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"VII School on Non-Accelerator Astroparticle Physics"

26 July - 6 August 2004

Neutrino Masses and Mixing

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Neutrino Masses, Mixing, Oscillations and the Nature of Massive Neutrinos

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7th School on Non-Accelerator
Astroparticle Physics
ICTP, Trieste
26 July - 6 August, 2004

2001– Remarkable progress in the studies of $\nu-$ mixing and oscillations

- **June, 2001: SNO CC data + SK data $\rightarrow \nu_{\mu,\tau}$ and/or $\bar{\nu}_{\mu,\tau}$ in $\Phi_E(\nu_\odot)$**

- **April, 2002: SNO NC data $\rightarrow$ evidences for $\nu_{\mu,\tau}$ and/or $\bar{\nu}_{\mu,\tau}$ in $\Phi_E(\nu_\odot)$ strengthen**

- **December, 2002: KamLAND**
  - First compelling evidence for $\nu-$oscillations in an experiment with terrestrial ν's
  - Evidence for ν_e-mixing in vacuum
  - $\nu_\odot$: LMA solution (CPT)
  - KamLAND "massacre":

    VO, QVO, LOW, SMA MSW, RSFP, FCNC, WEPV, LIV,...

- **September, 2003: SNO salt phase data,**

  higher precision measurement of $\Phi_E(\nu_\odot)$

Evidences for $\nu-$Oscillations

$-\nu_{\text{atm}}$: SK UP-DOWN ASYMMETRY

θ_Z-, $L/E-$ dependences of $\mu-$like events

Dominant $\nu_\mu \rightarrow \nu_\tau$ K2K; MINOS, CNGS

$-\nu_\odot$: Homestake, Kamiokande, SAGE, GALLEX/GNO
Super-Kamiokande, SNO; KamLAND

Dominant $\nu_e \rightarrow \nu_{\mu,\tau}$ BOREXINO,...

– LSND

Dominant $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ MiniBOONE

$$\nu_{l\text{L}} = \sum_{j=1} U_{lj}\, \nu_{j\text{L}} \qquad l = e, \mu, \tau. \qquad (1)$$

$\nu_{\odot}$ - Oscillations in Vacuum

(Pontecorvo 1958; 1967 - relevance to $\nu_{\odot}$ - experiment)

ν - Oscillations in vacuum are possible if ν_i's with $m(\nu_i) \neq 0$ ($m_i \neq m_j$, $i \neq j$) and nontrivial lepton (ν-) mixing exist in vacuum.

(idea of ν-os. Pontecorvo; Maki et al, '6.)

Consider the simplest case:

$$\begin{cases} |\nu_e\rangle = |\nu_1\rangle \cos\theta + |\nu_2\rangle \sin\theta, \\ |\nu_x\rangle = -|\nu_1\rangle \sin\theta + |\nu_2\rangle \cos\theta. \end{cases}$$

in vacuum

$\vec{p}_1, E_1 (m_1)$ $\qquad$ $\vec{p}_2, E_2 (m_2)$ $\qquad$ $m_2 \neq m_1$, $\theta \neq \frac{\pi}{2} k$, $k = 0, 1, ...$

$\nu_x = \nu_\mu$ or ν_τ - active, or

$\nu_x = \nu_s$ - sterile;

$\boxed{\nu_e \rightleftarrows \nu_\mu \;\; x = \mu}$ possible in vacuum.

θ - neutrino mixing angle in vacuum,

ν_1, ν_2 - neutrinos with definite mass in vacuum (vacuum mass-eigenstate ν's)

IN THE MORE GENERAL CASE

$$|\nu_\ell\rangle = \sum_{j=1}^{3} U^*_{\ell j} |\nu_j\rangle \quad \text{- } \nu \text{ WITH FLAVOR } \ell \text{ AT } x = 0$$

m_j

$$\langle \nu_j | \nu_k \rangle = \delta_{jk}$$

ASSUMING THAT

- ν_j PROPAGATES AS PLANE WAVE
- $|\nu_j\rangle$ ARE STATIONARY STATES WITH ENERGY E

WE GET:

$$|\nu_\ell, x\rangle = \sum_{j=1}^{3} U^*_{\ell j} \, e^{-i(Et - p_j x)} |\nu_j\rangle$$

$$p_j = \sqrt{E^2 - m_j^2} \cong E - \frac{m_j^2}{2E}, \quad E \gg m_j$$

$$P(\nu_\ell \to \nu_{\ell'} ; x = L) \equiv |\langle \nu_{\ell'} | \nu_\ell, x\rangle|^2$$

$$= \left| \sum_{j=1}^{3} U_{\ell' j} U^*_{\ell j} e^{-i \frac{m_j^2}{2E} L} \right|^2$$

Assuming that ν_1, ν_2 are stable and relativistic, $\boxed{E_i = \sqrt{\vec{p}^{\,2} + m_i^2} \cong p + \frac{m_i^2}{2p}}$

it is not difficult to derive

$$P(\nu_e \to \nu_\mu; t, t_0) = \frac{1}{2} \sin^2 2\theta \left[1 - \cos 2\pi \frac{R}{L_\nu}\right] =$$

$$= \bar{P} - P^{osc}$$

$\Delta E R = E_2 t - E_1 t$, $t_0 = 0$

$$P(\nu_e \to \nu_e; t, t_0) = (1 - \bar{P}) + P^{osc}$$

$$\boxed{\bar{P} = \frac{1}{2} \sin^2 2\theta}, \quad \boxed{P^{osc} = \frac{1}{2} \sin^2 2\theta \cos 2\pi \frac{R}{L_\nu}}$$

[$R = (t - t_0)$ – the distance traveled by the ν's

[$L_\nu = 4\pi E/\Delta m^2 = 2.48$ m $E[\mathrm{MeV}]/\Delta m^2[\mathrm{eV}^2]$ – the oscillation length in vacuum

$$\Delta m^2 = m_2^2 - m_1^2 > 0$$

$P(\nu_e \to \nu_\mu; t, t_0)$ can be "large" ($\gtrsim 0.5$) if

both $\boxed{\begin{array}{l} \sin^2 2\theta \text{ is "large" } (\gtrsim 0.5) \\ 2\pi R \gtrsim L_\nu \end{array}}$

$\nu_\odot$: $R_\odot \cong 1.5 \times 10^8$ km, $\bar{p} \sim 1$ MeV :

$$\boxed{\Delta m^2 \gtrsim 10^{-11}\ \mathrm{eV}^2.}$$

SOURCE	$\bar{E}$ [MeV]	R [m]	$(\Delta m^2)_{min}$ [eV^2]
REACTOR $\bar{\nu}_e$	2	100	$\sim 10^{-2}$
		1000	$\sim 10^{-3}$
ACCELERATOR $\nu_\mu, \overset{(-)}{\nu}_\mu ; \overset{(-)}{\nu}_e$	$5 \cdot 10^3$	10^3	~ 1
		10^6	$\sim 10^{-3}$
ATMOSPHERIC ν's $\nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e$	10^4	$\sim 1.3 \cdot 10^7$	$\sim 10^{-3}$
SOLAR ν's ν_e	~ 1	$\sim 1.5 \cdot 10^{11}$	$\sim 10^{-11}$

Averaging over the region of ν-productio

$$\boxed{\bar{P}}(\nu_\ell \rightarrow \nu_{\ell'})_{\ell \neq \ell'} = \frac{1}{2\Delta r} \int_{R-\Delta r}^{R+\Delta r} P(\nu_\ell \rightarrow \nu_{\ell'}; x)\, dx \cong$$

$$\cong \underbrace{\frac{1}{2} \sin^2 2\theta} \quad \text{if} \quad \boxed{\frac{L_\nu}{2\pi \cdot 2\Delta r} \ll 1}$$

Averaging over Δp:

$$\bar{P}(\nu_\ell \rightarrow \nu_{\ell'})_{\ell \neq \ell'} = \frac{1}{\Delta p} \int_{p-\frac{1}{2}\Delta p}^{p+\frac{1}{2}\Delta p} P(\nu_\ell \rightarrow \nu_{\ell'}; R, \Delta m^2/2p', \theta)\, dp'$$

$$\cong \underbrace{\frac{1}{2} \sin^2 2\theta} \quad \text{if} \quad \boxed{2\pi \frac{R}{L_\nu} \frac{\Delta p}{p} \gg 1}$$

