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Homological algebra, abelian and derived categories

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Chapter I

Reminder of some notions about general categories.

General categories: settheory

The theory of categories fits somewhat unconfomfortably inside set theory. Luckily one can usually ignore set theory.

Possible foundation

Godel-Bernays axioms: “sets” and “classes”.

There is no set of all sets, but there is a class of all sets.

Mild extension of classical set theory (Zermelo Fraenkel axioms).

Alternative

“Universes”: more flexible, but requires a more serious extension of set theory.

General categories: axioms

A category $\mathcal{C}$ consist of the following data:

- A *class* of “objects” $\mathrm{Ob}(\mathcal{C})$.
- For every $X, Y \in \mathrm{Ob}(\mathcal{C})$ a *set* of “maps”: $X \to Y$ denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For every $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a “composition”

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z) : (f,g) \mapsto g \circ f$$

with the following axioms:

- Composition is associative.
- For every $X \in \mathrm{Ob}(\mathcal{C})$ there is an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ behaving as a left and right identity for composition.

Remark If follows that id_X is unique.

Examples

$\mathbf{Set}$: The category of (all) sets.

$\mathbf{Grp}$: The category of groups.

$\mathbf{Ab}$: The category of abelian groups.

$\mathbf{Rng}$: The category of rings (with unit).

$\mathbf{Top}$: The category of topological spaces with continuous maps.

etc. . .

Note Any set can viewed as a category with no (non-identity) arrows.

Properties

Standard properties of maps in these concrete categories can be mimicked in abstract categories.

Example A map $f : X \to Y$ is an *isomorphism* if it has an *inverse* i.e. there is a map $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

Two objects are *isomorphic* if there is an isomorphism between them.

We will see a systematic way of doing this below using representable functors.

Size matters

- A category is *small* if $\mathrm{Ob}(\mathcal{C})$ is a set.
- A category is *essentially small* if the class of isomorphism classes of objects forms a set.
- A "big" category is a category without the restriction that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a set.

Size is often implied by context.

Functors

Definition Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{C})$ maps
$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.

such that

- For all $X \in \mathrm{Ob}(\mathcal{C})$ one has $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
- For all maps $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ one has $F(g \circ f) = F(f) \circ F(g)$.

Notation

$\mathbf{Fun}(\mathcal{C}, \mathcal{D})$: functors $\mathcal{C} \to \mathcal{D}$ (a class in general, a set if $\mathcal{C}$, $\mathcal{D}$ are small).

Cat : the category of *small* categories.

Examples

Very common functors are *Forgetful functors* (which forget part of a structure).

$$\mathbf{Ab} \to \mathbf{Set}$$
$$\mathbf{Rng} \to \mathbf{Ab}$$
$$\mathbf{Top} \to \mathbf{Set}$$

Examples of non-forgetful functors:

- $U : \mathbf{Rng} \to \mathbf{Grp} : R \mapsto R^*$ where $R^* = \{x \in R \mid \exists y \in R : yx = xy = 1\}$
- $\mathbf{Set} \to \mathbf{Ab} : S \mapsto \mathbb{Z}S$ where $\mathbb{Z}S$ is the free abelian group with basis S.

Standard properties of functors

Definition Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is faithful (resp. full, resp. fully faithful) if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the map

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective (resp. surjective, resp. bijective).

Example The forgetful functors on the previous transparency are all faithful, but not full.

Definition A full subcategory of a category $\mathcal{D}$ is a category $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{C}) \subset \mathrm{Ob}(\mathcal{D})$ and such that for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(X,Y)$

Note A full subcategory is uniquely determined by its set of objects. We say that $\mathcal{C}$ is *spanned* by its set of objects.

Note If $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ then the obvious inclusion functor $I : \mathcal{C} \to \mathcal{D}$ is fully faithful.

Contravariant functors

A functor as defined above is often called a *covariant* functor. We also use *contravariant* functors which invert the direction of arrows.

Definition If $\mathcal{C}$ is a category then $\mathcal{C}^\circ$ is the category with

- $\mathrm{Ob}(\mathcal{C}^\circ) = \mathrm{Ob}(\mathcal{C})$.
- For all $X, Y \in \mathrm{Ob}(\mathcal{C}) : \mathrm{Hom}_{\mathcal{C}^\circ}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

Definition Let $\mathcal{C}, \mathcal{D}$ be categories. A *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^\circ \to \mathcal{D}$.

Remark A functor $F : \mathcal{C}^\circ \to \mathcal{D}$ is the same thing as a functor $\mathcal{C} \to \mathcal{D}^\circ$.

Natural transformations

Definition Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\theta : F \to G$ consists of, for all $X \in \mathrm{Ob}(\mathcal{D})$, a map $\theta(X) : F(X) \to G(X)$ in $\mathcal{D}$ such that for every map $f : X \to Y$ in $\mathcal{C}$ the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta(X)} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\theta(Y)]{} & G(Y) \end{array}$$

Notation $\mathrm{Hom}(F, G)$: the natural transformations $F \to G$.

Note If $\mathcal{C}, \mathcal{D}$ are categories then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is itself a category (the maps are the natural transformations).

Thus in particular the Hom-sets in $\mathbf{Cat}$ are categories.

$\mathbf{Cat}$ is more than just a category. It is a so-called 2-category.

Isomorphism and equivalence

$F : \mathcal{C} \to \mathcal{D}$ a functor.

Definition A F is an isomorphism if it has an *inverse* for F i.e. a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$, $FG = \mathrm{id}_{\mathcal{D}}$.

Isomorphisms between categories are quite rare.

Definition F is an equivalence if it has a quasi-inverse, i.e. a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, $FG \cong \mathrm{id}_{\mathcal{D}}$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

Definition

- The essential image of F consist of the objects in $\mathrm{Ob}(\mathcal{D})$ which are isomorphic to objects in the image of F.
- F is essentially surjective if its essential image is $\mathrm{Ob}(\mathcal{D})$.

Theorem F is an equivalence if and only if it is fully faithful and essentially surjective.

Isomorphism and equivalence II

Example Let $\mathbf{FinSet}$ be the full subcategory of $\mathbf{Set}$ spanned by the finite sets and let I be the full subcategory of $\mathbf{Set}$ spanned by $\emptyset$ and the intervals $\{1, \ldots, n\}$ for $n = 1, 2, 3, \ldots$.

Then I and $\mathbf{FinSet}$ are equivalent.

Indeed the obvious map

$$F : I \to \mathbf{FinSet}$$

is clearly fully faithful and essentially surjective.

Note F has no canonical quasi-inverse!

Representable functors

Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$. We define functors

$$\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\circ} \to \mathbf{Set} : Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$$
$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set} : Z \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

Theorem (Yoneda) Let $F : \mathcal{C}^{\circ} \to \mathbf{Set}$ be a contravariant functor. Then the map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\circ}, \mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(-, X), F) \to F(X) : \theta \to \theta(\mathrm{id}_X)$$

is a bijection.

Theorem (dual version) Let $G : \mathcal{C} \to \mathbf{Set}$ be a covariant functor. Then the map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(X, -), G) \to G(X) : \theta \to \theta(\mathrm{id}_X)$$

is a bijection.

Corollary The functors (the “Yoneda embeddings”)

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\circ}, \mathbf{Set}) : X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$$
$$\mathcal{C}^{\circ} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set}) : X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -)$$

are fully faithful.

Representable functors II

Definition A contravariant functor $F : \mathcal{C}^{\circ} \to \mathbf{Set}$ is *representable* if

$$F \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$$

for some $X \in \mathrm{Ob}(\mathcal{C})$.

In that case : The object X, together with the isomorphism $F \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$ is called a *representing object* for F.

Analogously : a covariant functor $G : \mathcal{C} \to \mathbf{Set}$ is representable if

$$G \cong \mathrm{Hom}_{\mathcal{C}}(Y, -)$$

for $Y \in \mathrm{Ob}(\mathcal{C})$.

Representable functors III

Note By Yoneda's theorem: a representing object is unique, up to unique isomorphism.

Given natural isomorphisms

$$\theta : \mathrm{Hom}_{\mathcal{C}}(-, X) \to F$$

and

$$\theta' : \mathrm{Hom}_{\mathcal{C}}(-, X') \to F$$

there is a unique isomorphism $f : X \to X'$ in $\mathcal{C}$ such that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(-, X) \arrow[r, "\theta"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,f)"'] & F \arrow[d, equal] \\ \mathrm{Hom}_{\mathcal{C}}(-, X') \arrow[r, "\theta'"] & F \end{tikzcd}$$

Monomorphisms and epimorphisms

General principle Use Yoneda embeddings to define properties of objects and maps.

Definition $f : X \to Y$ is a *monomorphism* if

$$\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$$

is an injection for all $Z \in \mathrm{Ob}(\mathcal{C})$.

Traditional (equivalent) definition

A map $f : X \to Y$ is a *monomorphism* if for all diagrams

$$\begin{tikzcd} Z \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

such that $fp = fq$ one has $p = q$.

Monomorphisms and epimorphisms II

Definition $f : X \to Y$ is an *epimorphism* if

$$\mathrm{Hom}_{\mathcal{C}}(f, Z) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is an injection for all $Z \in \mathrm{Ob}(\mathcal{C})$.

Traditional defintion

A map $f : X \to Y$ is an *epimorphism* if for all diagrams

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & Z \end{tikzcd}$$

such that $pf = qf$ one has $p = q$.

Remark An isomorphism is both a mono- and an epimorphism but the converse is not generally true. Counter examples: $\mathbf{Rng}$ and $\mathbf{Top}$.

Split maps

If we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

then f is mono and g is epi.

We say

- A monomorphism f is *split* if g exists as in the diagram.
- An epimorphism g is *split* if f exists as in the diagram.

Generators

Definition An object G in a category $\mathcal{C}$ is a *generator* if

$$\mathrm{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \to \mathbf{Set}$$

is faithful (i.e. is injective on Hom-sets).

Traditional definition:

G is a generator if for all pairs with $p \neq q$

$$\begin{tikzcd} X \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & Y \end{tikzcd}$$

there is a map $f : G \to X$ such that $pf \neq qf$.

Dual notion : *cogenerator.*

Examples

- The singleton is a generator for $\mathbf{Set}$ and $\mathbf{Top}$.
- The two element set is a cogenerator for $\mathbf{Set}$.
- $\mathbb{Z}$ is a generator for $\mathbf{Ab}$ and $\mathbf{Grp}$.
- $\mathbb{Z}[X]$ is a generator for $\mathbf{Rng}$.

Limits

Let I be a small category and $N : I \to \mathcal{C}$ a functor.

A *cone* over N is an object X in $\mathcal{C}$ together with maps for all $i \in \mathrm{Ob}(I)$

$$a_i : X \to N(i)$$

such that for all $q : i \to j$ in I there is a commutative diagram

$$\begin{tikzcd} N(i) \arrow[r, "N(q)"] & N(j) \\ X \arrow[u, "a_i"] \arrow[ur, "a_j"'] & \end{tikzcd}$$

$\lim N$ is a universal cone over N (unique if existing).

$$\begin{tikzcd} & N(i) \arrow[r, "N(p)"] & N(j) \\ \lim N \arrow[ur, "p_i"] \arrow[urr, "p_j"'] & X \arrow[l, dotted, "\exists!"] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The p_i are sometimes called the "projection maps".

Alternative (ambiguous) notation

$$\lim_{i \in I} N(i)$$

Limits exist in $\mathbf{Set}$

Let $N : I \to \mathbf{Set}$ be a functor. Then

$$\lim N$$

will stand for the following *concrete* construction.

