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ICTP 40th Anniversary

**SMR 1564 - 26**

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**SPRING COLLEGE ON SCIENCE AT THE NANOSCALE**
**(24 May - 11 June 2004)**

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# Theory / Transport - Part I

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*These are preliminary lecture notes, intended only for distribution to participants.*

# Introduction to theory of mesoscopic systems

**Boris Altshuler**

***Princeton University***

***& NEC Laboratories America***

# Electrons in nanostructures

## Clean systems without boundaries:

- Electrons are characterized by their momenta or quasimomenta
  ⇨ electronic wave functions are plane waves

- Physics is essentially local
  Example – conductivity

$$j_\alpha(\vec r) = \sigma_{\alpha\beta}(\vec r) E_\beta(\vec r)$$

- Often interaction between electrons is (apparently) not important

## In mesoscopic systems:

- Due to the scattering of the electrons off disorder (impurities) and/or boundaries the momentum is not a good quantum number

- Response to external perturbation is usually nonlocal

$$j_\alpha(\vec r) = \int \sigma_{\alpha\beta}(\vec r, \vec r') E_\beta(\vec r') d\vec r'$$

- Interaction between electrons is often crucial

# Introduction to theory of mesoscopic systems

**Boris Altshuler**

*Princeton University*

*& NEC Laboratories America*

**Part 1** Without interactions

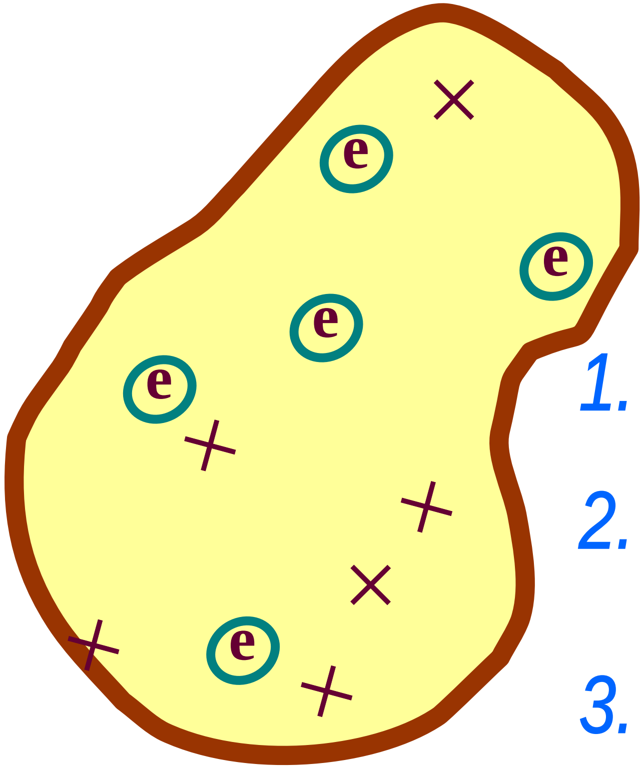
Random Matrices, Anderson Localization, and Quantum Chaos

# Finite size quantum physical systems

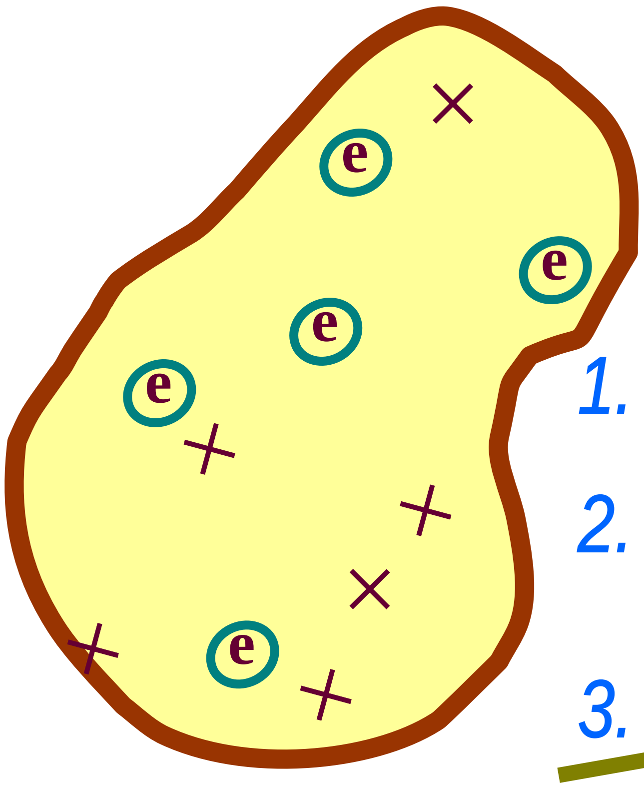
Atoms
Nuclei
Molecules
.
.
.

Quantum
Dots

# Quantum Dot




1. *Disorder (× – impurities)*
2. *Complex geometry*
3. ***e-e** interactions*

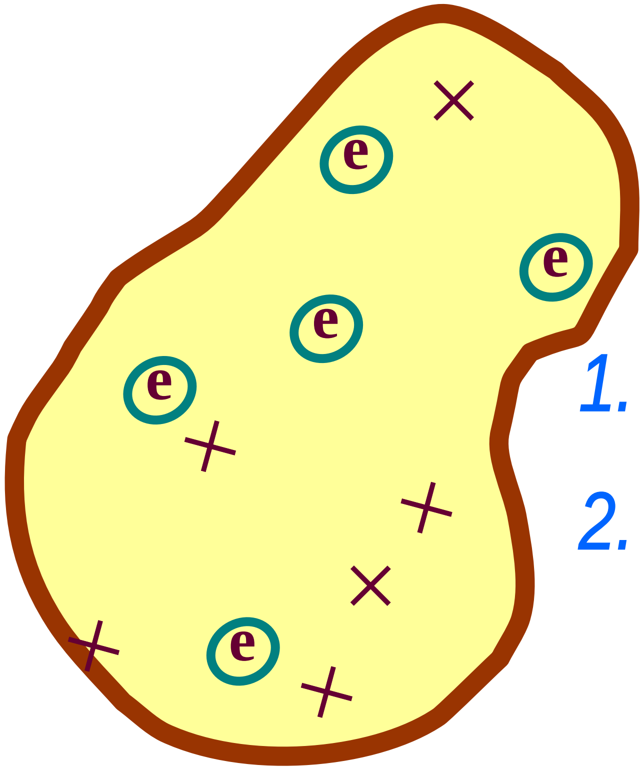



# Quantum Dot

1. *Disorder (× – impurities)*
2. *Complex geometry*
3. ~~*e-e interactions*~~ for a while

## Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-



1. *Disorder (× – impurities)*

2. *Complex geometry*

How to deal with disorder?

- ~~Solve the Shrodinger equation~~ exactly
- Start with plane waves, introduce the mean free path, and . . .

How to take quantum interference into account

?

## Idea:

Instead of thinking in terms of plane waves or solving exactly the Shrodinger equation let us substitute exact one-particle wavefunctions by eigenvectors of a random matrix !?

# RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ ***ensemble of Hermitian matrices with random matrix element*** $N \to \infty$

$E_\alpha$ **- spectrum (set of eigenvalues)**

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ **- mean level spacing**

# RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

$E_\alpha$ - spectrum (set of eigenvalues)

$\delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle$ - mean level spacing

