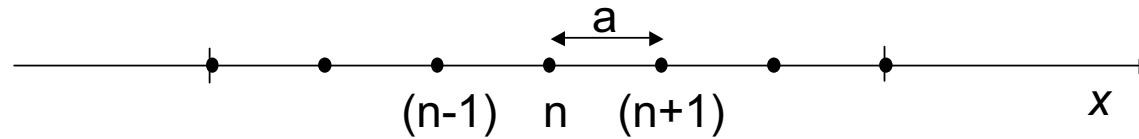
$$k_1 = \sqrt{2m(E - E_{C0})} / \hbar$$

$$k_2 = \sqrt{2m(E - E_{CL})} / \hbar$$

**there is NO  $E(k)$   
in the device!  
 $k$  refers to the  
lead**

## II) Solving the Wave Equation: $E_k$ in the leads

*infinite lead*



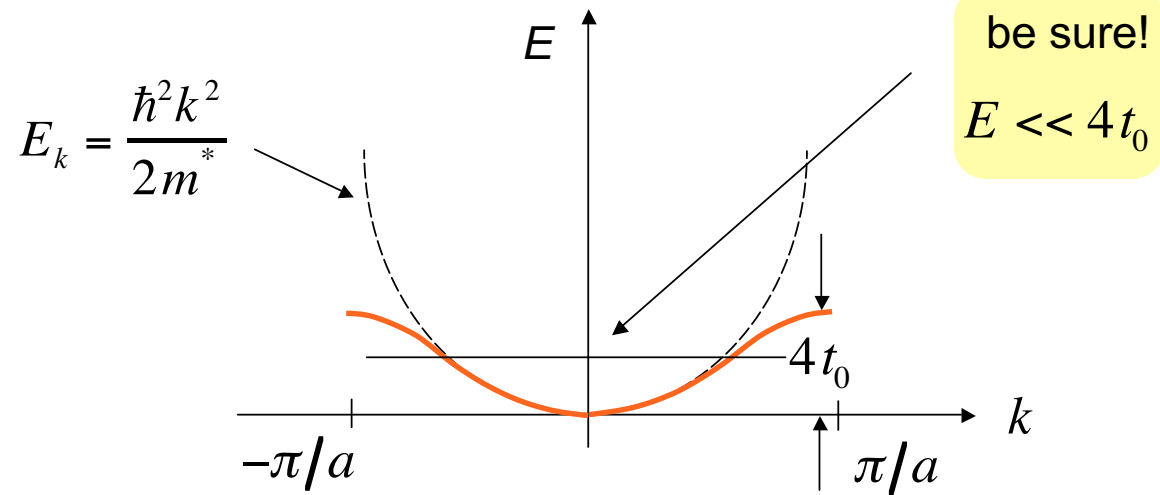
$$-\frac{\hbar^2}{2m^*} \frac{d^2\psi}{dx^2} + E_C\psi = E\psi \quad \longrightarrow \quad -\frac{\hbar^2}{2m^*} \left[ \frac{\psi_{n+1} - 2\psi_n + \psi_{n-1}}{a^2} \right] = (E - E_C)\psi_n$$

$$\psi_n = e^{ikna}$$

$$\longrightarrow E_k = E_C + 2t_0(1 - \cos ka)$$

$$t_0 \equiv \frac{\hbar^2}{2m^* a^2}$$

## II) Solving the Wave Equation: $E_k$ in the leads



$$E = E_C + 2t_0(1 - \cos ka)$$

$$t_0 \equiv \frac{\hbar^2}{2m^* a^2}$$



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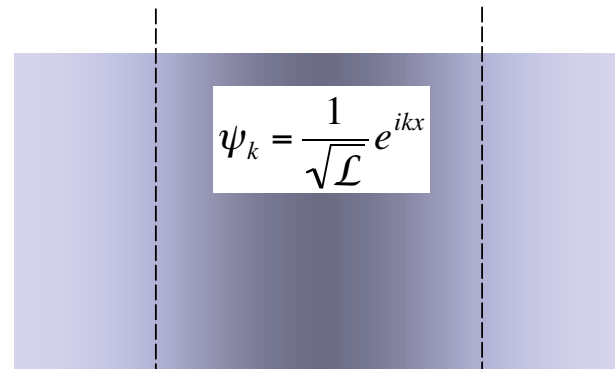


