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ICTP 40th Anniversary

SMR 1564 - 27

SPRING COLLEGE ON SCIENCE AT THE NANOSCALE
(24 May - 11 June 2004)

Theory / Transport - Part II

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These are preliminary lecture notes, intended only for distribution to participants.

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Introduction to theory of mesoscopic systems

Boris Altshuler
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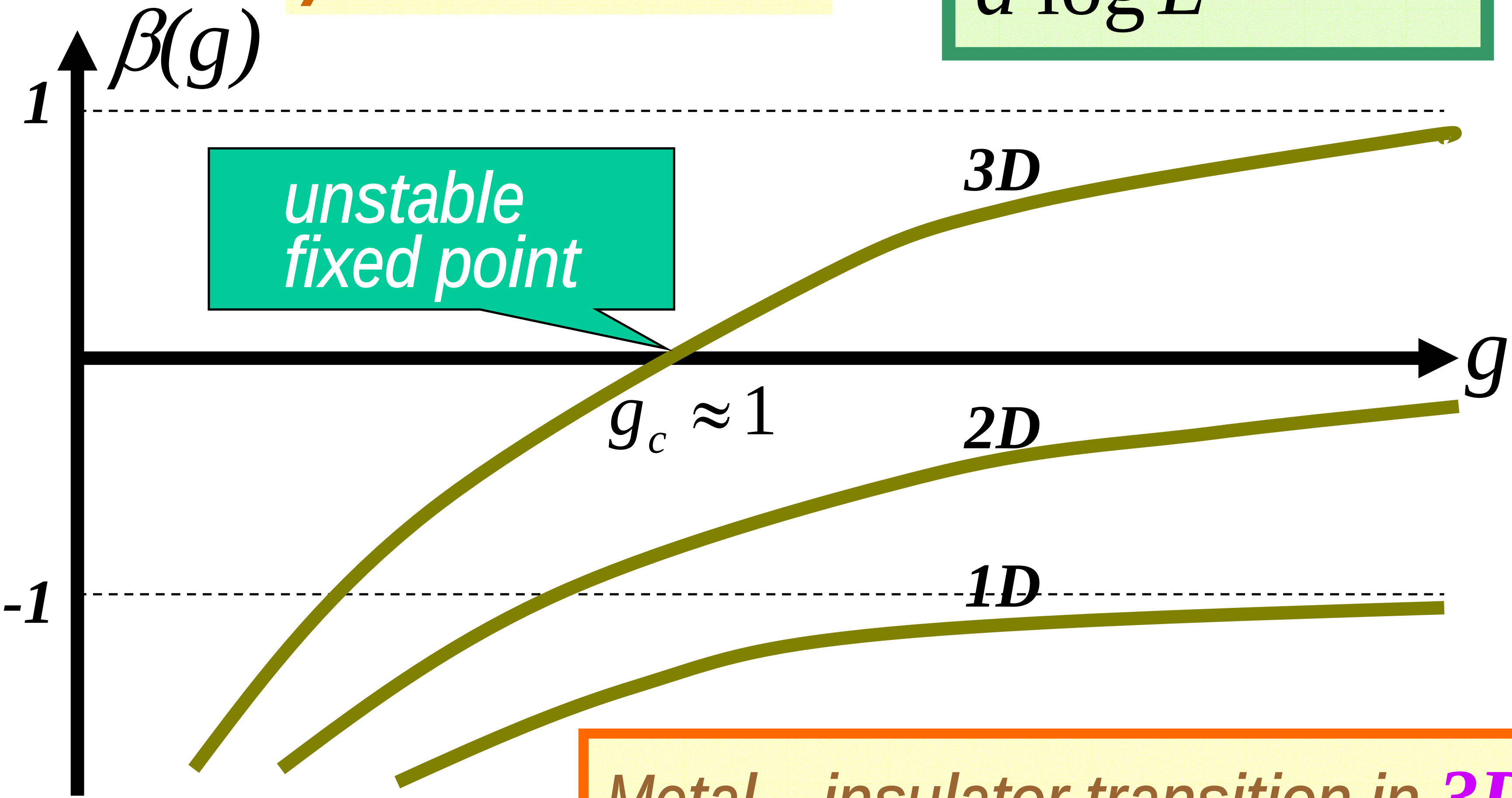
Part 1 Without interactions continued

Weak Localization,
Mesoscopic Fluctuations

β - function

$$\frac{d \log g}{d \log L} = \beta(g)$$

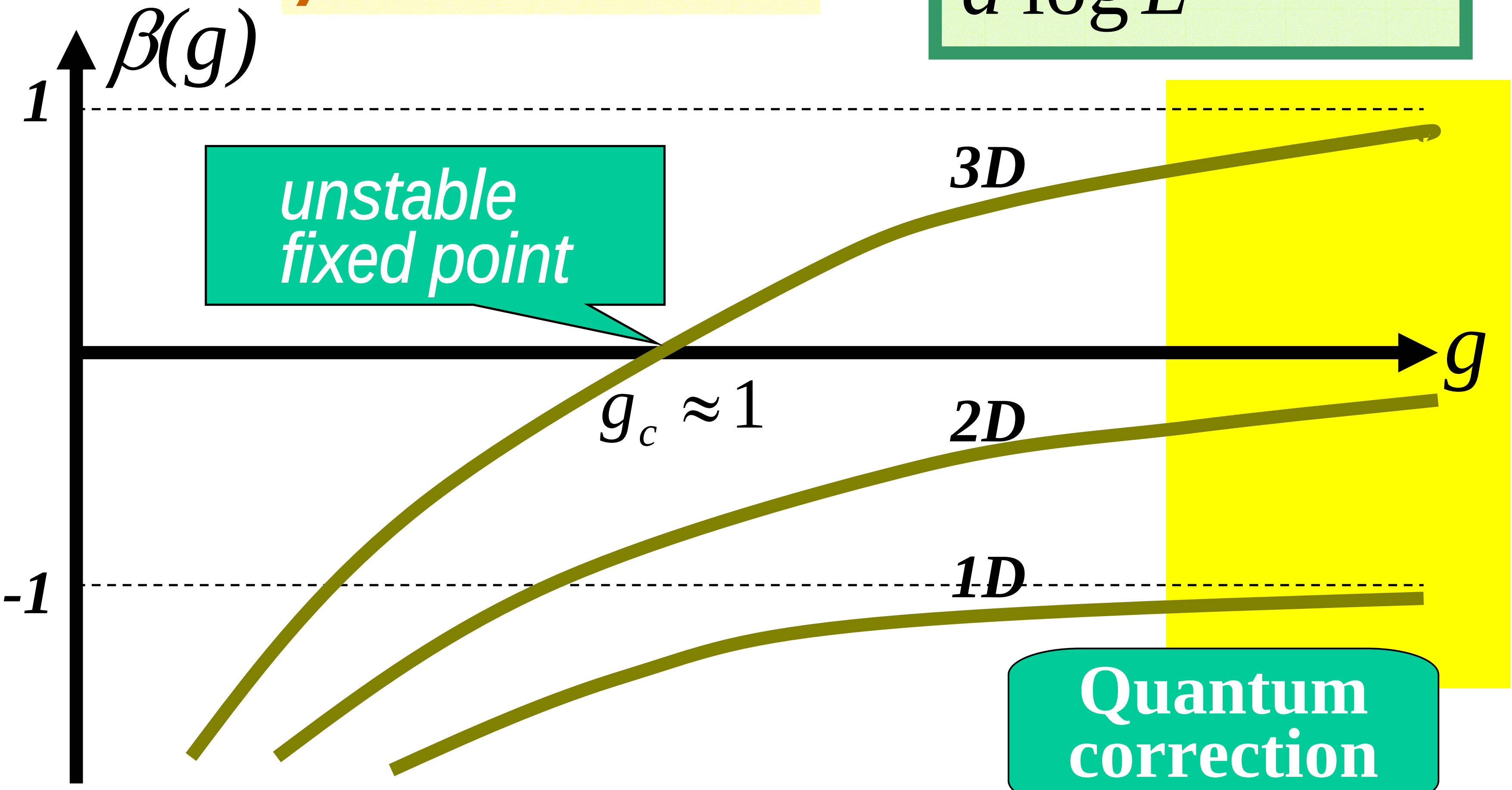


Metal – insulator transition in **3D**

All states are localized for **d=1,2**

β - function

$$\frac{d\log g}{d\log L} = \beta(g)$$

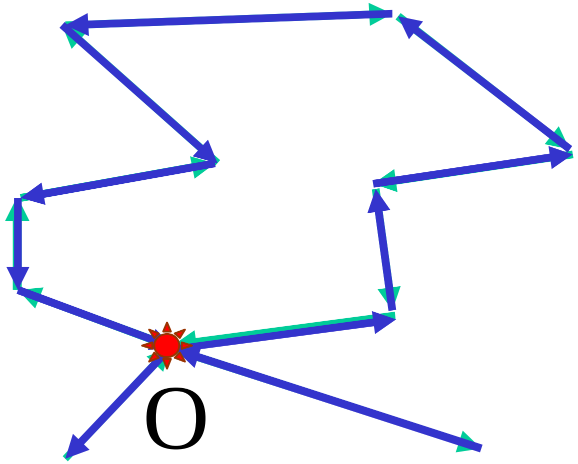


$$g(L) = \sigma_{cl} L^{d-2} + \frac{const}{g} \qquad const = ? \quad \pm ?$$

WEAK LOCALIZATION

$$\varphi = \oint \vec{p}\, d\vec{r}$$

Phase accumulated when traveling along the loop



The particle can go around the loop in two directions

$$\varphi_1 = \varphi_2$$

Constructive interference $\Rightarrow$ ***probability to return to the origin gets enhanced*** $\Rightarrow$ ***diffusion constant gets reduced. Tendency towards localization***

β ***- function is negative for d=2***

$$\frac{\delta g}{g} \approx -\frac{\lambda v_F}{D}\log\frac{L^2}{D\tau} = -\frac{2\lambda v_F}{D}\log\frac{L}{l}$$

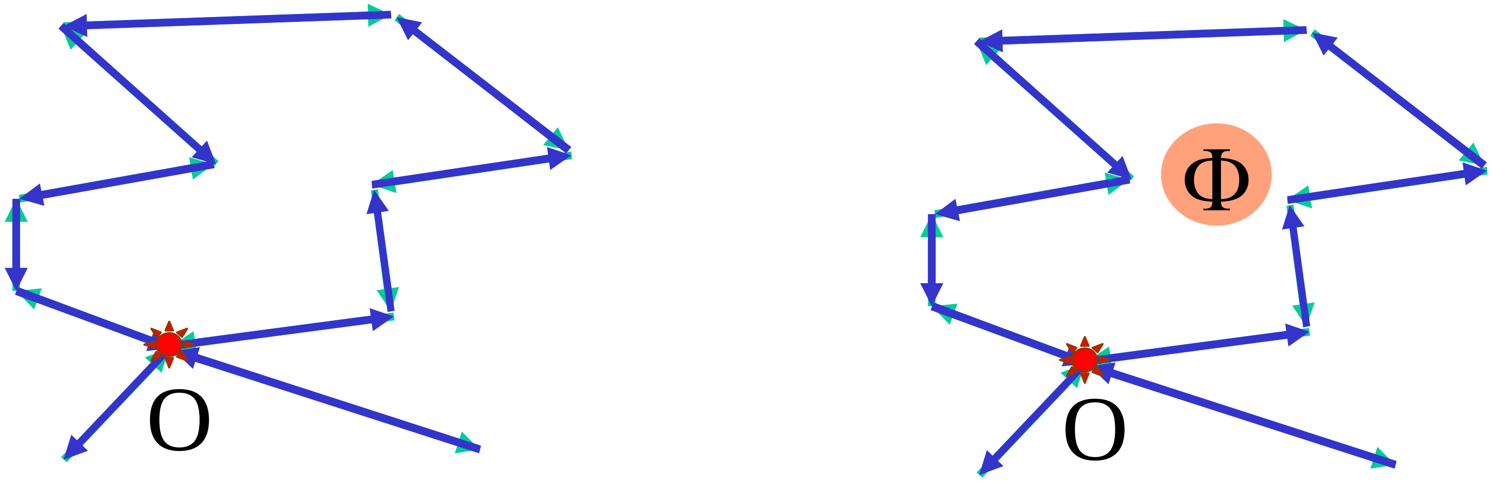
$$\left.\begin{aligned} \lambda v_F &= \frac{1}{\pi\nu} \\ g &= \nu D\hbar \end{aligned}\right\} \Rightarrow$$

$$\delta g = -\frac{2}{\pi}\log\frac{L}{l}$$

$$\beta(g) = -\frac{2}{\pi g}$$

Universal !!!