$\langle \ldots\ldots \rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

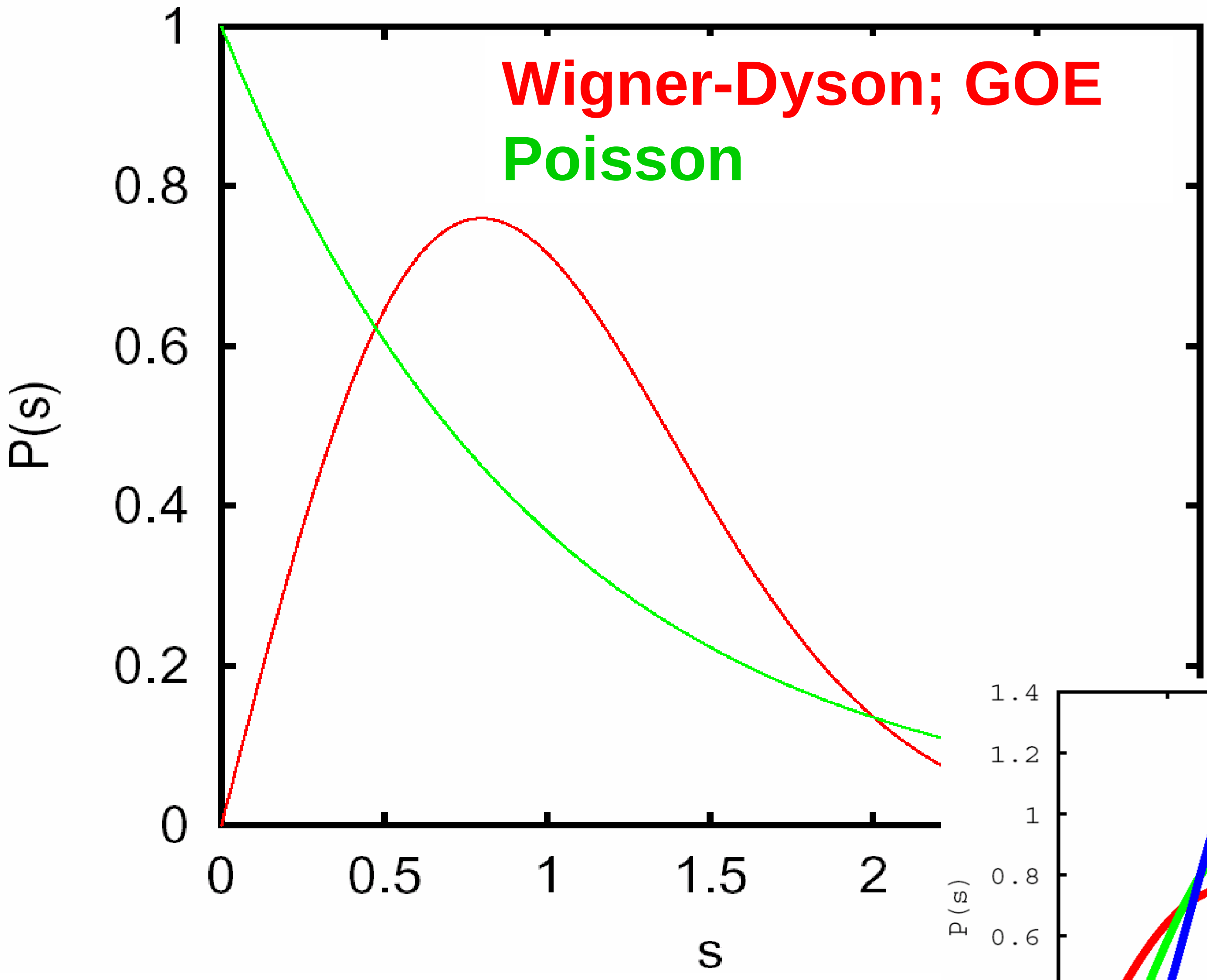
$P(s)$ - distribution function of nearest neighbors spacing between

**Spectral Rigidity**

**Level repulsion**

$$P(s = 0) = 0$$

$$P(s \ll 1) \propto s^\beta \qquad \beta = 1,2,4$$

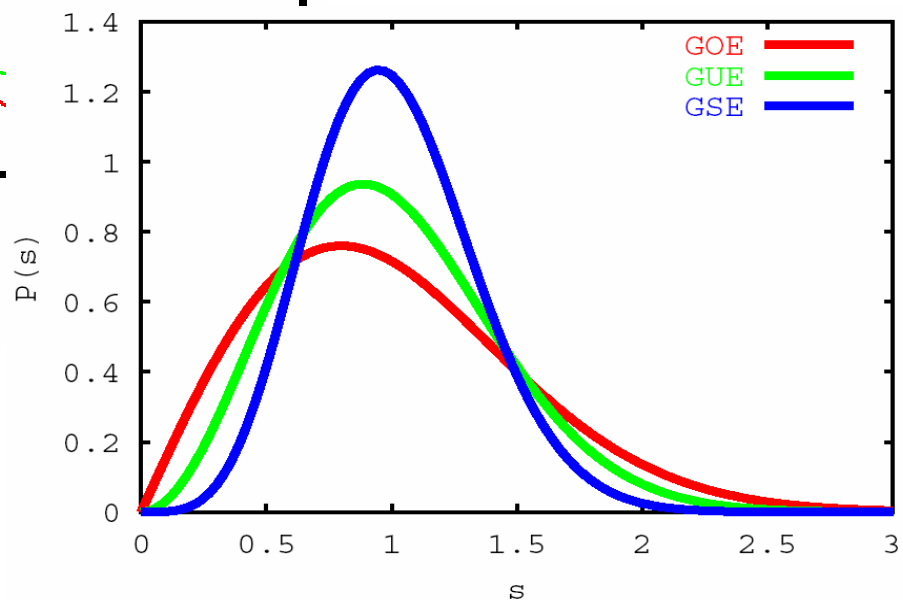



Gaussian
Orthogonal
Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$




Poisson – completely uncorrelated levels

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}$$

small

small

small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If $H_{12}$ is real (orthogonal ensemble), then for $s$ to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and $H_{12}$) should be small and thus $P(s) \propto s$ $\beta = 1$

3. Complex $H_{12}$ (unitary ensemble) $\Rightarrow$ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent $\Rightarrow$ three independent random variables should be small $\Rightarrow$ $P(s) \propto s^2$ $\beta = 2$

# RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

## Dyson Ensembles

| Matrix elements | Ensemble | $\beta$ | realization |
| --- | --- | --- | --- |
| real | orthogonal | 1 | T-inv potential |
| complex | unitary | 2 | broken T-invariance (e.g., by magnetic field) |
| $2 \times 2$ matrices | simplectic | 4 | T-inv, but with spin-orbital coupling |

# Finite size quantum physical systems

Atoms
Nuclei
Molecules
.
.
.

Quantum
Dots

**ATOMS** Main goal is to classify the eigenstates in terms of the quantum numbers

**NUCLEI** For the nuclear excitations this program does not work

***E.P. Wigner:*** Study spectral **statistics** of a **particular** quantum system – a given nucleus

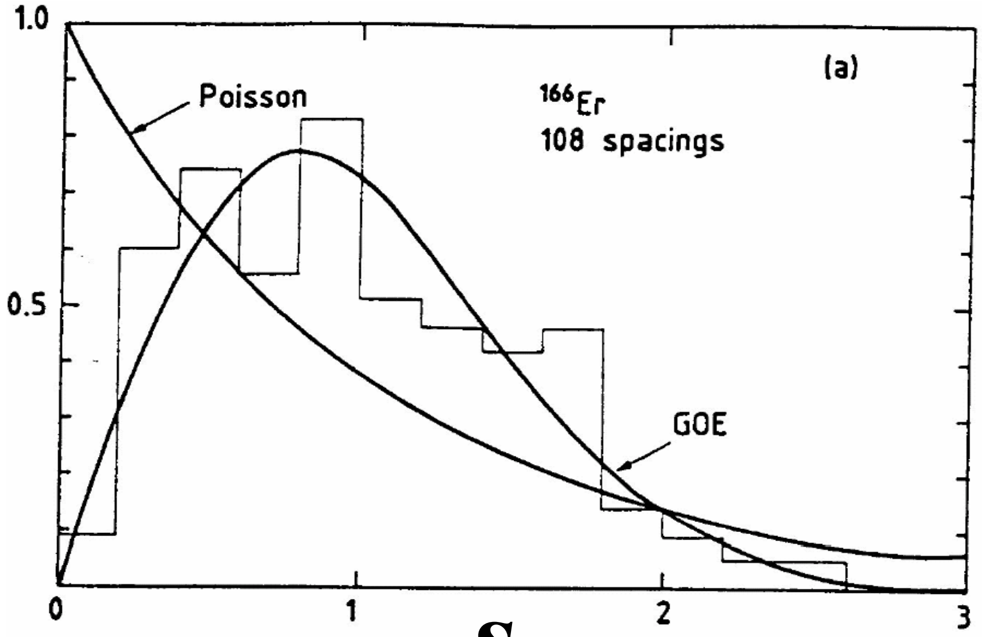
| Random Matrices | Atomic Nuclei |
|---|---|
| • ***Ensemble*** | • ***Particular quantum system*** |
| • ***Ensemble averaging*** | • ***Spectral averaging (over $\alpha$)*** |

