H4.SMR/1574-8

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Theory of Gravitational Waves

Transparencies - I

J. Miller

SISSA, Trieste
&
University of Oxford
U.K.

How does information about <u>change</u> in a gravitational field propagate?

Newton's theory:

$$\nabla^2 \Phi = 4\pi G \rho \quad \text{and} \quad \nabla^2 \Phi = 0$$

$$\rightarrow \quad \frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}$$

$\Rightarrow$ information about changes in the field propagates instantaneously

In GR laws must be written in a covariant form

– 4-vectors and tensors

$\Rightarrow$ – equation of motion must be in a 4D form

– the Laplacian would need to be replaced by a 4D operator

$\Box$ - the d'Alembertian

$$\rightarrow \; -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

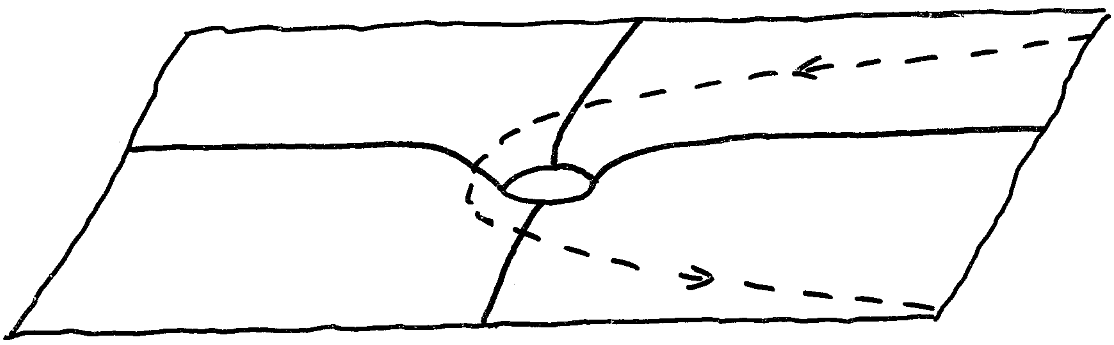
If one just replaced $\nabla^2 \rightarrow \Box$ then the speed of propagation becomes <u>finite</u> and $\rightarrow c$ in vacuum

$\rightarrow$ Concept of gravitational waves

Analogy with electromagnetic waves

- carry energy out as well as information about field changes

"Rubber sheet" analogy



- Introduce gravitating mass
  → curves the space-time
- Test particles follow geodesic paths
- Move the gravitating mass
  → waves in the space-time

What does GR tell us about gravitational waves?

Ripples in space-time – linked to curvature

The source might be a strong-field (or weak-field object

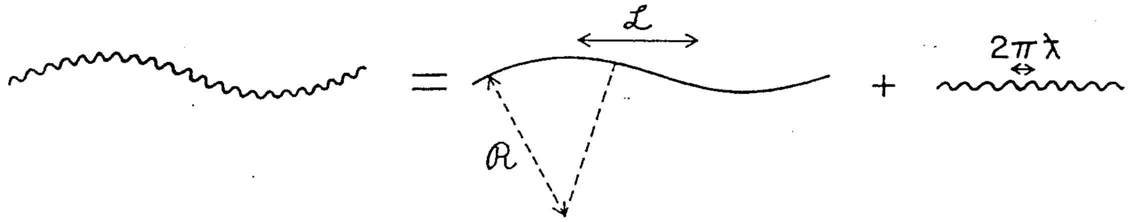
– (but expect that strong-field objects will produce stronger waves!)

Focus on weak field

- applies to region distant from source anyway

- can apply to source region as well in some cases

Distinction between waves and curvature of the background:



Treat waves as perturbations about <u>flat</u> space

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$\underline{x} \equiv (x^0, x^1, x^2, x^3)$$

Write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

flat space small perturbation

Our <u>gravitational field equation</u> is the <u>Einstein equation</u>:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $R_{\mu\nu}$ is the <u>Ricci tensor</u> (contraction of the Riemann curvature tensor)

R is the <u>Ricci scalar</u> (contraction of $R_{\mu\nu}$)

$T_{\mu\nu}$ is the <u>energy-momentum tensor</u> (telling us about the source)

We will write this out and <u>linearize</u> (retain only first-order terms in h)

- use convention of taking $c = G = 1$ (masses and times $\rightarrow$ lengths)