Magnetoresistance



No magnetic field

$$\varphi_1 = \varphi_2$$

With magnetic field H

$$\varphi_1 - \varphi_2 = 2 * 2\pi\, \Phi / \Phi_0$$

$\Phi = HS$ - ***magnetic flux through the loop***

$\Phi_0 = hc/e$ - ***flux quantum***

Length Scales

Magnetic length $L_H = (hc/eH)^{1/2}$

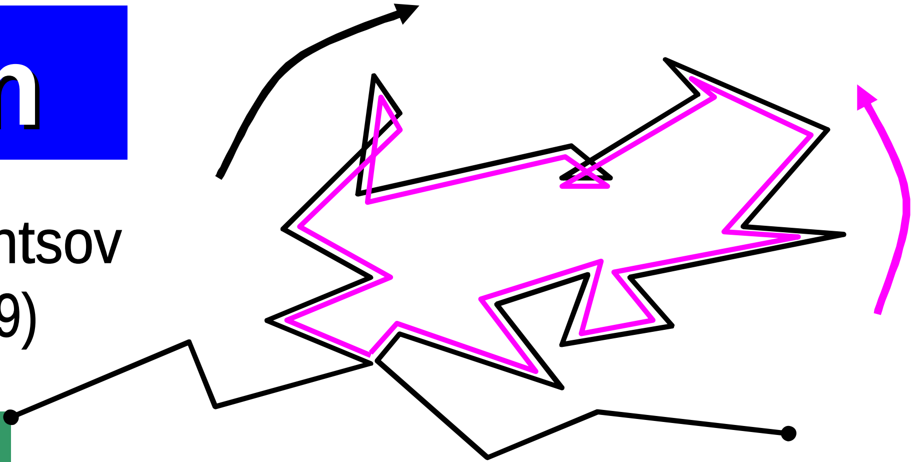
Dephasing length $L_\varphi = (D\,\tau_\varphi)^{1/2}$

$$\delta g(H) = f_d\left(\frac{L_H}{L_\varphi}\right)$$

Universal functions

Magnetoresistance measurements allow to study inelastic collisions of electrons with phonons and other electrons

Weak Localization



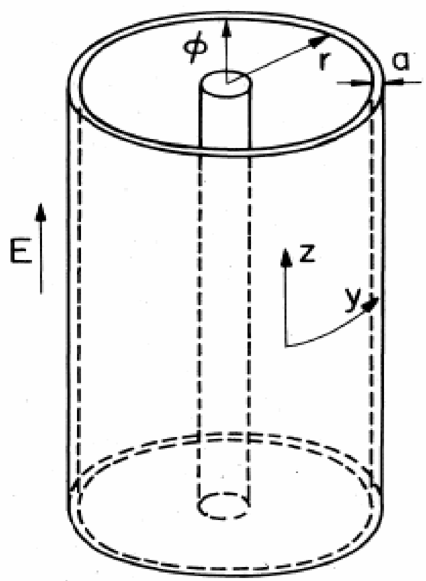
Negative Magnetoresistance

Chentsov (1949)

Aharonov-Bohm effect

Theory

B.A., Aronov & Spivak (1981)



Experiment

Sharvin & Sharvin (1981)

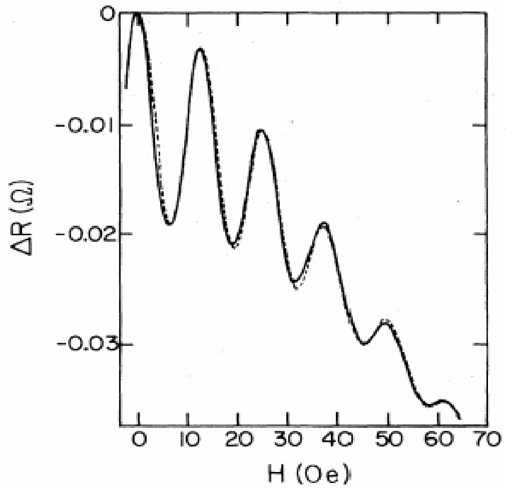
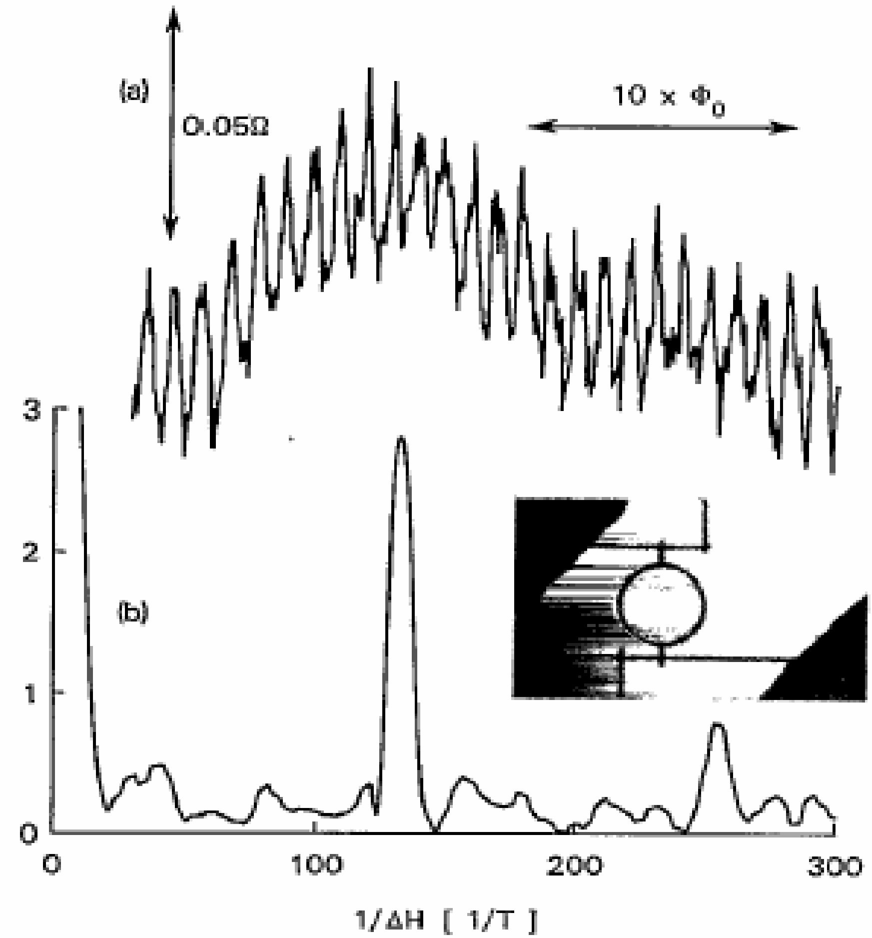
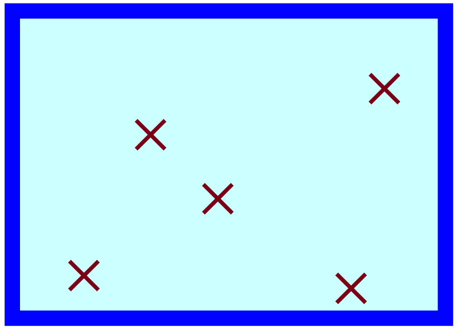


FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{4.2}=2$ kΩ, $R_{300}/R_{4.2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_\varphi=2.2$ μm, $\tau_\varphi/\tau_{so}=0$, filament diameter $d=1.31$ μm, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ μm (Altshuler *et al.*, 1982; Sharvin, 1984).

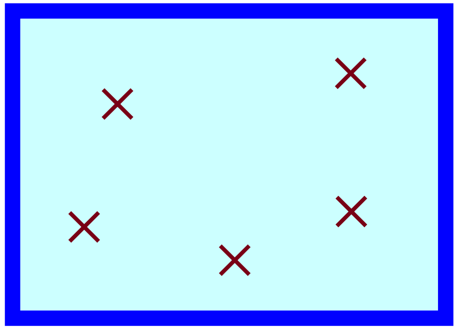
R.A. Webb et al
(1984)



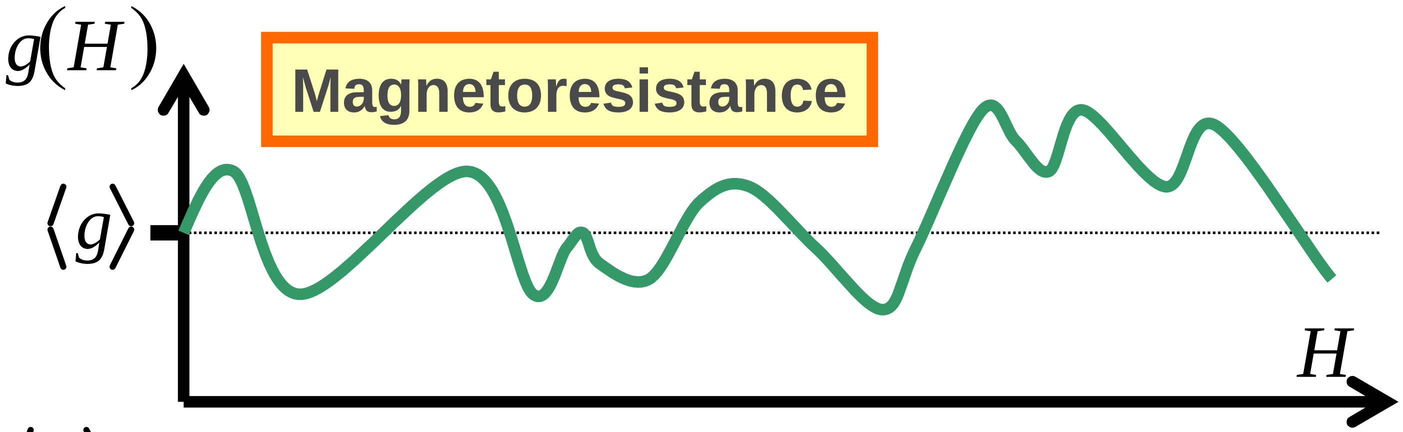
Mesoscopic Fluctuations.



$$g_1 \neq g_2$$



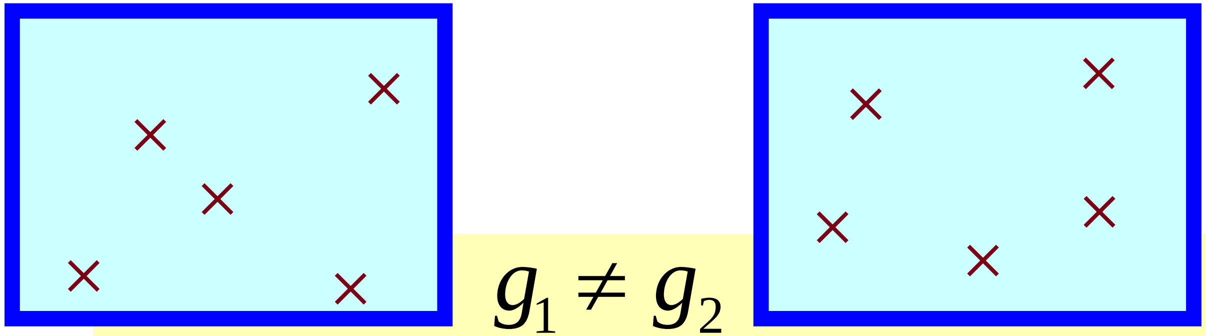
Properties of systems with identical set of macroscopic parameters but different realizations of disorder are different!