It is the set of all

$$(a_i)_i \in \prod_{i \in \mathrm{Ob}(I)} N(i)$$

subject to the condition :

$$\forall p : i \to j \quad \text{in } I : N(p)(a_i) = a_j.$$

The p_i are the restrictions of the projection maps

$$\prod_{i \in \mathrm{Ob}(I)} N(i) \to N(i)$$

Construction works in other "concrete" categories (e.g. $\mathbf{Ab}$, $\mathbf{Rng}$).

Definition using representable functors

We work again in a category $\mathcal{C}$. The universal property may be restated as:

$$\mathrm{Hom}_{\mathcal{C}}(-, \lim_{i \in I} N(i)) \cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(-, N(i))$$

(limit in $\mathbf{Set}$)

($\cong$ as contravariant functors).

Usual formulation : there are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, \lim_{i \in I} N(i)) \cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X, N(i))$$

"natural in X"

Notation If

$$f : X \to \lim_{i \in I} N(i)$$

is a map then we write f_k for the composition

$$X \xrightarrow{f} \lim_{i \in I} N(i) \xrightarrow{p_k} N(k)$$

It is the projection of f on $\mathrm{Hom}_{\mathcal{C}}(X, N(k))$.

Special limits

Products If I is a set (no arrows) then a functor

$$N : I \to \mathcal{C}$$

is the same as a set of objects $N(i) \in \mathrm{Ob}(\mathcal{C})$.

The limit is denoted by

$$\prod_{i \in I} N(i)$$

and is called the *product* of the $N(i)$.

Finite products (i.e. if $I = \{1, \ldots, n\}$) are written as:

$$N(1) \times \cdots \times N(n)$$

Limit (or product) over the empty set

The limit over (the unique) functor $N : \emptyset \to \mathcal{C}$ is a *final* object in $\mathcal{C}$, i.e. an object F such that for all $X \in \mathcal{C}$:

$$|\mathrm{Hom}_{\mathcal{C}}(X, F)| = 1$$

Functors and diagrams

Principle Diagrams can be viewed as functors.

Example Let I be the category.

$$p \xrightarrow[\alpha]{} q$$

- two objects p, q;
- one (non-identity) arrow $\alpha : p \to q$

A functor $M : I \to \mathcal{C}$ is determined by

- Objects $X = M(p)$, $Y = M(q)$.
- A map $f : X \to Y$, given by $f = M(\alpha)$.

I.e. a functor $M : I \to \mathcal{C}$ is the same as a (very small) diagram

$$X \xrightarrow{f} Y$$

in $\mathcal{C}$.

Notation : $\mathrm{Maps}(\mathcal{C}) = \mathrm{Fun}(\bullet \to \bullet, \mathcal{C})$.

Functors and diagrams II

Example Let J be the category.

$$\begin{tikzcd} p \arrow[r, "\alpha"] \arrow[d, "\beta"'] & q \arrow[d, "\delta"] \\ r \arrow[r, "\gamma"'] & s \end{tikzcd}$$

with "relation" $\delta\alpha = \gamma\beta$.

(i.e. there are 4 objects p, q, r, s and 5 non-identity arrows $\alpha, \beta, \gamma, \delta, \delta\alpha = \gamma\beta$).

A functor $J \to \mathcal{C}$ is a *commutative diagram* in $\mathcal{C}$.

Special limits II

Equalizers The limit of a pair of arrows

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is called the *equalizer* of f, g.

Pullbacks (fiber products) The limit of a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is called the *pullback* (or fiber product) of (f, g).

Notation : $X \times_Z Y$

Completeness

A category $\mathcal{C}$ is complete if it has all limits.

Equivalent with : All products and equalizers exist.

Proof Assume products and equalizers exist.

If $N : I \to \mathcal{C}$ is a functor then $\lim N$ is the equalizer of

$$\prod_{i\in I} N(i) \underset{s}{\overset{r}{\rightrightarrows}} \prod_{\phi:i\to j \text{ in } I} N(j)$$

where
$$r_{\phi:i\to j} = p_j$$
$$s_{\phi:i\to j} = \phi \circ p_i$$

Definition A functor is continuous if it commutes with all (existing) limits.

Example The representable functors $\mathrm{Hom}_{\mathcal{C}}(X, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X)$ are continuous.

Colimits

The dual notion of a limit is a colimit.

Universal property

$$\begin{tikzcd}
N(i) \arrow[r, "N(p)"] \arrow[dr] \arrow[drr, "q_i"] & N(j) \arrow[d] \arrow[dr, "q_j"] & \\
& Y & \mathrm{colim}\, N \arrow[l, dotted, "\exists!"]
\end{tikzcd}$$

a co-cone

Equivalent condition : There are isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{i\in I} N(i), Y) \cong \lim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(N(i), Y)$$

natural in Y.

Note We only refer to *limits* in $\mathbf{Set}$ (not colimits).

Dual notions : Coproduct ($\coprod$), initial object, coequalizer, pushout, cocomplete, cocontinuity.

Chapter II

(Pre-)addititive and abelian categories.

Pre-additive categories

Definition A *pre-additive* category is a category where the Hom-sets have the additional structure of an abelian group and compositions are bilinear.

Example $\mathbf{Ab}$ is pre-additive.

Example If R is a ring then $\mathrm{Mod}(R)$, the category of left R-modules is pre-additive. Note: $\mathbf{Ab} = \mathrm{Mod}(\mathbb{Z})$.

Example Assume $\mathcal{C}$ pre-additive and $|\,\mathrm{Ob}(\mathcal{C})| = 1$ e.g. $\mathrm{Ob}(\mathcal{C}) = \{*\}$. Then $\mathcal{C}$ is determined by $R = \mathrm{End}_{\mathcal{C}}(*)$. It is easy to see that R is a ring (always with unit).

A pre-additive category with one object is the same as a ring!

Alternative name for a pre-additive category (Mitchell)

"A ring with many objects."

Excercise If $\mathcal{C}$ is pre-additive and I is small then $\mathrm{Fun}(I, \mathcal{C})$ is pre-additive (in a natural way) as well.

Additive functors

Definition A functor $F : \mathcal{C} \to \mathcal{D}$ between pre-additive categories is *additive* if the maps

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are linear.

Notation $\mathrm{Add}(\mathcal{C}, \mathcal{D})$: additive functors $\mathcal{C} \to \mathcal{D}$.

Basic example

- Let R be a ring, considered as pre-additive category with one object $*$.
- Let $F : R \to \mathbf{Ab}$ be an additive functor.

F is determined by

- An abelian group $M = F(*)$.
- A map of abelian groups:

  $$F : R \to \mathrm{Hom}_{\mathbf{Ab}}(M, M)$$

  compatible with composition. I.e. it should be a ring map.

Putting for $r \in R$, $m \in M$: $r \cdot m = F(r)m$ defines a left R-module structure on M.

This construction yields an *isomorphism* between
$\mathrm{Add}(R, \mathbf{Ab})$ and $\mathrm{Mod}(R)$

For a *small* pre-additive category $\mathcal{C}$ we put
$$\mathrm{Mod}(\mathcal{C}) = \mathrm{Add}(\mathcal{C}, \mathbf{Ab})$$

Special properties

$\mathcal{C}$ pre-additive category.

For $X \in \mathrm{Ob}(\mathcal{C})$ put

1_X : the identity id_X in $\mathrm{Hom}_{\mathcal{C}}(X, X)$

0_X : the zero map in $\mathrm{Hom}_{\mathcal{C}}(X, X)$

Definition A *zero object* in $\mathcal{C}$ is an object X such that

$$1_X = 0_X$$

Proposition The properties of being an initial, final or zero object are equivalent.

Notation A zero object is denoted by ... 0.

Special properties continued

Let $(M_i)_{i \in I}$ be objects in $\mathcal{C}$ and assume $\prod_{i \in I} M_i$ exists.

As for general categories we have "projection maps"
$$p_k : \prod_i M_i \to M_k$$

For $f : X \to \prod_i M_i$ put $f_k = p_k f$. Under the isomorphism
$$\mathrm{Hom}_{\mathcal{C}}(X, \prod_i M_i) \cong \prod_i \mathrm{Hom}_{\mathcal{C}}(X, M_i)$$

f_k is the image of f under the projection on $\mathrm{Hom}_{\mathcal{C}}(X, M_k)$

We now also have "inclusion maps"
$$q_j : M_j \to \prod_i M_i$$

which are defined by

$$(q_j)_k = \begin{cases} 1_{M_j} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

Note Here we notice the importance of having a *canonical* element 0 in every $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Special properties continued

The coproduct in a pre-additive category is usually denoted by $\oplus$.

Assume $\oplus_i M_i$ exists.

We again have canonical maps

$$M_j \xrightarrow{q_j} \oplus_i M_i \xrightarrow{p_j} M_j$$

There is also a canonical map

$$c : \oplus_i M_i \to \prod_i M_i$$

Defined as follows

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\oplus_i M_i, \prod_j M_j) &\cong \prod_j \mathrm{Hom}_{\mathcal{C}}(\oplus_i M_i, M_j) \\ &\cong \prod_j \prod_i \mathrm{Hom}_{\mathcal{C}}(M_i, M_j)\end{aligned}$$

Denote the projection on $\mathrm{Hom}_{\mathcal{C}}(M_i, M_j)$ by $(-)_{ij}$. Then

$$c_{ij} = \begin{cases} 1_{M_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Biproducts

The similarity between products and coproducts in pre-additive categories leads to the notion of a *biproduct*.

Fix a *finite* set of objects $M_1, \dots, M_n$.

Definition A *biproduct* of $M_1, \dots, M_n$ is

- an object N
- maps $q_i : M_i \to N, p_i : N \to M_i$

satisfying

$$\sum_i q_i p_i = 1_N$$

$$p_i q_j = \begin{cases} 1_{M_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

One easily proves.

- Biproducts are unique up to unique isomorphism.
- The coproduct and the product of the M_i (if they exist) are biproducts.
- A biproduct is both a product and a coproduct.

Special properties continued

Using biproducts one obtains.

Theorem Let $M_1, \dots, M_n$ be objects in $\mathcal{C}$. Then $M_1 \times \cdots \times M_n$ exists if and only if $M_1 \oplus \cdots \oplus M_n$ exists. In that case the canonical map

$$c : M_1 \oplus \cdots \oplus M_n \to M_1 \times \cdots \times M_n$$

is an isomorphism.

Functors

Since biproducts are defined by equations, the following is clear.

Theorem An additive functor between pre-additive categories preserves biproducts (and hence finite products and coproducts).

Additive categories

Definition A pre-additive category is *additive* if

- it has a zero object and
- finite products (or equivalently coproducts or biproducts) exist.

Remark It is of course sufficient that *binary* products exist.

Remark A zero object is a final object so it is a product over the empty set.

Example A (non-zero) ring viewed as a pre-additive category is not additive. In fact it has no zero object.

Example If R is a ring then the category $\mathrm{Mod}(R)$ is additive.

Example More generally if $\mathcal{C}$ is a small pre-additive category then $\mathrm{Mod}(\mathcal{C})$ is additive.

Kernels

$\mathcal{C}$ an additive category, $f : M \to N$ a map in $\mathcal{C}$.