Need to calculate the Ricci tensor

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu}$$
$$= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

The comma indicates a standard partial derivative

eg $$\Gamma^{\alpha}_{\mu\nu,\alpha} \equiv \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}$$

A repeated index implies summation

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}\left(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}\right)$$

$$\rightarrow \frac{1}{2}\left(h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu}\right)$$

Note: indices are raised and lowered with $\eta^{\mu\nu}$

Then

$$R_{\mu\nu} \to \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu}$$

$$= \frac{1}{2}\left(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu}\right)$$

with $h \equiv h_{\alpha}{}^{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}$

Contract $R_{\mu\nu}$ to get the Ricci scalar

$$R = \eta^{\alpha\beta} R_{\alpha\beta}$$

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

then becomes

$$h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}\right) = 16\pi T_{\mu\nu}$$

On the left-hand-side see 4D second derivatives of $h_{\mu\nu}$ which describes the field, including

$$h_{\mu\nu,\alpha}{}^{\alpha} \equiv \Box h_{\mu\nu}$$

It is convenient to define

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h$$

– the "gravitational potentials"

The field equation then becomes

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} - \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} - \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu}$$

$$= -16\pi T_{\mu\nu}$$

Now use a <u>gauge freedom</u> (infinitessimal redefinition of coordinates $(x^\mu)' = x^\mu + \xi^\mu(\underline{x})$)

to set $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$

(<u>Note</u>: this is the tensor analogue of the Lorentz gauge in electromagnetism $A^\alpha{}_{,\alpha} = 0$)

Then get

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} \equiv \Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

and in vacuum

$$\Box\bar{h}_{\mu\nu} = 0$$

$\Rightarrow$ <u>If</u> $\bar{h}_{\mu\nu}$ changes with time then changes propagate at velocity c.

How do gravitational waves interact with matter?

Consider here a <u>plane-fronted</u> wave propagating in the $x^3 = z$ direction (using rectangular Cartesian coordinates)

Can write the Riemann tensor

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(t - z)$$

It satisfies the <u>Bianchi identities</u>:

$$R_{\alpha\beta[\gamma\delta;\varepsilon]} = 0$$

where the semi-colon denotes a <u>covariant derivative</u>

For a plane wave on a flat background, these give:

$$R_{\alpha\beta 12,0} = 0 \qquad \Rightarrow \qquad R_{\alpha\beta 12} = 0$$

$$R_{\alpha\beta 13,0} - R_{\alpha\beta 10,3} = 0 \qquad \Rightarrow \qquad R_{\alpha\beta 13} = -R_{\alpha\beta 10}$$

$$R_{\alpha\beta 23,0} - R_{\alpha\beta 20,3} = 0 \qquad \Rightarrow \qquad R_{\alpha\beta 23} = -R_{\alpha\beta 20}$$

Therefore (using symmetries

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \text{ and } R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$$

any pair of purely spatial indices (12, 13 or 23)

<u>either</u>

- gives a vanishing componen

<u>or</u>

- can be converted to a space-time pair 10 or 20

$\Rightarrow$ there are six independent components:

$$R_{1010} \quad R_{1020} \quad R_{1030}$$

$$R_{2020} \quad R_{2030} \quad R_{3030}$$

However, Einstein's equation for vacuum $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = 0$ then reduces this to two:

$$R_{x0x0} = -R_{y0y0} = -\frac{1}{4}\left(\bar{h}_{xx} - \bar{h}_{yy}\right)_{,00}$$

$$R_{x0y0} = R_{y0x0} = -\frac{1}{2}\bar{h}_{xy,00}$$

The only relevant components of $\bar{h}_{\mu\nu}$ are purely <u>transverse</u> (only x, y involved) and a further gauge freedom can be used to make it <u>traceless</u> (the TT gauge)

$$h^{TT}_{xx} = \frac{1}{2}(\bar{h}_{xx} - \bar{h}_{yy}) = -h^{TT}_{yy}$$

$$h^{TT}_{xy} = \bar{h}_{xy}$$

– all other components are zero

Then $R_{j0k0}(t-z) = -\frac{1}{2} h^{TT}_{jk,00}$

$(j, k = 1 \text{ or } 2)$

Can write h^{TT}_{jk} in terms of two <u>polarizations</u>

Introduce <u>polarization tensors</u> $\underset{\sim}{e}^{+}$ and $\underset{\sim}{e}^{\times}$ such that