$g(H)$ is sample -dependent

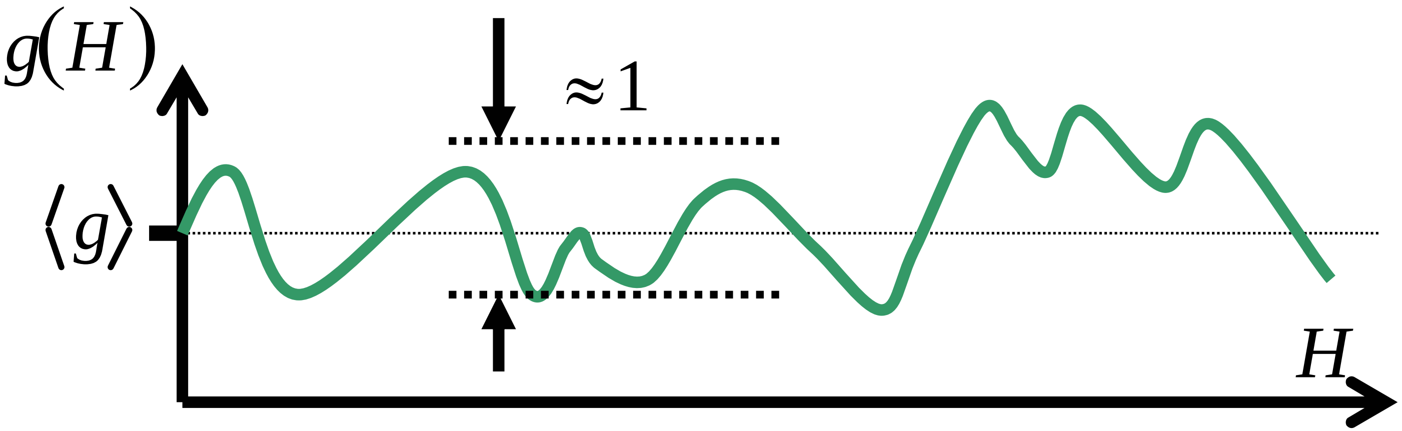
$\langle \ldots \rangle$ - ensemble averaging

$\langle g \rangle >> 1$



$g_1 - g_2 \cong 1$ $\qquad$ $G_1 - G_2 \cong e^2/h$

Magnetoresistance



Statistics of random function(s) $g(H)$ are universal !!!

Statistics of random function(s) $g(H)$ are universal !!!

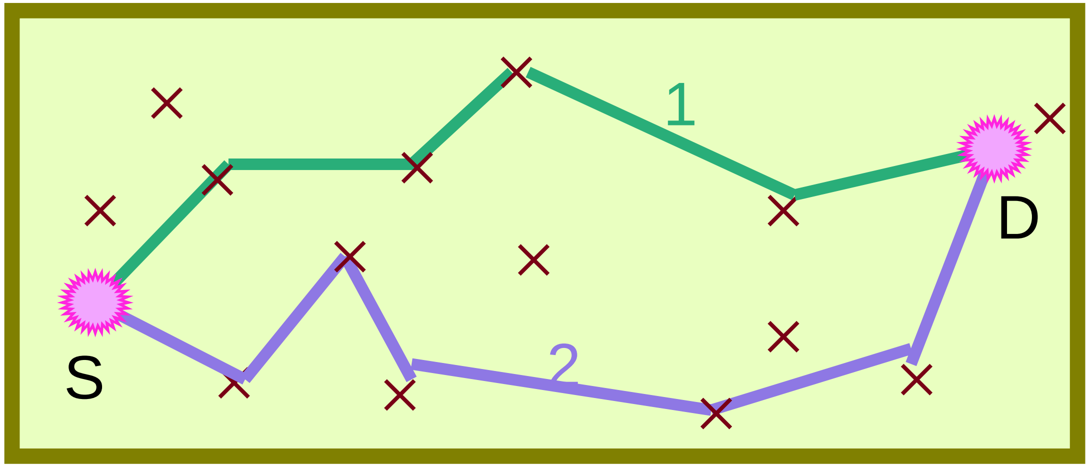
In particular,

$$\left\langle (\delta g)^2 \right\rangle \sim 1$$

$$g \propto L^{d-2} \quad \rightarrow \quad \frac{\left\langle (\delta g)^2 \right\rangle}{g^2} \propto L^{4-2d} >> L^{-d}$$

Fluctuations are large and nonlocal

Waves in Random Media



W_1, W_2
probabilities

A_1, A_2
probability amplitudes

$$W_{1,2} = |A_{1,2}|^2$$

$$A_{1,2} = |A_{1,2}| e^{i\varphi_{1,2}}$$

Total probability

$$W = |A_1 + A_2|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^*\right)$$

interference term:

$$2\,\mathrm{Re}\left(A_1 A_2^*\right) = 2\sqrt{W_1 W_2}\cos\left(\varphi_1 - \varphi_2\right)$$

$$W = \left|A_1 + A_2\right|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\ *}\right)$$

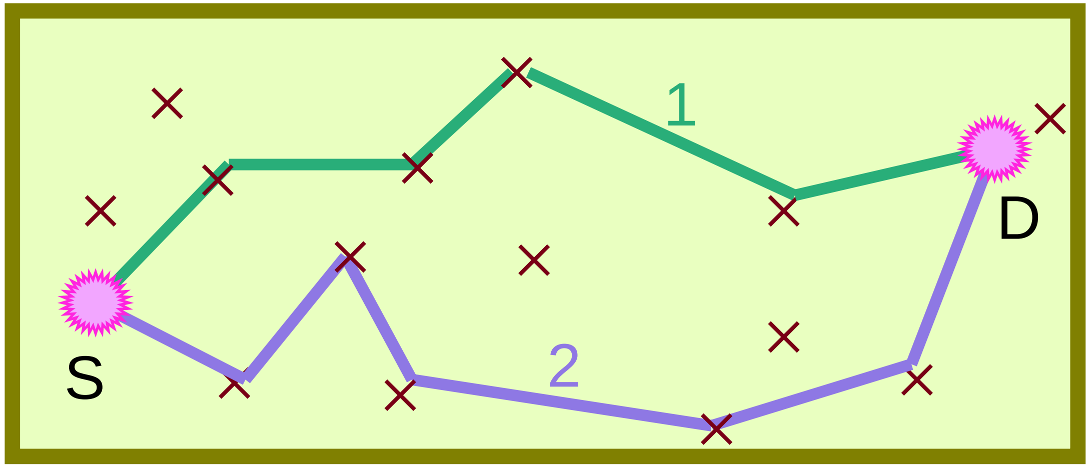
$$2\,\mathrm{Re}\left(A_1 A_2^{\ *}\right) = 2\sqrt{W_1 W_2}\cos\left(\varphi_1 - \varphi_2\right)$$

1. $A_{1,2} = \sqrt{W_{1,2}}\exp\left(i\varphi_{1,2}\right)$
2. Phases $\varphi_{1,2}$ are random
3. $\left|\varphi_1 - \varphi_2\right| >> 2\pi$

The interference term disappears after averaging

$$\left\langle \cos\left(\varphi_1 - \varphi_2\right)\right\rangle = 0$$

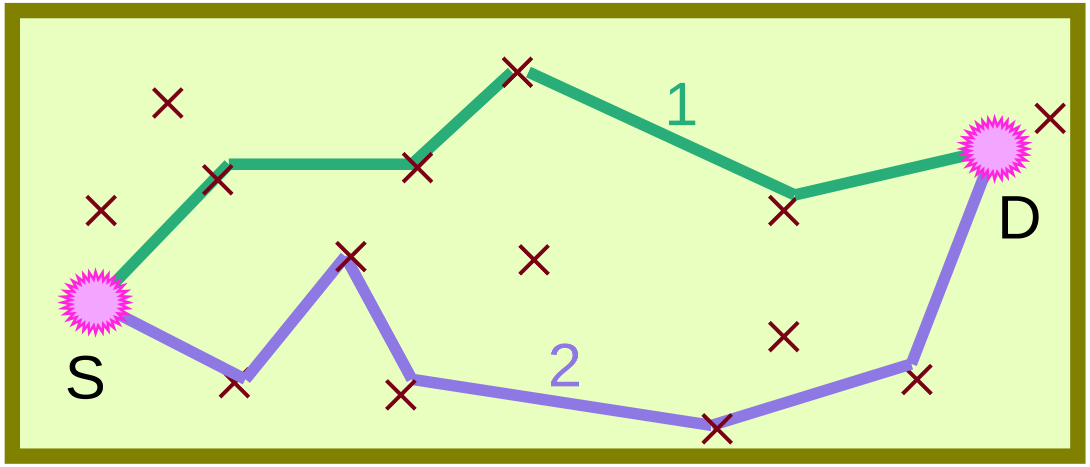
$$\left\langle W \right\rangle = \left\langle W_1 \right\rangle + \left\langle W_2 \right\rangle$$



$$\boxed{W = |A_1 + A_2|^2 = W_1 + W_2 + 2\,\mathrm{Re}\left(A_1 A_2^{\,*}\right)}$$