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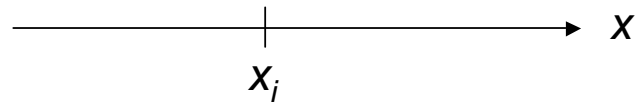
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### III) Computing $n(x)$ : **in the bulk**

---



(normalization length)

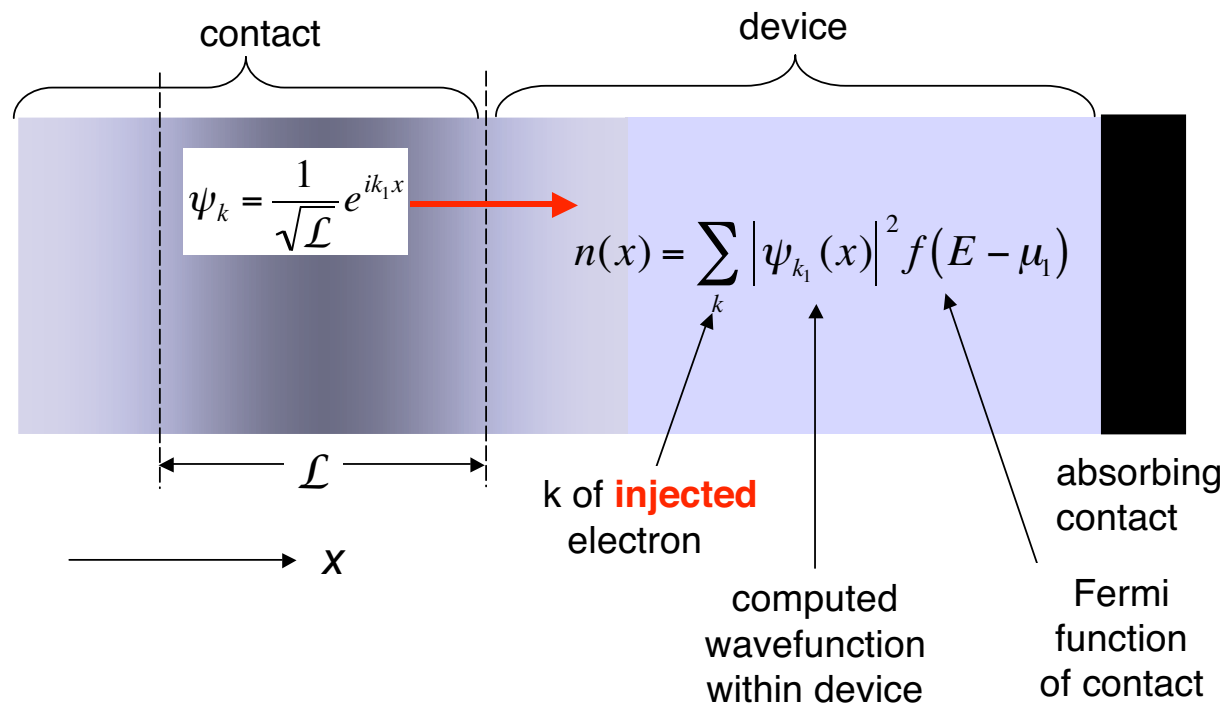


$$n(x_i) = \sum_k |\psi_k(x_i)|^2 f(E - \mu)$$

$$n(x_i) = \frac{1}{\mathcal{L}} \sum_k f(E - \mu)$$

### III) Computing $n(x)$

*now attach a device to a bulk contact .....*



### III) Computing $n(x)$

---

sum over the distribution  
of injected  $k$ 's from  
source contact

$$n_1(x_i) = \frac{1}{\mathcal{L}} \sum_{k_1} |\psi_{k_1}(x_i)|^2 f(E - \mu_1)$$

normalization length  
in the **contact**

$$\psi_k = \frac{1}{\sqrt{\mathcal{L}}} e^{ik_1 x}$$

computed assuming  
unit amplitude  
injected wave

weight by Fermi  
function of the  
source

### III) Computing $n(x)$

---

$$n_1(x_i) = \frac{1}{\mathcal{L}} \sum_{k_1 > 0} |\psi_{k_1}(x_i)|^2 f(E - \mu_1)$$

$$n_1(x_i) = \frac{1}{\mathcal{L}} \frac{\mathcal{L}}{2\pi} \times 2 \int_0^{\infty} dE \frac{dk_1}{dE} |\psi_{k_1}(x_i)|^2 f(E - \mu_1)$$

Electron density in the device due to injection from contact 1.