**Nevertheless** Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics

Particular nucleus

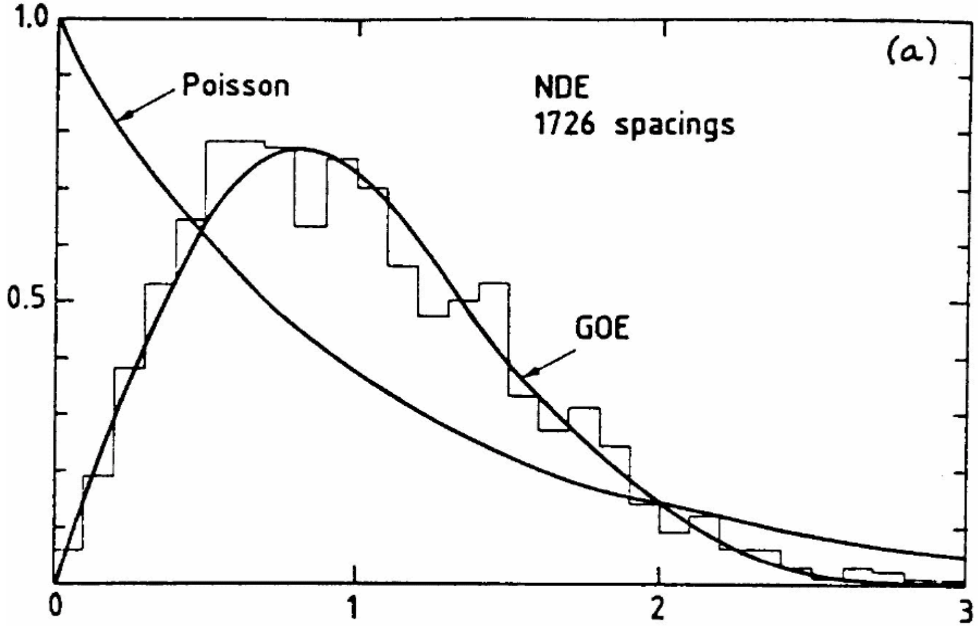
$^{166}Er$

$P(s)$




$s$

$P(s)$




Spectra of several nuclei combined (after rescaling by the mean level spacing)

# Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

**Original answer:**

*These are systems with a large number of degrees of freedom, and therefore the "complexity" is high*

**Later it became clear that**

*there exist very "simple" systems with as many as 2 degrees of freedom* (d=2), *which demonstrate RMT - like spectral statistics*

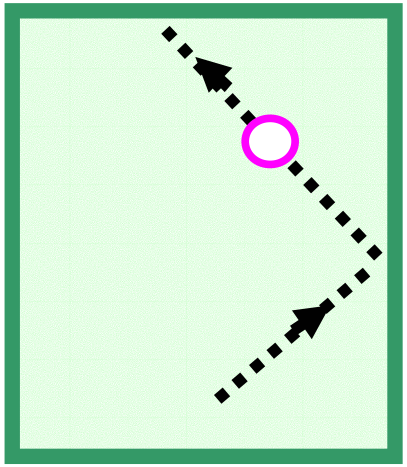
# Classical ($\hbar$ =0) Dynamical Systems with $d$ degrees of freedom

**Integrable Systems**

The variables can be separated and the problem reduces to $d$ one-dimensional problems ⇒ $d$ integrals of motion

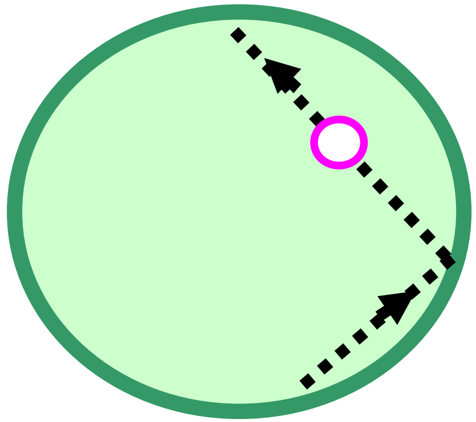
## Examples

### 1. A ball inside rectangular billiard; $d=2$



- Vertical motion can be separated from the horizontal one
- Vertical and horizontal components of the momentum, are both integrals of motion

### 2. Circular billiard; $d=2$



- Radial motion can be separated from the angular one
- Angular momentum and energy are the integrals of motion

# Classical Dynamical Systems with $d$ degrees of freedom

**Integrable Systems**

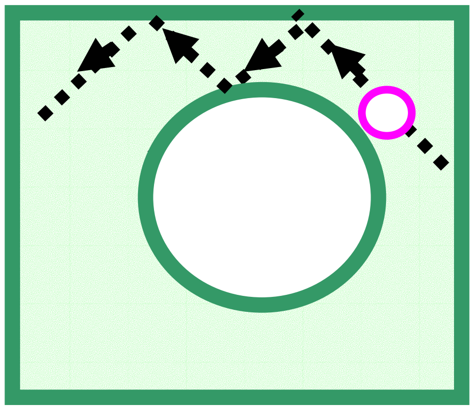
The variables can be separated ⇨ $d$ one-dimensional problems ⇨ $d$ integrals of motion

Rectangular and circular billiard, Kepler problem, . . . , 1d Hubbard model and other exactly solvable models, . .

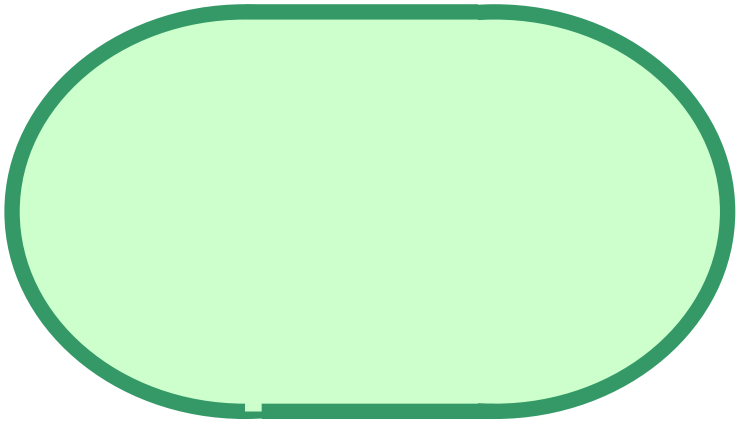
**Chaotic Systems**

The variables can not be separated ⇨ there is only one integral of motion - energy

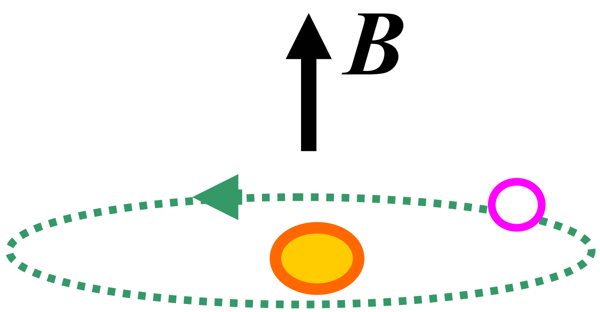
**Examples**



Sinai billiard



Stadium



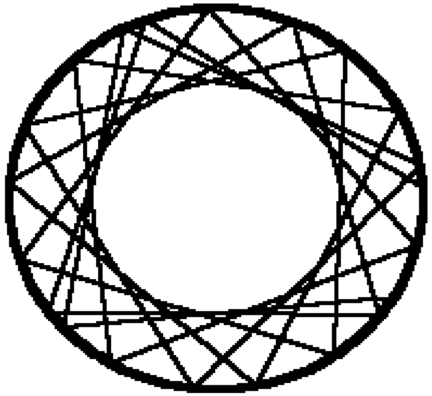

Kepler problem in magnetic field

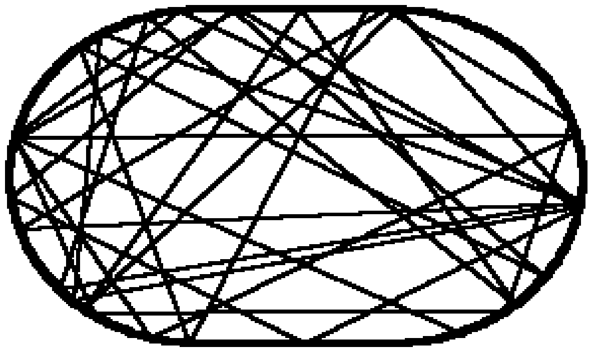
Classical Chaos
$\hbar = 0$

•*Nonlinearities*

•*Exponential dependence on the original conditions (Lyapunov exponents)*

•*Ergodicity*





**Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation**

Q: What does it mean Quantum Chaos ?