Classical result for average probability:

$$\boxed{\langle W \rangle = W_1 + W_2}$$



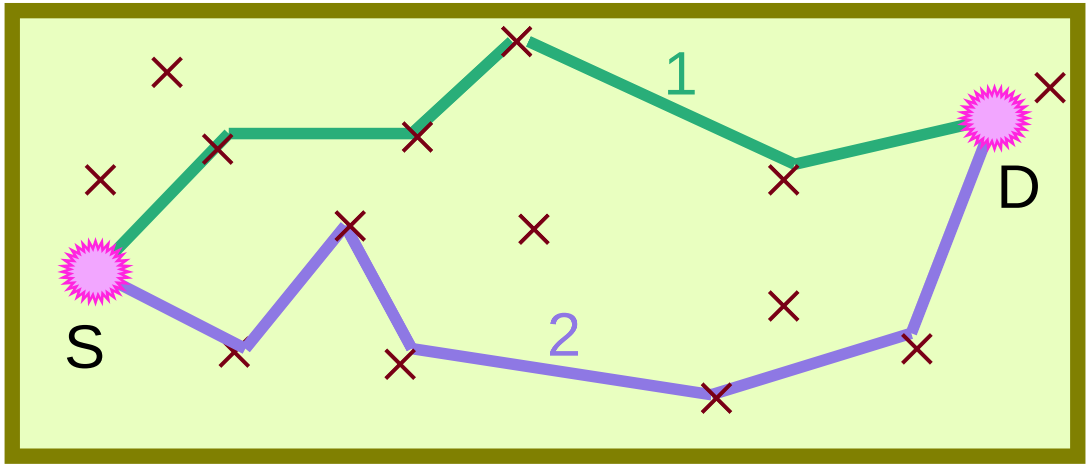
Consider now square of the probability

$$\boxed{\left\langle W^2 \right\rangle = \left(W_1 + W_2\right)^2 + 2 W_1 W_2}$$

Reason:

$$\left\langle \cos\left(\varphi_1 - \varphi_2\right)\right\rangle = 0$$
$$\left\langle \cos^2\left(\varphi_1 - \varphi_2\right)\right\rangle = 1/2$$

$$\boxed{\left\langle W^2 \right\rangle \neq \left\langle W \right\rangle^2}$$



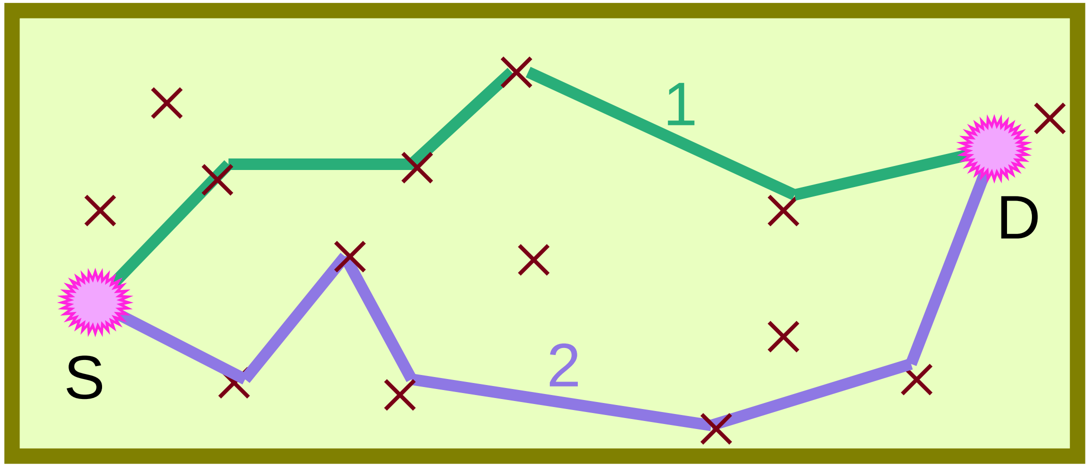
$$\langle W^2 \rangle \neq \langle W \rangle^2$$

CONCLUSIONS:

1. There are fluctuations!

2. Effect is nonlocal.

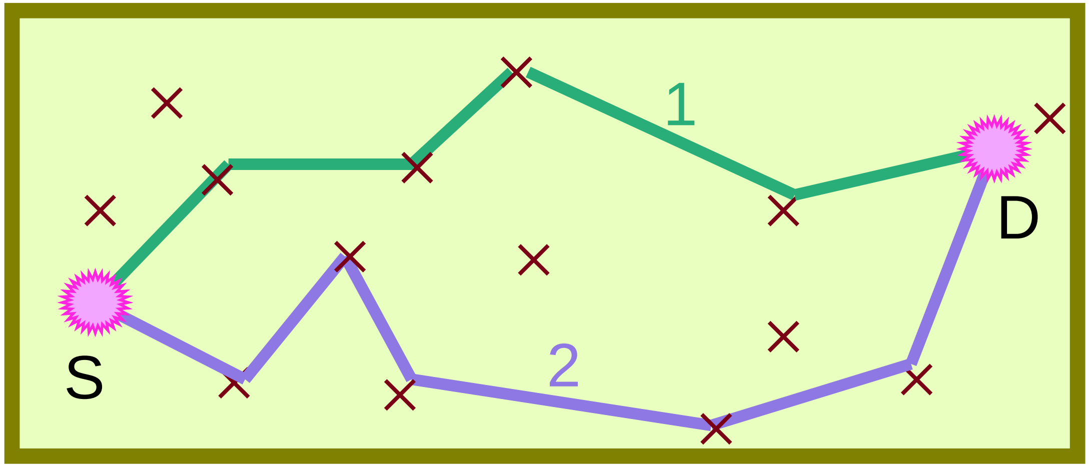
Now let us try to understand the effect of magnetic field. Consider the correlation function



$$\left\langle W(H)W(H+h)\right\rangle = \left\langle W(H)\right\rangle\left\langle W(H+h)\right\rangle$$
$$+2W_1W_2\left\langle\cos\left(\delta\varphi(H)\right)\cos\left(\delta\varphi(H+h)\right)\right\rangle$$

$$\delta\varphi \equiv \varphi_1-\varphi_2$$

Now let us try to understand the effect of magnetic field. Consider the correlation function



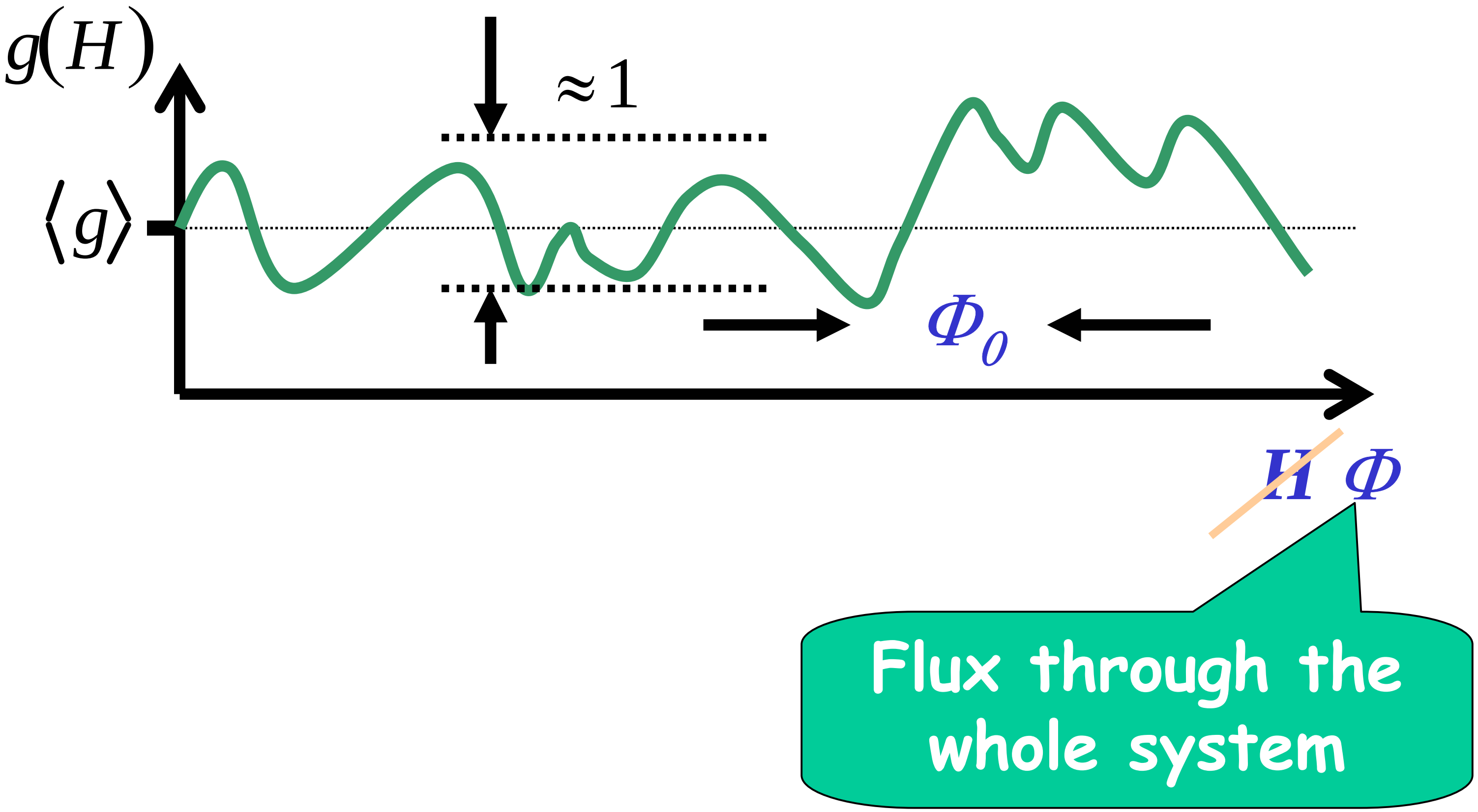
$$\langle W(H)W(H+h)\rangle = \langle W(H)\rangle\langle W(H+h)\rangle + 2W_1W_2\left\langle \cos\left(\delta\varphi(H)\right)\cos\left(\delta\varphi(H+h)\right)\right\rangle$$

$$\delta\varphi \equiv \varphi_1 - \varphi_2$$

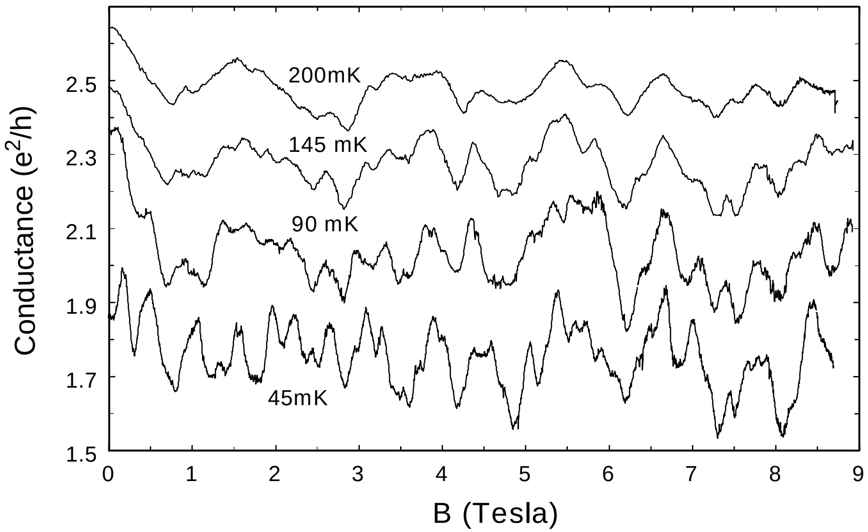
$$\left\langle \cos\left(\delta\varphi(H)\right)\cos\left(\delta\varphi(H+h)\right)\right\rangle \Rightarrow \begin{cases} \frac{1}{2} & \text{for } h \to 0 \ \left(\Phi(h) \ll \Phi_0\right) \\ 0 & \text{for } \Phi(h) \gg \Phi_0 \end{cases}$$