We can write this as:

### III) Computing $n(x)$

---

$$n_1(x_i) = \int_0^\infty \underbrace{\left[ \frac{1}{\pi} \frac{dk_1}{dE} \left| \psi_{k_1}(x_i) \right|^2 \right]}_{\text{LDOS}_1(x_i, E_k)} f_0(E - \mu_1) dE$$

$$LDOS_1(x_i, E_k) \equiv 2 \times \frac{A_1(x_i, E_k)}{2\pi}$$

this is the **portion** of the local density-of-states within the device due to **injection from the source**.

### III) Computing $n(x)$

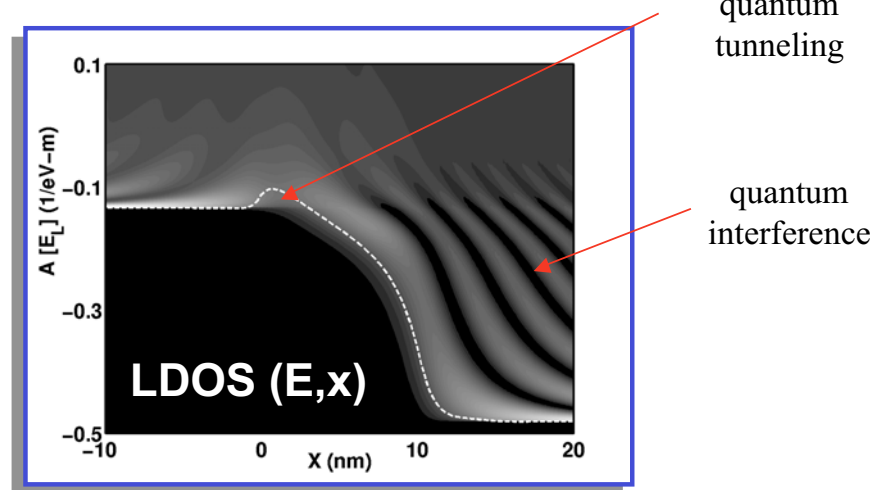
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$$n(x_i) = \int_0^{\infty} LDOS_1(x_i, E) f_0(E - \mu_1) dE + \int_0^{\infty} LDOS_2(x_i, E) f_0(E - \mu_2) dE$$

can be computed semi-classically  
or  
quantum mechanically

## II) Computing $n(x)$ : local DOS in a MOSFET

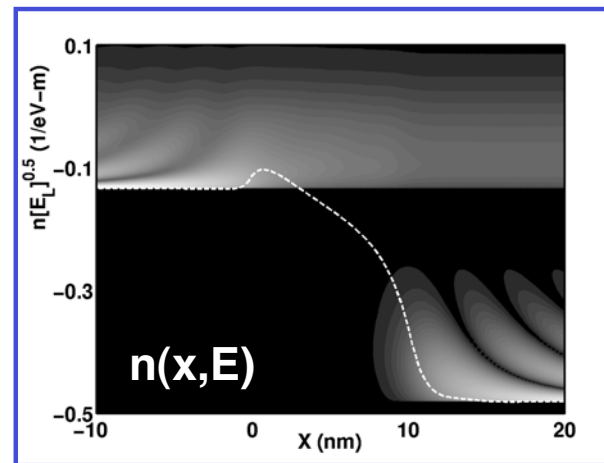
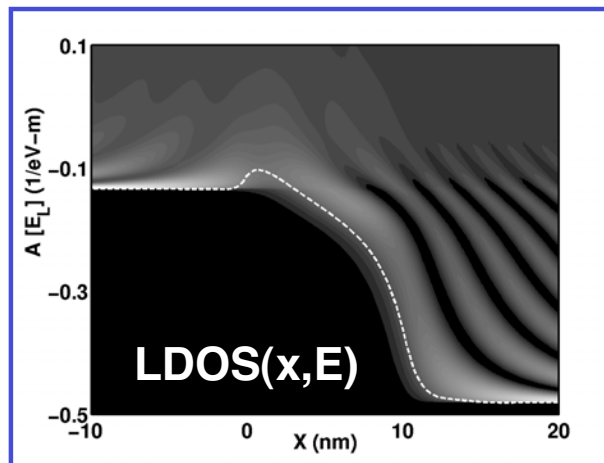
$$LDOS_1(x_i, E) = \left[ \frac{1}{\pi} \frac{dk_1}{dE} |\psi_{k_1}(x_i)|^2 \right] \quad LDOS_2(x_i, E) = \left[ \frac{1}{\pi} \frac{dk_2}{dE} |\psi_{k_2}(x_i)|^2 \right]$$