# $\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

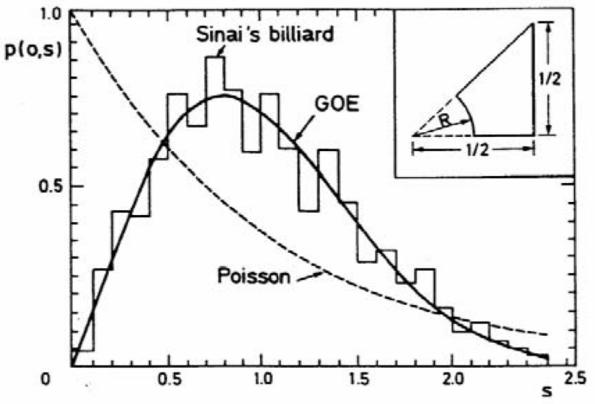
VOLUME 52 2 JANUARY 1984 NUMBER 1

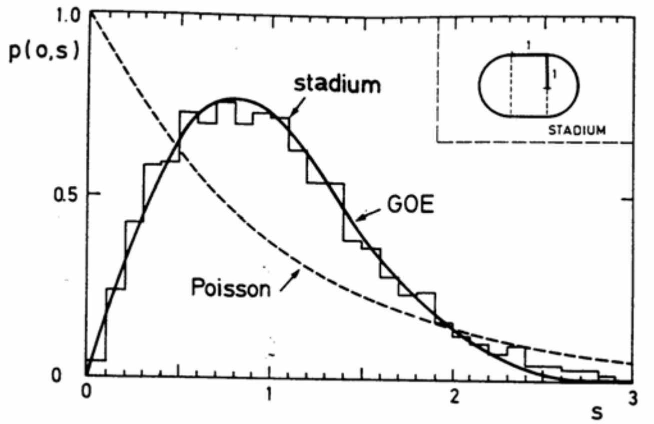
**Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws**

O. Bohigas, M. J. Giannoni, and C. Schmit
*Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France*
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal-invariant systems whose classical analogs are $K$ systems show the same fluctuation properties as predicted by GOE




Chaotic classical analog

Wigner- Dyson spectral statistics

No quantum numbers except energy

# Q: What does it mean Quantum Chaos ?

## *Two possible definitions*

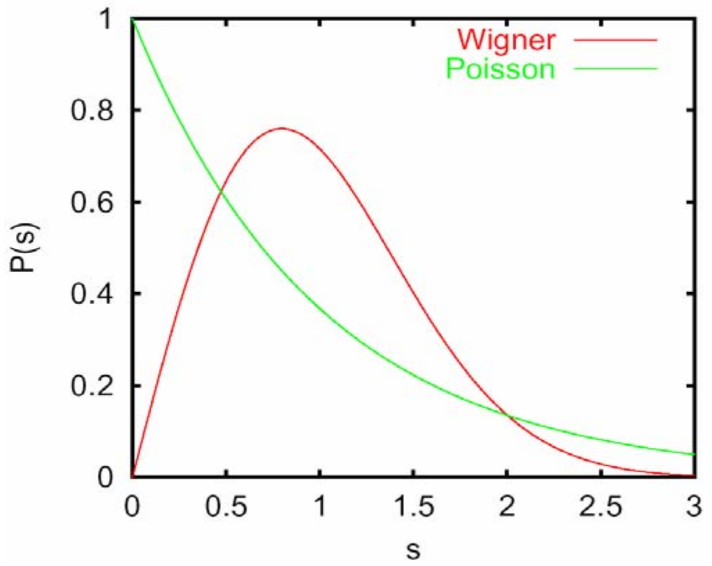
Chaotic classical analog

Wigner - Dyson-like spectrum

Classical Quantum

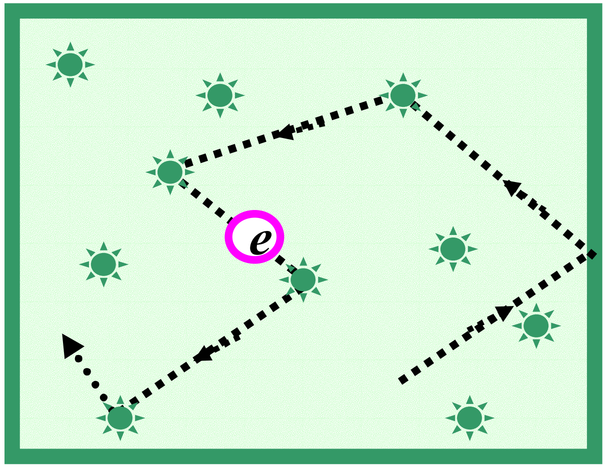
**Integrable** ⟺ **Poisson**

**Chaotic** ⟺ **Wigner-Dyson**



# Poisson to Wigner-Dyson crossover

**Important example: quantum particle subject to a random potential – disordered conductor**




*Scattering centers, e.g., impurities*

- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.
- The problem is much richer than RM theory
- There is still a lot of universality.

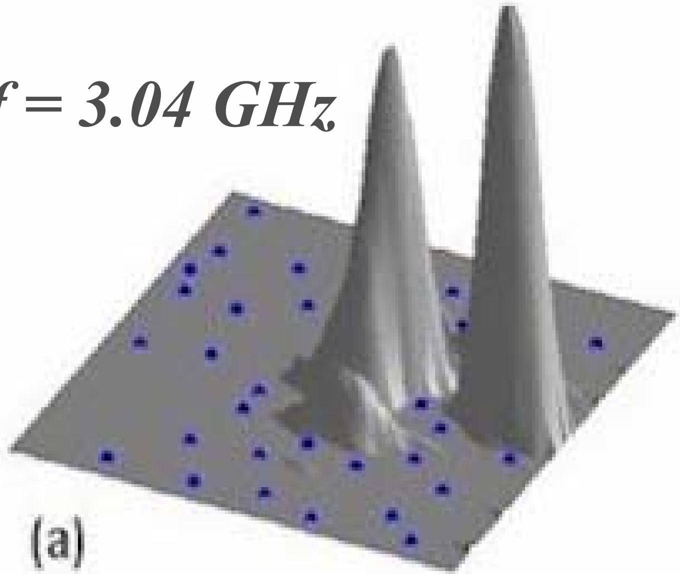
## Anderson localization (1958)

At strong enough disorder all eigenstates are localized in space

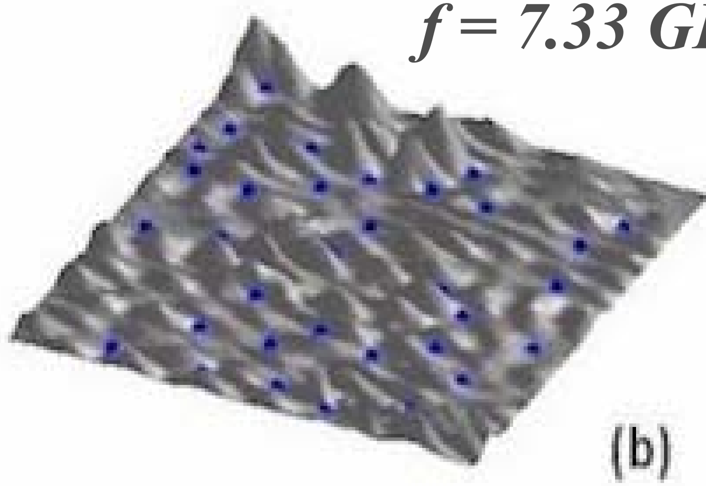
# Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
*Department of Physics, Northeastern University, Boston, Massachusetts 02115*
(Received 28 February 2000)