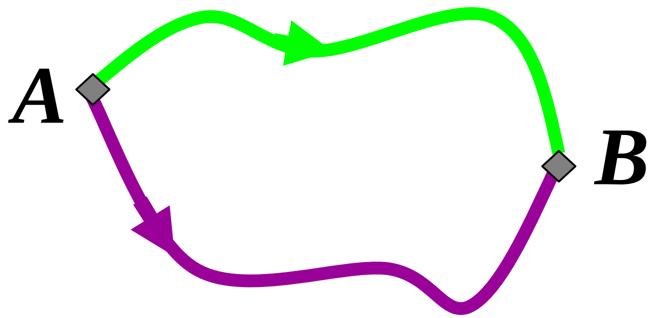
$$\Phi(h) = h \bullet (\text{area of the loop})$$

Magnetoresistance



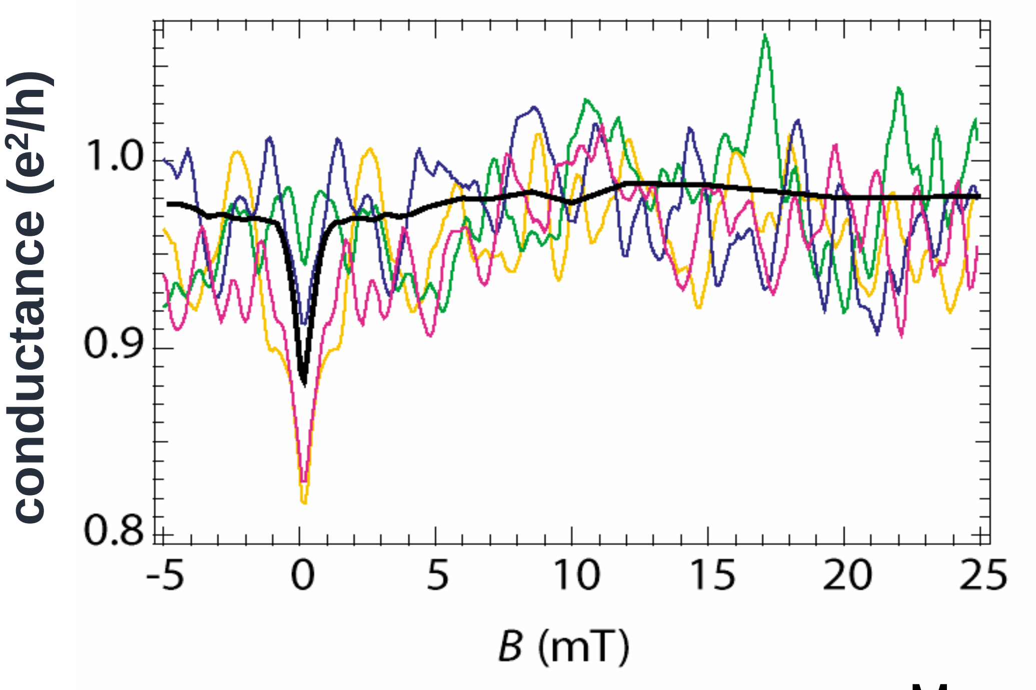
Mesoscopic fluctuations in metallic wires

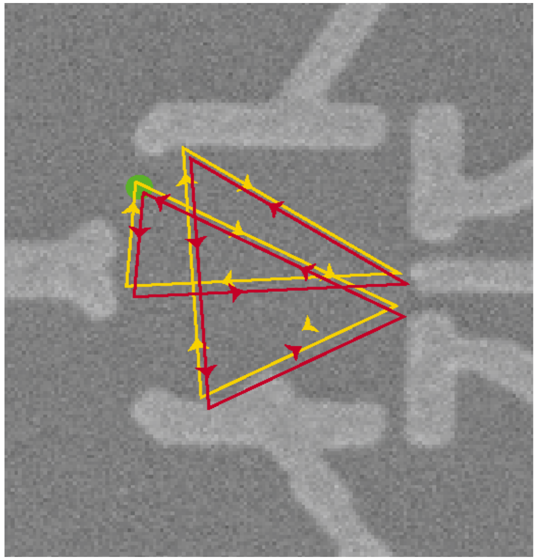


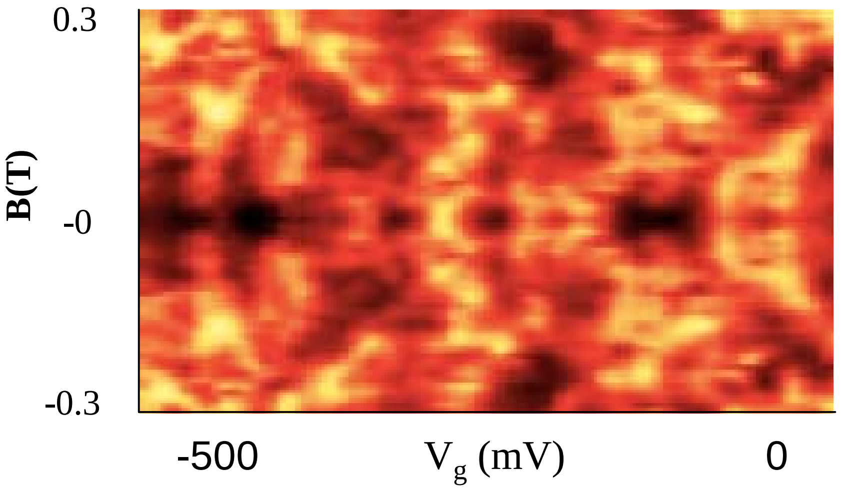


R.A. Webb et al
(1984)