$$e^{+}_{xx} = -e^{+}_{yy} = 1 \qquad e^{\times}_{xy} = e^{\times}_{yx} = 1$$

All other components zero

Can then write:

$$h^{TT}_{jk} = h_{+} e^{+}_{jk} + h_{\times} e^{\times}_{jk}$$

Now calculate <u>effect</u> of the wave on matter

- consider two adjacent particles (separation ξ^j) hit by a gravitational wave

Use the <u>equation of geodesic deviation</u> to calculate the relative acceleration:

$$\frac{d^2 \xi^j}{dt^2} = - R_{joko}(t)\, \xi^k$$

$$= \frac{1}{2} h^{TT}_{jk,tt}\, \xi^k$$

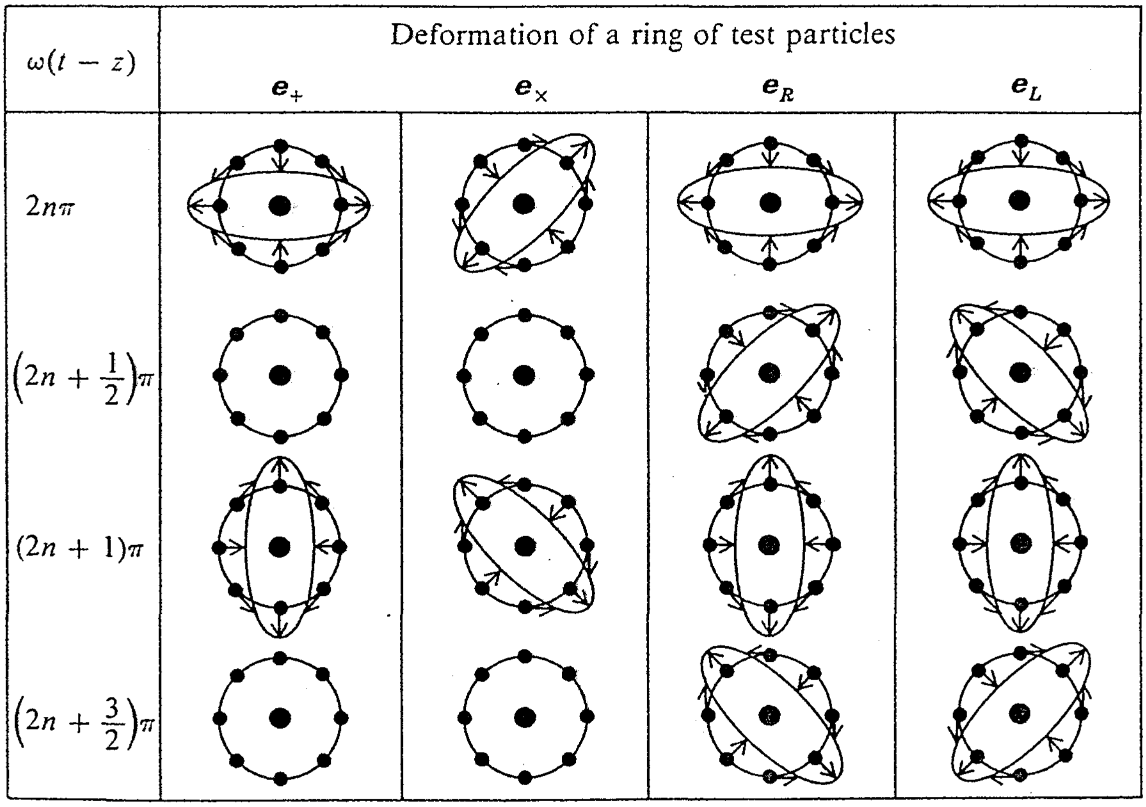
This produces a change in their separation:

$$\delta \xi^j = \frac{1}{2} h^{TT}_{jk}\, \xi^k$$

and so $\quad \dfrac{\delta \xi}{\xi} \sim h$

Now consider the effect on a ring of test particles perpendicu to the direction of propagation of the wave

– effect depends on polarizatio



Connection with a spin 2 particle

– invariance under rotation of

$$\pi = \frac{2\pi}{S} \quad \text{with} \quad S = 2$$