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Theory of Gravitational Waves - II

J. Miller

SISSA, Trieste
&
University of Oxford
U.K.

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John Miller
SISSA & Oxford

How is gravitational wave emission linked to changes in the source?

Changes in a spherically symmetric source which maintain its spherical symmetry, cause no change in the external field. This result is known as Birkhoff's theorem and holds both in Newtonian theory and also in GR. Since there is no change in the external field in this case, it follows that no gravitational waves are produced. For getting gravitational-wave emission, it is necessary to have *time-dependent*, *non-spherical* behaviour of the source of the gravitational field.

There are some notable similarities between gravity and electromagnetism which are relevant for what we are discussing here:

$$\begin{array}{llll} \text{em:} & A_0 = 0 & A_{i,i} = 0 & \Box A_i = 0 \\ \text{GR:} & h^{TT}_{0\mu} = 0 & h^{TT}_{jk,k} = 0 & \Box h^{TT}_{jk} = 0 \end{array} \tag{1}$$

where A_i is the vector potential of electromagnetism. However, there are also some important differences.

The leading order multipole radiation in electromagnetism is *dipole* radiation:

$$A_j\,(t, \mathbf{x}) = \frac{1}{cr}\,\dot{d}_j\,(t - \frac{r}{c}) \tag{2}$$

where $r \equiv |\mathbf{x}|$, $\mathbf{d}$ is the electric dipole moment and the dot indicates a derivative with respect to time. Substituting the $\mathbf{B}$ and $\mathbf{E}$ fields obtained from this into the Poynting vector and integrating over solid angle, we obtain the luminosity:

$$L_{em} \;=\; \frac{2}{3c^3}\,\ddot{d}_j\,\ddot{d}_j \tag{3}$$

In GR there is no dipole radiation; the leading order is *quadrupole*.

Recall that

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{4}$$

Integrating this, one gets after some manipulation

$$h^{TT}_{jk}(t, \mathbf{x}) = \frac{2}{r} \frac{G}{c^4} \ddot{\mathcal{I}}^{TT}_{jk}(t - \frac{r}{c}) \tag{5}$$

where $\mathcal{I}_{jk}$ is the mass quadrupole moment given by:

$$\mathcal{I}_{jk} = \sum_A m_A \left[x^A_j x^A_k - \tfrac{1}{3} \delta_{jk} (x^A)^2 \right] \tag{6}$$

(following the definition of Misner, Thorne & Wheeler [2]).

The energy flux is given by

$$T_{0r} = \frac{c^4}{32\pi G} \left\langle h^{TT}_{jk,0} \, h^{TT}_{jk,r} \right\rangle \tag{7}$$

where $\langle \;\; \rangle$ indicates the average over several cycles.

Inserting the expression for h^{TT}_{jk} into this and integrating over the solid angle, we get the luminosity:

$$L_{GW} \equiv -\frac{\mathrm{d}E}{\mathrm{d}t} \tag{8}$$

$$= \frac{1}{5} \frac{G}{c^5} \left\langle \dddot{\mathcal{I}}_{jk} \, \dddot{\mathcal{I}}_{jk} \right\rangle \tag{9}$$

This is known as the *quadrupole formula* for gravitational radiation. Although, as presented here, it applies for just the weak-field regime, it does, in fact, have a wider range of validity if $\mathcal{I}_{jk}$ is suitably defined.

If the source is *non-axisymmetric*, gravitational waves can also carry away angular momentum:

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{2}{5} \frac{G}{c^5} \varepsilon_{ijk} \left\langle \ddot{\mathcal{I}}_{jm} \, \dddot{\mathcal{I}}_{km} \right\rangle \tag{10}$$

where ε_{ijk} is the permutation tensor.

Order of magnitude estimates

In this section, we make some order of magnitude estimates of gravitational wave emission to show how detectability of gravitational waves is linked to the characteristics of the source. The third time derivative of $\mathcal{I}$ can be very roughly approximated by

$$\dddot{\mathcal{I}}_{jk} \sim \frac{MR^2}{T^3} \sim \frac{Mv^3}{R} \tag{11}$$

where M, R and T are the characteristic mass, size and timescale of the source and v is a characteristic velocity.

From Eq. (9) we then get

$$L_{GW} \sim \frac{G}{c^5}\left(\frac{M}{R}\right)^2 v^6 \tag{12}$$

$$\sim L_0 \left(\frac{R_s}{R}\right)^2 \left(\frac{v}{c}\right)^6 \tag{13}$$

where $L_0 \equiv c^5/G \to 3.6\times10^{59}\, erg/s$ and R_s $(= 2GM/c^2)$ is the Schwarzschild radius of the source. It follows that the most powerful sources will be *compact* (with $R \sim R_s$) and *fast moving* (with $v \sim c$.)

For detectors, the important quantity is h since it is this which indicates the relative strain produced by an incident gravitational wave. From Eq. (5):

$$h_{jk}^{TT} = \frac{2}{r}\frac{G}{c^4}\,\ddot{\mathcal{I}}_{jk}^{TT} \tag{14}$$

we have that

$$h \sim \left(\frac{R_s}{R}\right)\left(\frac{v}{c}\right)^2 \frac{R}{r}. \tag{15}$$

The behaviour $h \propto 1/r$ is a general feature of gravitational waves and indicates that distant sources are more easily observable than one would expect on the basis of a normal $(1/r^2)$ fall-off.