*f = 3.04 GHz*




*f = 7.33 GHz*




***Anderson Insulator***

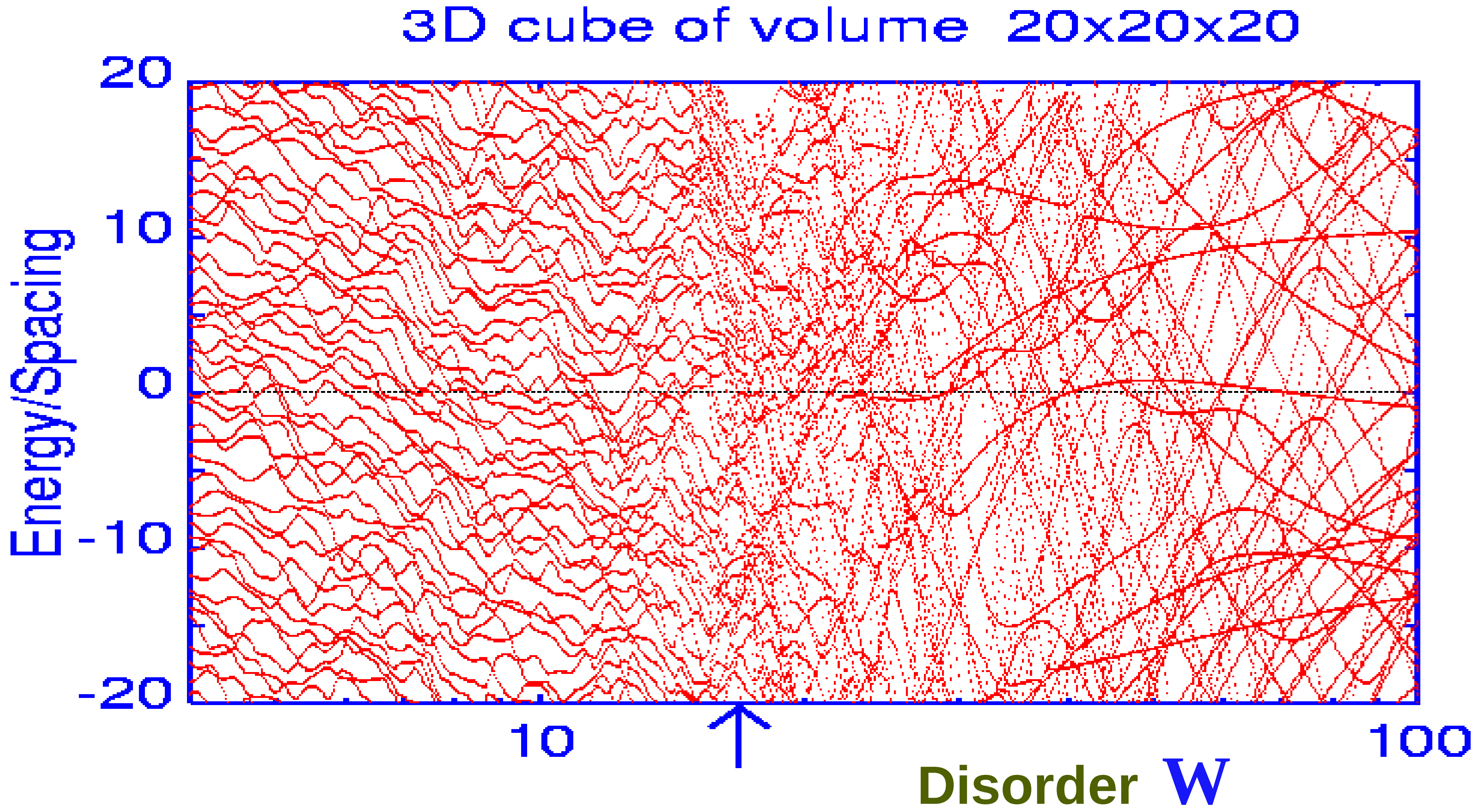
***Anderson Metal***

# Anderson Transition

| $I < I_c$ | $I > I_c$ |
|---|---|
| **Insulator** | **Metal** |
| *All eigenstates are localized* | *There appear states extended all over the whole system* |
| *Localization length* $\xi$ | |
| *The eigenstates, which are localized at different places will not repel each other* | *Any two extended eigenstates repel each other* |
| ⇩ | ⇩ |
| *Poisson spectral statistics* | *Wigner – Dyson spectral statistics* |

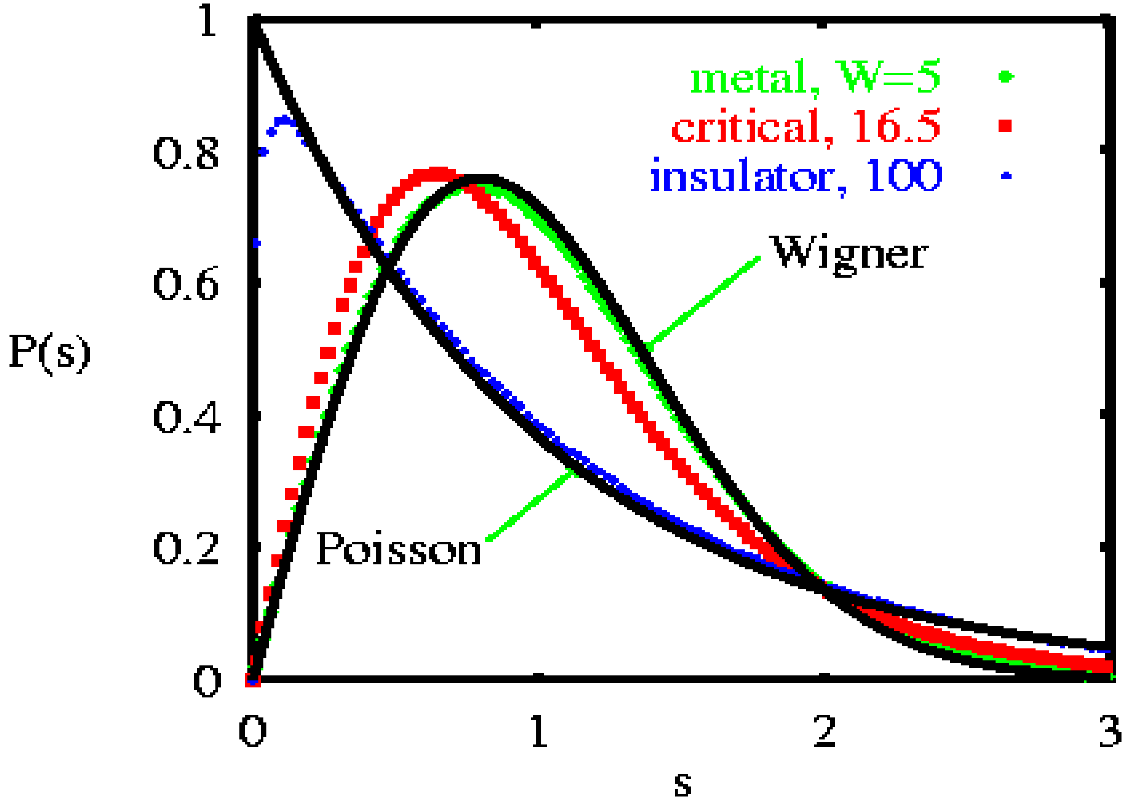
**Zharekeschev & Kramer.**

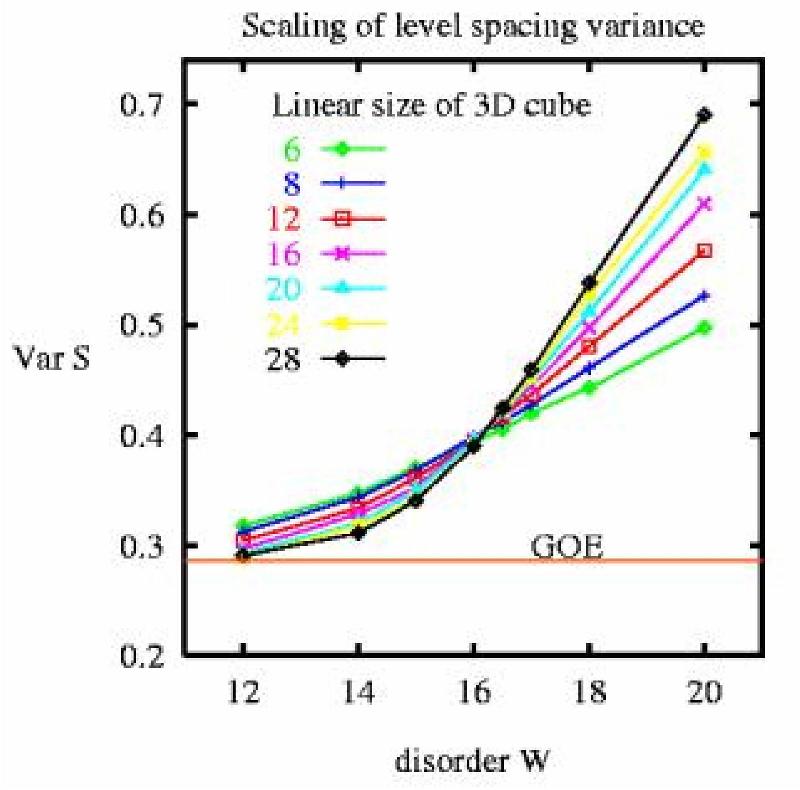
## *Exact diagonalization of the Anderson model*



# Anderson transition in terms of pure level statistics

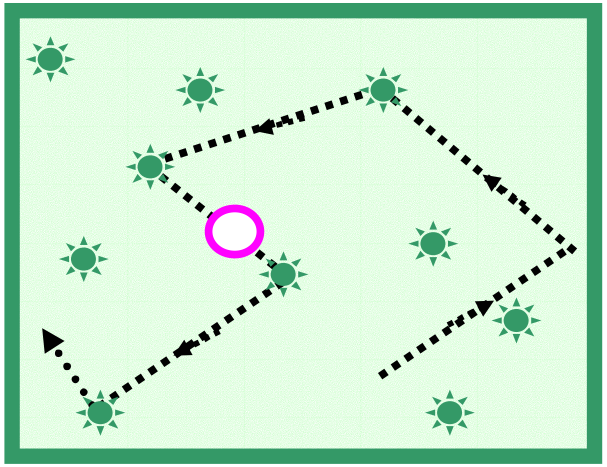
***P(s)***



# Classical particle in a random potential

# Diffusion



1 particle - random walk

Density of the particles $\rho$

Density fluctuations $\rho(\mathbf{r},t)$ at a given point in space $\mathbf{r}$ and time $t$.