(P. I. KRASTEV, S.T.P. '95

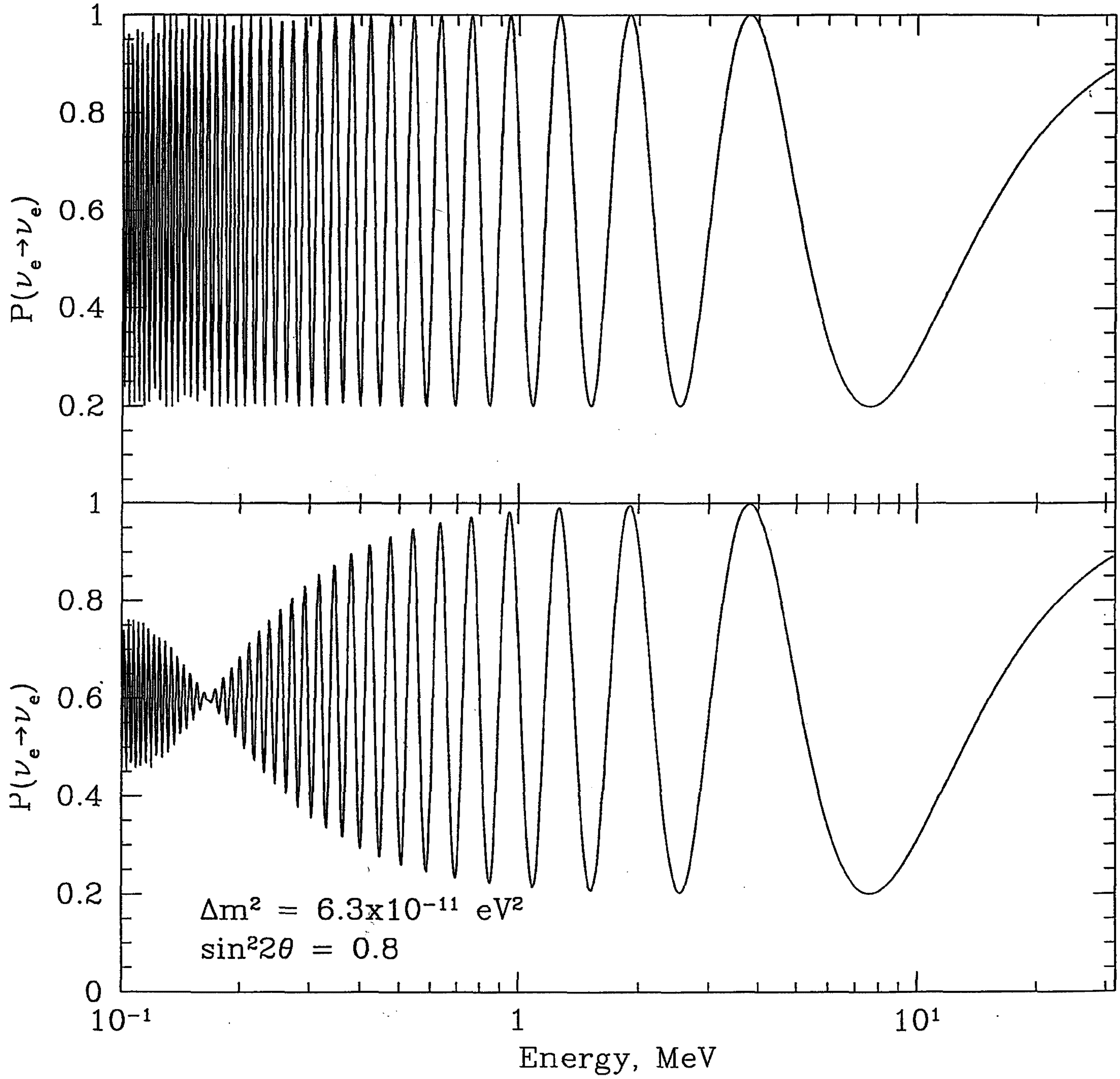
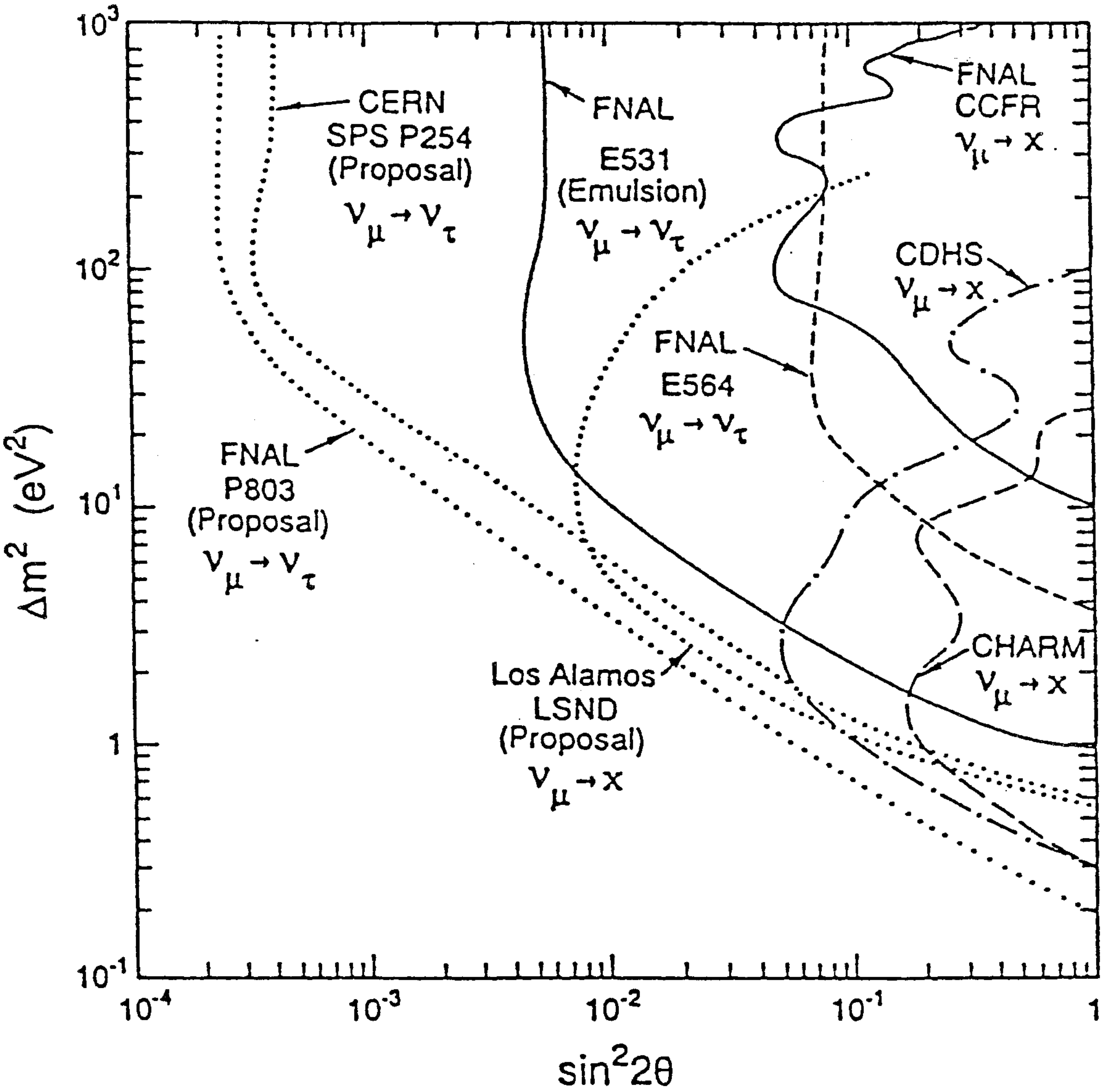
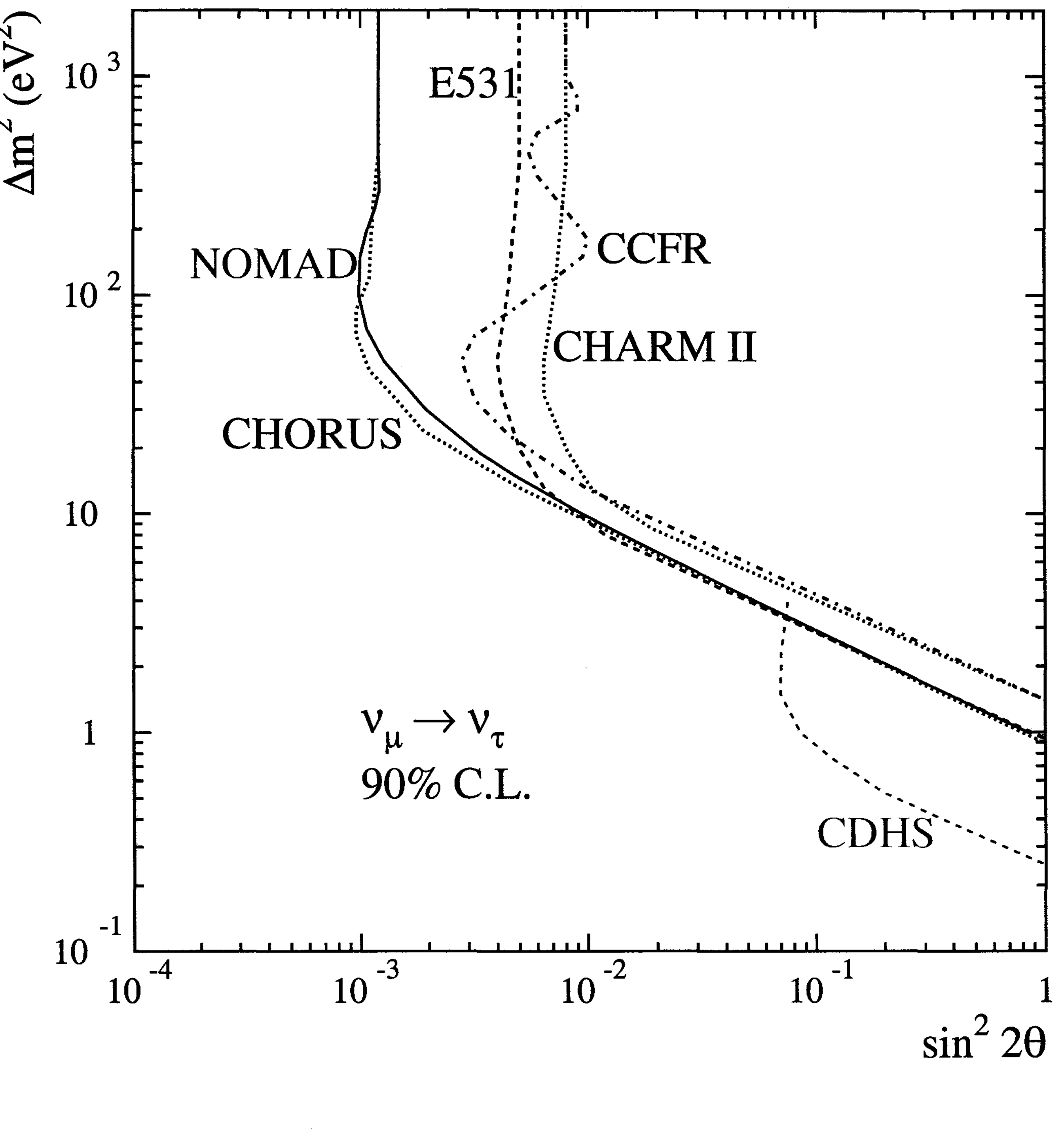


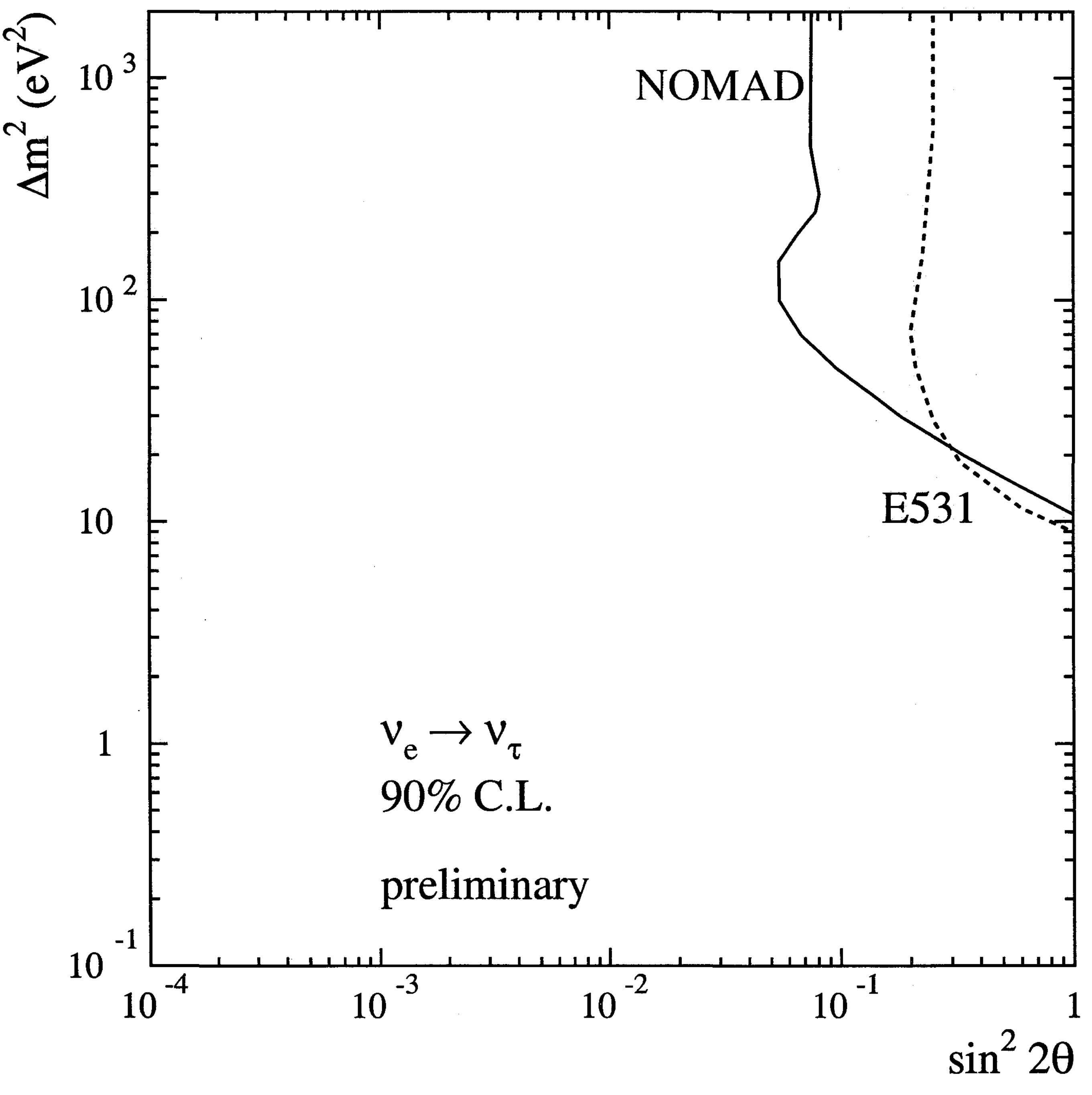
Fig.2



g. 3 Exclusion plot for neutrino oscillation parameters summarizing the results that volve coupling to ν_μ. The excluded regions are to the right of the curves. Sensitivi- :s of the proposed experiments are shown by dotted curves. For references see text.



Δm^2 (eV^2)
10^3
10^2
10
1
10^{-1}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
$\sin^2 2\theta$
E531
NOMAD
CCFR
CHARM II
CHORUS
CDHS
$\nu_\mu \rightarrow \nu_\tau$
90% C.L.


Δm^2 (eV^2)
10^3
10^2
10
1
10^{-1}
NOMAD
E531
$\nu_e \rightarrow \nu_\tau$
90% C.L.
preliminary
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
$\sin^2 2\theta$

Neutrino Oscillations in Matter

$$|\nu_{eL}\rangle = |\nu_{1L}\rangle \cos\theta + |\nu_{2L}\rangle \sin\theta$$

$$|\nu_{\mu L}\rangle = -|\nu_{1L}\rangle \sin\theta + |\nu_{2L}\rangle \cos\theta$$

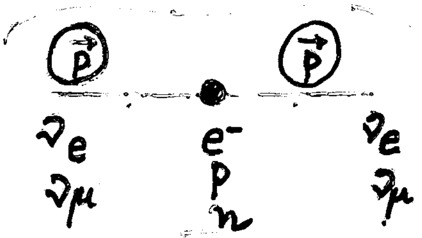
$$\vec{p} \qquad\qquad m_1\ \vec{p} \qquad\qquad m_2\ \vec{p}$$

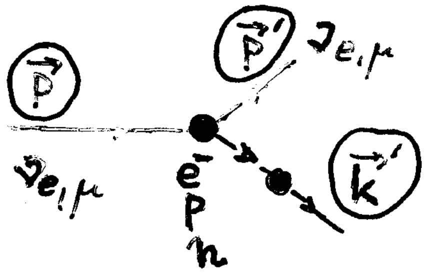
H_{vac} : $\quad E_{1,2} = \sqrt{\vec{p}^{\,2} + m_{1,2}^2}\,,\quad \nu_1, \nu_2$ - stable, relativistic

$$H_{vac}|\nu_{1,2}\rangle = E_{1,2}|\nu_{1,2}\rangle$$

The presence of matter can change drastically the pattern of ν-oscillations

$$\boxed{H_m = H_{vac} + H_{int}}$$



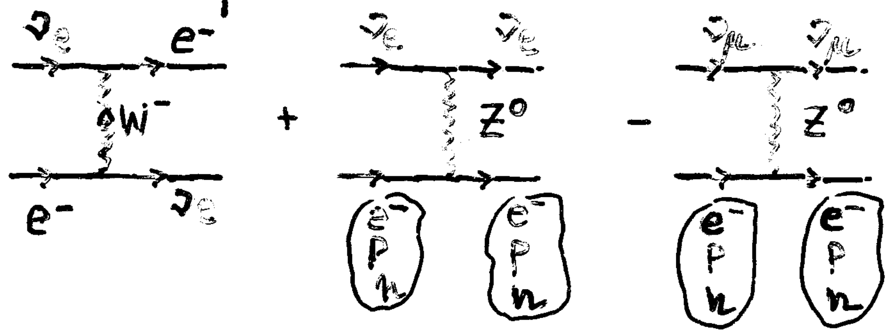


- negligible! for the Sun

$$n(\nu_e) \neq 1,\quad n(\nu_\mu) \neq 1$$

$$n(\nu_e) - n(\nu_\mu) = \frac{2\pi}{p^2}\left[F^V_{\nu_e e^-}(0) - F^V_{\nu_\mu e^-}(0)\right] =$$

$$= +\frac{2\pi}{p^2}\Big\{$$



$$\Big\}$$

$$= -\frac{1}{p}\sqrt{2}\, G_F\, N_e$$

$$\langle \nu_1 | H_m | \nu_2 \rangle \neq 0 \quad ; \quad H_m |\nu_{1,2}^m\rangle = E_{1,2}^m \cdot |\nu_{1,2}^m\rangle$$