Definition The *kernel* $\ker f$ of f is the pullback of the diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ & 0 \arrow[u] \end{tikzcd}$$

(need not exist)

Universal property

$$\begin{tikzcd} \ker f \arrow[r] & M \arrow[r, "f"] & N \\ & X \arrow[u] \arrow[ur, "0"'] \arrow[ul, dotted, "\exists!"] & \end{tikzcd}$$

Kernels II

Definition using representable functors

$\ker f \to M$ is the kernel of $f : M \to N$ if for all $X \in \mathrm{Ob}(\mathcal{C})$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, \ker f) \to \mathrm{Hom}_{\mathcal{C}}(X, M) \to \mathrm{Hom}_{\mathcal{C}}(X, N)$$

is exact

In particular

- $\ker f \to M$ is a monomorphism.
- f is a monomorphism if and only if $\ker f = 0$.

Cokernels

$\mathcal{C}$ an additive category, $f : M \to N$ a map in $\mathcal{C}$.

Definition The *cokernel* $\operatorname{coker} f$ of f is the pushout of the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \\ 0 & \end{tikzcd}$$

(need not exist)

Universal property

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "0"'] & N \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[dl, dotted, "\exists!"] \\ & X & \end{tikzcd}$$

Cokernels II

Definition using representable functors

$N \to \operatorname{coker} f$ is the cokernel of $f : M \to N$ if for all $X \in \operatorname{Ob}(\mathcal{C})$ the sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}}(\operatorname{coker} f, X) \to \operatorname{Hom}_{\mathcal{C}}(N, X) \to \operatorname{Hom}_{\mathcal{C}}(M, X)$$

is exact.

In particular

- $N \to \operatorname{coker} f$ is an epimorphism.
- f is an epimorphism if and only if $\operatorname{coker} f = 0$.

Abelian categories

Assume that $f : M \to N$ has both a kernel and a cokernel.

Consider the following commutative diagram.

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & M \arrow[rrr, "f"] \arrow[dr] & & & N \arrow[r, "j"] & \operatorname{coker} f \\
& & \operatorname{coker} i \arrow[r, dotted, "\exists!"'] & \ker j \arrow[ur] & &
\end{tikzcd}$$

The universal properties for kernel and cokernel imply that the dotted arrow exists and is unique.

Definition An additive category is *abelian* if

- Every map f has a kernel and a cokernel.
- The canonical map $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$ is an isomorphism.

We call $\operatorname{coker} \ker f \cong \ker \operatorname{coker} f$ the *image* of f and denote it by $\operatorname{im} f$.

Examples

Here are some examples of abelian categories.

- If $\mathcal{A}$ is abelian then so is $\mathcal{A}^\circ$ ($\ker$ and coker are exchanged).
- If R is a ring then $\operatorname{Mod}(R)$ is an abelian category.
- Assume R is a ($\mathbb{Z}$-)graded ring. I.e. R comes with a decomposition

  $$R = \oplus_{n \in \mathbb{Z}} R_n$$

  such that $R_m R_n \subset R_{m+n}$. A graded R-module is an R-module with a (given) decomposition $M = \oplus_n M_n$ such that $R_n M_m \subset M_{n+m}$. The category $\operatorname{Gr}(R)$ of graded R-modules is abelian.
- (For those who know) If X is a topological space then

  $\operatorname{Pre}(X)$: Presheaves on X

  $\operatorname{Sh}(X)$: Sheaves on X

  are abelian categories.

A non-example

Let $\mathcal{F}$ be the category of torsion free abelian groups (as a full subcategory of $\mathbf{Ab}$).

- $\mathcal{F}$ has arbitrary products and coproducts (computed as in $\mathbf{Ab}$).
- $\mathcal{F}$ clearly has kernels (computed as in $\mathbf{Ab}$).

Less obvious : $\mathcal{F}$ also has cokernels.

$$\operatorname{coker}_{\mathcal{F}} f = \operatorname{coker}_{\mathbf{Ab}} f / \{\text{torsion}\}$$

However the identity $\operatorname{coker} \ker = \ker \operatorname{coker}$ does not hold.

Example $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.

$$\begin{tikzcd}
0 \arrow[r, "i"] & \mathbb{Z} \arrow[rrr, "\times 2"] \arrow[dr, "\cong"'] & & & \mathbb{Z} \arrow[r, "j"] & 0 \\
& & \mathbb{Z} \arrow[r, dotted, "\times 2"'] & \mathbb{Z} \arrow[ur, "\cong"'] & &
\end{tikzcd}$$

The dotted arrow is *not* an isomorphism.

A non-example II

Note $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ has zero kernel and cokernel. So it is both a monomorphism and an epimorphism. However it is clearly not an isomorphism.

Remark Conversely in an abelian category, the identity $\operatorname{coker}\ker = \ker\operatorname{coker}$ implies that a map which is both a monomorphism and an epimorphism is an isomorphism.

Epi-mono factorization

We have a commutative diagram

$$\begin{tikzcd} \ker f \arrow[r, "i"] & M \arrow[rr, "f"] \arrow[dr, "p"'] & & N \arrow[r, "j"] & \operatorname{coker} f \\ & & \operatorname{im} f \arrow[ur, "q"'] & & \end{tikzcd}$$

Note

- p is epi (being a cokernel).
- q is mono (being a kernel).

We say : $f = pq$ is the epi-mono factorization of f.

Exact sequences

We work in an abelian category $\mathcal{A}$.

Consider a diagram

$$M \xrightarrow{f} N \xrightarrow{g} P \qquad (*)$$

with $gf = 0$.

We obtain a commutative diagram

$$\begin{tikzcd} \ker g \arrow[r] & N \arrow[r, "g"] & P \\ M \arrow[r, "p"'] & \operatorname{im} f \arrow[u, "q"'] \arrow[ul, dotted, "q'"] & \end{tikzcd}$$

(since $gqp = 0$ and p is epi we obtain $gq = 0$).

Definition The diagram (*) is exact (at N) if the canonical map

$$\operatorname{im} f \to \ker g$$

is an isomorphism.

Generalization

Definition A sequence of maps

$$P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} P_n \xrightarrow{d_n} P_{n+1}$$

is *exact* if

$$d_{i+1}d_i = 0$$

for all i and

$$\begin{aligned} \operatorname{im} d_0 &= \ker d_1 \\ \operatorname{im} d_1 &= \ker d_2 \\ &\vdots \\ \operatorname{im} d_{n-1} &= \ker d_n \end{aligned}$$

Special cases

$$0 \to M \xrightarrow{f} N$$

is exact iff f is a monomorphism.

$$N \xrightarrow{g} P \to 0$$

is exact iff g is an epimorphism.

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P$$

is exact iff $gf = 0$ and the canonical map $M \to \ker g$ is an isomorphism. We call this *a (short) left exact sequence.*

$$M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

is exact iff $gf = 0$ and the canonical map $\operatorname{coker} f \to P$ is an isomorphism. We call this a *(short) right exact sequence.*

Short exact sequences

A diagram of the form

$$0 \to M \to N \to P \to 0$$

which is exact, is called *a short exact sequence.*

Below we frequently encounter the category $\mathrm{Ex}(\mathcal{A})$ of short exact sequences in $\mathcal{A}$. The morphisms in $\mathrm{Ex}(\mathcal{A})$ are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Ex}(\mathcal{A})$ is an additive category in a natural way.

Split exact sequences

Assume that we have a short exact sequence in an abelian category

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "q"] & B \arrow[r, shift left, "p"] & C \arrow[l, dotted, shift left, "q'"] \arrow[r] & 0 \end{tikzcd}$$

with p split by q'. I.e. $pq' = \mathrm{id}_C$.

One proves : There is a (unique) left splitting p' of q such that

$$\begin{tikzcd} A \arrow[r, shift left, "q"] & B \arrow[l, shift left, "p'"] \arrow[r, shift right, "p"'] & C \arrow[l, shift right, "q'"'] \end{tikzcd}$$

is a biproduct of A and C.

Proposition A short exact sequence is split on the left if and only if is split on the right. In that case the middle object is a biproduct of the outer objects.

Corollary Split exact sequences remain exact under application of any additive functor.

Note : Any biproduct yields a split short exact sequence.

Left and right exact functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

We know : F preserves finite (co)products.

Definition

- F is *left exact* if it preserves left exact sequences (or equivalently: kernels).
- F is *right exact* if it preserves right exact sequences (or equivalently: cokernels).
- F is *exact* if it preserves short exact sequences (or equivalently, if it is both left and right exact).

The Freyd-Mitchell embedding theorem

Theorem Let $\mathcal{A}$ be an essentially small abelian category. Then there exist a fully faithful exact functor

$$\mathcal{A} \to \mathrm{Mod}(R)$$

where R is some ring.

This is the basis for the technique of *diagram chasing*. I.e. to prove theorems in an abelian category we may assume that we are dealing with objects in a module category.

Warning We need to be careful applying this principle. For example a product of exact sequences is exact in a module category but not in a general abelian category.

Reason: the Freyd-Mitchell embedding functor does not need to preserve products.

However: being additive the F-M functor preserves *finite* (co)products.

The five lemma

Start with a commutative diagram with exact rows.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

Theorem If

- β, δ are isomorphisms.
- ϵ is mono.
- α is epi.

then γ is an isomorphism.

The snake lemma

(exact rows)

$$\begin{tikzcd}
& \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, dotted] & \\
& A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
& \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

The snake lemma asserts that

- The dotted arrow exists (in a canonical way).
- We obtain an exact sequence

$$\ker f \to \ker g \to \ker h \to$$
$$\operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

Additional info

- If $A \to B$ is mono then so is $\ker f \to \ker g$.
- If $B' \to C'$ is epi then so is $\operatorname{coker} g \to \operatorname{coker} h$.

Projective objects

$\mathcal{A}$ abelian category.

Definition An object $P \in \mathrm{Ob}(\mathcal{A})$ is projective if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is an exact functor.

Since $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is always left exact this is equivalent to

- An object $P \in \mathrm{Ob}(\mathcal{A})$ is projective if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ sends epimorphisms to surjections.

This leads to the usual definition in terms of commutative diagrams:

$$\begin{tikzcd}
A \arrow[r] & B \arrow[r] & 0 \\
& P \arrow[u] \arrow[ul, dotted, "\exists"] &
\end{tikzcd}$$

Other characterization : P is projective if and only if any epimorphism

$$A \longrightarrow P \longrightarrow 0$$

splits.

Projective objects: properties

Proposition If $(P_i)_{i\in I}$ are projective objects then so is $\oplus_{i\in I} P_i$ (if the latter exists).

Follows from

$$\mathrm{Hom}_{i\in I}(\oplus_i P_i, -) = \prod_i \mathrm{Hom}_i(P_i, -)$$

which is exact.

Proposition If $P \oplus Q$ is projective then so are P and Q.

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & 0 \\ & P \arrow[u, "f"'] & \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r] & B \arrow[r] & 0 \\ & P \oplus Q \arrow[u, "{(f,0)}"'] \arrow[ul, dotted] & \\ & P \arrow[u, "{(1,0)}"'] \arrow[uul, dotted] & \end{tikzcd}$$

Definition If $X \cong P \oplus Q$ then P, Q are said to be *summands* of X.

Modules

Let R be a ring.

Defintion A free R module is one which is of the form $R^{\oplus I}$ for some set I.

Proposition The projective modules are precisely the direct summands of free modules.

$$\begin{tikzcd} R^{\oplus P} \arrow[r] & P \arrow[r] \arrow[l, dotted, bend left, "\exists"] & 0 \end{tikzcd}$$

Injective objects

Short definition : The injective objects in $\mathcal{A}$ are the projective objects in $\mathcal{A}^\circ$.

