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Theory of Gravitational Waves - I

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Basic ideas

As an entry into our discussion of gravitational radiation, we start by asking the question: How does information about a change in a gravitational field propagate?

In Newtonian theory, the gravitational potential Φ is given by

$$\nabla^2\Phi = 4\pi G\rho, \tag{1}$$

in the presence of matter (where ρ is the mass density), or

$$\nabla^2\Phi = 0 \tag{2}$$

in vacuum. The acceleration of a test particle moving freely in the field produced is then given by

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\frac{\partial\Phi}{\partial x^i}. \tag{3}$$

Suppose now that there is a change in the matter distribution giving rise to the field. According to the above equations, the effect of this as felt in the acceleration of the test particle, would occur *instantaneously* even if the test particle were very distant from the source, and this corresponds to an instantaneous transmission of information about the change in the source which is contrary to the ideas of relativity theory.

In general relativity (GR), all physical laws must be written in a *covariant* form, i.e. in terms of scalars, 4-vectors and 4-tensors. In line with this, the equation of motion needs to be written in a four-dimensional form (rather than a three-dimensional form as above), and the Laplacian needs to be replaced by a four-dimensional operator. If the modification were simply to replace ∇^2 by the d'Alembertian:

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2, \tag{4}$$

then it is easy to see that the speed of propagation of information would become *finite*, going to the speed of light, c, in vacuum. The concept of gravitational waves then arises as the finite-speed carriers of information about changes in the source of the gravitational field. In analogy with electromagnetism, these waves carry *energy* away from the source as well as *information* about the field changes.

What does GR tell us about gravitational waves?

General relativity is a geometrical theory describing gravity in terms of curvature of spacetime and, within this context, gravitational waves appear as *ripples* in spacetime.

The *source* of the gravitational waves might be either a strong-field object (such as a dynamically-changing black hole or neutron star) or a weak-field object (such as a normal stellar binary system). Clearly, strong-field objects will normally give rise to the larger-amplitude gravitational waves although this is not certain because the amplitude also depends on the degree of dynamical motion and asymmetry of the object concerned. Here, we will be focussing on a *weak-field* approach which will certainly be appropriate for regions far enough away from the source and can also be appropriate for the source itself in some cases. Treating these problems in strong field is much more complicated and many important features appear already in a weak-field treatment.

In weak-field situations, the wavelength of gravitational waves is usually much shorter than the length-scale associated with the curvature of the background space-time and so it is an excellent approximation to treat the waves as a perturbation about *flat space*. The general *metric line element* is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{5}$$

where ds is the space-time interval, $\mathbf{x} = (x^0, x^1, x^2, x^3)$ is the space-time position vector and $g_{\mu\nu}$ is the metric tensor. (We will be using the conventions that Greek indices range from 0 to 3, referring to both time and space dimensions, and Latin indices range from 1 to 3, referring to spatial dimensions only.) In the weak-field limit we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where $\eta_{\mu\nu}$ is the metric tensor for flat space and $h_{\mu\nu}$ is a small perturbation.

The gravitational field equation in general relativity is the *Einstein field equation*:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{7}$$

where: $R_{\mu\nu}$ is the *Ricci tensor* (contraction of the Riemann curvature tensor)
R is the *Ricci scalar* (contraction of $R_{\mu\nu}$)
$T_{\mu\nu}$ is the *Energy-momentum tensor* (telling us about the source)

We will now write this out and *linearize* it, retaining only first order terms in h. (We use the convention of taking $c = G = 1$ which allows masses and times to be expressed in terms of lengths). First, we need to calculate Ricci tensor:

$$\begin{aligned} R_{\mu\nu} &\equiv R^{\alpha}_{\mu\alpha\nu} & (8) \\ &= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} & (9) \end{aligned}$$

where a *comma* indicates a standard partial derivative (for example, $\Gamma^{\alpha}_{\mu\nu,\alpha} = \partial\Gamma^{\alpha}_{\mu\nu}/\partial x^{\alpha}$), a *repeated index* implies summation over that index and the Γs (the *Christoffel symbols*) are given by

$$\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= \tfrac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) & (10) \\ &\to \tfrac{1}{2}(h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu}) & (11) \end{aligned}$$