Quantum Chaos

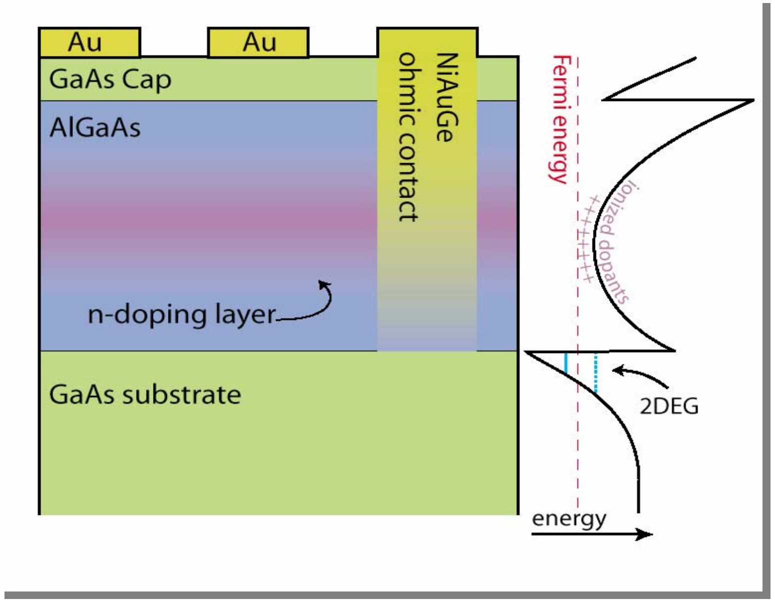




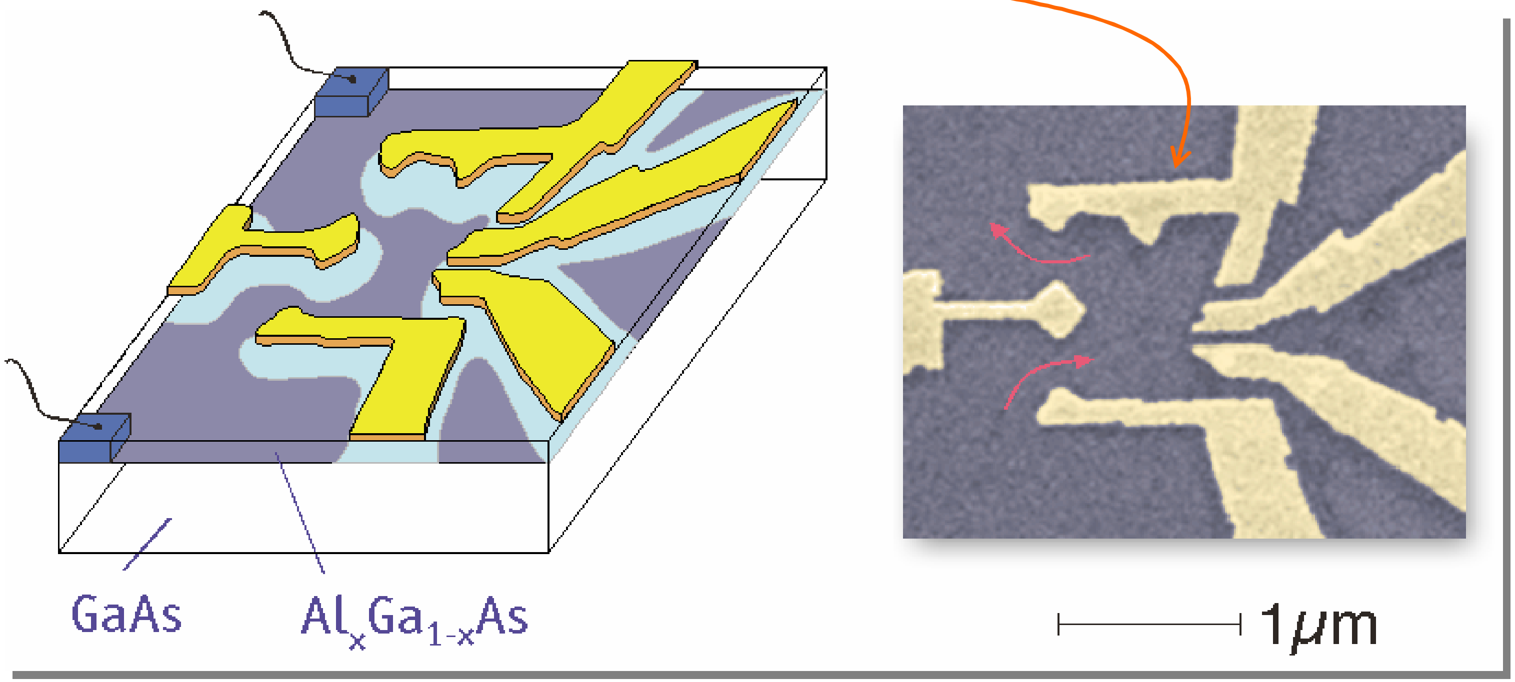


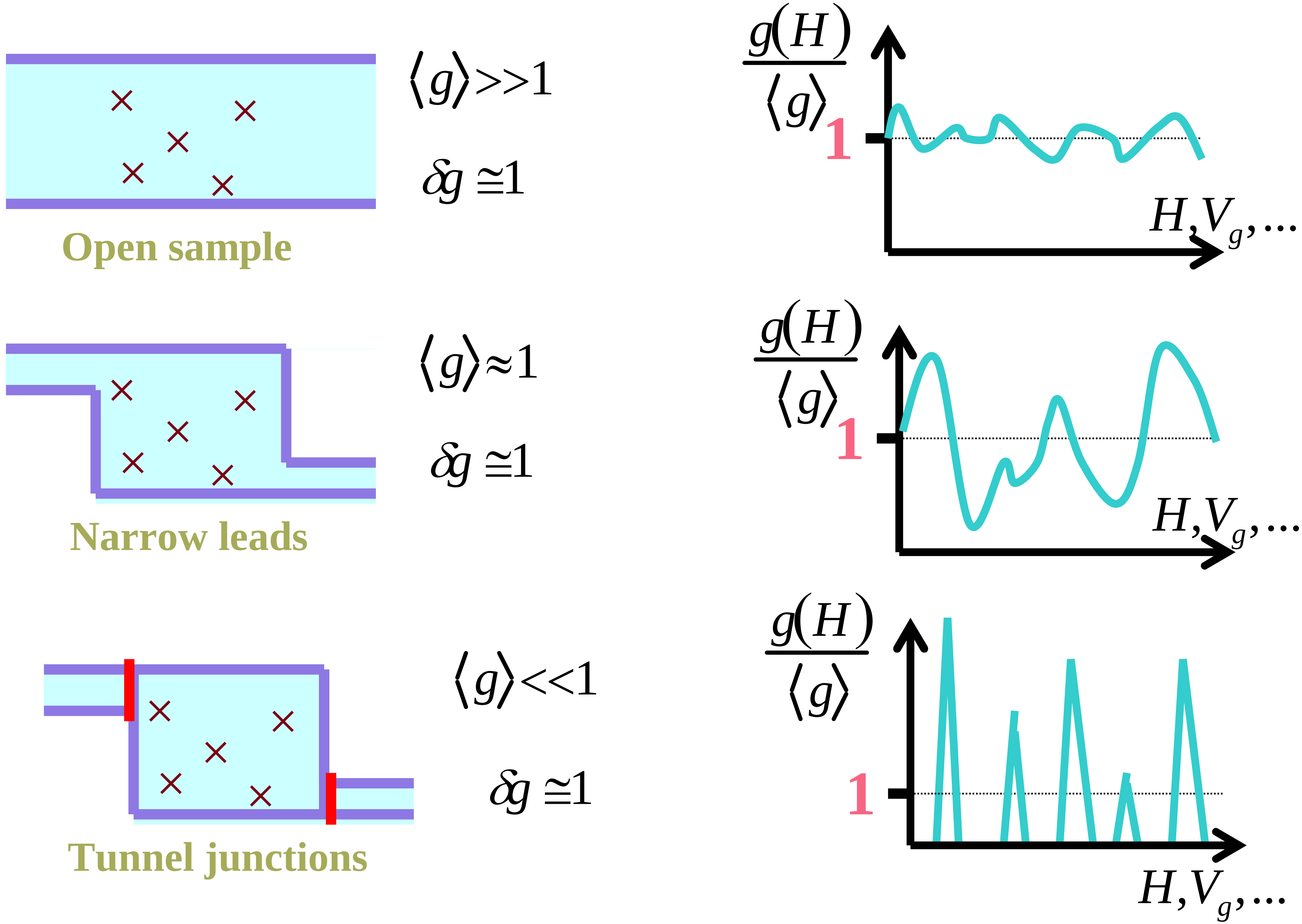
Marcus et al

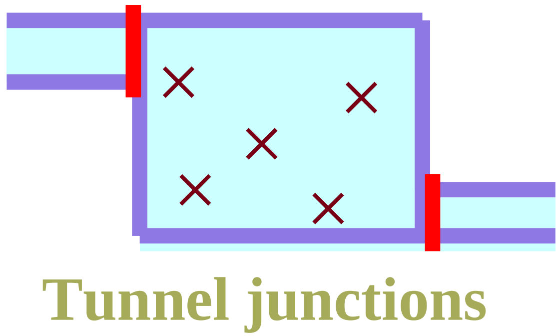
Lateral Quantum Dots
A well-controlled box of electrons



irregular shape for
generic behavior

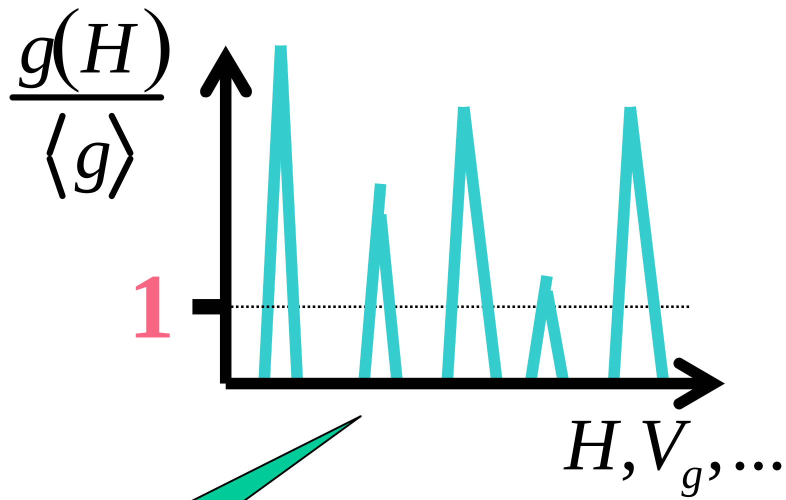



$\langle g\rangle >> 1$
$\delta g \cong 1$
Open sample
$\frac{g(H)}{\langle g\rangle}$
1
$H, V_g, \ldots$
$\langle g\rangle \approx 1$
$\delta g \cong 1$
Narrow leads
$\frac{g(H)}{\langle g\rangle}$
1
$H, V_g, \ldots$
$\langle g\rangle << 1$
$\delta g \cong 1$
Tunnel junctions
$\frac{g(H)}{\langle g\rangle}$
1
$H, V_g, \ldots$