$$\frac{\partial\rho}{\partial t} - D\nabla^2\rho = 0$$

Diffusion Equation

$D$ - Diffusion constant

$$D = \frac{l\tau}{d}$$

$l$ mean free path

$\tau$ mean free time

$d$ # of dimensions

## Conductivity

$$\sigma = \frac{e^2\tau}{m}\rho$$

## Density of States

$$\nu = \frac{1}{\delta_1 * \mathrm{Volume}}$$

## Einstein Relation

$$\sigma = e^2 \nu D$$

Conductivity

Density of States

$\}$ local quantities

$$j = \sigma E$$

## Conductance

$$I = GV$$

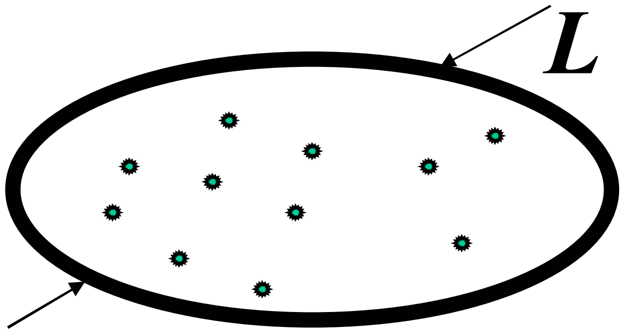
$$G = \sigma L^{d-2}$$

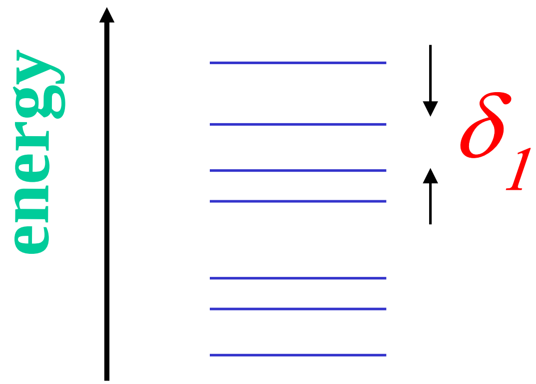
for a cubic sample of the size $L$

# Energy scales (*Thouless, 1972*)

## 1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$




$L$ *is the system size;*

$d$ *is the number of dimensions*

## 2. Thouless energy

$$E_T = hD/L^2$$

$D$ *is the diffusion const*

$E_T$ *has a meaning of the* ***inverse diffusion time*** *of the traveling through the system or the* ***escape rate*** *(for open systems)*

$$g = E_T / \delta_1$$

dimensionless
Thouless
conductance

$$g = Gh/e^2$$

# Thouless Conductance and One-particle Spectral Statistics


***Localized states***
***Insulator***

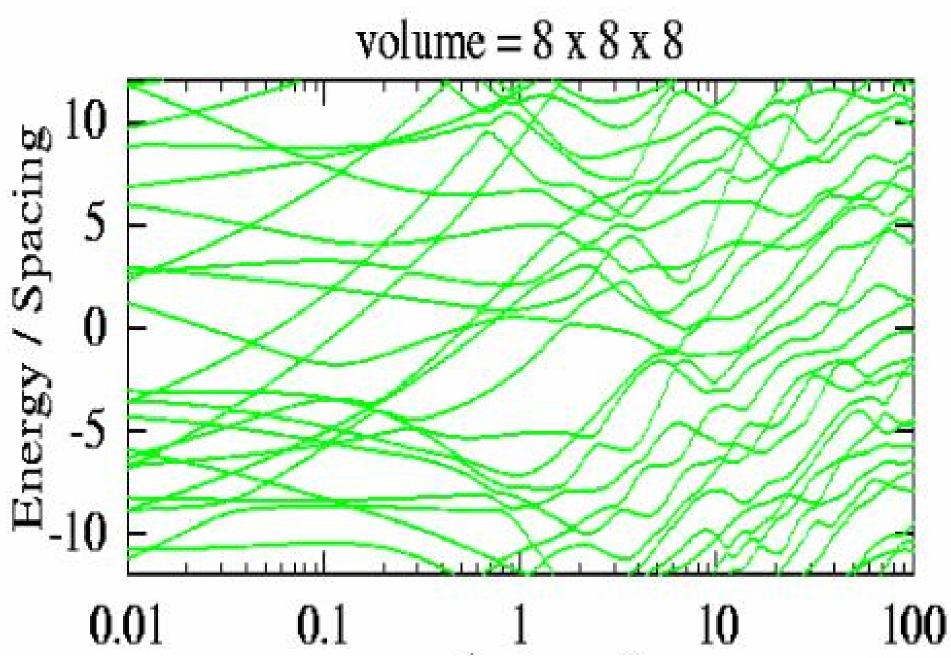
**Poisson spectral statistics**

***Extended states***
***Metal***

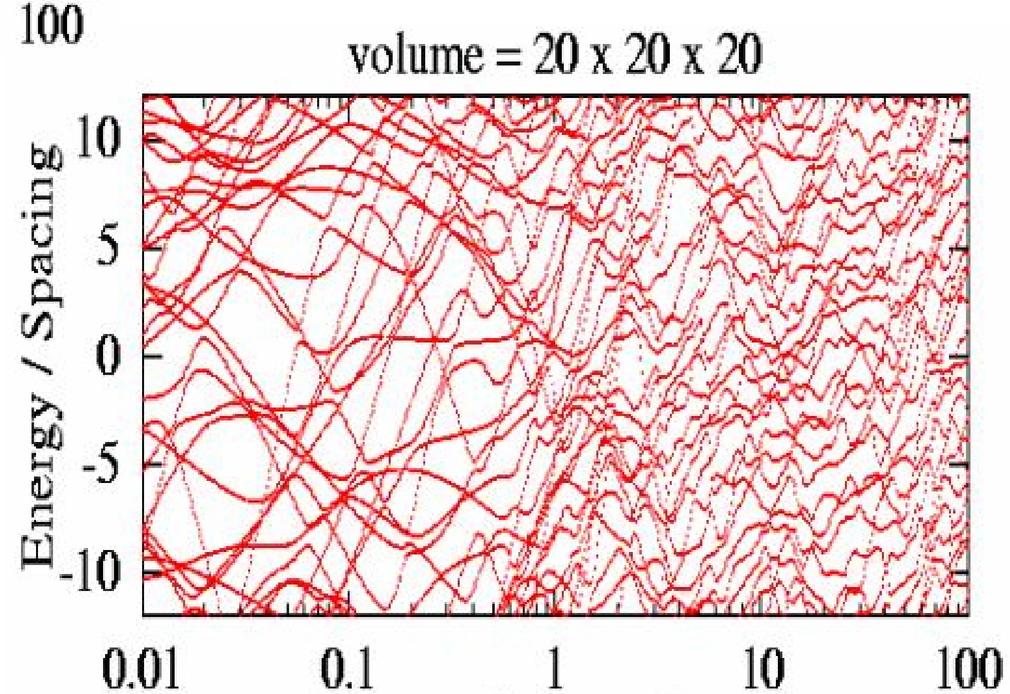
**Wigner-Dyson spectral statistics**

Transition at $g$~1.

Is it sharp?

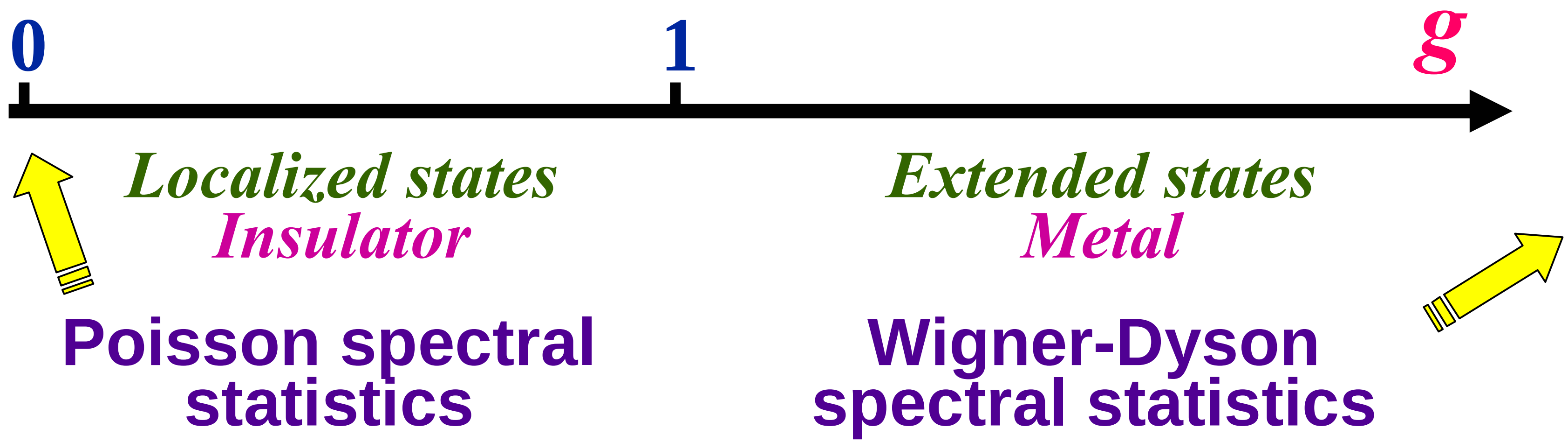



Conductance *g*




Conductance *g*

# Thouless Conductance and One-particle Spectral Statistics




Q: How the Thouless conductance $g$ depends on the size of the system ?