(note that indices are here raised and lowered with $\eta^{\mu\nu}$). Then

$$\begin{aligned} R_{\mu\nu} &\to \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} & (12) \\ &= \tfrac{1}{2}(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu}) & (13) \end{aligned}$$

where $h \equiv h_{\mu}{}^{\mu} = \eta^{\mu\nu}h_{\mu\nu}$. Contracting $R_{\mu\nu}$ gives the Ricci scalar:

$$R = \eta^{\mu\nu} R_{\mu\nu} \tag{14}$$

The Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{15}$$

then becomes

$$h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}) = 16\pi\, T_{\mu\nu} \tag{16}$$

On the left-hand-side we see four-dimensional second derivatives of $h_{\mu\nu}$, which describes the field, including

$$h_{\mu\nu,\alpha}{}^{\alpha} \equiv \Box h_{\mu\nu}. \tag{17}$$

It is convenient to define

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\, h \tag{18}$$

which may be referred to as "gravitational potentials". The Einstein field equation then becomes

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\, \bar{h}_{\alpha\beta,}{}^{\alpha\beta} - \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} - \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} \;=\; -16\pi\, T_{\mu\nu} \tag{19}$$

We now use a *gauge freedom* (an infinitesimal redefinition of coordinates, $(x^\mu)' = x^\mu + \xi^\mu(\mathbf{x})$) to set $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$. (This gauge is analogous to the Lorentz gauge in electromagnetism: $A^{\alpha}_{,\alpha} = 0$). Eq. (19) then becomes

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} \equiv \Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu} \tag{20}$$

and in vacuum (where $T_{\mu\nu} = 0$):

$$\Box \bar{h}_{\mu\nu} = 0 \tag{21}$$

which implies that if $\bar{h}_{\mu\nu}$ changes with time, then the changes propagate at velocity c.

How do gravitational waves interact with matter?

Consider here a *plane-fronted* wave propagating in the $x^3 = z$ direction (using rectangular Cartesian coordinates). For this, we can write the Riemann tensor as

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(t - z) \tag{22}$$

This satisfies the *Bianchi identities*:

$$R_{\alpha\beta[\gamma\delta;\epsilon]} = 0 \tag{23}$$

where the semi-colon denotes a *covariant derivative* and the square brackets [ ] denote anti-symmetrization. For a plane wave on a flat background, these give:

$$R_{\alpha\beta12,0} = 0 \quad \Rightarrow \quad R_{\alpha\beta12} = 0 \tag{24}$$
$$R_{\alpha\beta13,0} - R_{\alpha\beta10,3} = 0 \quad \Rightarrow \quad R_{\alpha\beta13} = -R_{\alpha\beta10} \tag{25}$$
$$R_{\alpha\beta23,0} - R_{\alpha\beta20,3} = 0 \quad \Rightarrow \quad R_{\alpha\beta23} = -R_{\alpha\beta20} \tag{26}$$

Therefore, using the symmetries of the Riemann tensor:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \tag{27}$$
$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{28}$$

it follows that any pair of purely spatial indices $(12, 13, 23, etc.)$ *either* gives a vanishing component *or* can be converted to a spacetime pair 10 or 20. There are then six independent components:

$$\begin{matrix} R_{1010} & R_{1020} & R_{1030} \\ R_{2020} & R_{2030} & R_{3030} \end{matrix} \tag{29}$$

However, Einstein's equation for vacuum $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = 0$ then reduces this to two:

$$\begin{aligned} R_{x0x0} &= -R_{y0y0} = -\tfrac{1}{4}(\bar{h}_{xx} - \bar{h}_{yy})_{,00} \\ R_{x0y0} &= R_{y0x0} = -\tfrac{1}{2}\,\bar{h}_{xy,00} \end{aligned} \tag{30}$$

The only relevant components of $\bar{h}_{\mu\nu}$ are purely *transverse* (i.e. only x and y are involved) and a further gauge freedom can be used to make it *traceless*. This is called the *Transverse Traceless* (TT) gauge:

$$\begin{aligned} h^{TT}_{xx} &= \tfrac{1}{2}\,(\bar{h}_{xx} - \bar{h}_{yy}) = -h^{TT}_{yy} \\ h^{TT}_{xy} &= \bar{h}_{xy} \end{aligned} \tag{31}$$

with all of the other components being zero. Then

$$R_{j0k0}\,(t-z) = -\tfrac{1}{2}h^{TT}_{jk,00} \qquad (\text{for } j,k = 1 \text{ or } 2) \tag{32}$$