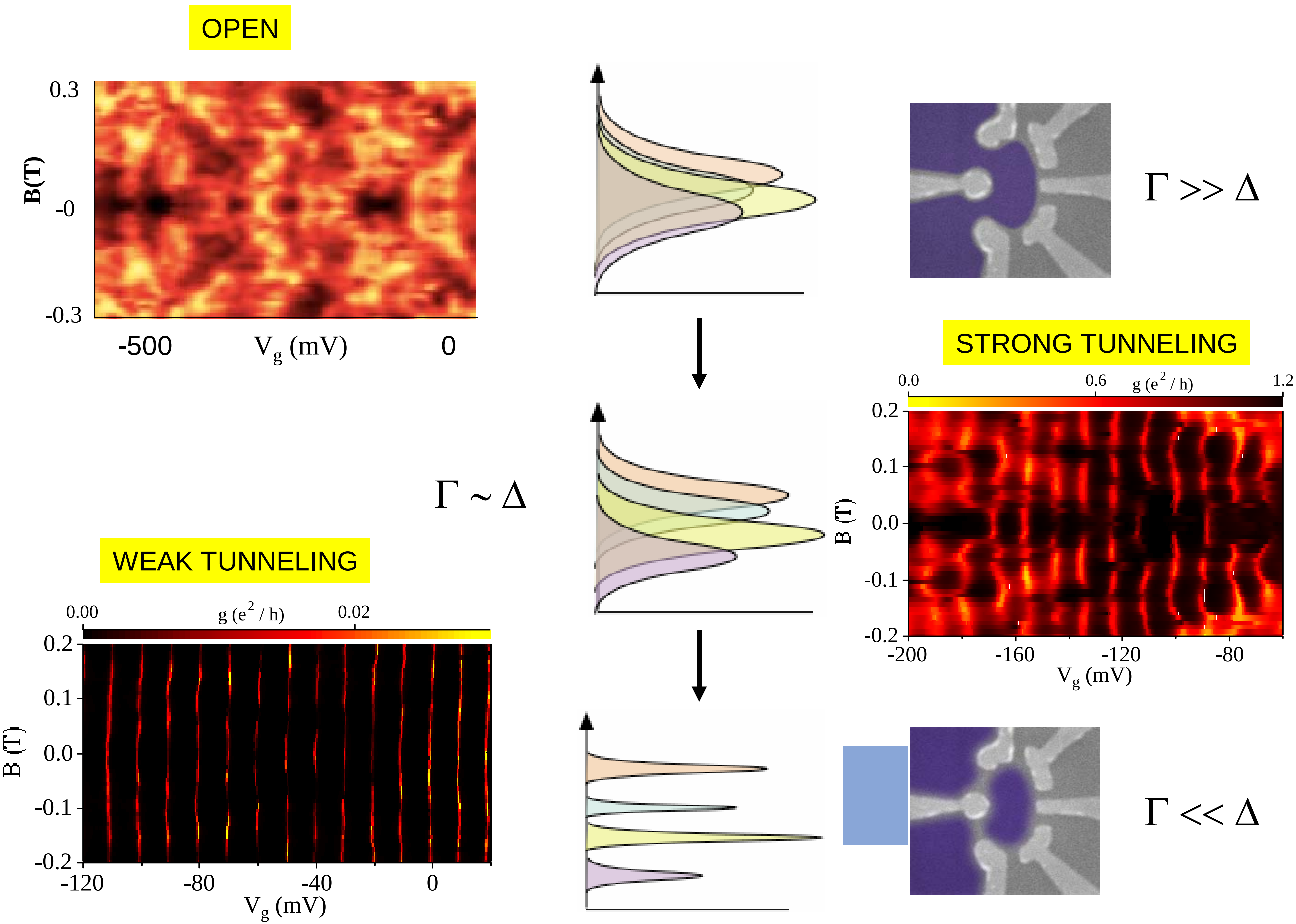
$\langle g \rangle << 1$

$\delta g \cong 1$



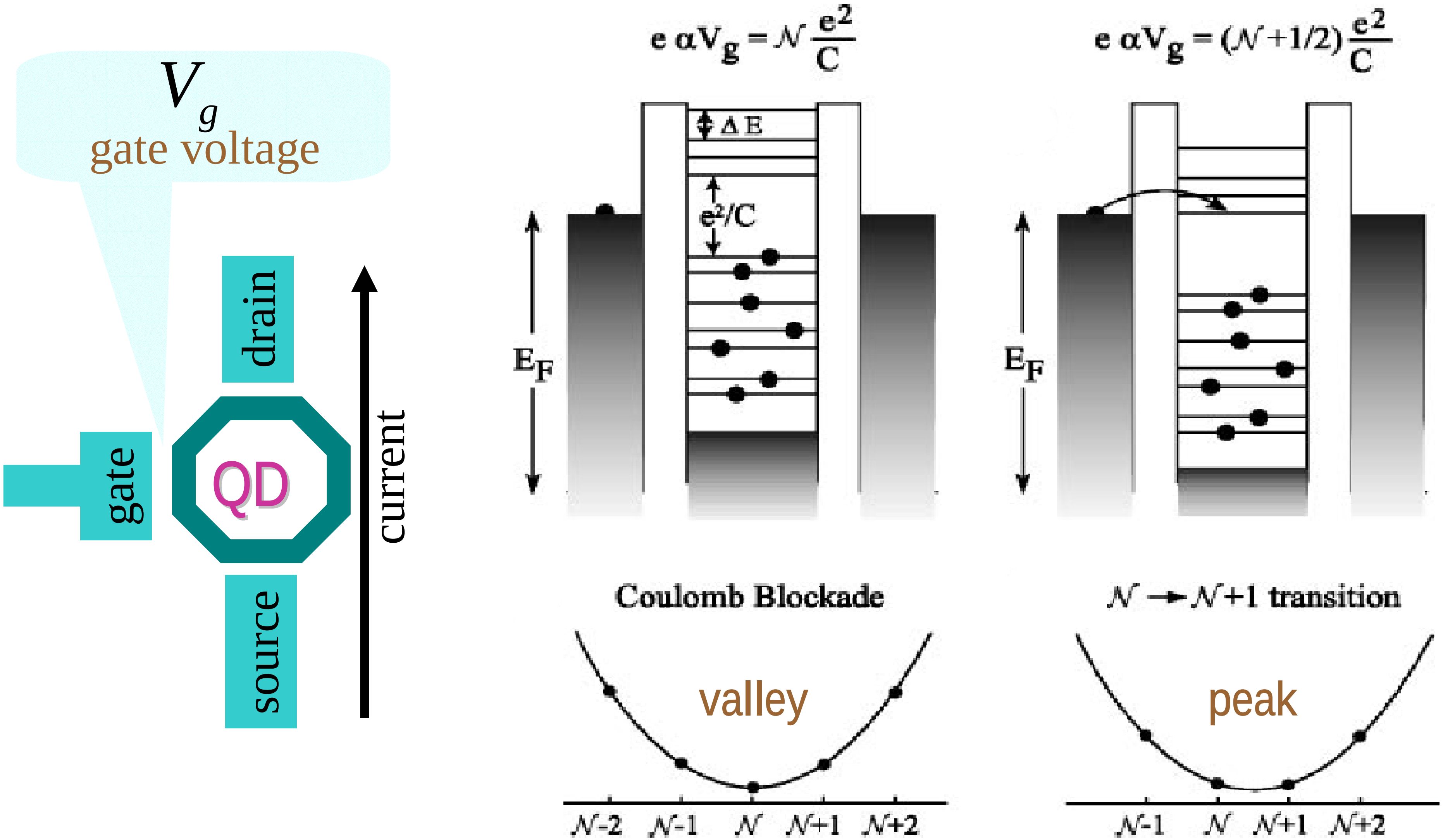
Resonance tunneling

Interaction between electrons becomes crucial !



OPEN
0.3
B(T)
-0
-0.3
-500
V_g (mV)
0
$\Gamma >> \Delta$
STRONG TUNNELING
0.0
0.6
g (e^2 / h)
1.2
0.2
0.1
B (T)
0.0
-0.1
-0.2
-200
-160
-120
-80
V_g (mV)
$\Gamma \sim \Delta$
WEAK TUNNELING
0.00
g (e^2 / h)
0.02
0.2
0.1
B (T)
0.0
-0.1
-0.2
-120
-80
-40
0
V_g (mV)
$\Gamma << \Delta$

Coulomb Blockade



Introduction to theory of mesoscopic systems

Boris Altshuler

Princeton University

& NEC Laboratories America

Part 2 With interactions

Zero dimensional Fermi liquid

~~What does it mean non-Fermi liquid ?~~

Q: What is the difference between Fermi-liquid and non-Fermi liquid ?

A: The difference is the same as between bananas and non-bananas.

What does it mean Fermi liquid ?

Fermi Liquid

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

$\Rightarrow$ *Fermi Liquid*

What does it mean?

- Fermi statistics
- Low temperatures
- Not too strong interactions
- Translation invariance

$\Rightarrow$ Fermi Liquid

It means that

1. Excitations are similar to the excitations in a Fermi-gas:

   a) the same quantum numbers – momentum, spin ½ , charge e

   b) decay rate is small as compared with the excitation energy

2. Substantial renormalizations. For example, in a Fermi gas

$$\partial n/\partial\mu, \quad \gamma = c/T, \quad \chi/g\mu_B$$

are all equal to the one-particle density of states ν.

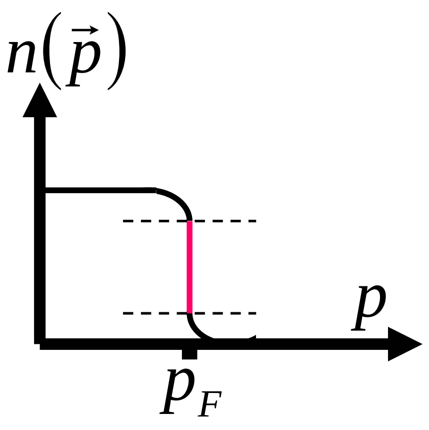
These quantities are different in a Fermi liquid

Signatures of the Fermi - Liquid state ?!

1. *Resistivity is proportional to* T^2 :

L.D. Landau & I.Ya. Pomeranchuk *"To the properties of metals at very low temperatures"*; Zh.Exp.Teor.Fiz., 1936, v.10, p.649

Umklapp electron – electron scattering dominates the charge transport (?!)



2. *Jump in the momentum distribution function at* $T=0$.

2a. *Pole in the one-particle Green function*

$$G(\varepsilon, \vec{p}) = \frac{Z}{i\varepsilon_n - \xi(\vec{p})}$$

Fermi liquid = $0<Z<1$ *(?!)*

Landau Fermi - Liquid theory

Momentum	$\vec{p}$
Momentum distribution	$n(\vec{p})$
Total energy	$E\{n(\vec{p})\}$
Quasiparticle energy	$\xi(\vec{p}) \equiv \delta E / \delta n(\vec{p})$
Landau f-function	$f(\vec{p}, \vec{p}\,') \equiv \delta\xi(\vec{p}) / \delta n(\vec{p}\,')$

Q:

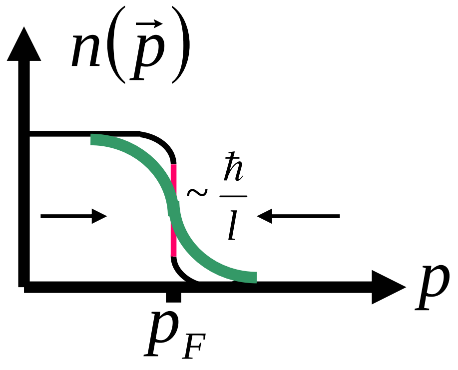
Can Fermi – liquid survive without the momenta

Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder

?

Q: Does it make sense to speak about the *Fermi – liquid* state in the presence of a *quenched disorder* ?

1. *Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, **l**. The step in the momentum distribution function is broadened by this uncertainty*



2. *Neither resistivity nor its temperature dependence is determined by the umklapp processes and thus does not behave as* T^2

3. *Sometimes (e.g., for random quenched magnetic field) the disorder averaged one-particle Green function even without interactions does not have a pole as a function of the energy,* ε*. The residue ,* Z*, makes no sense.*

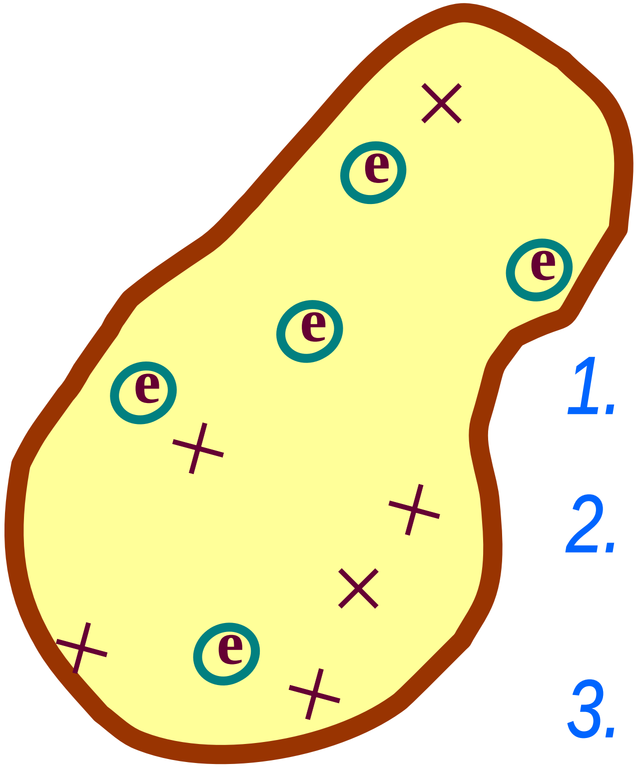
Nevertheless even in the presence of the disorder

I. *Excitations are similar to the excitations in a disordered Fermi-gas.*

II. *Small decay rate*

III. *Substantial renormalizations*

Quantum Dot



1. Disorder (×impurities)
2. Complex geometry

} chaotic one-particle motion

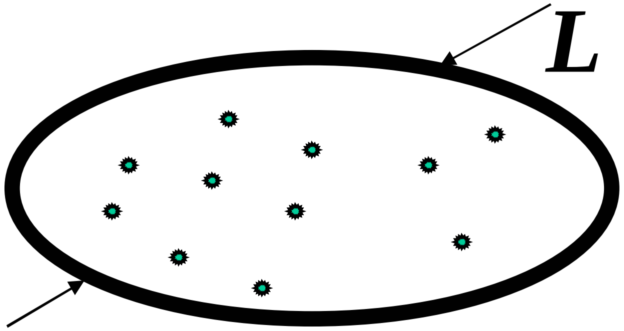
3. ***e-e*** interactions

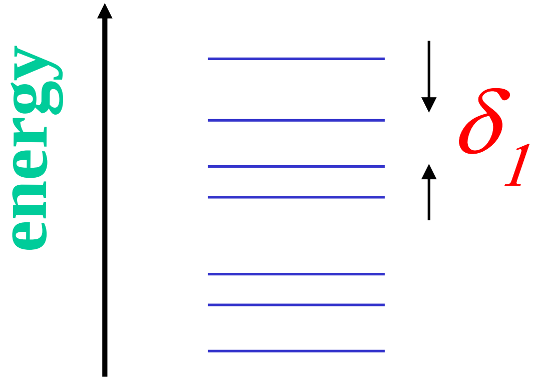
One-particle problem (*Thouless, 1972*)