### III) Computing $n(x)$

(on-state,  $V_{DD} = 0.4$  V)



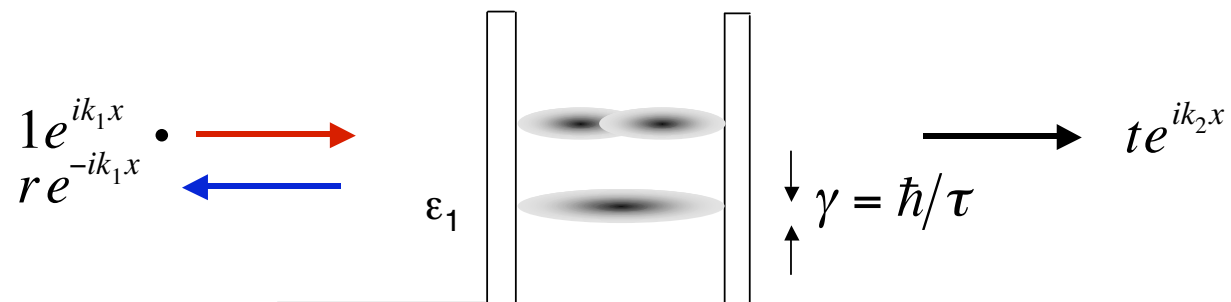
$$LDOS(x,E) = LDOS_1(x,E) + LDOS_2(x,E)$$

$$n_1(x,E) = f_1(E)LDOS_1(x,E)$$

$$n_2(x,E) = f_2(E)LDOS_2(x,E)$$

### III) Computing $n(x)$ : bound states again

---



“broadening”

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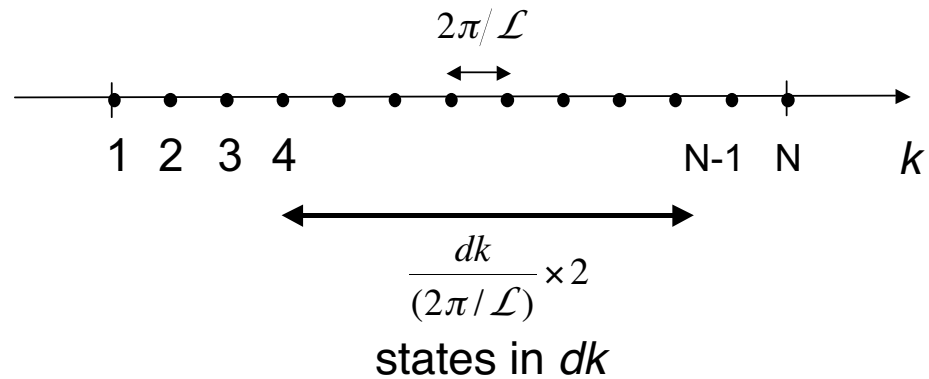
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#### IV) Computing $I_D$