Proposition The following are equivalent for $E \in \mathrm{Ob}(\mathcal{A})$.

- E is injective.
- $\mathrm{Hom}_{\mathcal{A}}(-, E)$ is exact.
- $\mathrm{Hom}_{\mathcal{A}}(-, E)$ sends monomorphims to surjections.
- $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$
- Any monomorphism

  $$0 \longrightarrow E \longrightarrow A$$

  splits.

Example $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective objects in $\mathbf{Ab}$.

In general : Injective objects are quite complicated.

Chapter III

Grothendieck's AB properties.

Note about limits and colimits

An abelian category has all finite limits and colimits.

This follows from the fact that (co)equalizers and finite (co)products exist.

Example : The equalizer of

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is the kernel of $f - g$.

An abelian category is (co) complete if and only if it has all (co) products.

Grothendieck's list

Grothendieck made up a list of possible good properties of abelian category $\mathcal{A}$.

The relevant properties are (AB3-5) and their duals (AB3*-5*).

(AB3) $\mathcal{A}$ is cocomplete.

(AB4) $\mathcal{A}$ satisfies (AB3) and coproducts are exact. I.e. if we have a family of exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

indexed by a set I then

$$0 \to \oplus_i A_i \to \oplus_i B_i \to \oplus_i C_i \to 0$$

is also exact.

Filtered partially ordered sets

Definition Let $(I, \leq)$ be a partially ordered set. We say that I is *filtered* if for all $i, j \in I$ there exists $k \in I$ such that $i \leq k, j \leq k$.

Construction A partially ordered set may be viewed as a category as follows.

$$\mathrm{Hom}_I(i,j) = \begin{cases} \{*\} & \text{if } i \leq j \\ \emptyset & \text{otherwise} \end{cases}$$

($\{*\}$ is a fixed singleton). The compositions are defined by $* \circ * = *$.

Colimits over filtered posets

Fact Let $I \to \mathrm{Mod}(R)$ be a functor with I a filtered partially ordered set.

$$\mathrm{colim}_{i \in I} M(i) = \coprod_{i \in I} M(i) / \sim$$

where

$$(m \in M(i)) \sim (n \in M(j)) \Leftrightarrow$$
$$\exists k \in I, i, j \leq k, M(i \to k)(m) = M(j \to k)(n)$$

Terminology If $\mathcal{C}$ is a category then a set of objects $(M_i)_{i \in I}$ indexed by a partially ordered set is a functor $M : I \to \mathcal{C}$ with $M(i) = M_i$.

The (AB5) axiom

(AB5) $\mathcal{A}$ satifies (AB3) and filtered colimits are exact. I.e. if we have a family of exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

indexed by a partially ordered set I then

$$0 \to \operatorname{colim}_{i\in I} A_i \to \operatorname{colim}_{i\in I} B_i \to \operatorname{colim}_{i\in I} C_i \to 0$$

is also exact.

Fact $\operatorname{Mod}(R)$ satisfies (AB5) and (AB4*). Typical categories in algebraic geometry (i.e. sheaves) satisfy (AB5) and (AB3*).

Note about generators

Proposition

Let $\mathcal{A}$ be a cocomplete abelian category. Then $G \in \operatorname{Ob}(\mathcal{A})$ is a generator if and only if for any $A \in \operatorname{Ob}(\mathcal{A})$ there is an epimorphism

$$G^{\oplus I} \to A$$

for some I.

Proof Excercise.

Grothendieck categories

Definition A *Grothendieck category* is an abelian category which has a generator and which satisfies (AB5).

Grothendieck categories have some highly non-obvious properties.

Proposition A Grothendieck category satisfies (AB3*).

Proposition A Grothendieck category has enough injectives. I.e. any object has a monomorphism to an injective object.

The Gabriel-Popescu theorem

The deepest result about Grothendieck categories is the *Gabriel-Popescu theorem.*

Proposition Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{G}$ be a generator of $\mathcal{A}$. Put $S = \operatorname{End}_{\mathcal{A}}(G)$. Then the functor

$$\operatorname{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \to \operatorname{Mod}(S^\circ)$$

is fully faithful (and has an exact left adjoint).

Note Let $A \in \operatorname{Ob}(\mathcal{A})$. The right S module structure on $\operatorname{Hom}_{\mathcal{A}}(G, A)$ is obtained from the composition

$$\operatorname{Hom}_{\mathcal{A}}(G,A)\times\operatorname{Hom}_{\mathcal{A}}(G,G)\to\operatorname{Hom}_{\mathcal{A}}(G,A):(f,g)\mapsto fg$$

The embedding theorem

For $\mathcal{A}$ an essentially small abelian category put

$$\mathrm{Lex}(\mathcal{A}) = \{\text{left exact additive functors } \mathcal{A}^\circ \to \mathbf{Ab}\}$$

Theorem $\mathrm{Lex}(\mathcal{A})$ is a Grothendieck category and the functor

$$\mathcal{A} \mapsto \mathrm{Lex}(\mathcal{A}) : A \mapsto \mathrm{Hom}_{\mathcal{A}}(-, A)$$

is fully faithul (Yoneda!) and exact.

Remark One may show that $\mathrm{Lex}(\mathcal{A})$ is in a certain sense the formal closure $\mathcal{A}$ under filtered colimits.

Alternative notation

$$\mathrm{Ind}(\mathcal{A})$$

Chapter IV

Morita theory

General Morita theory

Let $\mathcal{A}$ be a cocomplete abelian category.

Definition $A \in \mathrm{Ob}(\mathcal{A})$ *small* if for any family of objects $(B_i)_{i\in I}$ the canonical map

$$\oplus_i \mathrm{Hom}_{\mathcal{A}}(A, B_i) \to \mathrm{Hom}_{\mathcal{A}}(A, \oplus_i B_i)$$

is an isomorphism.

One proves : A projective object $P \in \mathrm{Mod}(R)$ is small if and only if it is finitely generated.

Definition A small projective generator in $\mathcal{A}$ is called a *progenerator*.

Theorem Assume that $P \in \mathrm{Ob}(\mathcal{A})$ is a progenerator. Put $S = \mathrm{End}_{\mathcal{A}}(P)$. Then the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Mod}(S^\circ)$$

is an equivalence of categories.

Note This result may be deduced from the Gabriel-Popescu theorem but here the proof is much easier.

Proof

We have to prove that $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is fully faithful and essentially surjective.

Full faithfulness

We have to prove that the natural map

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_S(\mathrm{Hom}_{\mathcal{A}}(P, M), \mathrm{Hom}_{\mathcal{A}}(P, N)) :$$
$$f \mapsto \mathrm{Hom}_{\mathcal{A}}(P, f)$$

is an isomorphism for all $M, N \in \mathcal{A}$.

We use the fact that the functor

$$F : \mathrm{Hom}_S(\mathrm{Hom}_{\mathcal{A}}(P, -), \mathrm{Hom}_{\mathcal{A}}(P, N)) : \mathcal{A}^\circ \to \mathbf{Ab}$$

is left exact (since P is projective), and sends sums to products (since P is small).

Furthermore we have

$$\begin{aligned} F(P) &= \mathrm{Hom}_S(\underbrace{\mathrm{Hom}_{\mathcal{A}}(P, P)}_{S}, \mathrm{Hom}_{\mathcal{A}}(P, N)) \\ &= \mathrm{Hom}_{\mathcal{A}}(R, N) \end{aligned}$$

Proof cont'd

Since P is a generator we may construct a right exact sequence

$$P^{\oplus J} \to P^{\oplus I} \to M \to 0$$

for sets I, J. This yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P,N)^{\times I} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P,N)^{\times J} \arrow[d] \\
0 \arrow[r] & F(M) \arrow[r] & F(P)^{\times I} \arrow[r] & F(P)^{\times J}
\end{tikzcd}$$

Since $F(P) = \mathrm{Hom}_{\mathcal{A}}(P, N)$ the two rightmost vertical maps are iso's. Hence so is the leftmost one.

Proof cont'd

Essential surjectivity

Let $Z \in \mathrm{Mod}(S^\circ)$. We have to write it (up to isomorphism) as $\mathrm{Hom}_{\mathcal{A}}(P, X)$.

Idea : We can do this if $Z = S^{\oplus I}$. Take $X = P^{\oplus I}$.

For general Z construct a short exact sequence

$$S^{\oplus J} \xrightarrow{g} S^{\oplus I} \to Z \to 0$$

Since $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is fully faithful there is some

$$f : P^{\oplus J} \to P^{\oplus I}$$

such that

$$g = \mathrm{Hom}_{\mathcal{A}}(P, f)$$

It now suffices to take

$$X = \operatorname{coker} f$$

A converse result

If there is any equivalence

$$H : \mathcal{A} \to \mathrm{Mod}(S^\circ)$$

between a cocomplete abelian category and a module category then $S \cong \mathrm{End}_{\mathcal{A}}(P)$ for a progenerator P in $\mathcal{A}$, and furthermore the equivalence is of the form $\mathrm{Hom}_{\mathcal{A}}(P, -)$.

Hint : Take a (quasi-)inverse H^{-1} for H and let $P = H^{-1}(S)$.

Morita equivalent rings

Definition Rings R, S are *Morita equivalent* if $\mathrm{Mod}(R^\circ) \cong \mathrm{Mod}(S^\circ)$.

Applying the above theory with $\mathcal{A} = \mathrm{Mod}(R^\circ)$ we obtain a second equivalent definition.

Definition Rings R, S are *Morita equivalent* if $S \cong \mathrm{End}_R(P)$ for a progenerator $P \in \mathrm{Mod}(R^\circ)$.

Example If $P = R^n$ then $S = M_n(R)$ ($n \times n$-matrices over R).

Notes on duality

If $M \in \mathrm{Mod}(R)$ then we define the *dual* $M^* \in \mathrm{Mod}(R^\circ)$ of M as:

$$M^* = \mathrm{Hom}_R(M, R)$$

As usual there is a canonical map

$$\mathrm{ev}_M : M \to M^{**} : m \mapsto (\phi \mapsto \phi(m))$$

One proves

- If P is finitely generated projective then ev_P is an isomorphism.
- $(-)^*$ defines a (contravariant) equivalence between the categories of finitely generated left and right projective R-modules.

Duality for progenerators

Note If P is a finitely generated projective in $\mathrm{Mod}(R)$ then P is a progenerator if and only if R is a summand of some $P^{\oplus n}$.

We obtain : P is a progenerator if and only if this is the case for P^*.

Morita equivalence for left modules

- Assume $\mathrm{Mod}(R^\circ) \cong \mathrm{Mod}(S^\circ)$.
- This equivalence corresponds to a progenerator $P \in \mathrm{Mod}(R^\circ)$ with $S = \mathrm{End}_R(P)$.
- Then $P^* = \mathrm{Hom}_R(P, R)$ is a progenerator of $\mathrm{Mod}(R)$.
- Hence
$$\mathrm{Mod}(R) \cong \mathrm{Mod}(T^\circ)$$
for $T = \mathrm{End}_R(P^*)$.
- But one has
$$T = \mathrm{End}_R(P^*) \cong \mathrm{End}_R(P)^\circ = S^\circ$$
- Thus we obtain
$$\mathrm{Mod}(R) \cong \mathrm{Mod}(S^{\circ\circ}) = \mathrm{Mod}(S)$$

Conclusion : One has $\mathrm{Mod}(R^\circ) \cong \mathrm{Mod}(S^\circ)$ if and only if $\mathrm{Mod}(R) \cong \mathrm{Mod}(S)$.