We can write h_{jk}^{TT} in terms of two *polarizations*, introducing the polarization tensors $\mathbf{e}^{+}$ and $\mathbf{e}^{\times}$ such that

$$\begin{aligned} e_{xx}^{+} &= -e_{yy}^{+} = 1 \\ e_{xy}^{\times} &= e_{yx}^{\times} = 1 \end{aligned} \tag{33}$$

with all other components being zero. Then

$$h_{jk}^{TT} = h_{+}\, e_{jk}^{+} + h_{\times}\, e_{jk}^{\times} \tag{34}$$

We are now ready to calculate the effect of the wave on matter.

Consider two adjacent particles (with separation vector, ξ^{j}) hit by a gravitational wave. Their relative acceleration can be calculated using the *equation of geodesic deviation*:

$$\begin{aligned} \frac{\mathrm{d}^2 \xi^j}{\mathrm{d}t^2} &= -R_{j0k0}\,(t)\,\xi^k \\ &= \tfrac{1}{2}\, h_{jk,tt}^{TT}\, \xi^k \end{aligned} \tag{35}$$

and the change produced in their separation is then

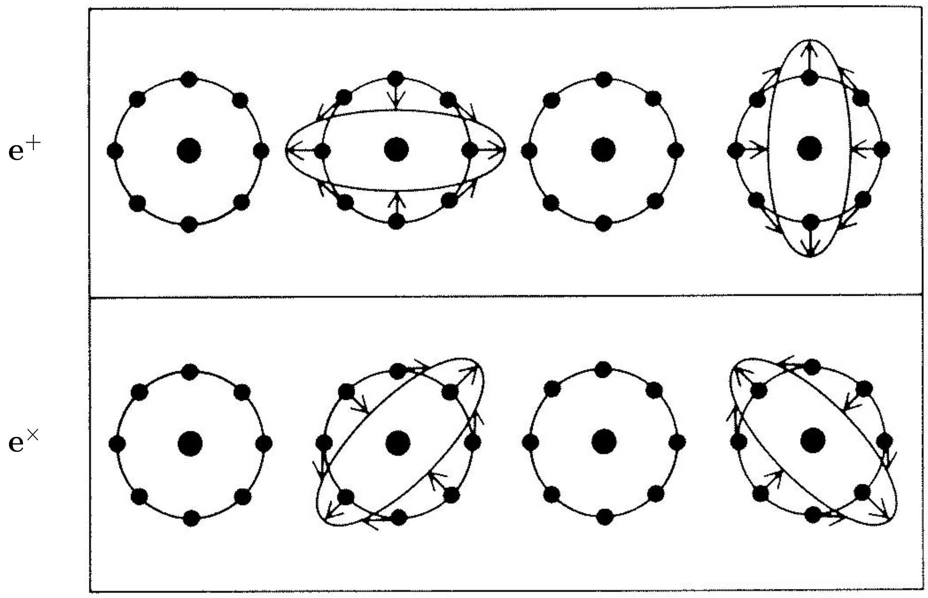
$$\delta\xi^j = \tfrac{1}{2}\, h_{jk}^{TT}\, \xi^k \tag{36}$$

giving an overall fractional change of

$$\frac{\delta\xi}{\xi} \sim h \tag{37}$$

where h is the amplitude of the metric perturbation.

We now consider the effect produced by a periodic gravitational wave on a ring of test particles oriented perpendicular to the direction of propagation of the wave, looking separately at the effects of the two polarization modes $\mathbf{e}^{+}$ and $\mathbf{e}^{\times}$ (see the figure on the next page). In both cases, the originally circular ring first becomes elliptical, then returns to being circular again, then becomes elliptical again with the semi-major axis being in a perpendicular direction to before and then becomes circular again. The cycle is then repeated. A bar of material placed perpendicular to the direction of propagation of the wave would experience oscillations in its length. Note that the behaviour shown in the figure indicates that the particle seen as mediating the gravitational interaction (the graviton) should have spin 2 since there is an invariance under rotation through π and one would expect invariance under rotation through $2\pi/S$, where S is the spin of the intermediary particle.



Deformation of a ring of test particles under the influence of different gravitational wave polarizations