---

let's derive the current expression beginning with:

$$I_{1-2} = \frac{q}{\mathcal{L}} \sum_{k>0} T_{1-2}(E) v_k f(E - \mu_1)$$



#### IV) Computing $I_D$

---

$$I_{1-2} = \frac{q}{\mathcal{L}} \sum_{k>0} T_{1-2}(E) v_k f(E - \mu_1)$$

$$\sum_{k>0} \cdot \Rightarrow \left( \frac{\mathcal{L}}{\pi} \right) \int_{k>0} \cdot dk$$

$$\Rightarrow I_{1-2} = \frac{q}{\pi} \int T_{1-2}(E) v_k f(E - \mu_1) dk$$

#### IV) Computing $I_D$

---

$$\begin{aligned} I_{1-2} &= \frac{q}{\pi} \int_{k>0} T_{1-2}(E) v_k f(E - \mu_1) dk \\ &= \frac{q}{\pi} \int_{k>0} T_{1-2}(E) \frac{dE}{d(\hbar k)} f(E - \mu_1) dk \end{aligned}$$

$$I_{1-2} = \frac{2q}{h} \int_0 T_{1-2}(E) f(E - \mu_1) dE$$

#### IV) Computing $I_D$

---

$$I_D = I_{1-2} - I_{2-1}$$

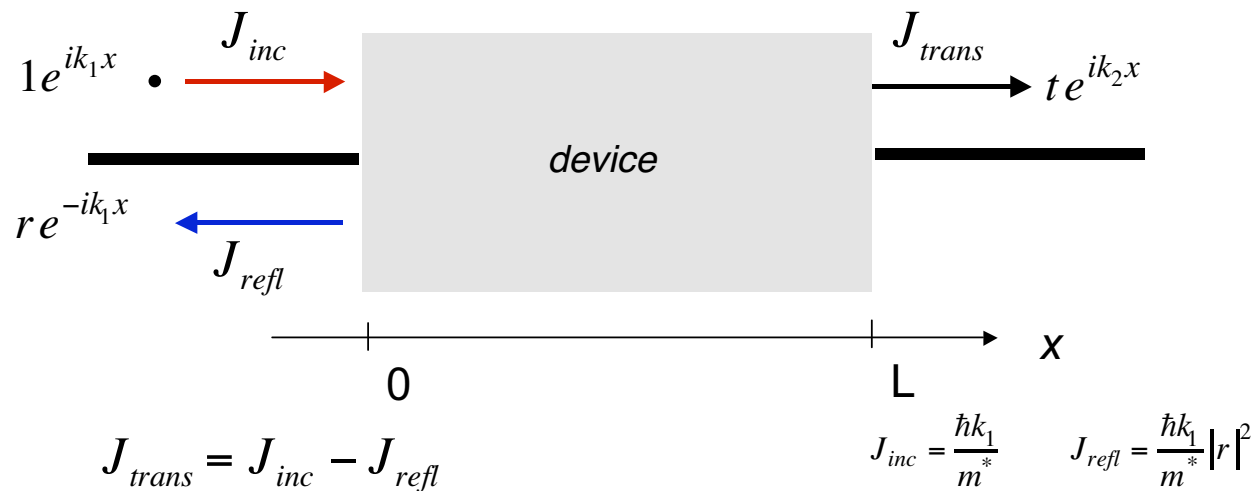
$$I_{1-2} = \frac{2q}{h} \int_0^\infty T_{1-2}(E) f(E - \mu_1) dE$$

$$I_{2-1} = \frac{2q}{h} \int_0^\infty T_{2-1}(E) f(E - \mu_2) dE$$

can show:  
 $T_{1-2}(E) = T_{2-1}(E)$   
**(ballistic)**

$$I_D = \frac{2q}{h} \int_0^\infty T(E) [f(E - \mu_1) - f(E - \mu_2)] dE$$

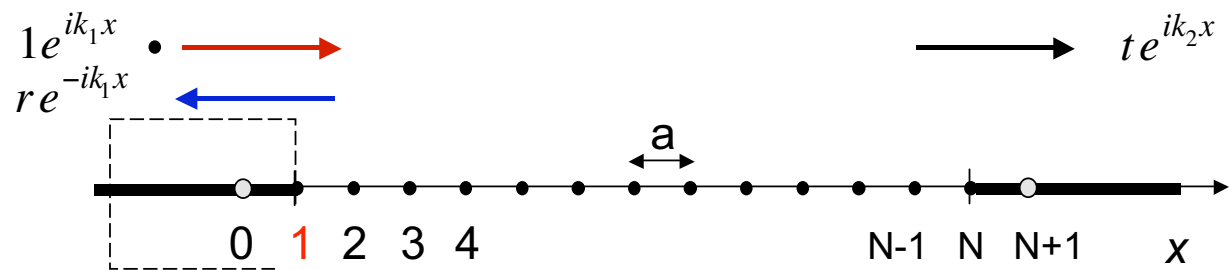
#### IV) Computing $I_D$ : Finding $T_{1-2}(E)$



$$T_{1-2}(E) = \frac{J_{trans}}{J_{inc}} = 1 - \frac{J_{refl}}{J_{inc}} = 1 - |r|^2$$