$\nu_{1,2} \to \nu_{1,2}^m$

$E_{1,2} \to E_{1,2}^m$

$\theta \to \theta_m$

$L_v \to L_m$

$$|\nu_e\rangle = |\nu_1^m\rangle \cos\theta_m + |\nu_2^m\rangle \sin\theta_m$$

$$|\nu_\mu\rangle = -|\nu_1^m\rangle \sin\theta_m + |\nu_2^m\rangle \cos\theta_m$$

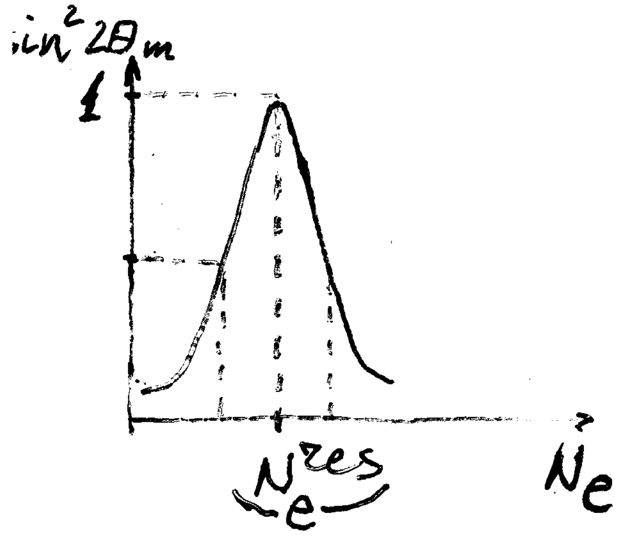
$$\sin^2 2\theta_m = \frac{tg^2 2\theta}{(1 - N_e/N_e^{res})^2 + tg^2 2\theta}$$

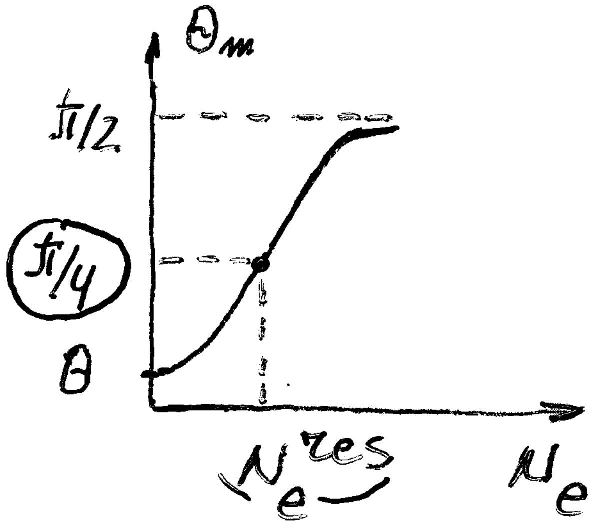
$$N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2E\sqrt{2}\,G_F}$$

$N_e >> N_e^{res}, \quad \theta_m \simeq \pi/2$

$N_e << N_e^{res}, \quad \theta_m \simeq \theta$

$N_e = N_e^{res}, \quad \theta_m \simeq \pi/4$





$$\Delta N_e^{res} = 2 N_e^{res}\, tg\, 2\theta$$

$$E_2^m - E_1^m \Big|_{res} = \min\,(E_2^m - E_1^m)$$

For $\sin\theta \ll 1$, $|\nu_e\rangle \cong |\nu_1^m\rangle$, if $N_e \ll N_e^{res}$

$|\nu_e\rangle \cong |\nu_2^m\rangle$, if $N_e \gg N_e^{res}$

$2\pi L_m^{-1} = \Delta E'$

$\Delta E' \equiv E_1^m - E_2^m$

$$P(\nu_e \to \nu_\mu) = \frac{1}{2}\sin^2 2\theta_m \left[1 - \cos 2\pi \frac{R}{L_m}\right]$$

$\Delta m^2 = m_2^2 - m_1^2 > 0$

$$L_m = \frac{L_v}{\sqrt{(1 - N_e/N_e^{res})^2 \cos^2 2\theta + \sin^2 2\theta}}, \quad L_v = 4\pi \frac{p}{\Delta m^2}$$

$$\boxed{L_m^{res} = \frac{L_v}{\sin 2\theta}}; \qquad L_m \begin{cases} \ll L_v, & N_e \gg N_e^{res} \\ \cong L_v, & N_e \ll N_e^{res} \end{cases}$$

So, for $N_e \ll N_e^{res}$ – $\nu_e \rightleftarrows \nu_\mu$ like in vacuum

$N_e \gg N_e^{res}$ – $\nu_e \rightleftarrows \nu_\mu$ damped

$N_e \cong N_e^{res}$ – the oscillations can be resonantly enhanced

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{\left(1 - \frac{N_e}{N_e^{res}}\right)^2 \cos^2 2\theta + \sin^2 2\theta}, \quad N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2p\sqrt{2}G_F}$$

$\max P(\nu_e \to \nu_\mu) = \sin^2 2\theta_m$, $\Delta E' R = \pi(2k+1)$

$\max P(\nu_e \to \nu_\mu) \cong 1$, $\sin^2 2\theta_m \cong 1$, $\Delta E' R = \pi(2k+1)$

MSW EFFECT

$$i\frac{d}{dt}\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0) \end{pmatrix} = \begin{pmatrix} E_1c^2+E_2s^2+\boxed{\sqrt{2}G_F\left(N_e-\frac{N_n}{2}\right)} & \frac{1}{2}(E_2-E_1)\sin 2\theta \\ \frac{1}{2}(E_2-E_1)\sin 2\theta & E_1s^2+E_2c^2+\boxed{\sqrt{2}G_F\left(-\frac{N_n}{2}\right)} \end{pmatrix}\begin{pmatrix} A_e \\ A_\mu \end{pmatrix}$$

$A_\ell(t,t_0) = \langle \nu_\ell | \nu(t)\rangle$ $\qquad\langle \nu_{\ell'} | H_m | \nu_\ell\rangle$, $\ell', \ell = e, \mu$

$c \equiv \cos\theta$, $s \equiv \sin\theta$

$$A_\ell(t,t_0) = A'_\ell(t,t_0)\, e^{-i\int_{t_0}^{t}\left\{E_1c^2+E_2s^2+\sqrt{2}G_F\left(N_e-\frac{N_n}{2}\right)\right\}dt'}$$

$$E_2-E_1 \cong \Delta m^2/2p, \quad \Delta m^2 = m_2^2-m_1^2$$

$$i\frac{d}{dt}\begin{pmatrix} A'_e(t,t_0) \\ A'_\mu(t,t_0) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\Delta m^2}{2p}\sin 2\theta \\ \frac{1}{2}\frac{\Delta m^2}{2p}\sin 2\theta & \frac{\Delta m^2}{2p}\cos 2\theta - \boxed{\sqrt{2}G_FN_e} \end{pmatrix}\begin{pmatrix} A'_e \\ A'_\mu \end{pmatrix}$$

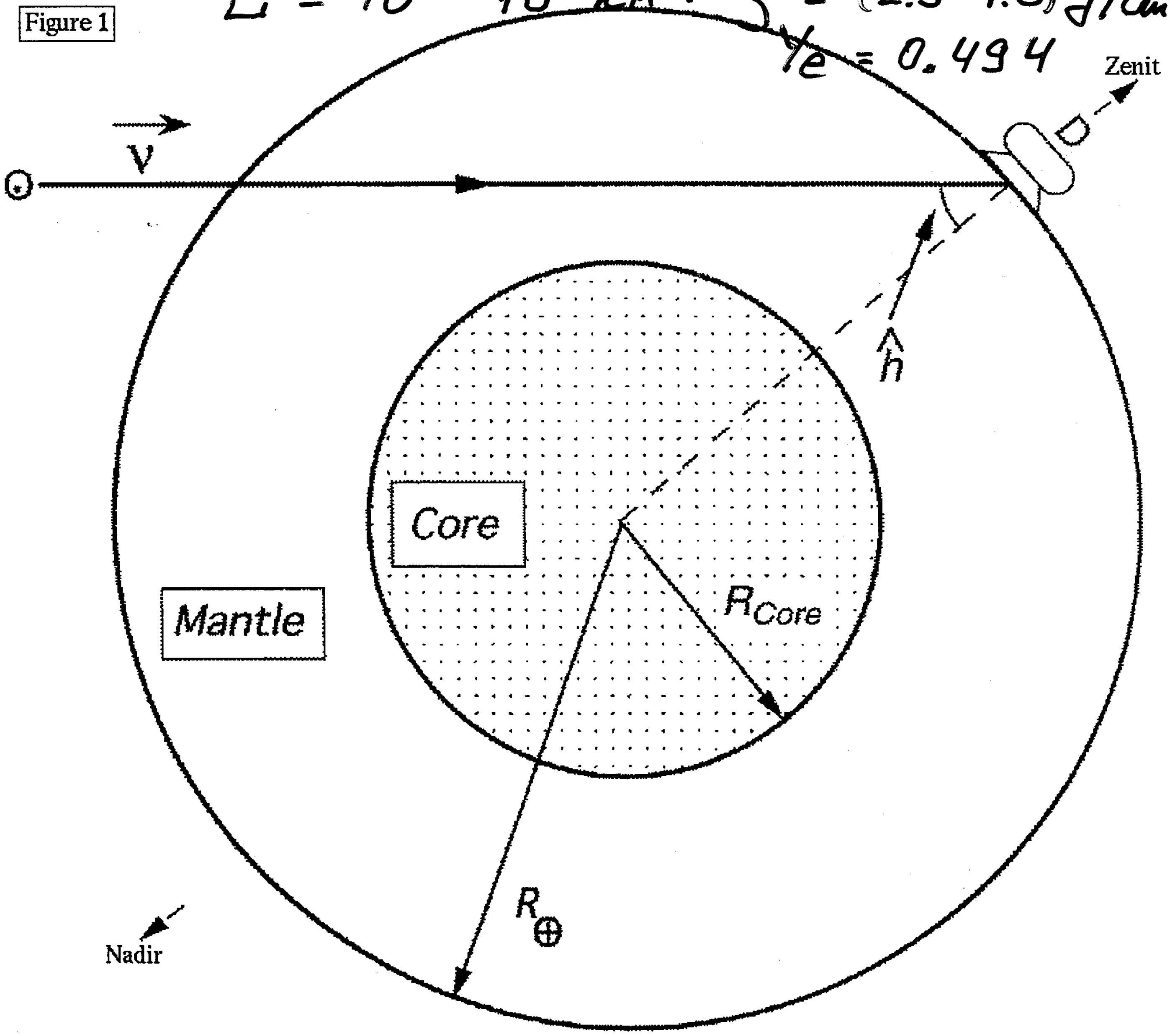
$A'_e(t_0,t_0) = A^0_e$, $\quad A'_\mu(t_0,t_0) = A^0_\mu$

WHEN THE ν'S CROSS ONLY THE EARTH MANTLE:

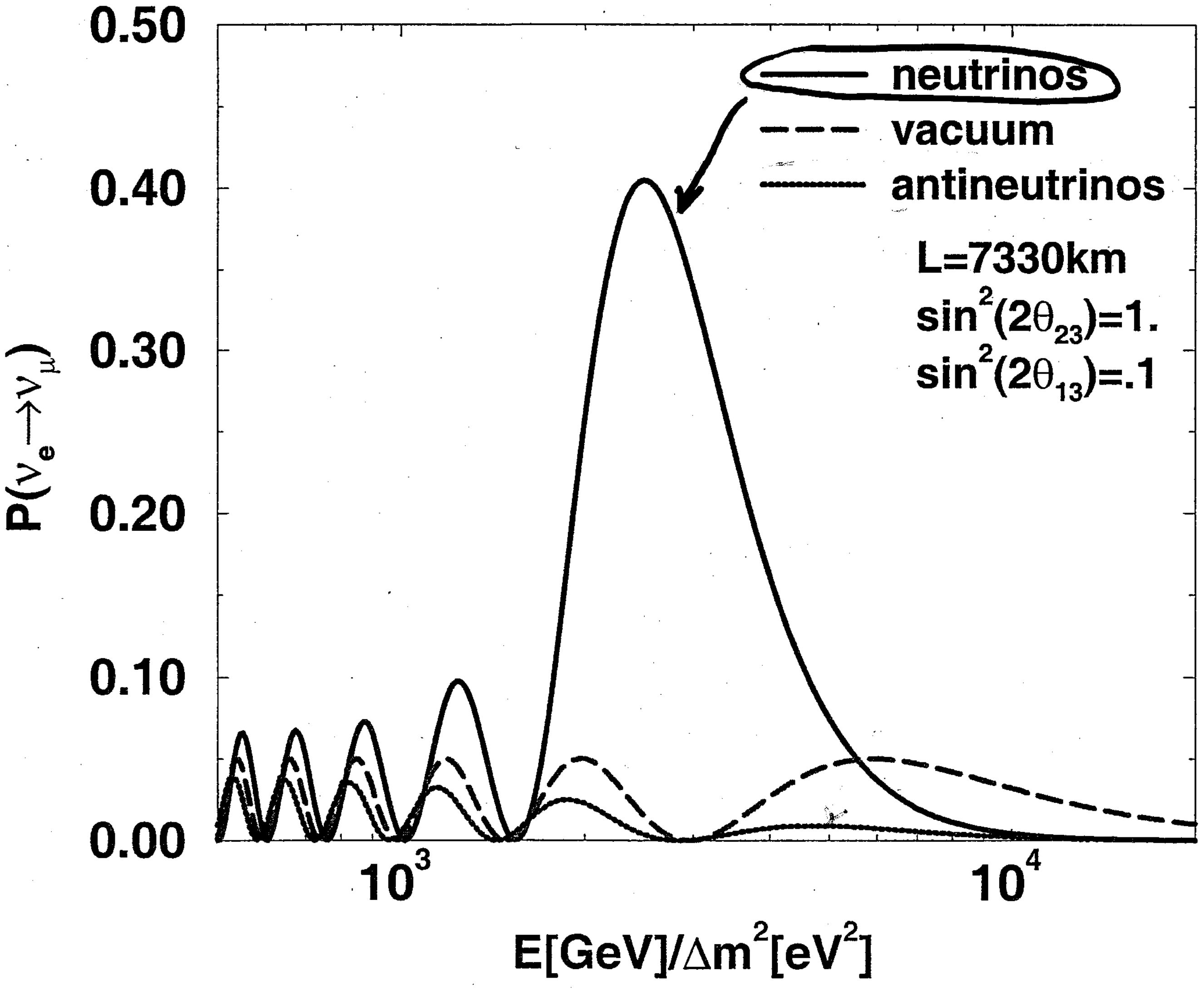
$$L = 2R_E \cos h, \quad R_E = 6371\ km$$
$$h \gtrsim 33^\circ, \quad L \lesssim 10600\ km$$

$$L = 10^3 - 10^4\ km: \quad \bar{\rho} \simeq (2.9-4.8)\ g/cm^3$$
$$Y_e = 0.494$$

Figure 1



MOCIOIU, SHROCK. '00



Suppose: $N_e = N_e(\vec{r}_\nu(t)) = N_e(t)$

In the Sun: $20\,cm^{-3} N_A \lesssim N_e(t_0) \lesssim 100\,cm^{-3} N_A$

N_e decreases monotonically (exponentially) along the ν path

$N_e(t_s) = 0$

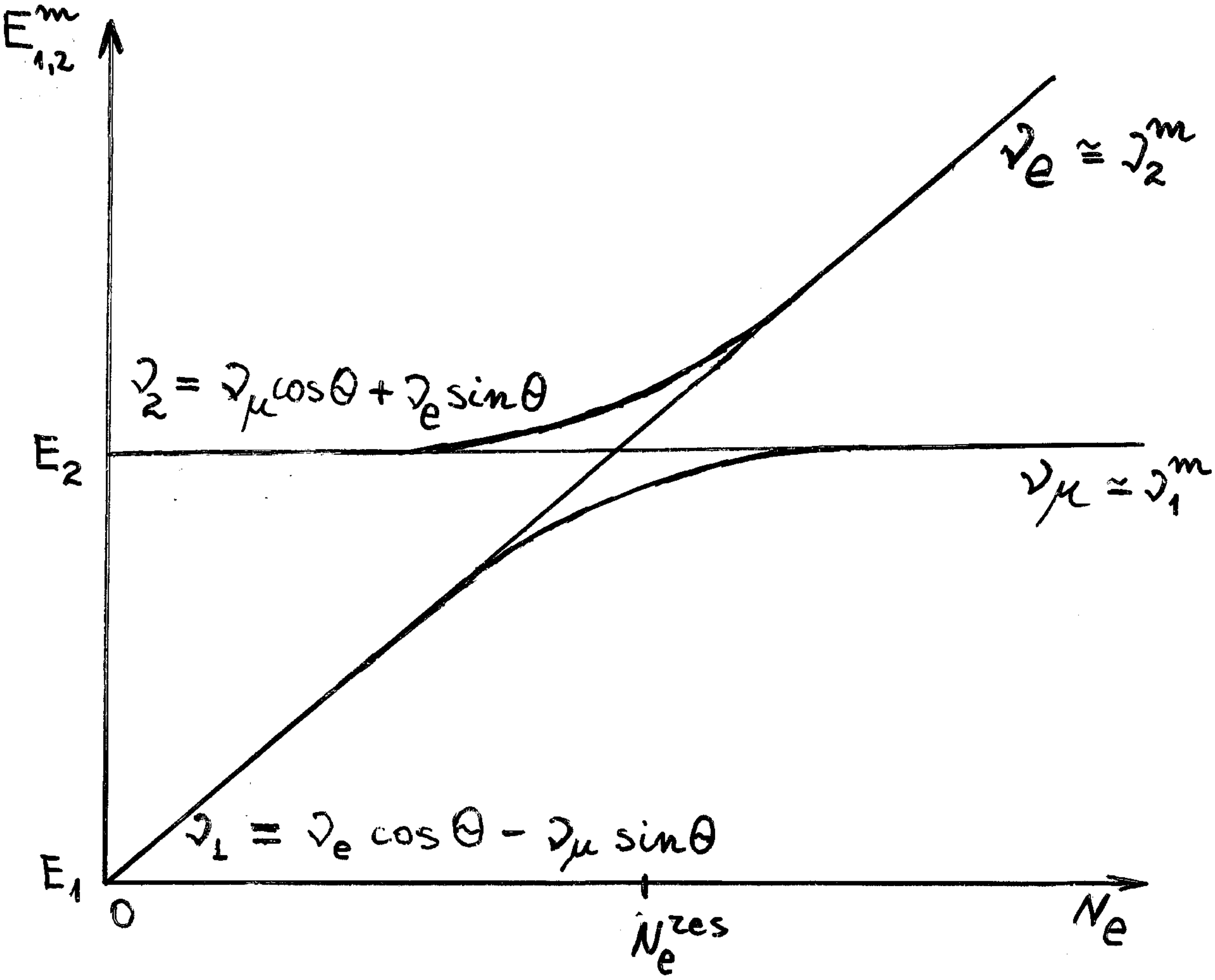
$\nu_\odot \equiv \nu_e$

In this case $\left|\begin{array}{l} \theta_m = \theta_m(t),\ |\nu^m_{1,2}\rangle = |\nu^m_{1,2}(t)\rangle, \\ L_m = L_m(t),\ E^m_{1,2} = E^m_{1,2}(t) \end{array}\right.$

Qualitatively,

$$H = H(t) = \begin{pmatrix} 0 & \frac{\Delta m^2}{4p}\sin 2\theta \\ \frac{\Delta m^2}{4p}\sin 2\theta & \frac{\Delta m^2}{2p}\cos 2\theta - \sqrt{2}G N_e \end{pmatrix}$$

$\boxed{\sin\theta \ll 1.}$



1. $P(\nu_2^m \to \nu_1; t_\odot, t_0) \equiv P'$ - negligible : adiabatic transition
2. P' - nonnegligible : nonadiabatic

NUMERICAL CALCULATIONS HAVE SHOWN THAT $\nu_\odot$ CAN UNDERGO BOTH TYPES OF TRANSITIONS AND THAT SOLAR MATTER EFFECTS IN $\nu_\odot$ - OSCILLATIONS CAN BE SUBSTANTIAL FOR

$$\boxed{\begin{array}{l} 10^{-8}\,eV^2 \lesssim \Delta m^2 \lesssim 10^{-4}\,eV^2 \\ \sin^2 2\theta \gtrsim 10^{-4} \end{array}}$$

(MIKHEYEV, SMIRNOV '85, P. VOGEL '86, J. BOUCHEZ ET AL. '86; ROSEN, GELB '86

ADIABATIC CONDITION: $4n(t) \equiv \gamma(t) \gg 1$

$$4n(t) = \frac{|E_2^m - E_1^m|}{|2\dot{\theta}_m(t)|} =$$

$$= \sqrt{2}\,G_F \frac{(N_e^{res})^2}{|dN_e(t)/dt|}\, tg^2 2\theta \left[1 + \frac{\left(1 - \frac{N_e(t)}{N_e^{res}}\right)^2}{tg^2 2\theta}\right]^{3/2}$$

$$N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2E\sqrt{2}\,G_F}$$

The Adiabatic Condition:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix} \begin{pmatrix} |\nu_1^m(t)\rangle \\ |\nu_2^m(t)\rangle \end{pmatrix}$$

$$\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0) \end{pmatrix} = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix} \begin{pmatrix} A_1^m(t,t_0) \\ A_2^m(t,t_0) \end{pmatrix}$$

$$i\frac{d}{dt}\begin{pmatrix} A_1^m \\ A_2^m \end{pmatrix} = \begin{pmatrix} 0 & -i\underbrace{\dot{\theta}_m}(t) \\ i\underbrace{\dot{\theta}_m}(t) & E_2^m(t)-E_1^m(t) \end{pmatrix} \begin{pmatrix} A_1^m \\ A_2^m \end{pmatrix}$$

$P(\nu_2^m(t_0) \longrightarrow \nu_1^m(t_0))$ – "small" (negligible)

if $$\boxed{[\gamma_m(t)]^{-1} \equiv \frac{2|\dot{\theta}_m(t)|}{|E_2^m(t)-E_1^m(t)|} \ll 1,}$$

for all t

The system of evolution equations:

$$i\frac{d}{dt}\begin{pmatrix} a_e(t,t_0) \\ a_\mu(t,t_0)\end{pmatrix} = \begin{pmatrix} 0 & \frac{\Delta m^2}{2p}\frac{1}{2}\sin 2\theta \\ \frac{\Delta m^2}{2p}\frac{1}{2}\sin 2\theta & \frac{\Delta m^2}{2p}\cos 2\theta - \sqrt{2}G_F N_e(t) \end{pmatrix}\begin{pmatrix} a_e \\ a_\mu \end{pmatrix}$$

$$|\nu\rangle_t = a_e(t,t_0)|\nu_e\rangle + a_\mu(t,t_0)|\nu_\mu\rangle$$

<u>The adiabatic condition:</u>

$$|\nu\rangle_t = a_1^m(t,t_0)|\nu_1^m(t_0)\rangle + a_2^m(t,t_0)|\nu_2^m(t_0)\rangle$$

$$\longrightarrow \quad \begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix}\begin{pmatrix} a_1^m \\ a_2^m \end{pmatrix}$$

$$i\frac{d}{dt}\begin{pmatrix} a_1^m(t,t_0) \\ a_2^m(t,t_0)\end{pmatrix} = \begin{pmatrix} M_1^m(t) & -i\dot{\theta}_m(t) \\ i\dot{\theta}_m(t) & M_2^m(t) \end{pmatrix}\begin{pmatrix} a_1^m(t,t_0) \\ a_2^m(t,t_0) \end{pmatrix}$$

$$M_2^m(t) - M_1^m(t) = E_2^m(t) - E_1^m(t)$$

<u>Adiabatic condition</u>: $4\kappa(t) = |(E_2^m(t) - E_1^m(t))/2\dot{\theta}_m(t)| \gg 1$.

$$\dot{\theta}_m(t) \sim dN_e(t)/dt$$

MSW Transitions of Solar Neutrinos in the Sun and the Hydrogen Atom

$$i\frac{d}{dt}\begin{pmatrix} A_\alpha(t,t_0) \\ A_\beta(t,t_0) \end{pmatrix} = \begin{pmatrix} -\epsilon(t) & \epsilon'(t) \\ \epsilon'(t) & \epsilon(t) \end{pmatrix}\begin{pmatrix} A_\alpha(t,t_0) \\ A_\beta(t,t_0) \end{pmatrix} \tag{11}$$

where

$\epsilon(t) = \frac{1}{2}\ [\frac{\Delta m^2}{2E}\cos 2\theta - \sqrt{2}G_F N_e(t)]$,

$\epsilon'(t) = \frac{\Delta m^2}{4E}\sin 2\theta$, **with** $\Delta m^2 = m_2^2 - m_1^2$.