Energy scales

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)*

$$g = E_T / \delta_1$$

dimensionless Thouless conductance

$$g = Gh/e^2$$

Zero Dimensional Fermi Liquid

Finite System $\Longrightarrow$ Thouless energy E_T

$$\varepsilon << E_T \xrightarrow{def} \mathbf{0D}$$

At the same time, we want the typical energies, ε, to exceed the mean level spacing, δ_1:

$$\delta_1 << \varepsilon << E_T$$

$$\boxed{g \equiv \frac{E_T}{\delta_1} >> 1}$$

Thouless Conductance and One-particle Quantum Mechanics

0 — **1** — ***g***

Localized states
Insulator

Poisson spectral statistics

Extended states
Metal

Wigner-Dyson spectral statistics

$N \times N$
Random Matrices

$N \to \infty$

Quantum Dots with dimensionless conductance g

$g \to \infty$

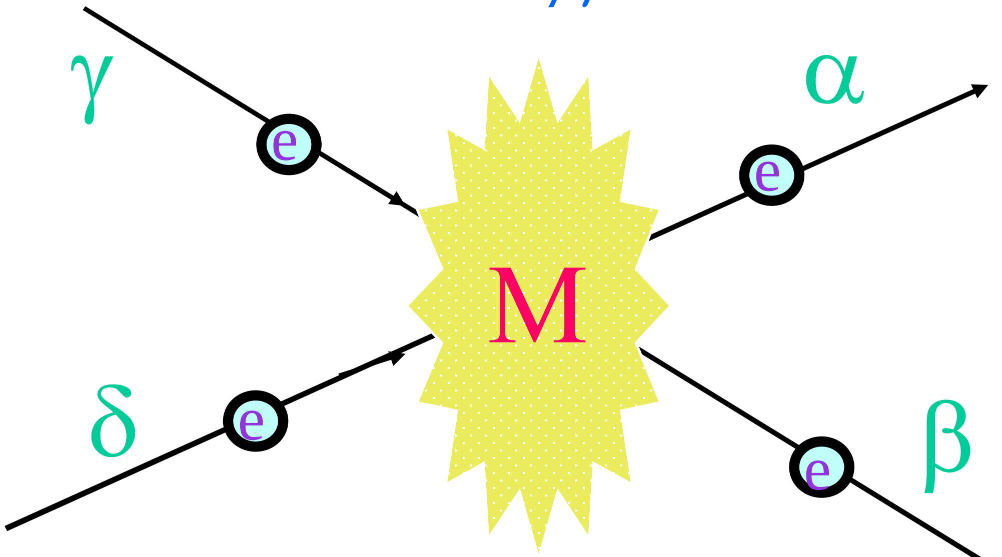
The same statistics of the random spectra and one-particle wave functions (eigenvectors)

Two-Body Interactions

$|\alpha,\sigma>$ *Set of one particle states. σ and α label correspondingly spin and orbit.*

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha a^+_{\alpha,\sigma} a_{\alpha,\sigma} \qquad \hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^+_{\alpha,\sigma} a^+_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

ε_α *-one-particle orbital energies* $\quad$ $\boldsymbol{M}_{\alpha\beta\gamma\delta}$ *-interaction matrix elements*



Two-Body Interactions

$|\alpha,\sigma>$ Set of one particle states. σ and α label correspondingly spin and orbit.

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha a^+_{\alpha,\sigma} a_{\alpha,\sigma} \qquad \hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta}\, a^+_{\alpha,\sigma} a^+_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

ε_α -one-particle orbital energies $\quad$ $M_{\alpha\beta\gamma\delta}$ -interaction matrix elements

Nuclear Physics	ε_α	are taken from the shell model
	$M_{\alpha\beta\gamma\delta}$	are assumed to be random
Quantum Dots	ε_α	RANDOM; Wigner-Dyson statistics
	$M_{\alpha\beta\gamma\delta}$	? ? ? ? ? ? ? ?

Matrix Elements

$$\hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta}\, a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

Matrix Elements $M_{\alpha\beta\gamma\delta}$

Diagonal - $\alpha,\beta,\gamma,\delta$ are equal pairwise

$\alpha=\gamma$ and $\beta=\delta$ or $\alpha=\delta$ and $\beta=\gamma$ or $\alpha=\beta$ and $\gamma=\delta$

Offdiagonal - otherwise

It turns out that in the limit $g \to \infty$

- Diagonal matrix elements are much bigger than the offdiagonal ones

$$M_{\text{diagonal}} \gg M_{\text{offdiagonal}}$$

- Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging

Toy model:

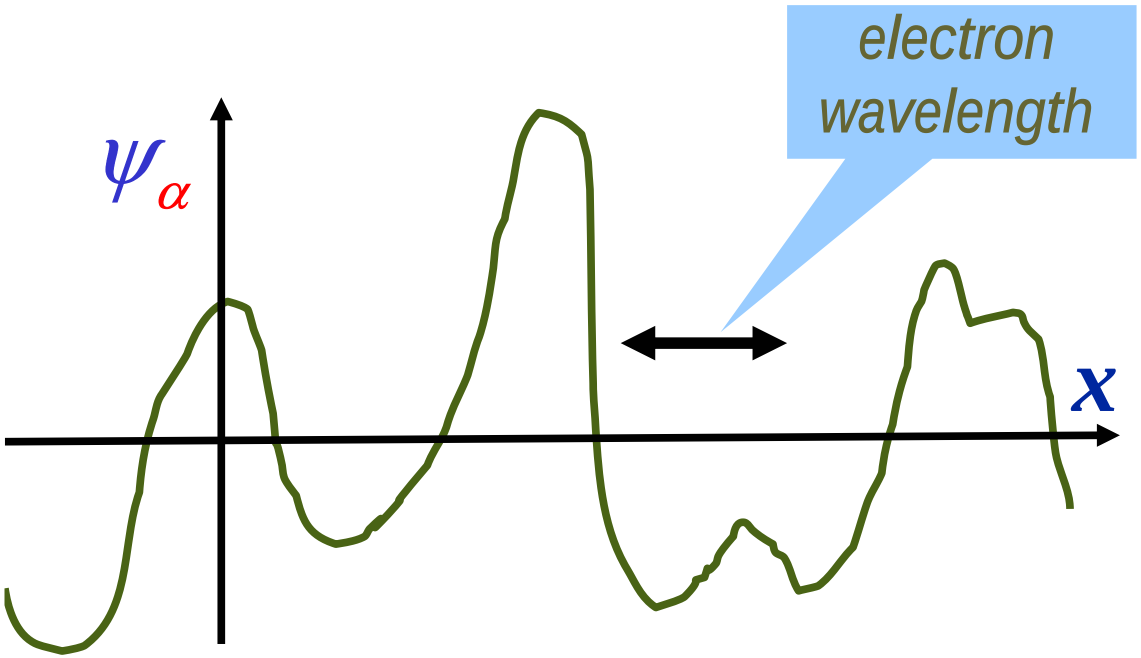
*Short range **e-e** interactions*

$$U(\vec{r}) = \frac{\lambda}{\nu}\delta(\vec{r})$$

λ is dimensionless coupling constant ν is the electron density of states

$$M_{\alpha\beta\gamma\delta} = \frac{\lambda}{\nu}\int d\vec{r}\,\psi^*_{\alpha}(\vec{r})\psi^*_{\beta}(\vec{r})\psi_{\gamma}(\vec{r})\psi_{\delta}(\vec{r})$$

$\psi_{\alpha}(\vec{r})$ one-particle eigenfunctions



$\Psi_\alpha(x)$ *is a random function that rapidly oscillates*

$$|\psi_\alpha(x)|^2 \geq 0$$

$$\psi_\alpha(x)^2 \geq 0$$

as long as T-invariance is preserved

In the limit $g \to \infty$

- *Diagonal matrix elements are much bigger than the offdiagonal ones*

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- *Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging*

$$M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu}\int d\vec{r}\,|\psi_\alpha(\vec{r})|^2 |\psi_\beta(\vec{r})|^2$$

$$|\psi_\alpha(\vec{r})|^2 \Rightarrow \frac{1}{\text{volume}}$$

$$\Rightarrow \boxed{M_{\alpha\beta\alpha\beta} = \lambda\delta_1}$$

More general: *finite range interaction potential* $U(\vec{r})$

$$\boxed{M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu}\int |\psi_\alpha(\vec{r}_1)|^2 |\psi_\beta(\vec{r}_2)|^2 U(\vec{r}_1 - \vec{r}_2)\,d\vec{r}_1 d\vec{r}_2}$$

The same conclusion

Random Matrices:

E_α - spectrum

$\psi_\alpha(i)$ – i-th component of α-th eigenvector

$$\boxed{\left\langle \psi_\alpha^*(i)\psi_\gamma(j)\right\rangle = \frac{1}{N}\delta_{\alpha\gamma}\delta_{ij}}$$

$$\boxed{\left\langle \psi_\alpha(i)\psi_\gamma(j)\right\rangle = \frac{2-\beta}{N}\delta_{\alpha\gamma}\delta_{ij}}$$

in the limit $N \to \infty$

Components of the different eigenvectors as well as different components of the same eigenvector are not correlated

Universal (Random Matrix) limit - Random Matrix **symmetry** of the correlation functions:

All correlation functions are invariant under arbitrary orthogonal transformation:

$$\boxed{\tilde{\psi}_\mu(\vec{r}) = \sum_\nu \int d\vec{r}_1 O_\mu^\nu(\vec{r},\vec{r}_1)\psi_\nu(\vec{r}_1)}$$

$$\boxed{\int d\vec{r}_1 O_\mu^\nu(\vec{r},\vec{r}_1) O_\nu^\eta(\vec{r}_1,\vec{r}\,') = \delta_{\mu\eta}\delta(\vec{r}-\vec{r}\,')}$$

There are *only* three operators, which are quadratic in the fermion operators a^+, a and invariant under *RM* transformations:

$$\hat{n} = \sum_{\alpha,\sigma} a^+_{\alpha,\sigma} a_{\alpha,\sigma}$$

total number of particles

$$\hat{S} = \sum_{\alpha,\sigma_1,\sigma_2} a^+_{\alpha,\sigma_1} \vec{\sigma}_{\sigma_1,\sigma_2} a_{\alpha,\sigma_2}$$

total spin

$$\hat{T}^+ = \sum_{\alpha} a^+_{\alpha,\uparrow} a^+_{\alpha,\downarrow}$$

????

Charge conservation (gauge invariance) -no $\hat{T}$ or $\hat{T}^{+}$ only $\hat{T}\hat{T}^{+}$

Invariance under rotations in spin space - no $\hat{S}$ only $\hat{S}^{2}$

Therefore, in a very general case

$$\boxed{\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.}$$

Only three coupling constants describe all of the effects of e-e interactions

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

I.L. Kurland, I.L.Aleiner & B.A., 2000

See also

P.W.Brouwer, Y.Oreg & B.I.Halperin, 1999

H.Baranger & L.I.Glazman, 1999

H-Y Kee, I.L.Aleiner & B.A., 1998

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.$$

Only one-particle part of the Hamiltonian, $\hat{H}_0$, ***contains randomness***

!

$$\boxed{\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}}$$

$$\boxed{\hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha}$$

$$\boxed{\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.}$$

E_c determines the charging energy (Coulomb blockade)

J describes the spin exchange interaction

λ_{BCS} determines effect of superconducting-like pairing

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

For a short range interaction with a coupling constant λ

$$E_c = \frac{\lambda\delta_1}{2} \qquad J = -2\lambda\delta_1 \qquad \lambda_{BCS} = \lambda\delta_1(2-\beta)$$

where δ_1 is the one-particle mean level spacing

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_\alpha \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.$$

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

III. Substantial renormalizations

Isn't it a Fermi liquid ?

Fermi liquid behavior follows from the fact that different wave functions are almost uncorrelated

CONCLUSIONS

One-particle chaos + moderate interaction of the electrons $\mapsto$ to a rather simple Hamiltonian of the system, which can be called Zero-dimensional Fermi liquid.

The main parameter that justifies this description is the Thouless conductance, which is supposed to be large

Excitations are characterized by their one-particle energy, charge and spin, but not by their momentum.

These excitations have the lifetime, which is proportional to the Thouless conductance, i.e., is long.

This approach allows to describe Coulomb blockade (renormalization of the compressibility), as well as the substantial renormalization of the magnetic susceptibility and effects of superconducting pairing