There is a connection between compactness of the source and typical velocities of motion since it is often the case that the *kinetic energy* of a system is of the same order as its *gravitational potential energy*, i.e.

$$\tfrac{1}{2}v^2 \sim \frac{GM}{R} \tag{16}$$

$$\Rightarrow \frac{v^2}{c^2} \sim \frac{R_s}{R} \tag{17}$$

Inserting this into Eq. (15) gives

$$h \sim \left(\frac{R_s}{R}\right)^2 \frac{R}{r} \tag{18}$$

Summary of astronomical sources of gravitational waves

In this section, we give a list of the main predicted astronomical sources of gravitational waves, focussing particularly on those which can be good candidates for detection by the new generation laser-interferometric detectors such as LIGO, VIRGO and GEO600 which are most sensitive to frequencies in the range from ten Hz to a few hundred Hz. We group the sources into three classes: *burst sources*, for which there is a sharp pulse of gravitational radiation emitted; *periodic sources*, where gravitational waves are emitted over a very large number of similar cycles; and *stochastic sources*, where the signals from many objects mix to form a "noise" background. Different detection strategies will be used for these different types of source and hence the detection thresholds are very different (see the article by Thorne [3] for more details). For a long-lived periodic source, there is the possibility of integrating over very many cycles in order to extract the signal from detector noise and this can give as much as six orders of magnitude enhancement in sensitivity as compared with burst sources. For stochastic sources, which give a background whose overall features change only very slowly with time, it is again possible to make use of time integration to enhance sensitivity and here there can be a gain of up to three orders of magnitude in sensitivity with respect to burst sources. In the lists that follow, rough sensitivity thresholds are given for each type of source, appropriate for the Advanced LIGO detector. Note that, in fact, these thresholds are dependent on frequency of the waves but the values given are typical ones for our frequency range from ten to a few hundred Hz.

Burst sources ($h \gtrsim 10^{-22}$ for detection by Advanced LIGO)

- Gravitational collapse to form stellar mass black holes and neutron stars (associated with supernovae)
- Coalescence of neutron star and black hole binaries
- Infall of a star into a large black hole (lower frequency than the LIGO range)

Periodic Sources ($h \gtrsim 10^{-28}$)

- Rotating neutron stars
  - Young neutron stars in supernovae which are non-axisymmetric as a result of the growth of unstable modes
  - Neutron stars which are non-axisymmetric as a result of misaligned strong magnetic fields
- Binary stars (lower frequency than the LIGO range)

Stochastic Sources ($h \gtrsim 10^{-25}$)

- Supernovae
- Binary stars (lower frequency than the LIGO range)
- Early universe, cosmic strings and phase transitions (mostly lower frequency than the LIGO range)
- Population III stars (mostly lower frequency than the LIGO range)

Note that the sensitivity thresholds (the minimum induced fractional strains measurable by the detector) are rather impressive numbers! The best possibilities for early detection by laser interferometers seem to be

- Coalescing neutron star binaries
- Rotating non-axisymmetric neutron stars
- Coalescing black hole binaries

For all of these, it is extremely important to produce *templates* of the expected wave signals to aid the extraction of signals from detector noise. This is an area of physics where experiment and theory need to proceed very closely together.

For us as astrophysicists, the greatest excitement in the search for gravitational waves concerns the possibility of opening a new window on the universe to enable us to get information about phenomena of relativistic astrophysics which are largely hidden from us for as long as we are constrained to make observations only by means of electromagnetic radiation. However, in addition to the interest for astronomers, there is also the aspect that gravitational wave observations are likely to produce output of great interest for *basic physics* as well. In particular, we can mention:

- Confirmation of the existence of black holes
- Better understanding of gravity
- Better understanding of neutron stars (information about the physics of high density matter)
- Better understanding of the early universe (information about ultra-high energy physics)

Gravitational waves and the binary pulsar PSR 1913+16

In conclusion, we turn to a brief discussion of this famous object which has presented the strongest observational evidence so far for gravitational waves actually being emitted. It was discovered by Hulse and Taylor in 1974. They saw a single pulsar (a rotating neutron star) with a period of 59 ms and, from Doppler shifts in the frequency, they were able to infer that it was in orbital motion around an unseen companion with an orbital period of $\sim$ 8 hours. The orbital velocity was measured at $\sim 300\,km/s$, giving $v/c \sim 10^{-3}$. Now, after twenty-five years of observations, all of the parameters of the system are known to high accuracy. In particular, the orbital period is seen to be decreasing at a rate $\dot{P} = -2.425 \times 10^{-12}\,s\,s^{-1}$ ($\sim$ 0.1 $ms/year$). For a relativistic binary system such as this, GR predicts that gravitational radiation will carry away orbital angular momentum and cause the two components of the binary to spiral towards each other with a progressively shortening orbital period, exactly as observed for PSR 1913+16. The observed period change is in excellent quantitative agreement with the theoretical prediction using GR giving strong circumstantial evidence that this object is indeed emitting gravitational waves. In 1991, a second rather similar object was discovered (called PSR 1534+12). This again shows evidence for orbital angular momentum being carried away by gravitational waves and seems to be in good agreement with GR predictions.

These results are very encouraging. However, the really exciting moment will come when there is the first direct evidence for gravitational waves from an astronomical source actually being picked up by a detector.

Further reading

[1] S.L. Shapiro and S.A. Teukolsky, *Black holes, white dwarfs and neutron stars: the physics of compact objects* (Wiley, New York, 1983); pp. 466 – 498.

[2] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, New York, 1973); pp. 941 – 1044.

[3] K.S. Thorne, in *Three hundred years of gravitation* (eds. S.W. Hawking and W. Israel - Cambridge University Press, 1987); pp. 330 – 458.