#### IV) Computing $I_D$ : Finding $T_{1-2}(E)$



$$\psi(x) = 1e^{ik_1x} + re^{-ik_1x} \quad x \leq 0$$

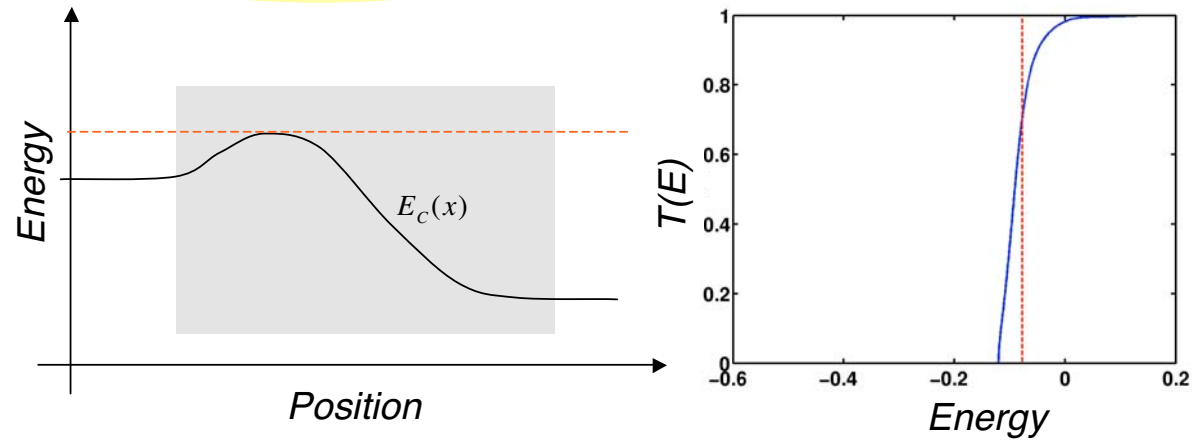
$$\psi(x=0) = 1 + r = \psi_1$$

$$T_{1-2}(E) = 1 - |\psi_1 - 1|^2$$

#### IV) Computing $I_D$ : $T_{1-2}(E)$ for a MOSFET

---

$$T_{1-2}(E) = 1 - |\psi_1 - 1|^2$$



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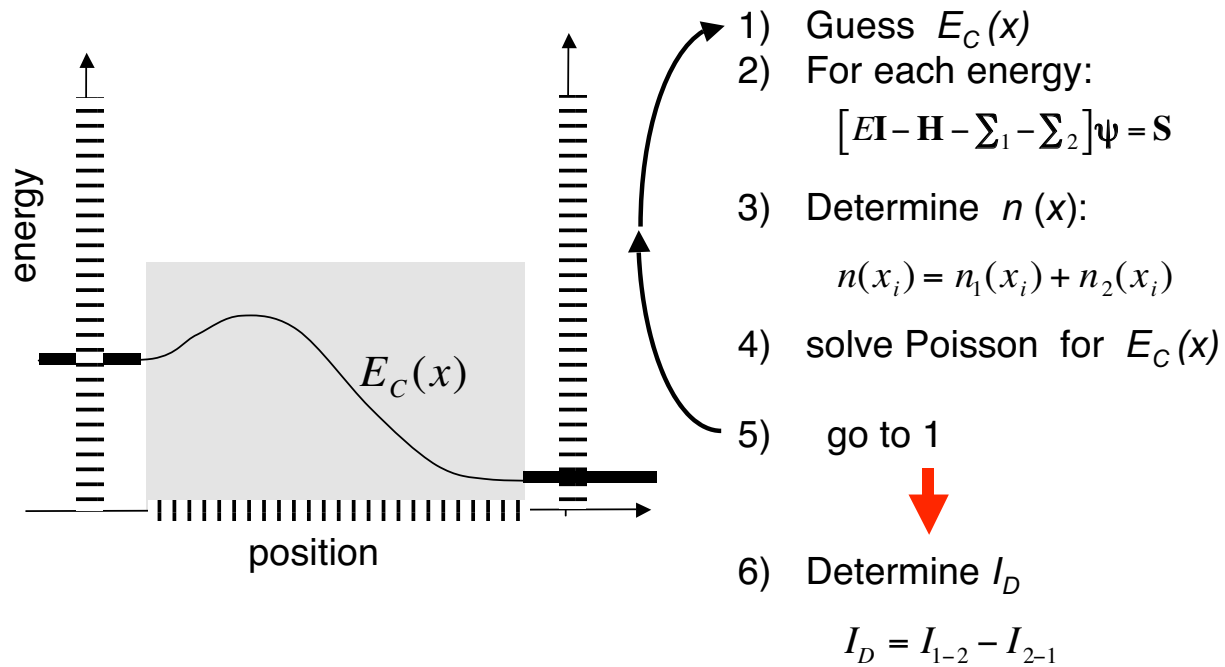
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## V) Discussion: The wavefunction approach