Chapter V

Presheaves and sheaves.

Presheaves

Throughout let X be a topologogical space.

Let $\mathrm{Open}(X)$ be the set of all open subsets of X. We view $\mathrm{Open}(X)$ as a partially ordered set (ordered by inclusion), and hence as a category.

We define the category of *presheaves of abelian groups* on X as

$$\mathrm{Pre}(X) = \mathrm{Fun}(\mathrm{Open}(X)^{\circ}, \mathbf{Ab})$$

Alternative definition

A pre-sheaf $\mathcal{F}$ (of abelian groups) consists of

- For every open $U \subset X$ an abelian group $\mathcal{F}(U)$.
- For every inclusion of opens $U \subset V$: *restriction maps* $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

such that for every inclusion of opens $U \subset V \subset W$ we have equality

$$\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$$

Notation : For $x \in \mathcal{F}(V)$ we write $x|U = \rho_{V,U}(x)$.

Terminology The elements of the $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$. The elements of $\mathcal{F}$ are called *global sections*.

Easy : $\mathrm{Pre}(X)$ is an abelian category. Kernels and cokernels can be computed on each open set.

$$\ker(\mathcal{F} \to \mathcal{G})(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$$
$$\mathrm{coker}(\mathcal{F} \to \mathcal{G})(U) = \mathrm{coker}(\mathcal{F}(U) \to \mathcal{G}(U))$$

Sheaves

Definition A sheaf is a presheaf $\mathcal{F}$ such that

- for every open $U \subset X$
- and every open covering $\bigcup_{i \in I} U_i = U$
- and every family of sections $s_i \in \mathcal{F}(U_i)_i$
- such that $s_i|U_i \cap U_j = s_j|U_i \cap U_j$
- there exists a unique section $s \in \mathcal{F}(U)$
- such that $s|U_i = s_i$ for all i.

For a presheaf $\mathcal{F}$ the "sheaffification" $a\mathcal{F}$ is defined by the following universal property for any sheaf $\mathcal{G}$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & a\mathcal{F} \arrow[dl, dotted, "\exists!"] \\
\mathcal{G} &
\end{tikzcd}$$

Roughly speaking : $a\mathcal{F}$ is constructed by first dividing out the sections which are locally zero and then by adjoining new sections which are defined on a covering.

Abelian structure

Notation : $\mathrm{Sh}(X)$ is the full subcategory of $\mathrm{Pre}(X)$ whose objects are sheaves.

Fact : $\mathrm{Sh}(X)$ is an abelian category.

However : The inclusion

$$\mathrm{Sh}(X) \subset \mathrm{Pre}(X)$$

is left exact (it preserves kernels), but *not right exact* (it does not preserve cokernels).

Principle : The non-exactness of this inclusion functor is the basis for the theory of *sheaf cohomology*.

Formula :

$$\mathrm{coker}_{\mathrm{Sh}(X)}(\mathcal{F} \to \mathcal{G}) = a(\mathrm{coker}_{\mathrm{Pre}(X)}(\mathcal{F} \to \mathcal{G}))$$

Fact : The sheaffification functor

$$a : \mathrm{Pre}(X) \to \mathrm{Sh}(X)$$

is exact.

Stalks

Let $\mathcal{F}$ be a presheaf on X and $x \in X$. Let

$$\mathrm{Open}(X,x) = \{U \in \mathrm{Open}(X) \mid x \in U\}$$

(viewed as poset and as category)

The *stalk* of $\mathcal{F}$ at x is defined as

$$\mathcal{F}_x = \mathrm{colim}_{U \in \mathrm{Open}(X,x)^\circ} \mathcal{F}(U)$$

One proves : $(\mathcal{F})_x = (a\mathcal{F})_x$.

Since $\mathrm{Open}(X,x)^\circ$ is filtered(!) one also sees that $(-)_x$ is an exact functor on $\mathrm{Pre}(X)$ and on $\mathrm{Sh}(X)$.

One proves : A diagram in $\mathrm{Sh}(X)$

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact at $\mathcal{F}$ if and only if for all $y \in X$

$$\mathcal{F}_y \xrightarrow{f_y} \mathcal{G}_y \xrightarrow{g_y} \mathcal{H}_y$$

is exact at $\mathcal{G}_y$.

Example: the exponential sequence

X : manifold.

$\mathcal{O}$: the sheaf of complex valued continuous functions on X (with the additive abelian group structure).

$\mathcal{O}^*$: functions which are everywhere non-zero (with the multiplicative group structure).

$\underline{\mathbb{Z}}^p$: the *constant presheaf* with values in $\mathbb{Z}$.

$\underline{\mathbb{Z}} = a(\mathbb{Z}^p)$ (the constant *sheaf* with values in $\mathbb{Z}$).

Fact : There is an exact sequence of sheaves.

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

(look at stalks).

However this sequence is usually not exact as presheaves.

Example : Let $X = \mathbb{C}^*$ and let $f \in \mathcal{O}^*(X)$ be the non-zero function $z \mapsto z$. f is not of the form $\exp(g)$, as $\log f$ cannot be made continuous on $\mathbb{C}^*$.

Grothendieck categories

Fact : Both $\mathrm{Pre}(X)$ and $\mathrm{Sh}(X)$ are Grothendieck categories. Hence they have enough injectives.

Formula :

$$\mathrm{colim}_{i,\mathrm{Sh}(X)} \mathcal{F}_i = a(\mathrm{colim}_{i,\mathrm{Pre}(X)} \mathcal{F}_i)$$

Generators for $\mathrm{Pre}(X)$

$U \subset X$ open.

$$\mathbb{Z}_U^p(V) = \begin{cases} \mathbb{Z} & \text{if } V \subset U \\ 0 & \text{otherwise} \end{cases}$$

Formula :

$$\mathrm{Hom}_{\mathrm{Pre}(X)}(\mathbb{Z}_U^p, \mathcal{F}) = \mathcal{F}(U)$$

Fact : The $\mathbb{Z}_U^p$ are generators for $\mathrm{Pre}(X)$.

Generators for $\mathrm{Sh}(X)$

$$\mathbb{Z}_U = a(\mathbb{Z}_U^p)$$

Fact : The $\mathbb{Z}_U$ are generators for $\mathrm{Sh}(X)$.

Special properties of $\mathrm{Sh}(X)$

- $\mathrm{Pre}(X)$ has enough projectives, but $\mathrm{Sh}(X)$ usually has not.
- $\mathrm{Pre}(X)$ satisfies (AB4*) but $\mathrm{Sh}(X)$ only satisfies (AB34*).

Chapter VI

Classical homological algebra.

Complexes

Definition A *graded category* is a category $\mathcal{C}$ with an automorphism $s : \mathcal{C} \longrightarrow \mathcal{C}$ (the shift functor).

Convention : We write

$$A[n] = s^n A$$

and

$$\mathrm{Hom}^n_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B[n])$$

Terminology : $\mathrm{Hom}^n_{\mathcal{C}}(A, B)$ are the maps of *degree* n.

Notations : $|f| = \deg f = n$.

We obtain compositions

$$\mathrm{Hom}^m_{\mathcal{C}}(B,C) \times \mathrm{Hom}^n_{\mathcal{C}}(A,B) \to \mathrm{Hom}^{m+n}(A,C):$$
$$(f, g) \to s^n(f)g$$

Graded objects

If $\mathcal{D}$ is any category then the category $\mathrm{Gr}(\mathcal{D})$ of *graded objects* over $\mathcal{D}$ is the category of sequences of objects in $\mathcal{D}$

$$(A_n)_{n \in \mathbb{Z}}$$

with Hom-sets.

$$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{D})}((A_n)_n, (B_n)_n) = \prod_n \mathrm{Hom}_{\mathcal{D}}(A_n, B_n)$$

Shift functor

$$s((A_n)_n) = (A_{n+1})_n$$

(shift to left).

Complexes II

If $\mathcal{A}$ is abelian then so is $\mathrm{Gr}(\mathcal{A})$ and $\ker, \mathrm{coker}$ may be computed componentwise.

Definition A complex over $\mathcal{A}$ is

- An object A in $\mathrm{Gr}(\mathcal{A})$.
- A map $d \in \mathrm{Hom}^1_{\mathrm{Gr}(\mathcal{A})}(A, A)$ with $d^2 = 0$.

Standard view

$$\cdots \longrightarrow A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} A_{n+2} \longrightarrow \cdots$$

with $d_{n+1}d_n = 0$.

Complexes III

The category $\mathrm{C}(\mathcal{A})$ has as objects the complexes over $\mathcal{A}$ and Hom-sets:

$$\mathrm{Hom}_{\mathrm{C}(\mathcal{A})}((A,d),(A',d')) =$$
$$\{f \in \mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}((A,d),(A',d')) \mid d'f = fd\}$$

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_{n+2} \arrow[r] \arrow[d, "f_{n+2}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n+1} \arrow[r, "d'_{n+1}"'] & A'_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

$\mathrm{C}(\mathcal{A})$ is also abelian and $\ker, \operatorname{coker}$ may be computed termwise.

Grading $(A, d_A)[1] = (A[1], \overset{(!)}{-}d_A)$.

Homology

$(A, d) \in \mathrm{C}(\mathcal{A})$.

$$H(A) \overset{\text{def}}{=} \ker d / \operatorname{im} d$$

Homology functor

$$H : \mathrm{C}(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$$

Notation : $H^n(A) = H(A)_n$.

$$\cdots \to A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \to \cdots$$

$$H^n(A) = \ker d_n / \operatorname{im} d_{n-1}$$

The long exact sequence for homology

Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

Then there exists a *connecting morphism*

$$\delta : H(C) \to H(A)[1]$$

such that there is a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[r, "H^{i-1}(\beta)"] & H^{i-1}(C) \arrow[dlll, dotted, "\delta^{i-1}"] \\
H^i(A) \arrow[r, "H^i(\alpha)"] & H^i(B) \arrow[r, "H^i(\beta)"] & H^i(C) \arrow[dll, dotted, "\delta^i"] & \\
H^{i+1}(A) \arrow[r, "H^{i+1}(\alpha)"] & \cdots & &
\end{tikzcd}$$

The long exact sequence II

Remarks

- The result is proved using diagram chasing.
- The statement must be completed by saying that any map between short exact sequences yields a map between the corresponding long exact sequences (viewed as complexes). The only problem is the connecting morphism which requires some diagram chasing.
- Will give a better proof later using derived categories.

Homotopy

For $A, B \in \mathrm{Gr}(\mathcal{A})$ we define

$$\underline{\mathrm{Hom}}_{\mathrm{Gr}(\mathcal{A})}(A,B) = (\mathrm{Hom}^n_{\mathrm{Gr}(\mathcal{A})}(A,B))_n \in \mathrm{Gr}(\mathbf{Ab})$$

If $A, B \in \mathrm{C}(\mathcal{A})$ then $\underline{\mathrm{Hom}}_{\mathrm{Gr}(\mathcal{A})}(A,B)$ becomes an element of $\mathrm{C}(\mathcal{A})$ by defining

$$d(f) = d_B f - (-1)^n f d_A$$

for $f \in \mathrm{Hom}^n_{\mathrm{Gr}(\mathcal{A})}(A,B)$.

We denote this complex by $\underline{\mathrm{Hom}}_{\mathrm{C}(\mathcal{A})}(A,B)$

Note : If $|f| = 0$ then $f \in \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A,B)$ if and only if $d(f) = 0$.

