NONCOMMUTATIVE ALGEBRAS IV–VI: PI RINGS

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8. Prime right Goldie rings

We know from the previous sections that a prime right Goldie ring has a semisimple ring of fractions. As might be expected, that ring of fractions is actually prime, hence simple artinian (i.e., of the form $M_n(D)$). In fact, more is true.

Proposition 8.1. *Let R be a right order in Q. If R is prime (resp. semiprime) then Q is prime (resp. semiprime).*

Proof. We do only the prime case. Let R be prime. Let $as^{-1}, bt^{-1} \in Q$ (where $a, s, b, t \in R$, with s, t regular) with $as^{-1}Qbt^{-1} = 0$. Then $as^{-1}Qb = 0$. Since $sR \subset Q$, we have $as^{-1}sRb = aRb = 0$. Thus $a = 0$ or $b = 0$. Then $as^{-1} = 0$ or $bt^{-1} = 0$. Thus Q is prime. □

On the other hand,

Proposition 8.2. *If R is a right order in a simple artinian ring Q, then R must be prime right Goldie.*

Proof. We already know from Proposition 6.4 that R is semiprime right Goldie. We need only show that R is prime.

Let I be a non-zero ideal of R and let J be an ideal such that $IJ = 0$. Then $J \subset \mathrm{r.ann}(I)$, so if we can show $\mathrm{r.ann}(I) = 0$, we are done.

Well, since $I \neq 0$, we have $QIQ \neq 0$. Since Q is simple, $QIQ = Q$. Thus $1 = \sum_{i=1}^{n} q_i a_i t_i s^{-1}$ with $q_i \in Q, a_i \in I, t_i \in R$ and s regular in R. (Just one s is necessary since we can find a "common denominator.") So $s = \sum_{i=1}^{n} q_i a_i t_i \in QI$. Since $I\,\mathrm{r.ann}(I) = 0$, we have $QI\,\mathrm{r.ann}(I) = 0$. Thus $s\,\mathrm{r.ann}(I) = 0$. But s is regular, so $\mathrm{r.ann}(I) = 0$. □

9. Radicals

As far as I have seen in the literature, for noncommutative rings we do not say that the intersection of all prime ideals containing an ideal I is called the radical of I. However, there are "radical ideals" out there. Here are the main ones, with some of their basic properties.

Recall that an ideal is nil if all of its elements are nilpotent.

Definition 9.1. Let R be a ring. The *lower nilradical*, $\mathrm{nil}(R)$, is the intersection of all prime ideals of R. The *upper nilradical*, $\mathrm{Nil}(R)$, is the sum of all nil ideals.

If R is commutative, then $\mathrm{nil}(R) = \mathrm{Nil}(R)$. But in general, we have only $\mathrm{nil}(R) \subseteq \mathrm{Nil}(R)$.

Perhaps the most widely used radical is the Jacobson radical, $J(R)$.

Definition 9.2. Let R be a ring. The *Jacobson radical*, $J(R)$, is the intersection of all primitive ideals.

There are various other important ways of representing the Jacobson radical.

Proposition 9.3. *Let R be a ring. Then $J(R)$ is equal to the intersection of all maximal right ideals. Symmetrically, it is equal to the intersection of all maximal left ideals.* □

Proposition 9.4. *Let $y \in R$. Then $y \in J(R)$ if and only if $1 - xyz$ is invertible for all $x, z \in R$.*

Just as in the commutative case, the above proposition leads to

Lemma 9.5 (Nakayama's Lemma)**.** *Let M be a finitely generated right R-module. (That is, there exist $m_1, \dots, m_k$ such that for any $m \in M$, $m = \sum m_i r_i$ for some $r_i \in R$.) Let $J = J(R)$. Then*

(1) *If $M = MJ$, then $M = 0$.*
(2) *If $N \leq M$ and $M = N + MJ$, then $N = M$.*
(3) *If M/MJ is generated by $x_1 + MJ, \dots, x_k + MJ$, then $x_1, \dots, x_k$ generate M.* □

The main use of these radicals is to mod out by one them and then study the resulting simpler ring. That is, given a ring R one wishes to study, one method of attack would be to first study $R/J(R)$ (or R modulo another radical). Then $R/J(R)$ is a semiprime ring. Try to prove your theorem for $R/J(R)$. Then using Nakayama's Lemma or other theorems, try to bring your results back to R.

10. PI-rings

In this section we will briefly survey the class of rings which are arguably closest to commutative rings. Consider the noncommutative polynomial $f(x, y) = xy - yx$. Then if R is commutative, for any $a, b \in R$, we have $f(a, b) = 0$. So what happens if other polynomials are used instead?

First we'll need some definitions.

Definition 10.1. Let C be a commutative ring and let $= C\langle X \rangle = C\langle x_1, x_2, \dots \rangle$ be the ring of noncommutative polynomials in countably many variables (also known as the free associative algebra on countably many variables). Let $f(x_1, \dots, x_n) \in C\langle X \rangle$. Then

(1) We say f is an *identity* if $f(r_1, \dots, r_n) = 0$ for all $r_i \in R$.
(2) If f is an identity and the coefficient of one of its monomials of highest degree is 1, then f is a *polynomial identity*.