## V) Discussion: Key Results

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$$[E\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]_{N \times N} \Psi_{N \times 1} = i\gamma_{N \times 1}$$

$$n_1(x_i) = \int_0^{E_{top}} 2 \times \frac{A_1(E_k, x_i)}{2\pi} f(E_F - E_k) dE_k$$

$$\frac{A_1(x_i, E)}{2\pi} = \left( \frac{1}{2\pi} \frac{dk}{dE} |\psi_k(x_i)|^2 \right)$$

$$I^{1-2} = \frac{2q}{h} \int_0^{E_{top}} T(E) f(E_k) dE_k$$

$$T(E) = 1 - |\psi_1 - 1|^2$$

$$\Psi_{N \times 1} = i\mathbf{G}_{N \times N} \gamma_{N \times 1}$$

$$\mathbf{G}_{N \times N} = [E\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]_{N \times N}^{-1}$$

**Can we do everything  
with the Green's  
function?**

## V) Discussion: density matrix

---

We need  $\psi^* \psi$  at each node.  $\Psi \Psi^H = (\mathbf{G} \boldsymbol{\gamma}_1) (\boldsymbol{\gamma}_1^H \mathbf{G}^H)$

We have:

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} \quad \left( \begin{array}{cccc} \psi_1^* \psi_1 & \psi_1^* \psi_2 & \psi_1^* \psi_3 & \dots \\ \psi_2^* \psi_1 & \psi_2^* \psi_2 & & \\ \psi_3^* \psi_1 & & \psi_3^* \psi_3 & \\ \vdots & & & \\ & & & \psi_N^* \psi_N \end{array} \right)$$

## V) Discussion: density matrix

---

$$(N \times 1) \quad n_1(x_i) = \int_0^{E_{Top}} \frac{1}{\pi} \frac{dk}{dE_k} |\psi_k(x_i)|^2 f(E_k) dE$$

$$(N \times N) \quad \rho_1 = \int_0^{E_{Top}} \frac{1}{\pi} \frac{dk}{dE_k} \{ \mathbf{G} \gamma_1 \gamma_1^H \mathbf{G}^H \} f(E) dE$$

$$\gamma_1 = \begin{bmatrix} i[\Sigma_1(1,1) - \Sigma_1^*(1,1)] \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 2t_0 \sin ka \\ \vdots \\ 0 \end{bmatrix} \quad \gamma_1 \gamma_1^H = \begin{bmatrix} 4t_0^2 \sin^2 ka & \dots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$$

## V) Discussion: density matrix

---

$$\rho_1 = \int_0^{E_{\text{top}}} \frac{1}{\pi} \frac{dk}{dE_k} \{ \mathbf{G} \gamma_1 \gamma_1^H \mathbf{G}^H \} f(E) dE$$

$$E(k) = 2t_0(1 - \cos ka)$$

$$\frac{dk}{dE} = \frac{1}{2at_0 \sin ka}$$

$$\gamma_1 \gamma_1^H = \begin{bmatrix} 4t_0^2 \sin^2 ka & \dots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$$