Definition $f, g \in \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A,B)$ are *homotopic* if $f - g = dh$ for $h \in \mathrm{Hom}^{-1}_{\mathrm{C}(\mathcal{A})}(A,B)$. I.e. if

$$f - g = d_B h + h d_A$$

Notation : $f \sim g$ (h is called the *homotopy* connecting f and g).

Note : Two homotopic maps induces the same map on homology (since in homology, d_A, d_B become zero).

The homotopy category.

We define the *homotopy category* $K(\mathcal{A})$ of $\mathcal{A}$ as follows:

- $\mathrm{Ob}(K(\mathcal{A}) = \mathrm{Ob}(\mathrm{C}(\mathcal{A}))$.
- For $A, B \in \mathrm{Ob}(K(\mathcal{A})$

$$\begin{aligned}\mathrm{Hom}_{K(\mathcal{A})}(A,B) &= H^0(\underline{\mathrm{Hom}}_{\mathrm{C}(\mathcal{A})}(A,B)) \\ &= \mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A,B)/\sim\end{aligned}$$

The composition in $\mathrm{Hom}_{\mathrm{C}(\mathcal{A})}(A,B)$ induces a composition in $K(\mathcal{A})$ which is well defined because of the following identity:

Identity : For arrows of degrees m, n

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we have

$$d(gf) = d(g)f + (-1)^m g\, d(f)$$

Note : The category $K(\mathcal{A})$ is (almost never) abelian (in contrast to $\mathrm{C}(\mathcal{A})$).

Fundamental diagram

$$\begin{tikzcd} \mathrm{C}(\mathcal{A}) \arrow[r, "H"] \arrow[d] & \mathrm{Gr}(\mathcal{A}) \\ K(\mathcal{A}) \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

General principle Homological algebra takes place in the homotopy category (and later : in the derived category).

Note : Any additive

$$F : \mathcal{A} \to \mathcal{B}$$

functor can be lifted to a functor

$$\mathrm{C}(F) : \mathrm{C}(\mathcal{A}) \to \mathrm{C}(\mathcal{B})$$

(evaluating termwise). This yields a well defined functor

$$K(F) : K(\mathcal{A}) \to K(\mathcal{B})$$

The above diagram is compatible with these functors.

Analogy with topology

Principle The notion of homotopy equivalence for complexes is analogous to the notion of homotopy in algebraic topology.

Put $I = [0,1]$. Let $X, Y \in \mathbf{Top}$.

Definition $f, g : X \to Y$ are *homotopy equivalent* if there is a map

$$h : X \times I \to Y$$

such that there are commutative diagrams

$$\begin{tikzcd} & X \times I \arrow[dr, "h"] & \\ X \arrow[ur, "x \mapsto (x,0)"] \arrow[rr, "f"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} & X \times I \arrow[dr, "h"] & \\ X \arrow[ur, "x \mapsto (x,1)"] \arrow[rr, "g"'] & & Y \end{tikzcd}$$

In algebraic topology one constructs a functor

$$C : \mathbf{Top} \to \mathrm{C}(\mathbf{Ab})$$

(the singular chain complex) such that if f, g are homotopy equivalent then so are $C(f), C(g)$.

Projective resolutions

Definition $\mathcal{A}$ has *enough projectives* if for any $A \in \mathrm{Ob}(\mathcal{A})$ there is an epimorphism:

$$P \longrightarrow A \longrightarrow 0$$

with P projective.

Definition A *projective resolution* of $A \in \mathrm{Ob}(\mathcal{A})$ is a complex of projective objects

$$\cdots \to P_{-2} \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \to 0$$

together with a map $P_0 \to A$ such that

$$\cdots \to P_{-2} \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \to A \to 0$$

is exact.

Note : If $\mathcal{A}$ has enough projectives then a projective resolution always exists.

$$P_1 \twoheadrightarrow \ker p \hookrightarrow P_0 \overset{p}{\twoheadrightarrow} A$$

Note : If we drop the requirement that the P_i are projective then we speak of a (left) resolution of A.

Uniqueness of projective resolutions

Principle

- Maps between objects lift to maps between projective resolutions.
- Such a lifted map is far from unique but it is unique up to homotopy.

projective resolution of A

$$\begin{tikzcd}
\cdots \arrow[r] & P_{-2} \arrow[r, "d_{-2}"] \arrow[d, dotted, "\exists f_{-2}"'] & P_{-1} \arrow[r, "d_{-1}"] \arrow[d, dotted, "\exists f_{-1}"'] & P_0 \arrow[r] \arrow[d, dotted, "\exists f_0"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_{-2} \arrow[r, "d_{-2}"'] & Q_{-1} \arrow[r, "d_{-1}"'] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

resolution of A

The lifted map $f : P \to Q$ is unique in $K(\mathcal{A})$.

Uniqueness II

Apply lifting of identity to two resolutions an object.

$$\begin{tikzcd}
\cdots \arrow[r] & P_{-2} \arrow[r, "d_{-2}"] \arrow[d, dotted, "f_{-2}"'] & P_{-1} \arrow[r, "d_{-1}"] \arrow[d, dotted, "f_{-1}"'] & P_0 \arrow[r] \arrow[d, dotted, "f_0"'] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & 0 \\
\cdots \arrow[r] & Q_{-2} \arrow[r, "d_{-2}"] \arrow[d, dotted, "g_{-2}"'] & Q_{-1} \arrow[r, "d_{-1}"] \arrow[d, dotted, "g_{-1}"'] & Q_0 \arrow[r] \arrow[d, dotted, "g_0"'] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & 0 \\
\cdots \arrow[r] & P_{-2} \arrow[r, "d_{-2}"'] & P_{-1} \arrow[r, "d_{-1}"'] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Now

$$gf : P \to P$$

is a lifting of the identity on A. Hence $gf \sim \mathrm{id}_P$. Likewise $fg \sim \mathrm{id}_Q$.

Conclusion Projective resolutions are unique up to (unique) isomorphism in $K(\mathcal{A})$.

Uniqueness III

Assume that $\mathcal{A}$ has enough projectives. Pick for any $A \in \mathrm{Ob}(\mathcal{A})$ a projective resolution $P(A)$ in $\mathrm{C}(\mathcal{A})$. We obtain a commutative diagram of functors.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "P"] \arrow[dr, "\cong"'] & K(\mathcal{A}) \arrow[d, "H^0"] \\
& \mathcal{A}
\end{tikzcd}$$

The diagonal arrow is naturally isomorphic to the identity functor.

Note : Different choices of projective resolutions yield naturally isomorphic functors $\mathcal{A} \to K(\mathcal{A})$.

The horseshoe lemma

Principle : Assume we have an exact sequence

$$0 \to A \to B \to C \to 0$$

and projective resulutions

$$\cdots \longrightarrow P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

$$\cdots \longrightarrow Q_{-2} \longrightarrow Q_{-1} \longrightarrow Q_0 \longrightarrow C \longrightarrow 0$$

Then we can construct a commutative diagram of projective resolutions

$$\begin{tikzcd}
\cdots \arrow[r] & P_{-2} \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & R_{-2} \arrow[r] \arrow[d] & R_{-1} \arrow[r] \arrow[d] & R_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & Q_{-2} \arrow[r] & Q_{-1} \arrow[r] & Q_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

such that

$$0 \to P \to R \to Q \to 0$$

is exact (necessarily split in $\mathrm{Gr}(\mathcal{A})$).

Construction

$$\begin{tikzcd}
0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker p \arrow[r] \arrow[d, dotted] & P_0 \arrow[r, "p"] \arrow[d, "(1,0)"'] & A \arrow[r] \arrow[d] & 0 \\
\ker(1,\phi) \arrow[r, dotted] \arrow[d, dotted] & P_0 \oplus Q_0 \arrow[r, dotted, "(1,\phi)"] \arrow[d, "(0,1)"'] & B \arrow[r] \arrow[d] & 0 \\
\ker p' \arrow[r] \arrow[d, dotted] & Q_0 \arrow[r, "p'"'] \arrow[d] \arrow[ur, dotted, "\exists\phi"'] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

The snake lemma (applied to the dotted rectangle) implies that

$$0 \to \ker p \to \ker(1,\phi) \to \ker p' \to 0$$

is exact.

Hence We may repeat to obtain the desired resolutions.

Left derived functors

Assume

- $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor.
- $\mathcal{A}$ has enough projectives.

For $A \in \mathrm{Ob}(\mathcal{A})$ fix a projective resolution

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \to A \to 0$$

We appy F to this projective resolution.

$$\cdots \to F(P_{-2}) \to F(P_{-1}) \to F(P_0) \to F(A) \to 0$$

If F is not exact then this will in general not be an exact sequence.

We define

$$L_i F(A) = H^{-i}(\mathrm{C}(F)(P))$$

Note : $L_0 F(A) \cong A$ (canonically).

Well definedness and functoriality

- We may als write
$$L_i F(A) = H^{-i}(K(F)(P))$$
(since homology may be computed in the homotopy category).
- Two projective resolutions P, Q of A are canonically isomorphic in $K(\mathcal{A})$.
- Thus $K(F)(P)$ and $K(F)(Q)$ are canonically isomorphic in $K(\mathcal{B})$.
- Hence $H^{-i}(K(F)(P))$ and $H^{-i}(K(F)(Q))$ are canonically isomorphic. We identify them and write them as $L_{-i}F(A)$.

By lifting maps in $\mathcal{A}$ to projective resolutions (unique in the homotopy category!) we obtain functors

$$L_i F : \mathcal{A} \to \mathcal{B}$$

The sequence of functors $(L_i F)_i$ is called the left derived functor of F.

Example: Tor-functors

Let R be a ring and let M be a right R-module.

Then we have a right exact functor.

$$M \otimes_R - : \mathrm{Mod}(R) \to \mathbf{Ab}$$

The derived functors are written as

$$\mathrm{Tor}_i^R(M, -)$$

Example

Assume $R = \mathbb{Z}$. We will compute $\mathrm{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$.

Projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

Tensoring...

$$M \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{(\times p)\otimes_{\mathbb{Z}}\mathbb{Z}} N \otimes_{\mathbb{Z}} \mathbb{Z}$$

But $M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M$, etc... Thus $\mathrm{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$ is the homology of the complex

$$M \xrightarrow{\times p} M$$

Hence

$$\mathrm{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} M/pM & \text{if } i = 0 \\ \ker(M \xrightarrow{\times p} M) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

explaining the notation "Tor(sion)".

The long exact sequence

$F : \mathcal{A}, \mathcal{B}$ right exact, $\mathcal{A}$ enough projectives.

Theorem Assume that we have a long exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

Then there are "connecting maps"

$$\delta_i : L_iF(C) \to L_{i-1}F(A)$$

such that there is a long exact sequence

$$\begin{array}{l}
\cdots \xrightarrow{L_{i+1}F(\beta)} L_{i+1}F(C) \xrightarrow{\delta_{i+1}} \\
\quad L_iF(A) \xrightarrow{L_iF(\alpha)} L_iF(B) \xrightarrow{L_iF(\beta)} L_iF(C) \xrightarrow{\delta_i} \\
\quad L_{i-1}F(A) \xrightarrow{L_{i-1}F(\alpha)} \cdots \\
\qquad \vdots \\
\cdots \longrightarrow F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0
\end{array}$$

Proof of theorem (sketch)

We start by using the horseshoe lemma to construct a commutative diagram of projective resolutions

$$\begin{tikzcd}
\cdots \arrow[r] & P_{-2} \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & R_{-2} \arrow[r] \arrow[d] & R_{-1} \arrow[r] \arrow[d] & R_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & Q_{-2} \arrow[r] & Q_{-1} \arrow[r] & Q_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

such that

$$0 \to P \to R \to Q \to 0$$

is exact in $\mathrm{C}(\mathcal{A})$

Since this sequence is split in $\mathrm{Gr}(\mathcal{A})$ we obtain an exact sequence

$$0 \to \mathrm{C}(F)(P) \to \mathrm{C}(F)(R) \to \mathrm{C}(F)(Q) \to 0$$

in $\mathrm{Gr}(\mathcal{B})$. This sequence is then also exact in $\mathrm{C}(\mathcal{B})$.