- **Standard Solar Models**

$N_e(t) = N_e(t_0)\exp\left\{-\frac{t-t_0}{r_0}\right\}$, $r_0 \sim 0.1 R_\odot$, $R_\odot = 6.96 \times 10^5\ km$

Introducing the dimensionless variable

$Z = ir_0\sqrt{2}G_F N_e(t_0)e^{-\frac{t-t_0}{r_0}}$, $Z_0 = Z(t=t_0)$,

and making the substitution

$A_e(t,t_0) = (Z/Z_0)^{c-a}\ e^{-(Z-Z_0)+i\int_{t_0}^{t}\epsilon(t')dt'}\ A'_e(t,t_0)$,

we find that the amplitude $A'_e(t,t_0)$ satisfies the confluent hypergeometric equation (CHE):

$\left\{\ Z\frac{d^2}{dZ^2} + (c-Z)\ \frac{d}{dZ} - a\right\} A'_e(t,t_0) = 0$,

where

$a = 1 + ir_0\ \frac{\Delta m^2}{2E}\sin^2\theta$, $c = 1 + ir_0\ \frac{\Delta m^2}{2E}$.

In the case of $\nu_\odot$, $N_e(t) = 0$, and

$A(\nu_e \to \nu_\mu)$

$= \frac{1}{2}\sin 2\theta \left\{ \Phi(a-c, 2-c; Z_0) - e^{i(t-t_0)\frac{\Delta m^2}{2E}} \Phi(a-1, c; Z_0) \right\}$,

S.T.P, 1988; T. Kaneko, 1987; S. Toshev, 1987.

$\Phi(a-1, c; Z)$ **and** $Z^{1-c}\ \Phi(a-c, 2-c; Z)$ -

linearly independent, specified by:

$\Phi(a', c'; Z \to 0) \to 1, \quad a', c' \neq 0, -1, -2, \ldots.$

- **Vacuum limit:** $N_e(t_0) = 0$, **or** $Z_0 = ir_0\sqrt{2}G_F N_e(t_0) = 0$,

$\Phi(a-1, c; 0) = 1$, $\Phi(a-c, 2-c; 0) = 1$,

and one recovers the vacuum expression for $A(\nu_e \to \nu_\mu)$.

- **Standard Solar Models**

$20\ \mathrm{cm}^{-3}\mathrm{N_A} \lesssim \mathrm{N_e(t_0)} \lesssim 100\ \mathrm{cm}^{-3}\mathrm{N_A}, \quad r_0 \gtrsim 0.1 R_\odot$,

which implies

$|Z_0| = r_0\sqrt{2}G_F N_e(t_0) \gtrsim 500$

Using the asymtotic series expansion of $\Phi(a', c'; Z_0)$ one finds:

S.T.P, 1988

$$P_{\odot}(\nu_e \to \nu_e; t, t_0) = \bar{P}_{\odot} + P_1^{osc} + P_2^{osc} + P_3^{osc} + P_A^{osc},$$

where

$$\bar{P}_{\odot} = \frac{1}{2} + (\frac{1}{2} - P')\cos 2\theta_m(t_0)\cos 2\theta,$$

is the average probability and

$$P_1^{osc} = -\sqrt{P'(1-P')}\cos 2\theta_m(t_0)\sin 2\theta \;\; \cos(\Phi_{21} - \Phi_{22}),$$

$$P_2^{osc} = \sqrt{P'(1-P')}\sin 2\theta_m(t_0)\cos 2\theta \;\; \cos(\Phi_{21} + \Phi_{22}),$$

$$P_3^{osc} = -\frac{1}{2}P'\sin 2\theta_m(t_0)\sin 2\theta \;\; (\cos 2\Phi_{21} + \cos 2\Phi_{22}),$$

$$P_A^{osc} = \frac{1}{2}\sin 2\theta_m(t_0)\sin 2\theta \;\; \cos 2\Phi_{22},$$

are oscillating terms,

$$P' \equiv |A(\nu_1^m(t_0) \to \nu_2)|^2 \equiv |A_{12}^m|^2 = |A_{21}^m|^2, \; \Phi_{2j} = arg(A_{2j}^m).$$

In the exponential density approximation

$$P' = \frac{e^{-2\pi r_0 \frac{\Delta m^2}{2E}\sin^2\theta} - e^{-2\pi r_0 \frac{\Delta m^2}{2E}}}{1 - e^{-2\pi r_0 \frac{\Delta m^2}{2E}}} \quad ,$$

$$\Phi_{21} - \Phi_{22} = \frac{\Delta m^2}{2E}L + \Phi = \frac{\Delta m^2}{2E}(L - R_{\odot}) + \xi_{\text{sun}},$$

$$-\Phi = \varphi_1 - \varphi_2 + 2\varphi_3 + r_0\frac{\Delta m^2}{2E}ln(r_0\sqrt{2}G_F N_e(t_0)),$$

$$\varphi_1 = arg\Gamma(a-1), \;\; \varphi_2 = arg\Gamma(a-c), \;\; \varphi_3 = arg\Gamma(1-c).$$

The expression for $P_{\odot}(\nu_e \to \nu_e; t, t_0)$ in terms of P', Φ_{21}, Φ_{22}, follows from general QM considerations.

S.T.P, 1997

The confluent hypergeometric equation describing the ν_e oscillations in the Sun, coincides in form with the Schroedinger (energy eigenvalue) equation obeyed by the radial part, $\psi_{kl}(r)$, of the non-relativistic wave function of the hydrogen atom,

$\Psi(\vec{r}) = \frac{1}{r}\psi_{kl}(r)Y_{lm}(\theta', \phi')$,

r, θ' and ϕ' are the spherical coordinates of the electron in the proton's rest frame, l and m are the orbital momentum quantum numbers ($m = -l, ..., l$), k is the quantum number labeling (together with l) the electron energy (the principal quantum number is equal to $(k + l)$), E_{kl} ($E_{kl} < 0$), and $Y_{lm}(\theta', \phi')$ are the spherical harmonics. The function

$\psi'_{kl}(Z) = Z^{-c/2}\ e^{Z/2}\ \psi_{kl}(r)$

satisfies the confluent hypergeometric equation in which the variable Z and the parameters a and c are in this case related to the physical quantities characterizing the hydrogen atom:

$$Z = 2\ \frac{r}{a_0}\sqrt{-E_{kl}/E_I}, \quad a \equiv a_{kl} = l + 1 - \sqrt{-E_I/E_{kl}}, \quad c \equiv c_l = 2(l + 1),$$

$a_0 = \hbar/(m_e e^2)$ is the Bohr radius and $E_I = m_e e^4/(2\hbar^2) \cong 13.6\ eV$ is the ionization energy of the hydrogen atom.

It is quite remarkable that the behavior of such different physical systems as solar neutrinos undergoing MSW transitions in the Sun and the non-relativistic hydrogen atom are governed by one and the same differential equation.

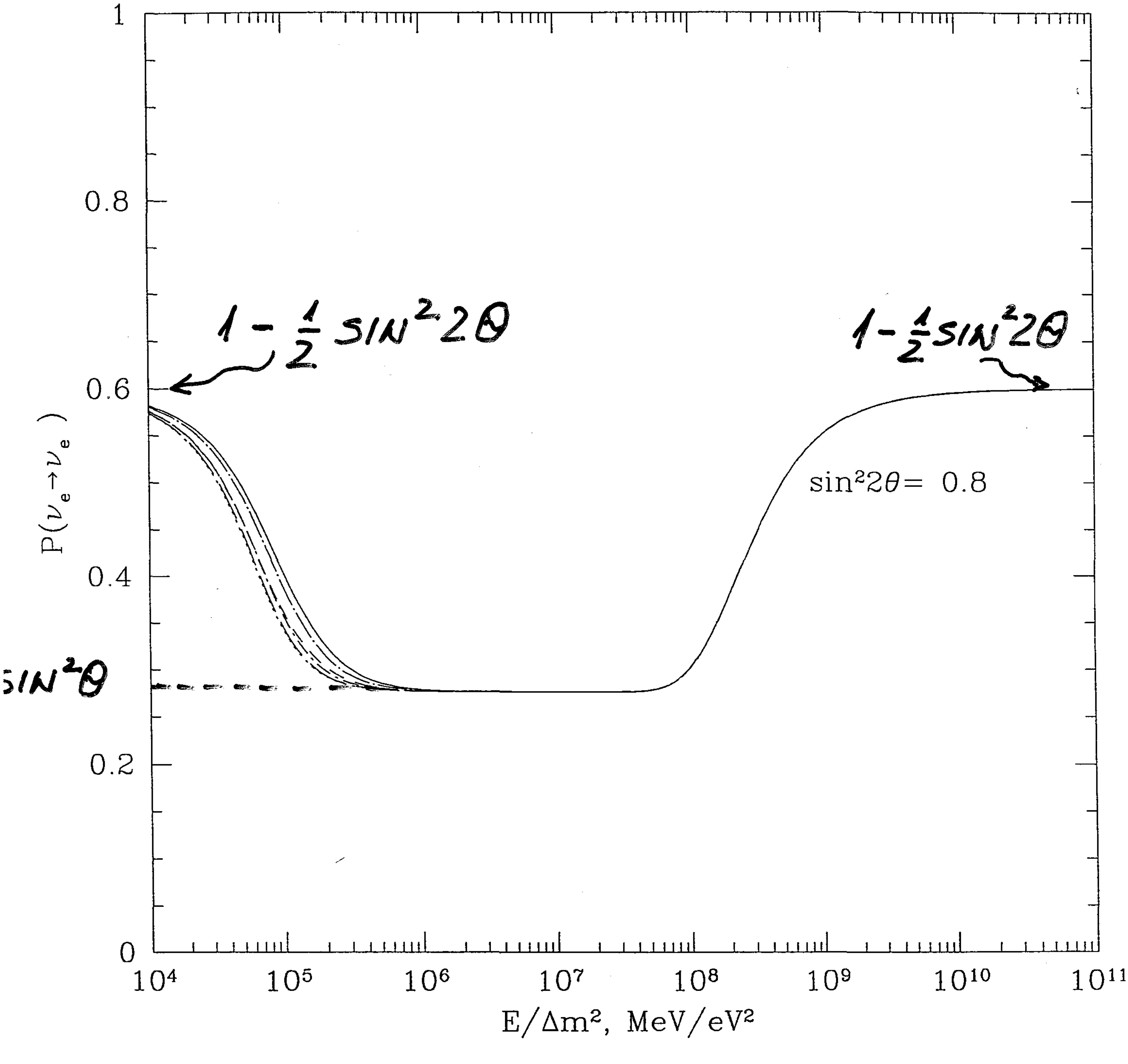
S.T.P., 1997

MSW Transitions :

S.T.P. '88

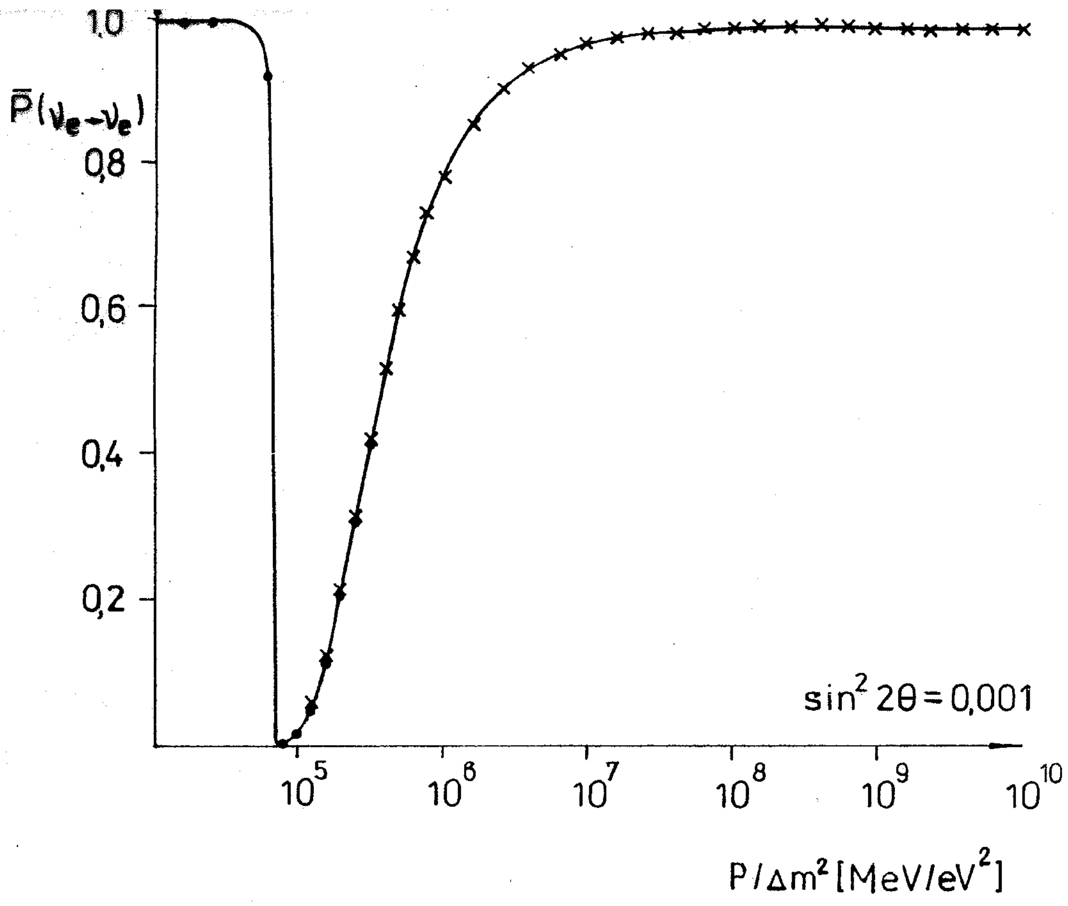
$$\boxed{\begin{aligned}\bar{P}(\nu_e \to \nu_e) &= \frac{1}{2} + \left(\frac{1}{2} - P'\right) \cos 2\theta_m(t_0) \cos 2\theta \\ P' &= \frac{e^{-2\pi r_0 \frac{\Delta m^2}{2p} \sin^2\theta} - e^{-2\pi r_0 \frac{\Delta m^2}{2p}}}{1 - e^{-2\pi r_0 \frac{\Delta m^2}{2p}}}\end{aligned}}$$