(3) A C-algebra R with a polynomial identity is called a *PI-algebra* (polynomial identity algebra).
(4) If $C = \mathbb{Z}$ and R has a polynomial identity, then R is called a *PI-ring*.

Note that if p is a prime number and R is a ring with characteristic p, then px is an identity for R, but not a polynomial identity. We do not wish these rings to (necessarily) be PI-rings, hence the need for the "coefficient 1" requirement.

Also, convince yourself that any subring and any homomorphic image of a PI-ring is a PI-ring.

As noted above, any commutative ring is a PI-ring. Also, any exterior algebra is as well, since $f(x) = x^2$ is a PI. Of great importance to PI theory is that matrices over a commutative ring are PI.

Definition 10.2. Let S_n be the symmetric group on n elements and let

$$s_n = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdot \cdots \cdot x_{\sigma(n)}.$$

Then s_n is called the *standard polynomial* in n variables.

The polynomials s_n are examples of *multilinear polynomials*. These are the polynomials which have each variable x_i appearing with degree exactly 1 in each monomial.

Theorem 10.3 (Amitsur-Levitski)**.** *Let C be a commutative ring. The matrix ring $M_n(C)$ is a PI-ring which satisfies s_{2n}. No polynomial of degree $< 2n$ is an identity for $M_n(C)$.* □

The fact that matrix rings are PI is not just an isolated curiosity. Amazingly, it turns out that any (semi)prime PI-ring is closely related to a matrix ring. We start with the primitive case.

Theorem 10.4 (Kaplansky)**.** *Let R be a primitive PI-ring. Then $R \cong M_t(D)$ where D is a division ring, and $n^2 = [R : Z(R)] = t^2[D : Z(D)]$.* □

So in other words, any primitive PI-ring is simple artinian.

Kaplansky's Theorem can be used to relate any semiprime PI-ring to matrix rings over commutative rings.

Proposition 10.5. *Let R be a semiprime PI-ring, satisfying a PI of degree d. Then R can be embedded as a subring of*

$$\prod_{k<n} M_k(H_k)$$

where each H_k is a direct product of fields and $n = [d/2]$. □

PI-rings are also connected to matrix rings (over division rings) via the Goldie theory we have studied.

Theorem 10.6 (Posner)**.** *Let R be a prime PI-ring. Then R is left and right Goldie, and hence has a quotient ring Q of the form $M_n(D)$ for some division ring D.*

Furthermore, $Q \cong R(Z(R)\backslash\{0\})^{-1}$ and $Z(Q) \cong Z(R)(Z(R)\backslash\{0\})^{-1}$. That is, one need only invert the non-zero elements of the center of R to get the quotient ring, and also the center of the quotient ring is the quotient ring of the center. □

(As far as I can tell from the literature, the 2nd paragraph was not originally proven by Posner, but it goes well with the 1st paragraph.)

Exercise 10.7. (i) Let R be a prime ring. Show that the center $Z(R)$ is a domain.
(ii) Let R be a simple ring. Show that $Z(R)$ is a field.

11. Azumaya algebras

Given the second part of the exercise above, we now define another important class of algebras. First we need another definition.

Definition 11.1. Let R be a ring. Then R^{op} is the *opposite ring*, where R^{op} has the same additive structure as R, but where $a \cdot b = ba$ when $\cdot$ denote multiplication in R^{op} and juxtaposition denotes multiplication in R.

If R is an algebra over a commutative ring C, then $R^e = R \otimes_C R^{op}$.

Definition 11.2. Let R be a simple ring which is finite dimensional over its center. Then R is a *central simple algebra*.

Notice that since a central simple algebra R is a finite dimensional vector space over $Z(R)$ (a field by Exercise 10.7, it must be that R is artinian. Thus $R \cong M_n(D)$ for some division ring D which is finite dimensional over $Z(D) \cong Z(R)$. In fact, central simple algebras are also tied to matrix rings as follows.

Proposition 11.3. *Let R be a central simple F-algebra with $[R : F] = n$. Then $R^e \cong M_n(F)$.* □

We wish to generalize the concept of central simple algebras to a much more general case.

Definition 11.4. Let C be a commutative ring and let R be a C-algebra. Then R is *Azumaya* if

(1) R is a faithful, finitely generated projective C-module.
(2) $R^e \cong \operatorname{End} R_C$

So we have that any central simple algebra is Azumaya, given Proposition 11.3. Some of the importance of Azumaya algebras comes from

Proposition 11.5. *Let R be a C-algebra. Then the following are equivalent:*

(1) *R is Azumaya,*
(2) *$C = Z(R)$ and R is a projective R^e-module,*
(3) *there is a category equivalence*

$$\{C\text{-}modules\} \to \{right\ R^e\text{-}modules\}$$

given by $R \otimes \bullet$. □

Azumaya algebras are tied to PI theory by the following theorem.

Theorem 11.6 (Artin-Procesi)**.** *Let R be a C-algebra. Then R is Azumaya of constant rank n^2 if and only if R satisfies all multilinear identities of $M_n(\mathbb{Z})$, but there is a multilinear identity f of $M_{n-1}(\mathbb{Z})$ such that no non-zero homomorphic image of R satisfies f.* □

References

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