It now suffices to apply the long exact sequence for homology.

Proof cont'd

The proof is incomplete as the constructed connecting maps may depend on the chosen projective resolutions. Proving that this is not the case is slightly tricky.

Main point : Suppose we have a commutative diagram of complexes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

and we have constructed projective resolutions of the top and the bottom row using the horseshoe lemma

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Then these resolutions may be assembled in one big commutative diagram.

Proof cont'd

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 & \\
 & 0 \arrow[r] & P \arrow[r] \arrow[uul] \arrow[d, dotted] & R \arrow[r] \arrow[uul] \arrow[d, dotted] & Q \arrow[r] \arrow[uul] \arrow[d, dotted] & 0 \\
 & 0 \arrow[r] & P' \arrow[r] \arrow[uul] & R' \arrow[r] \arrow[uul] & Q' \arrow[r] \arrow[uul] & 0
\end{tikzcd}$$

If we take the two exact sequences equal then the long exact sequence associated to the two projective resolutions yields that the resulting connecting maps are the same.

The proof also yields.

Proposition A map between short exact sequences give a map between the corresponding long exact sequences.

Another property

$F, \mathcal{A}, \mathcal{B}$ as above.

Proposition The functors L_iF for $i > 0$ are zero on projective objects.

Proof This is trivial since a projective object is its own projective resolution.

δ-functors

Definition Let $(F^i)_{i\in\mathbb{Z}}$ be a series of functors between abelian categories $\mathcal{A}$, $\mathcal{B}$ such that for any exact sequence

$$0 \to A \to B \to C \to 0$$

there is given a connecting morphism

$$\delta^i : F^i(C) \to F^{i+1}(A)$$

which fit in a long exact sequence

$$\cdots \xrightarrow{F^{i-1}(\beta)} F^{i-1}(C) \xrightarrow{\delta^{i-1}} F^i(A) \xrightarrow{F^i(\alpha)} F^i(B) \xrightarrow{F^i(\beta)} F^i(C) \xrightarrow{\delta^i} F^{i+1}(A) \xrightarrow{F^{i+1}(\alpha)} \cdots$$

Assume furthermore that any map between short exact sequences gives rise to a map between long exact sequences (in a functorial way). Then we say that the $(F^i)_i$ (together with the δ^i) form a δ-functor $\mathcal{A} \to \mathcal{B}$.

Examples

- Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Then the functors $(H^i)_i : \mathrm{C}(\mathcal{A}) \to \mathcal{B}$ form a δ-functor.
- Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between an abelian categories, such that $\mathcal{A}$ has enough projectives. Then the functors $(L_{-i}F)_i$ form a δ-functor.

Morphisms between δ-functors

Definition A morphism $(F^i)_i \to (G^i)_i$ between δ-functors is a sequence of natural transformation $\theta^i : F^i \to G^i$ compatible with the connecting maps. I.e. for any exact sequence

$$0 \to A \to B \to C \to 0$$

there is a commutative diagram

$$\begin{tikzcd} F^i(C) \arrow[r, "\delta^i"] \arrow[d, "\theta^i(C)"'] & F^{i+1}(A) \arrow[d, "\theta^{i+1}(A)"] \\ G^i(C) \arrow[r, "\delta^i"] & G^{i+1}(A) \end{tikzcd}$$

Example If $\theta : F \to G$ is a natural transformation then we obtain a corresponding morphism $L\theta : (L_{-i}F)_i \to (L_{-i}G)_i$ of δ-functors.

Universal δ-functors

Definition A δ-functor $(F^i)_i$ is a *universal homological δ-functor* if

- $F^i = 0$ for $i > 0$.
- For any δ-functor $(G^i)_i$ and any map $\theta^0 : G^0 \to F^0$ there is a *unique* extension of θ^0 to a map of δ-functors $\theta : (G^i)_i \to (F^i)_i$.

Note : Being given by a universal property, a universal homological δ-functors $(F^i)_i$ is determined, up to unique isomorphism, by F^0.

Characterization of universal homological δ-functors

Definion An additive functor $H : \mathcal{A} \to \mathcal{B}$ is *coeffaceable* if for any $A \in \mathrm{Ob}(\mathcal{A})$ there exists an epimorphism $u : P \to A$ such that $H(u) = 0$.

Theorem Let $(F^i)_i$ be a δ-functor $\mathcal{A} \to \mathcal{B}$. Assume that

- $F^i = 0$ for $i > 0$
- F^i is coeffaceable for $i < 0$.

Then $(F^i)_i$ is a universal homological δ-functor.

Example Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. Then the δ-functor $(L_{-i}F)_i$ satisfies the hypotheses of the theorem and hence is universal.

Proof

Assume $(F^i)_i$ satisfies the conditions of the theorem, let $(G^i)_i$ be an arbitrary δ-functor $\mathcal{A} \to \mathcal{B}$ and let θ^0 be a map $G^0 \to F^0$.

We need to construct (for $i \geq 0$)

$$\theta^{-i} : G^{-i} \to F^{-i}$$

compatible with the connecting morphisms, i.e. for any exact sequence

$$0 \to B \to C \to A \to 0$$

we should have a commutative diagram

$$\begin{tikzcd}
G^{-i-1}(A) \arrow[r, "\delta^{-i-1}"] \arrow[d, "\theta^{-i-1}(A)"'] & F^{-i}(B) \arrow[d, "\theta^{-i}(B)"] \\
F^{-i-1}(A) \arrow[r, "\delta^{-i-1}"'] & F^{-i}(B)
\end{tikzcd}$$

Proof cont'd

We proceed by induction. θ^0 is already given.

Assume that we have constructed $(\theta^{-i})_{i \leq n}$. Let $A \in \mathrm{Ob}(\mathcal{A})$ and construct a short exact sequence

$$0 \to B \to P \xrightarrow{u} A \to 0$$

such $F^{-n-1}(u) = 0$.

If $\theta^{-n-1}(A)$ exists then it should be equal to $\theta^{-n-1}(A, u)$, defined by the dotted arrow in the following diagram

$$\begin{tikzcd}
 & \cdots \arrow[r] & G^{-n-1}(A) \arrow[r, "\delta^{-n-1}"] \arrow[d, dotted, "\theta^{-n-1}(A,u)"'] & G^{-n}(B) \arrow[r] \arrow[d, "\theta^{-n}(B)"'] & G^{-n}(P) \arrow[d, "\theta^{-n}(P)"'] \\
F^{-n-1}(P) \arrow[rr, "0!", "F^{-n-1}(u)"'] & & F^{-n-1}(A) \arrow[r, "\delta^{-n-1}"'] & F^{-n}(B) \arrow[r] & F^{-n}(P)
\end{tikzcd}$$

We want

- $\theta^{-n-1}(A, u)$ is independent of u.
- If we put $\theta^{-n}(A) = \theta^{-n}(A, u)$ then $\theta^{-n}(A)$ is a natural transformation compatible with the connecting maps.

Proof cont'd

Consider first a commutative diagram of the form.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d] & P \arrow[r, "u"] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & B' \arrow[r] & P' \arrow[r, "u'"] & A' \arrow[r] & 0
\end{tikzcd}$$

with $F^{-n-1}(u) = F^{-n-1}(u') = 0$.

Then we get a diagram

$$\begin{tikzcd}
G^{-n-1}(A) \arrow[rrr, "G^{-n-1}(\alpha)"] \arrow[dr] \arrow[ddd, "\theta^{-n-1}(A,u)"'] & & & G^{-n-1}(A') \arrow[dl] \arrow[ddd, "\theta^{-n-1}(A',u')"] \\
 & G^{-n}(B) \arrow[r] \arrow[d] & G^{-n}(B') \arrow[d] & \\
 & F^{-n}(B) \arrow[r] & F^{-n}F(B') & \\
F^{-n-1}(A) \arrow[rrr, "F^{-n-1}(\alpha)"'] \arrow[ur, hook] & & & F^{-n-1}(A') \arrow[ul, hook]
\end{tikzcd}$$

where the trapezoids and the middle square are commutative. Since the lower diagonal arrows are monomorphims this implies that the outer square is commutative as well.

Proof cont'd

Thus we obtain a commutative diagram

$$\begin{tikzcd}
G^{-n-1}(A) \arrow[r, "G^{-n-1}(\alpha)"] \arrow[d, "\theta^{-n-1}(A,u)"'] & G^{-n-1}(A') \arrow[d, "\theta^{-n-1}(A',u')"] \\
F^{-n-1}(A) \arrow[r, "F^{-n-1}(\alpha)"'] & F^{-n-1}(A')
\end{tikzcd}$$

In particular :

- If $A = A'$ then $\theta^{-n-1}(A, u) = \theta^{-n-1}(A, u')$. Since two coeffacings of F^{-n-1} at A can be dominated by a third we obtain that $\theta^{-n-1}(A, u)$ is independent of u.
- Dropping the u's from the diagram we see that θ^{-n-1} is a natural transformation.

Proof cont'd

Now we prove that θ^{-n-1} is compatible with the connecting maps. We start with

$$\begin{tikzcd}
0 \arrow[r] & B'' \arrow[r] \arrow[d, dotted] & P \arrow[r, "u"] \arrow[d, "v"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

(given exact sequence)

with $F^{-n-1}(v) = 0$. If follows that $F^{-n-1}(u) = 0$ as well.

This gives a commutative diagram

$$\begin{tikzcd}
G^{-n-1}(A) \arrow[r] \arrow[d, equal] & G^{-n}(B'') \arrow[d] \\
G^{-n-1}(A) \arrow[r, "\delta^{-n-1}"'] & G^{-n}(B)
\end{tikzcd}$$

and a similar one for $(F^i)_i$.

Proof cont'd

The proof now ends with a final commutative diagram

$$\begin{tikzcd}
G^{-n-1}(A) \arrow[r] \arrow[d, "\theta^{-n-1}(A)"] \arrow[rr, bend left, "\delta^{-n-1}"] & F^{-n-1}(B'') \arrow[r] \arrow[d, "\theta^{-n}(B'')"] & G^{-n}(B) \arrow[d, "\theta^{-n}(B)"] \\
F^{-n-1}(A) \arrow[r] \arrow[rr, bend right, "\delta^{-n-1}"'] & F^{-n-1}(B'') \arrow[r] & F^{-n}(B)
\end{tikzcd}$$

The left square is commutative by the construction of θ^{-n-1} (and its independence of the coeffacing of F^{-n-1} at A). The right square is commutative since θ^{-n} is a natural transformation. Therefore the outer rectangle is commutative, finishing the proof.

Note : This proof illustrates the technique of *degree (or dimension) shifting*. By construction exact sequences with suitable middle term one reduces things to lower degree.