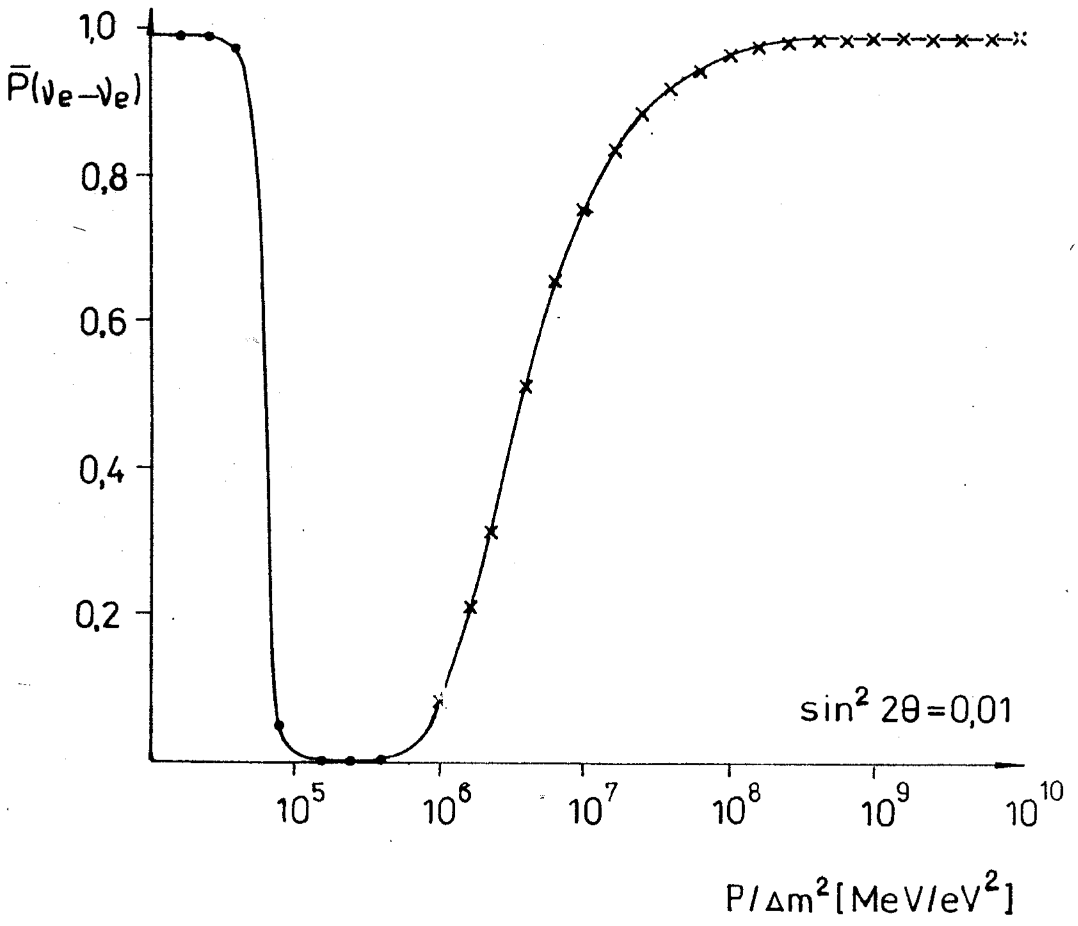
$$N_e(t) = N_e(t_0)\, e^{-\frac{t-t_0}{r_0}}, \quad r_0 \sim 0.1 R_\odot$$



1
0.8
0.6
0.4
0.2
0
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
10^{11}
$E/\Delta m^2$, MeV/eV^2
$P(\nu_e \rightarrow \nu_e)$
$\sin^2 2\theta = 0.8$
$1 - \frac{1}{2}\sin^2 2\theta$
$1 - \frac{1}{2}\sin^2 2\theta$
$\sin^2\theta$

(P. Krastev, S.T.P., '88





$$P'_{LZ} = e^{-2\pi n_0}, \tag{26}$$

where

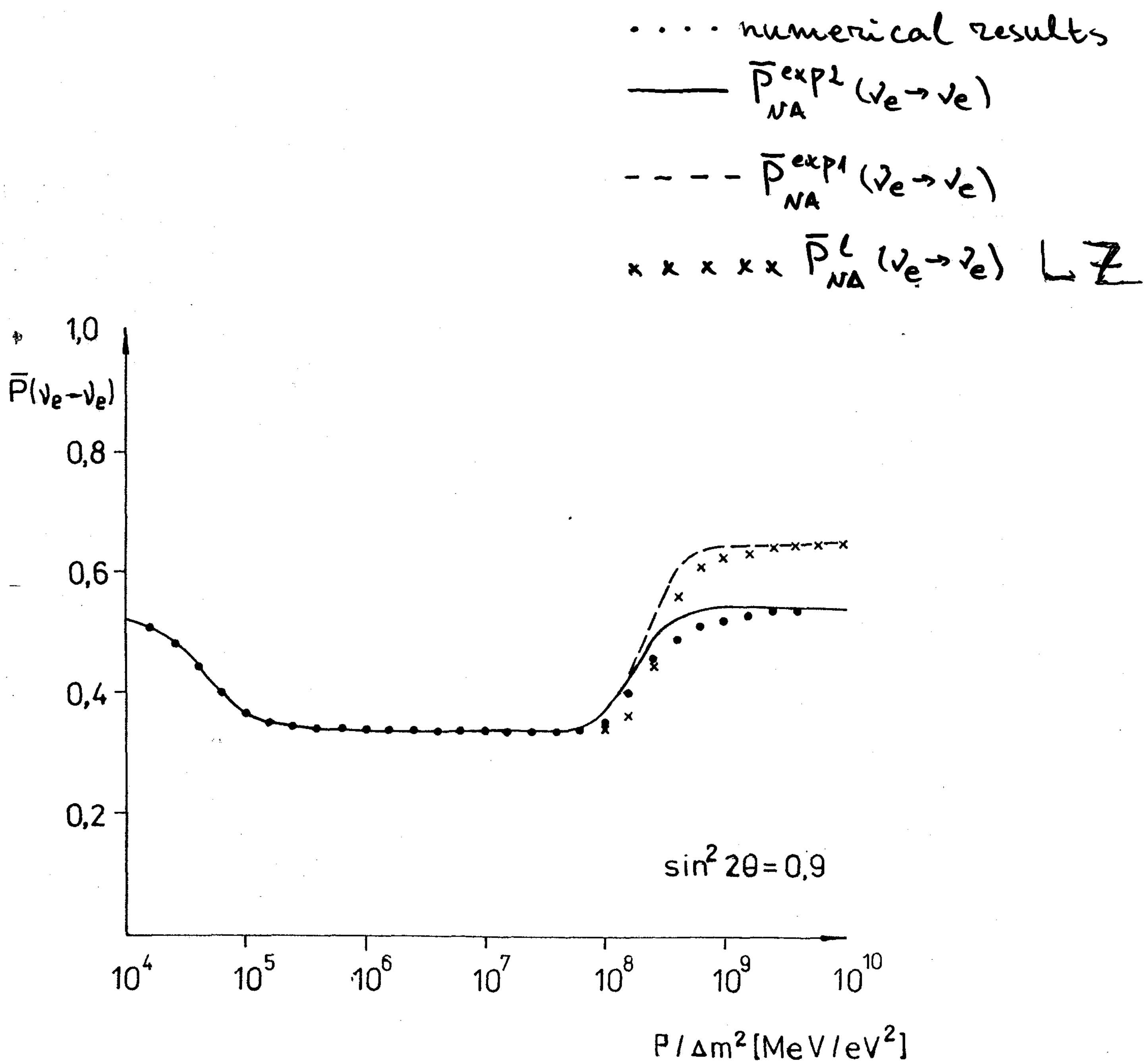
$$4n_0 = 4n(t = t_{res}) = 2\,\frac{\epsilon'^2(t = t_{res})}{|\dot{\epsilon}(t = t_{res})|} = r_0\,\frac{\Delta m^2}{2E}\,\frac{\sin^2 2\theta}{\cos 2\theta}\,, \tag{27}$$

is the adiabaticity parameter, which is just the value of the adiabaticity function $4n(t)$ at the resonance point,

$$4n(t) = \frac{E_2^m - E_1^m}{2\,|\dot{\theta}_m(t)|} = 2\,\frac{(\epsilon^2(t) + \epsilon'^2(t))^{\frac{3}{2}}}{|\epsilon(t)\dot{\epsilon}'(t) - \dot{\epsilon}(t)\epsilon'(t)|}$$

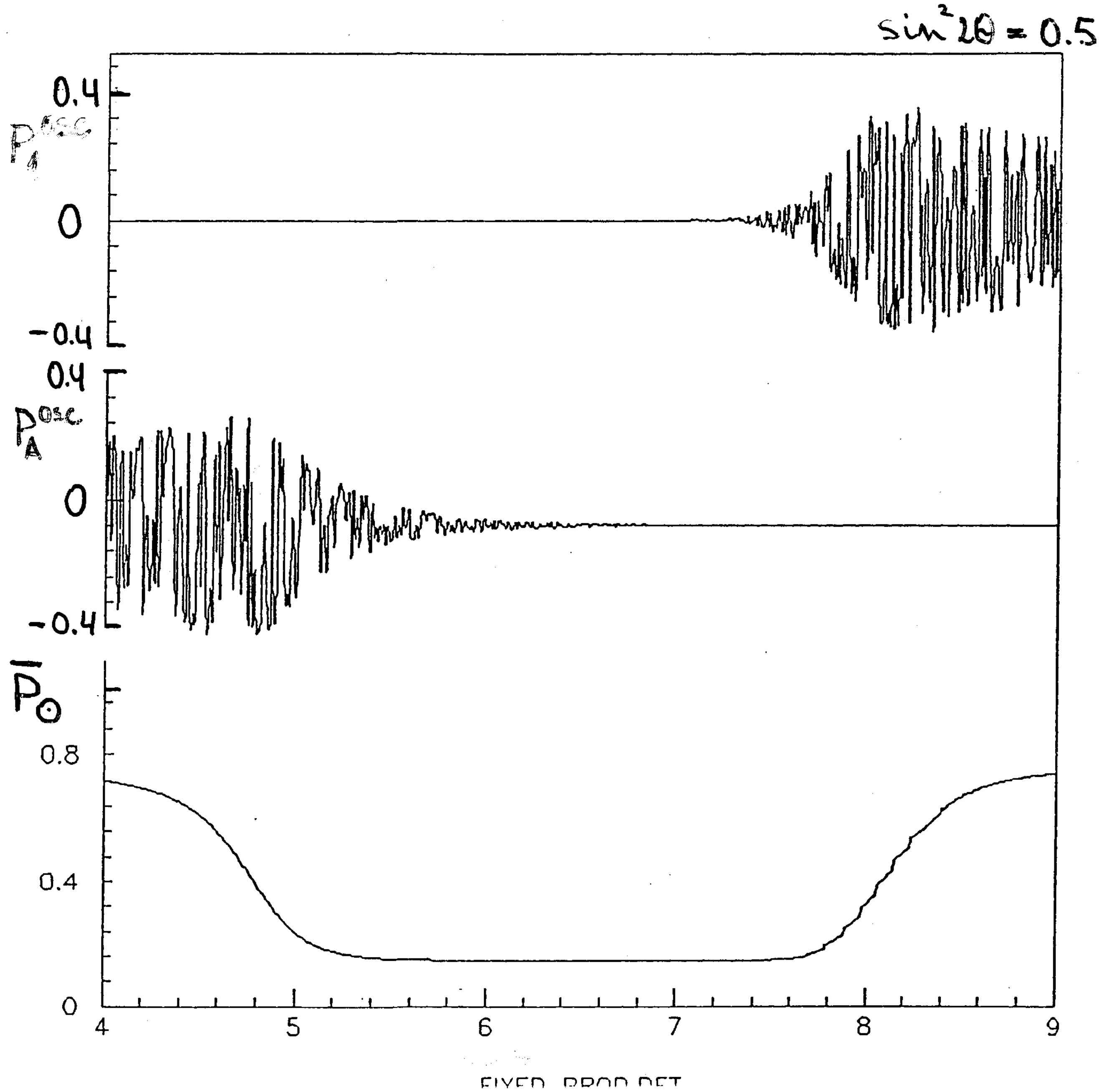
$$= \sqrt{2}G_F\frac{(N_e^{res})^2}{|\dot{N}_e(t)|}\tan^2 2\theta\left[1 + \frac{(1 - \frac{N_e(t)}{N_e^{res}})^2}{\tan^2 2\theta}\right]^{\frac{3}{2}}, \tag{28}$$

where $\dot{N}_e(t) = \frac{d}{dt}N_e(t) = \frac{d}{dx}N_e(x)$. In order for a given type of transition to be adiabatic the inequality $4n(t) \gg 1$ should be fulfilled at each point of the neutrino trajectory. The analytic expression for the "jump" probability, derived for matter (electron, neutron number) density changing exponentially along the neutrino trajectory P'_{exp}, provides a more accurate description of the matter- enhanced neutrino transitions in a medium (e.g., transitions of solar neutrinos in the Sun) than P'_{LZ}.