What happens with $g$ when $L \to$ infinity

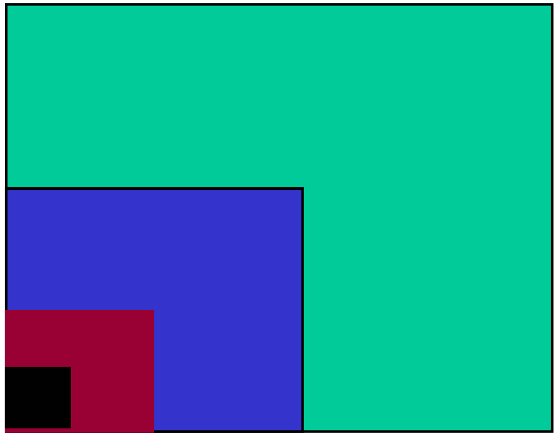
# Scaling theory of Localization

(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

$g = E_T / \delta_1$

Dimensionless Thouless conductance

$g = Gh/e^2$



$L = 2L = 4L = 8L$ ....

without quantum corrections

$E_T \propto L^{-2} \qquad \delta_1 \propto L^{-d}$

$\frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1} \longrightarrow \frac{E_T}{\delta_1}$

$\mathbf{g} \longrightarrow \mathbf{g} \longrightarrow \mathbf{g} \longrightarrow \mathbf{g}$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

$\beta -$ function is

Universal, i.e., material independent

**But**

It depends on the global symmetries, e.g., it is different with and without $T$-invariance (in orthogonal and unitary ensembles)

# Limits:

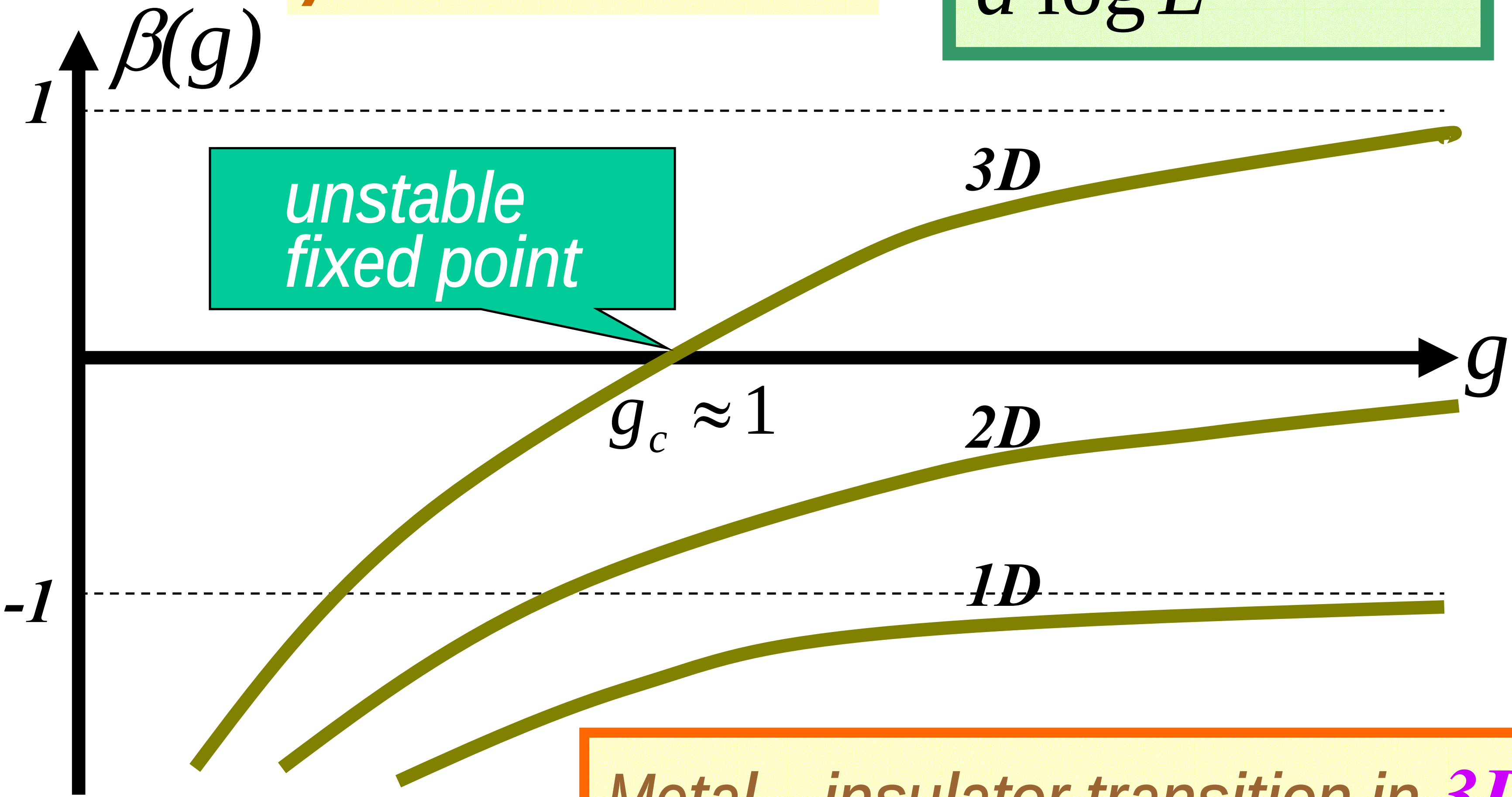
$$g >> 1 \quad g \propto L^{d-2} \qquad \beta(g) = (d-2) + O\left(\frac{1}{g}\right)$$

$$\boxed{\begin{array}{ll} < 0 & d > 2 \\ ?? & d = 2 \\ > 0 & d < 2 \end{array}}$$

$$g << 1 \quad g \propto e^{-L/\xi} \qquad \beta(g) \approx \log g < 0$$

# $\beta$ - function

$$\frac{d\log g}{d\log L} = \beta(g)$$




*Metal – insulator transition in* ***3D***

*All states are localized for* ***d=1,2***

Questions:

Why

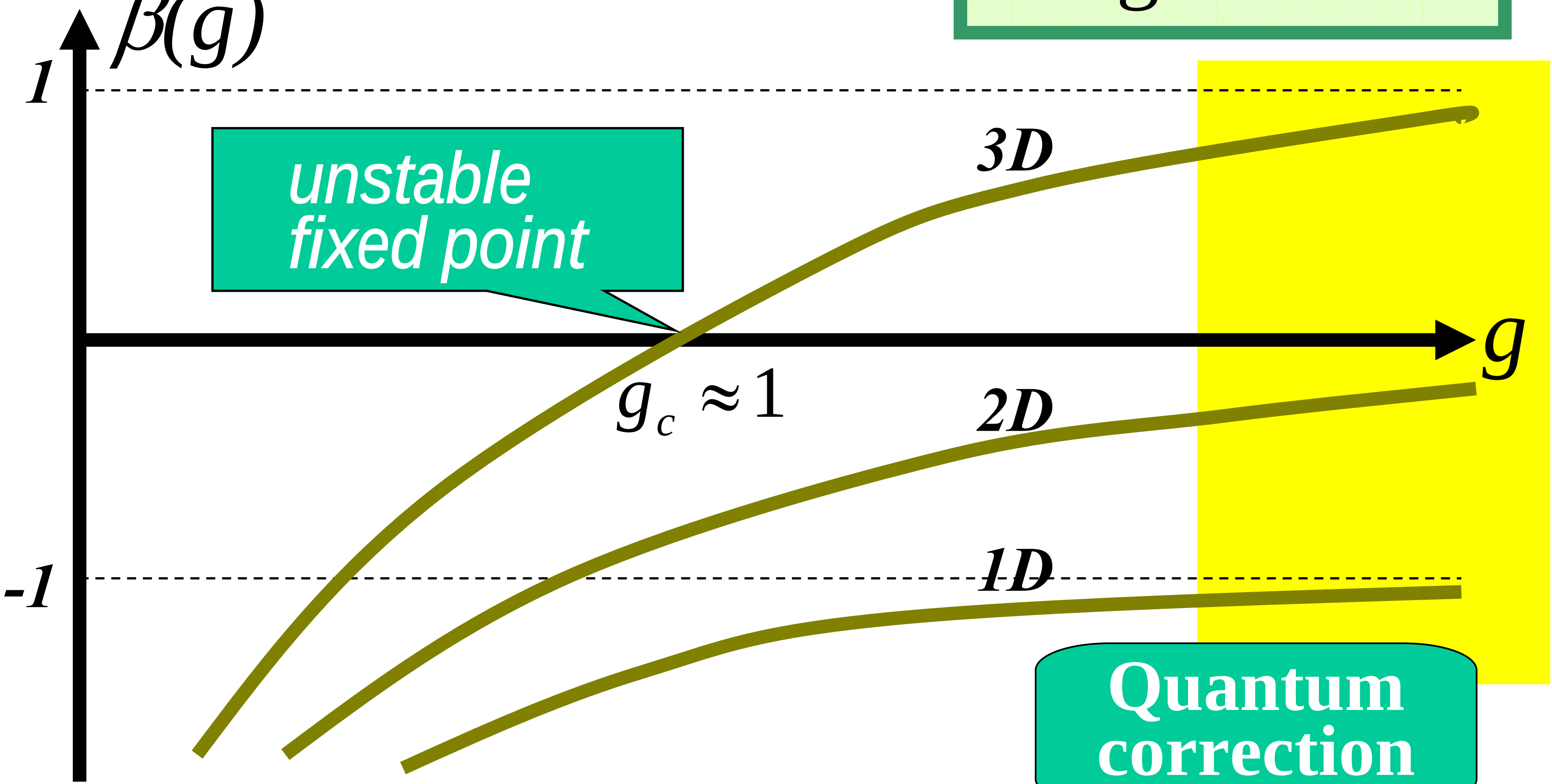
- the scaling theory is correct?
- the correction of the conductance is negative?

Quantum corrections at large Thouless conductance – **weak localization**

Universal description

# $\beta$ - function

$$\frac{d\log g}{d\log L} = \beta(g)$$



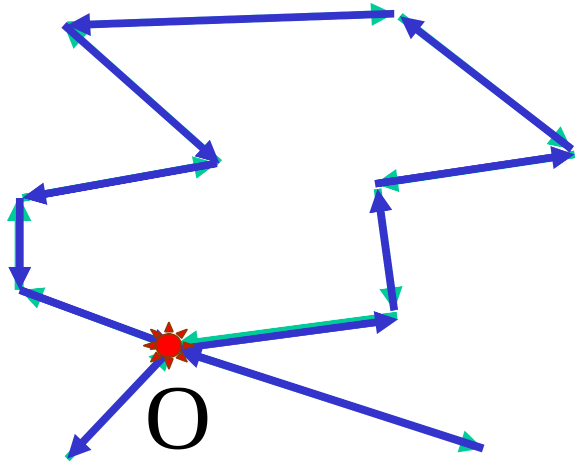

$$g(L) = \sigma_{cl} L^{d-2} + \frac{const}{g}$$

$const = ? \quad \pm ?$

# *WEAK LOCALIZATION*

$$\varphi = \oint \vec{p}\,d\vec{r}$$

Phase accumulated when traveling along the loop



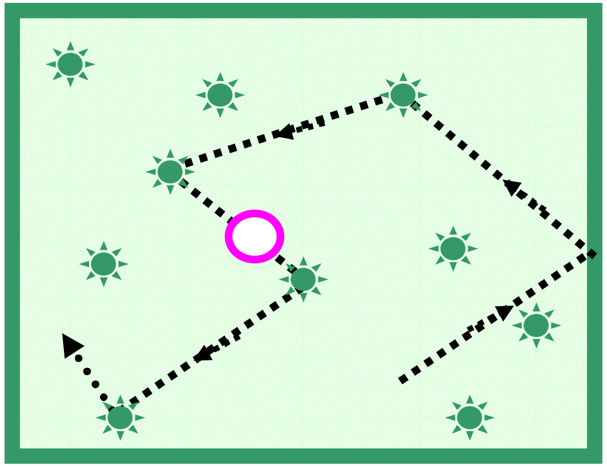

The particle can go around the loop in two directions

$$\varphi_1 = \varphi_2$$

***Constructive interference*** ⟹ ***probability to return to the origin gets enhanced*** ⟹ ***diffusion constant gets reduced. Tendency towards localization***

$\beta$ ***- function is negative for d=2***

# Diffusion



Random walk

Density fluctuations $\rho(\boldsymbol{r},t)$ at a given point in space $\boldsymbol{r}$ and time $t$.

$$\frac{\partial\rho}{\partial t} - D\nabla^2\rho = 0$$

Diffusion Equation

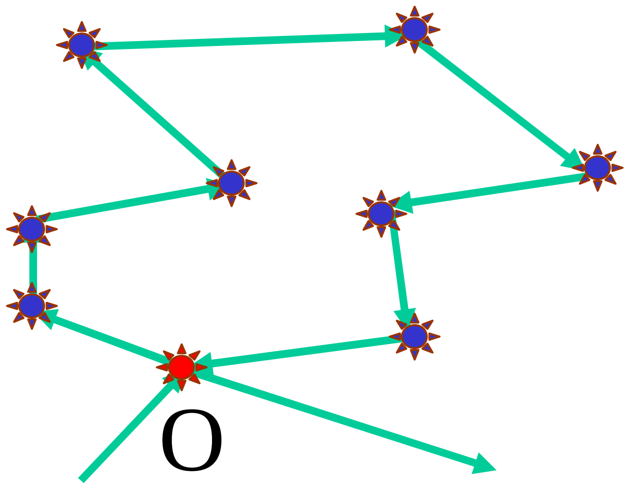
$D$ - Diffusion constant

Mean squared distance from the original point at time $t$

$$\langle r(t)^2\rangle = Dt$$

Probability to come back (to the element of the volume $dV$ centered at the original point)

$$P\left(r\left(t\right)=0\right)dV = \frac{dV}{\left(Dt\right)^{d/2}}$$




What is the probability $P(t)$ that such a loop is formed within a time $t$ ?

**Probability to come back (to the element of the volume $dV$ around the original point)**

$$P\big(r(t)=0\big)\,dV = \frac{dV}{(Dt)^{d/2}}$$

**Q:** $dV = ?$

**A:** $dV = \bar{\lambda}^{\,d-1}\,dt$

$$P(t) = -\bar{\lambda}^{\,d-1}\int_{\tau}^{t}\frac{v_F\,dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$P(t) = -\bar{\lambda}^{d-1} \int_{\tau}^{t} \frac{v_F \, dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

**Q:** $t_{\max} = ?$

**A:** $t_{\max} \sim \min\left\{ \frac{L^2}{D}, \frac{1}{\omega}, \tau_{\varphi}, \ldots \right\}$

$$P(t) = -\bar{\lambda}^{d-1}\int_{\tau}^{t}\frac{v_F\,dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$t_{\max} \sim \frac{L^2}{D} = \frac{h}{E_T}$$

$$d = 2$$

$$\frac{\delta g}{g} \approx -\frac{\bar{\lambda} v_F}{D}\log\frac{L^2}{D\tau}$$

$$P(t) = \lambda\!\!\!^{-} {}^{d-1} \int_{\tau}^{t} \frac{v_F \, dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\max})$$

$$\frac{\delta g}{g} \approx -\frac{\lambda\!\!\!^{-} v_F}{D} \log \frac{L^2}{D\tau} = -\frac{2\lambda\!\!\!^{-} v_F}{D} \log \frac{L}{l}$$

$$\left.\begin{array}{l} \lambda\!\!\!^{-} v_F = \dfrac{1}{\pi \nu} \\ \\ g = \nu D \hbar \end{array}\right\} \Rightarrow$$

$$\delta g = -\frac{2}{\pi} \log \frac{L}{l}$$

$$\beta(g) = -\frac{2}{\pi g}$$

Universal !!!

**Q:** What does it mean $d=2$ **?**

**A:** Transverse dimension is much less than

$$\sqrt{Dt_{\max}}$$