## V) Discussion: spectral function

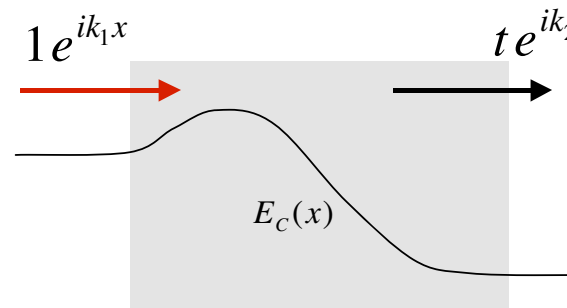
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$$\frac{a}{\pi} \frac{dk}{dE} \gamma \gamma^H \equiv \Gamma_1 = \begin{bmatrix} 2t_0 \sin ka & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix} = i(\Sigma_1 - \Sigma_1^H)$$

$$\rho_1 \Big|_{N \times N} = \int 2 \times \frac{\mathbf{A}_1 \Big|_{N \times N}}{2\pi} f(E_F - E) dE$$

$$\mathbf{A}_1 = \mathbf{G} \Gamma_1 \mathbf{G}^H$$

## V) Discussion: transmission



$$J_{inc} = 1v_1$$

$$J_{trans} = |t|^2 v_N$$

$$T = |t|^2 \left( \frac{v_N}{v_1} \right)$$

$$v_1 = \frac{2at_0}{\hbar} \sin k_0 a = \left( \frac{a}{\hbar} \right) \Gamma_1(1,1)$$

$$|t|^2 = |\psi_N|^2$$

$$v_N = \frac{2at_0}{\hbar} \sin k_N a = \left( \frac{a}{\hbar} \right) \Gamma_2(N,N)$$

$$\psi_N = G(N,1) \Gamma_1(1,1)$$

## V) Discussion: [transmission](#)

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$$T(E) = |t|^2 \frac{v_N}{v_1} = |\psi_N|^2 \frac{\Gamma_2(N,N)}{\Gamma_1(1,1)} = |G(N,1)\Gamma_1(1,1)|^2 \frac{\Gamma_2(N,N)}{\Gamma_1(1,1)}$$

$$T(E) = G(N,1)\Gamma_1^*(1,1)G^H(N,1)\Gamma_2(N,N)$$

*more generally, for transmission from contact 1 to 2:*

$$T(E) = \text{trace} \left[ \Gamma_1 \mathbf{G} \Gamma_2 \mathbf{G}^H \right]$$

## V) Discussion: key results in NEGF form

$$\mathbf{G} = [\mathbf{E}\mathbf{I} - \mathbf{H} - \Sigma_1 - \Sigma_2]^{-1}$$

$$\rho_1 = \int 2 \times \frac{dE}{2\pi} \mathbf{A}_1(E) f(E_F - E)$$

$$\mathbf{A}_1(E) = \mathbf{G}\Gamma_1\mathbf{G}^H$$

$$I^{1-2} = \frac{2q}{h} \int T^{1-2}(E) f(E_F - E_k) dE$$

$$T^{1-2}(E) = \text{trace}[\Gamma_1\mathbf{G}\Gamma_2\mathbf{G}^H]$$

$$\Sigma_1 = \begin{bmatrix} -t_0 e^{ik_1 a} & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix} \dots & \dots & \dots \\ \dots & 0 & 0 \\ \dots & 0 & -t_0 e^{ik_N a} \end{bmatrix}$$

$$\Gamma_1 = \Sigma_1 - \Sigma_1^H$$

$$\Gamma_2 = \Sigma_2 - \Sigma_2^H$$

## V) Discussion: **What else?**

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- 1) scattering
- 2) two and three dimensions
- 3) atomistic basis
- 4) boundary conditions
- 5) many body effects
- 6) ...

*Spring College on Science at the Nanoscale - Trieste, Italy, June 2004*

- 1) Introduction
- 2) Solving the Wave Equation
- 3) Computing  $n(x)$
- 4) Computing  $I_D$
- 5) Discussion
- 6) Summary



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- 
- 1) The NEGF provides a convenient formalism for solving the Schrödinger equation with open boundary conditions.
  - 2) NEGF also provides a rigorous basis for going beyond the single electron Schrödinger equation.
  - 3) NEGF is rapidly becoming an essential tool for device engineers.
  - 4) For more information, see

[www.nanohub.org](http://www.nanohub.org)