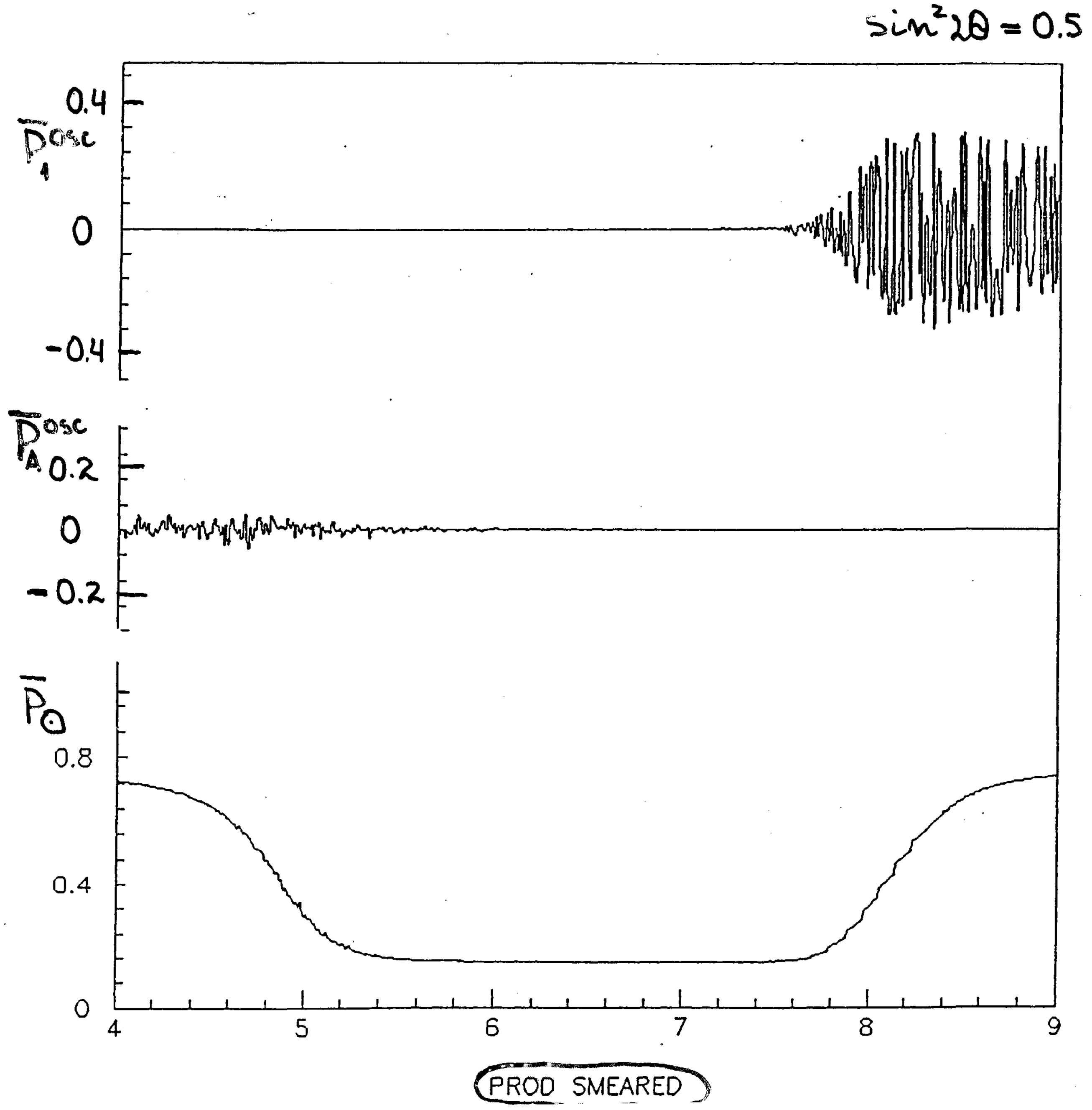
. . . . numerical results
———— $\bar{P}_{NA}^{exp2}(\nu_e \to \nu_e)$
- - - - $\bar{P}_{NA}^{exp1}(\nu_e \to \nu_e)$
x x x x x $\bar{P}_{NA}^{\ell}(\nu_e \to \nu_e)$ LZ
1,0
$\bar{P}(\nu_e \to \nu_e)$
0,8
0,6
0,4
0,2
$\sin^2 2\theta = 0,9$
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
$P/\Delta m^2$ [MeV/eV^2]

D.T.P., J. RICH, '89



$\sin^2 2\theta = 0.5$
0.4
$P_1^{osc.}$
0
-0.4
0.4
$P_A^{osc.}$
0
-0.4
$\bar{P}_\odot$
0.8
0.4
0
4
5
6
7
8
9
FIXED PROP DET

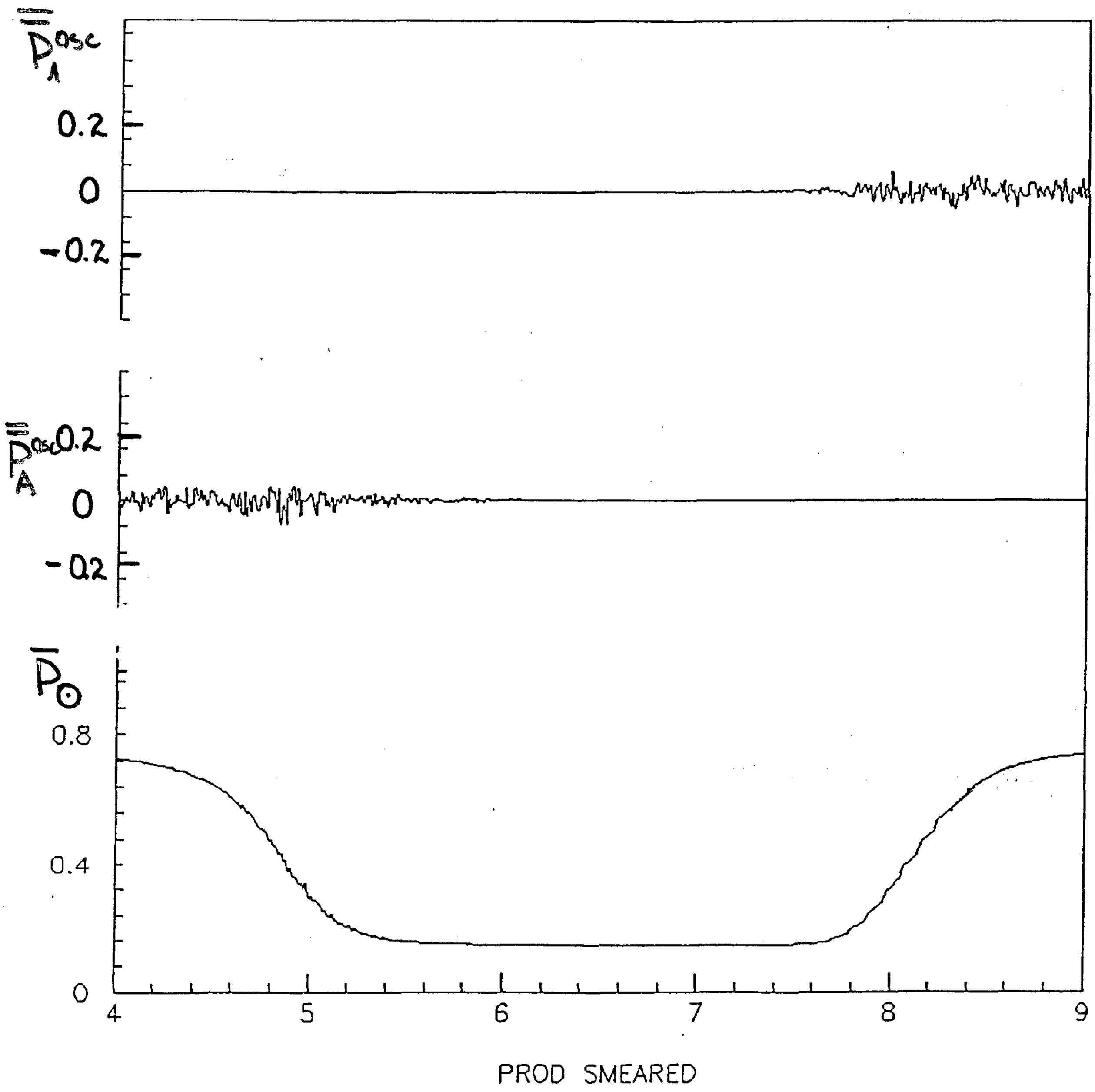
S.T.P., J. RICH, '89


$\sin^2 2\theta = 0.5$
0.4
$\bar{P}_1^{osc}$
0
-0.4
$\bar{P}_A^{osc}$
0.2
0
-0.2
$\bar{P}_\odot$
0.8
0.4
0
4
5
6
7
8
9
PROD SMEARED

J.I.F., J.RICH, '83

$\underline{\sin^2 2\theta = 0.5}$

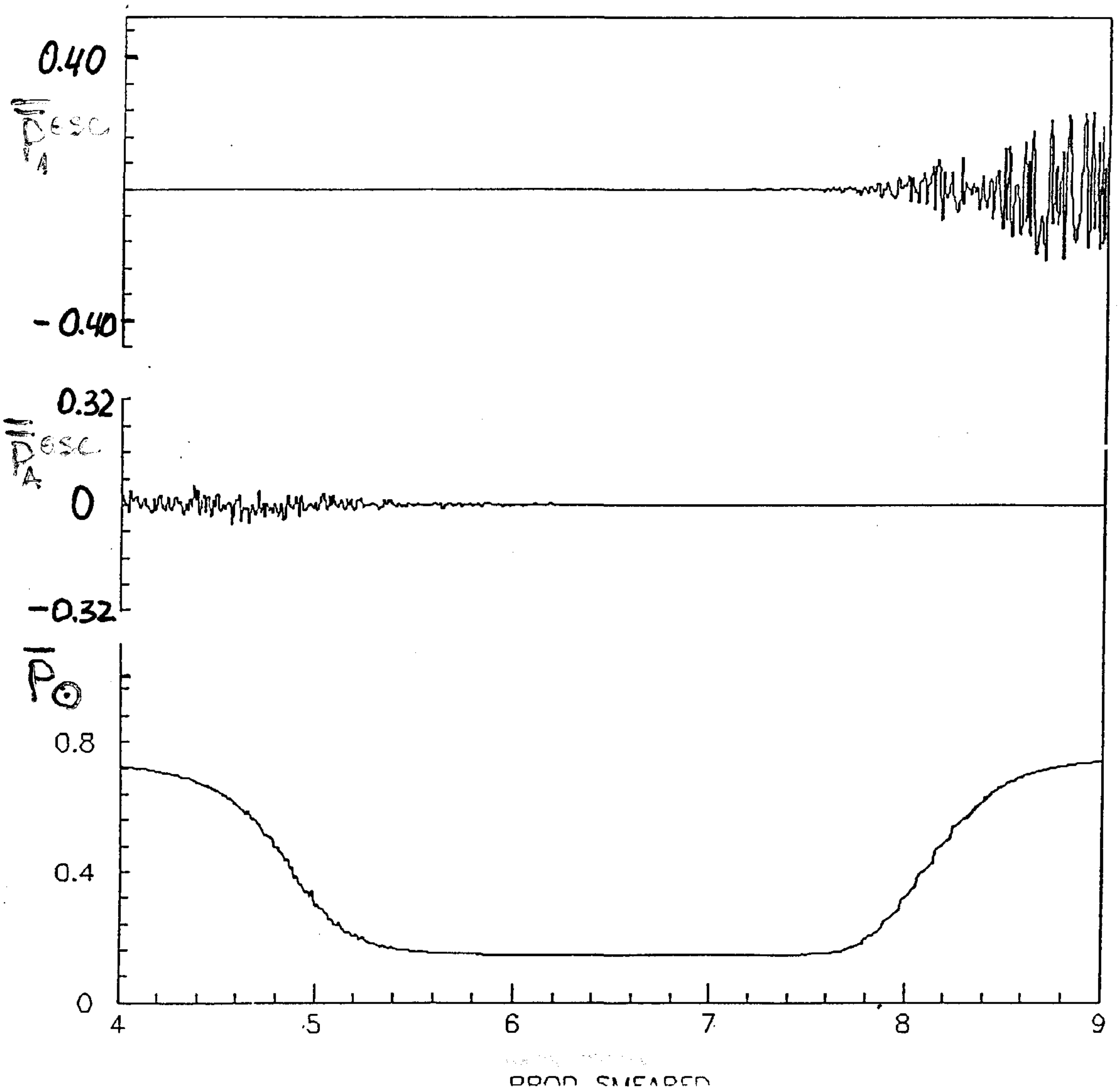
$\frac{\Delta p}{p} = 10^{-2}$



S.T.P., J. Rich, '89

$\sin^2 2\Theta = 0.5$

$\frac{\Delta p}{p} = 10^{-3}$



$$i\frac{d}{dt}\begin{pmatrix}A_e\\ A_\mu\end{pmatrix}=\begin{pmatrix}0 & \frac{\Delta m^2}{4E}\sin 2\theta\\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{2E}\cos 2\theta-\sqrt{2}G_F N_e\end{pmatrix}\begin{pmatrix}A_e\\ A_\mu\end{pmatrix}$$

The SUN :

$$20\,cm^{-3}N_A \lesssim N_e(t_0) \lesssim 100\,cm^{-3}N_A$$

$:=1+i\tau_0\frac{\Delta m^2}{2p}$

$:=1+i\tau_0\frac{\Delta m^2}{2p}\sin^2\theta$

$$\boxed{N_e(t)\cong N_e(t_0)\,e^{-\frac{x}{\tau_0}},\quad x\simeq t-t_0 \qquad \tau_0\simeq 0.1R_\odot}$$

S.T.P. '88

$$P_\odot(\nu_e\to\nu_e)\cong\bar{P}+P_{osc}$$

$$\bar{P}=\frac{1}{2}+\left(\frac{1}{2}-P'\right)\cos 2\theta\cos 2\theta_m(t_0)$$

$$P_{osc}=-\sqrt{P'(1-P')}\,\sin 2\theta\cos 2\theta_m(t_0)\cos\left(\frac{\Delta m^2 L}{2E}+\Phi\right)$$

known

$$P'=\frac{e^{-2\pi\tau_0\frac{\Delta m^2}{2E}\sin^2\theta}-e^{-2\pi\tau_0\frac{\Delta m^2}{2E}}}{1-e^{-2\pi\tau_0\frac{\Delta m^2}{2E}}}$$

VALID FOR ALL $\Delta m^2/E$, $\sin^2 2\theta$,
OF INTEREST AND ANY sign $(\cos 2\theta)$

$$-\Phi=2\varphi_3+\varphi_1-\varphi_2+\tau_0\frac{\Delta m^2}{2p}\ln\tau_0\sqrt{2}GN_e(t_0)$$

$\varphi_2\equiv\arg\Gamma(1-c)$, $\varphi_1=\arg\Gamma(a-1)$, $\varphi_3=\arg\Gamma(a-c)$

$$a = 1 + i r_0 \frac{\Delta m^2}{2E} \sin^2\theta, \quad C = 1 + i r_0 \frac{\Delta m^2}{2E}$$

$$\bar{P} + P_{osc}\Big|_{r_0 \frac{\Delta m^2}{2E} < 1} \overset{\simeq}{\text{QVO}} \qquad \Delta m^2 \simeq (5\cdot 10^{-10} - 5\cdot 10^{-8}) \; eV^2$$

$$\simeq 1 - \frac{1}{2}\sin^2 2\theta \left[1 + r_0 \frac{\Delta m^2}{2E}\cos 2\theta\right] \times$$

$$\times \left[1 - \cos\left(\frac{\Delta m^2}{2E} L + \phi\right)\right]$$

$$r_0 \frac{\Delta m^2}{2E} << 1 \qquad \boxed{\text{VO}} \qquad \Delta m^2 \lesssim 5\cdot 10^{-10} eV^2$$

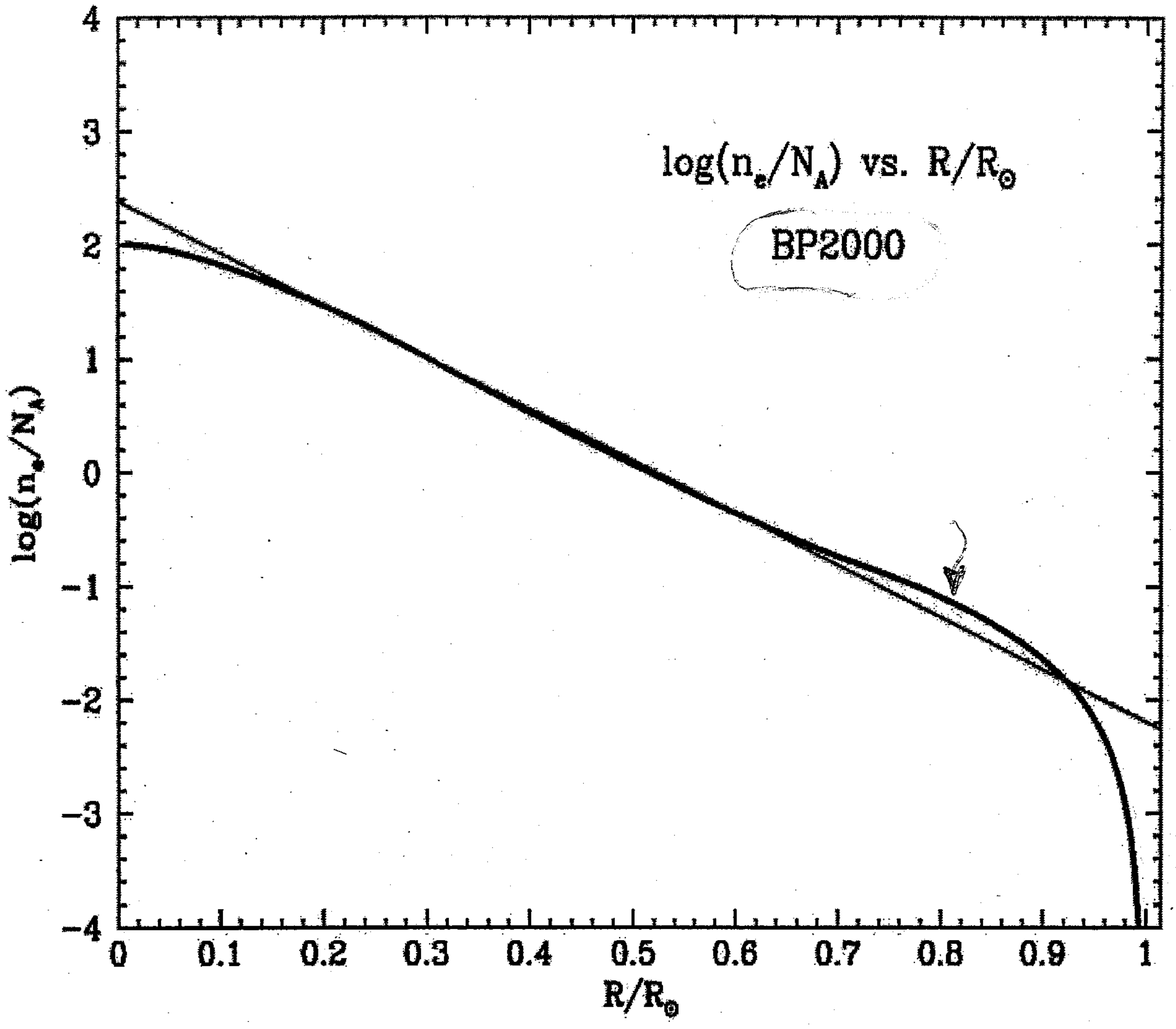
$$\simeq 1 - \frac{1}{2}\sin^2 2\theta \left[1 - \cos\frac{\Delta m^2}{2E} L\right] \equiv$$

$$\equiv P_{VAC}(\nu_e \to \nu_e)$$

$$\frac{\Delta m^2}{E} \gtrsim 2 \times 10^{-8} \; \frac{eV^2}{MeV} \qquad P_{osc} \simeq 0$$

$$\simeq \bar{P} \qquad \text{LOW, SMA + LMA MSW}$$

$$\simeq 1 - \frac{1}{2}\sin^2 2\theta, \qquad \Delta m^2 \gtrsim 10^{-3} eV^2$$


log(n_e/N_A) vs. $R/R_\odot$
BP2000
4
3
2
1
0
−1
−2
−3
−4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$R/R_\odot$
log(n_e/N_A)

THE PHASE:

$$\frac{\Delta m^2}{2E} L + \Phi = \underbrace{\Phi_{21} - \Phi_{22}}_{} = \underbrace{\xi_{SUN} + \frac{\Delta m^2}{2E}(L - R_{\odot})}_{}$$

$$\Phi_{2j}(x_0, L) = \arg\left[A\left(\nu_2^m(x_0) \to \nu_j(L)\right)\right], \; j = 1,2$$

$$\xi_{SUN} \cong 0.130 \frac{\Delta m^2}{2E} R_{\odot} + 1.67 \times 10^{-3} \left(\frac{\Delta m^2}{2E} R_{\odot}\right)^2 \times \cos 2\theta$$

HIGH PRECISION ANALYTIC DESCRIPTION:

"RUNNING" SCALE HEIGHT

$$r_0(x) = -\left[\frac{d}{dx} \ln N_e(x)\right]^{-1}$$

"STANDARD" CHOICE: $x = x_{res}$ IF $x_{res} \leq 0.904 R_{\odot}$

WORKS WELL FOR SMA + LMA MSW, LOW

$r_0 = R_{\odot}/18.9$ OTHERWISE

"UNIVERSAL" CHOICE: $x = x_{MVA}$, IF $x_{MVA} \leq 0.904 R_{\odot}$
+ QVO - VO

(LISI, MARRONE, MONTANINO, PALAZZO, S.L.P., 2000)

where $\rho = x/R_\odot$, and $V(x)$ is given by Eq. (5). In particular, the effective value of r_0 (in the limit of small k) is given by $2\pi^{-1}R_\odot\,\mathrm{Im}(I_\eta)$, namely,

$$\lim_{k\to 0} r_0 = \frac{2}{\pi} R_\odot \int_0^1 d\rho\, \sin\left(\int_\rho^1 d\rho' \sqrt{2}\, G_F\, N_e(\rho')\, R_\odot\right) , \tag{29}$$

independently of ω (i.e., both in the first and in the second octant).

The above perturbative results show that the asymptotic ($k \to 0$) effective value for r_0 depends upon a well-defined integral over the density profile $N_e(x)$. Using the SSM profile for $N_e(x)$ [9,10], we find that

$$\lim_{k\to 0} r_0 = R_\odot/18.9 \,. \tag{30}$$

The same value is obtained through exact numerical calculations.

The appearance of integrals over the whole density profile indicates that, for small k, the behavior of P_c becomes nonlocal, as was also recently noticed in [36]. We further elaborate upon the issue of nonlocality in Sec. VI, where we show that the $O(k)$ perturbative results are actually dominated by matter effects in the convective zone of the Sun ($x/R_\odot \gtrsim 0.7$), where the function $N_e(x)$ resembles a power law rather than an exponential.

In order to match the usual resonance prescription [$r_0 = r_0(x_{\rm res})$] with the value $r_0 = R_\odot/18.9$ in the regime of small k, we observe that, for the SSM density distribution [9,10] it is $r_0(x) = R_\odot/18.9$ at $x = 0.904\,R_\odot$. Thus, we are naturally led to the following "modified resonance prescription,"

$$r_0 = \begin{cases} r_0(x_{\rm res}) & \text{if } x_{\rm res} \le 0.904\,R_\odot\,, \\ R_\odot/18.9 & \text{otherwise}\,, \end{cases} \tag{31}$$

where "otherwise" includes cases with $\omega \ge \pi/4$, for which $x_{\rm res}$ is not defined. Such a simple recipe provides a description of P_c which is continuous in the mass-mixing parameters, and is reasonably accurate both in the QVO range ($\delta m^2/E \lesssim 10^{-8}$ eV^2/MeV) and in the lowest MSW decade ($\delta m^2/E \simeq$ few $\times\, 10^{-8}$–10^{-7} eV^2/MeV).

Figure 1 shows isolines of P_c in the bilogarithmic plane charted by the variables[9] $\delta m^2/E \in [10^{-10}, 10^{-7}]$ eV^2/MeV and $\tan^2\omega \in [10^{-3}, 10]$.[10] The solid lines refer to the exact numerical calculation of P_c, while the dotted lines are obtained through the analytical formula for P_c [Eq. (22)], supplemented with the modified resonance prescription [Eq. (31)]. Also shown are, in the first octant, isolines of resonance radius for $x_{\rm res}/R_\odot = 0.6$, 0.7, 0.8, and 0.904. The maximum difference between the exact (numerical) results and those obtained using the analytic expression for P_c, Eq. (22), amounts to $|\Delta P_c| \simeq 7.5\times 10^{-2}$, and is typically much smaller. Since P_c is not a directly measurable quantity, we propagate the results of

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[9] In all the figures of this work, we extend the $\delta m^2/E$ interval somewhat beyond the QVO range, in order to display the smooth transition to the MSW range.

[10] The variable $\tan^2\omega$ was introduced in [39] to chart both octants of the solar ν mixing angle ω in logarithmic scale.