Tor as a bi-functor

Write temporarily

$${}^{II}\,\mathrm{Tor}_i^R(M,-) = \mathrm{Tor}_i^R(M,-)$$

Let N be a *left* R-module and let $\mathrm{Mod}(R^\circ)$ be the category of right R-modules. Then we have a right exact functor

$$-\otimes_R N : \mathrm{Mod}(R^\circ) \to \mathrm{Mod}(R^\circ)$$

Denote the left derived functors by

$${}^{I}\,\mathrm{Tor}_i^R(-,N)$$

Theorem There are isomorphisms

$${}^{I}\,\mathrm{Tor}_i^R(M,N) \cong {}^{II}\,\mathrm{Tor}_i^R(M,N)$$

natural in M, N.

Proof

We first show that

$${}^{I}\,\mathrm{Tor}_i^R(-,N) \cong {}^{II}\,\mathrm{Tor}_i^R(-,N)$$

Since both functors have the same value for $i = 0$, it is sufficient to show that ${}^{II}\,\mathrm{Tor}_i^R(-,N)$ is a univeral homological δ-functor. I.e. it is sufficient that

- ${}^{II}\,\mathrm{Tor}_i^R(-,N)$ is a δ-functor.
- ${}^{II}\,\mathrm{Tor}_i^R(Q,N)$ is zero for $i > 0$ if Q is projective.

First assertion

We prove ${}^{II}\operatorname{Tor}_i^R(-,N)$ is a δ-fuctor.

Take a projective resolution of N

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \to N \to 0$$

and an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

Tensoring with a projective R-module is exact so we obtain a commutative diagram with exact columns

$$\begin{tikzcd}
\cdots \arrow[r] & M'\otimes_R P_2 \arrow[r] \arrow[d] & M'\otimes_R P_1 \arrow[r] \arrow[d] & M'\otimes_R P_0 \arrow[d] \\
\cdots \arrow[r] & M\otimes_R P_2 \arrow[r] \arrow[d] & M\otimes_R P_1 \arrow[r] \arrow[d] & M\otimes_R P_0 \arrow[d] \\
\cdots \arrow[r] & M''\otimes_R P_2 \arrow[r] & M''\otimes_R P_1 \arrow[r] & M''\otimes_R P_0
\end{tikzcd}$$

The long exact sequence for homology yields the connecting maps.

Remark The connecting maps are independent of the chosen resolution of N since this resolution is unique up to homotopy.

Second assertion

We prove ${}^{II}\operatorname{Tor}_i^R(Q,N) = 0$ for $i > 0$, Q projective (or flat).

Let the resolution of N be as above.

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \to N \to 0$$

Tensoring with Q is exact. So we obtain an exact sequence

$$Q\otimes_R P_{-2} \to Q\otimes_R P_{-1} \to Q\otimes_R P_0 \to Q\otimes_R N \to 0$$

So we have indeed $\operatorname{Tor}_i^R(Q,N) = 0$ for $i > 0$.

To finish the proof we need that the isomorphism ${}^{I}\operatorname{Tor}_i^R(-,N) \cong {}^{II}\operatorname{Tor}_i^R(-,N)$ is natural in N. This follows easily from the fact that they are universal homological δ-functors.

Acyclic objects

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives.

Definition An object $A \in \operatorname{Ob}(\mathcal{A})$ is *acyclic* for F if $L_i F(A) = 0$ for $i < 0$.

Note : Any projective object is acyclic but there are usually others.

Proposition Let $M \in \operatorname{Ob}(\mathcal{A})$ and assume there is a resolution by acyclic objects

$$\cdots A_{-2} \to A_{-1} \to A_0 \to M \to 0$$

Then $L_i F(M) \cong H^{-i}(\mathrm{C}(F)(A))$.

Homological characterization of flatness

Proposition The following are equivalent for $M \in \operatorname{Mod}(R^\circ)$.

(1) M is flat (i.e. $M \otimes_R -$ is exact).

(2) $\operatorname{Tor}_1^R(M,N) = 0$ for all $N \in \operatorname{Mod}(R)$.

(3) M is acyclic for all functors $- \otimes_R N$.

Proof (1)$\Rightarrow$ (3) has already been shown. (3) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1) follows from the long exact sequence for $M \otimes_R -$.

Flat dimension

Definition Let $M \in \mathrm{Mod}(R^\circ)$. The *flat dimension*

$$\operatorname{fd} M$$

of M is the minimal length of a resolution of M by right flat objects (by convention it is infinite if such a resolution does not exist).

Proposition The following are equivalent for $M \in \mathrm{Mod}(R^\circ)$.

- $\operatorname{fd} M \le n$.
- For any resolution
$$0 \to M_{-n} \to F_{-n+1} \to \cdots \to F_0 \to M \to 0 \qquad (*)$$
with $F_0, \ldots, F_{-n-1}$ flat we have that M_{-n} is flat.
- $\mathrm{Tor}^R_{n+1}(M, N) = 0$ for all $N \in \mathrm{Mod}(R)$.
- $\mathrm{Tor}^R_i(M, N) = 0$ for all $N \in \mathrm{Mod}(R)$ and all $i > n$.

Proof cont'd

(1)⇒ (4) We now have a resolution as in (*) with M_{-n} flat. It follows from (**) that $\mathrm{Tor}^R_i(M, N) = 0$ for $i > n$.

(4)⇒(3) This is trivial.

(3)⇒ (2) It follows from (**) that $\mathrm{Tor}^R_j(M_{-n}, N) = 0$ for $j > 0$ for all N. Hence M_{-n} is flat.

(2) ⇒(1) A resolution of length n as in (*) always exists. For example take the F_i projective.

Proof

We start with a preliminary computation. Assume that we have a resolution as in (*). If we break up this resolution in short exact sequences

$$0 \to M_{-1} \to F_0 \to M \to 0$$
$$0 \to M_{-2} \to F_{-1} \to M_{-1} \to 0$$
$$\cdots$$
$$0 \to M_{-n} \to F_{-n+1} \to M_{-n+1} \to 0$$

then from the long exact sequence for $- \otimes_R N$ we obtain for $j > 0$.

$$\begin{aligned} \mathrm{Tor}^R_j(M_{-n}, N) &\cong \mathrm{Tor}^R_{j+1}(M_{-n+1}, N) \\ &\cong \cdots \\ &\cong \mathrm{Tor}^R_{j+n}(M, N) \qquad (**) \end{aligned}$$

Weak dimension

Definition The *right weak dimension*

$$\mathrm{r.\,w.\,dim}\, R$$

of R is the maximum of the flat dimensions of all right R-modules. The *left weak dimension* is defined similarly.

Definition The Tor-dimension

$$\mathrm{Tdim}\, R$$

of R is the minimal number n such that $\mathrm{Tor}^R_i(M, N) = 0$ for $i \ge n + 1$ and all $M \in \mathrm{Mod}(R^\circ)$, $N \in \mathrm{Mod}(R)$ (infinite if such a number does not exist).

Theorem There is equality

$$\mathrm{l.\,w.\,dim}\, R = \mathrm{Tdim}\, R = \mathrm{r.\,w.\,dim}\, R$$

Proof The second equality follows from the proposition on flat dimension. The first equality follows by symmetry.

Below we write : $\mathrm{w.\,dim}\, R = \mathrm{r.\,w.\,dim}\, R$.

Dual notions

$\mathcal{A}$ abelian.

- We say that $\mathcal{A}$ has enough injectives if for any A there is a monomorphism $A \hookrightarrow E$ with E injective (example: $\mathrm{Mod}(R)$ has enough injectives).
- If $\mathcal{A}$ has enough in injectives then we may construct *injective resolutions*
$$0 \to A \to E_0 \to E_1 \to E_2 \to \cdots$$
- Such resolutions have the usual functoriality and uniqueness properties in the homotopy category.
- If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor then we define the right-derived functors of F as $R^i F(A) = H^i(\mathrm{C}(F)(E))$. This is well-defined and functorial in the usual sense.
- An exact sequence
$$0 \to A \to B \to C \to 0$$
yields a long exact seqeunce
$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{\delta^0} R^1F(A) \to R^1F(B) \to \cdots$$
with the usual naturality properties.

Dual notions II

- A δ-functor $(F^i)_i$ is a *universal cohomological δ-functor* if $F^i = 0$ for $i < 0$ and for any δ-functor $(G^i)_i$ and any map $\theta^0 : F^0 \to G^0$ there is a *unique* extension of θ^0 to a map of δ-functors $\theta : (F^i)_i \to (G^i)_i$.
- An additive functor $H : \mathcal{A} \to \mathcal{B}$ is *effaceable* if for any $A \in \mathrm{Ob}(\mathcal{A})$ there exists an monomorphism $u : A \to E$ such that $H(u) = 0$.
- If $(F^i)_i$ is a δ-functor such that $F^i = 0$ for $i < 0$ and F^i is effaceable for $i > 0$. then it is a universal cohomological δ-functor.
- If $F : \mathcal{A} \to \mathcal{B}$ is left exact then $(R^i F(A))_i$ is a universal cohomological δ-functor.

Ext-functors

Let $\mathcal{A}$ be an abelian category.

Definition Assume that $\mathcal{A}$ has enough injectives and let $A \in \mathrm{Ob}(\mathcal{A})$. Then we define
$$^{II}\operatorname{Ext}^i_{\mathcal{A}}(A, -)_i$$
as the right derived functor of
$$\operatorname{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$$

Definition Assume that $\mathcal{A}$ has enough projectives and let $B \in \mathrm{Ob}(\mathcal{A})$. Then we define
$$^{I}\operatorname{Ext}^i_{\mathcal{A}}(-, B)_i$$
as the right derived functors of
$$\operatorname{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^\circ \to \mathbf{Ab}$$

Note $\mathcal{A}$ has enough projectives if and only if $\mathcal{A}^\circ$ has enough injectives.

Ext-functors II

One proves as for the Tor-functors.

Theorem If $\mathcal{A}$ has both enough injectives and projectives then there are isomorphisms
$$^{I}\operatorname{Ext}^i_{\mathcal{A}}(A, B) \cong {}^{II}\operatorname{Ext}^i_{\mathcal{A}}(A, B)$$
natural in A, B.

Below we write $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ for both $^{I}\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ and $^{II}\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ whenever these are defined.

Not enough projectives or injectives

$\mathcal{A}$ is an essentially small abelian category then we may define

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Ext}^i_{\mathrm{Ind}(\mathcal{A})}(A, B)$$

One may show : This coincides with the earlier definitions if there are enough injectives or projectives.

Homological characterizations of projectives and injectives

Principle The results proved for Tor have analogs for Ext. For example:

Proposition Assume that $\mathcal{A}$ has enough projectives. Then the following are equivalent. for $P \in \mathrm{Ob}(\mathcal{A})$.

(1) P is projective.

(2) $\mathrm{Ext}^1_{\mathcal{A}}(P, B) = 0$ for all $B \in \mathrm{Ob}(\mathcal{A})$.

(3) P is acyclic for all functors $\mathrm{Hom}_{\mathcal{A}}(-, B) = 0$.

There is of course a dual result for injective objects.

Projective and injective dimension

Definition Let $\mathcal{A}$ be an abelian category with enough projectives.

- Assume that $\mathcal{A}$ has enough projectives and $A \in \mathrm{Ob}(\mathcal{A})$. The *projective dimension*

  $$\mathrm{pd}(A)$$

  of A is the minimal length of a finite projective resolution of A (as usual infinite if such a finite resolution does not exist).

- Assume that $\mathcal{A}$ has enough injectives and $B \in \mathrm{Ob}(\mathcal{A})$. The *injective dimension*

  $$\mathrm{id}(B)$$

  of B is the minimal length of a finite injective resolution of B

Global dimension

As for Tor one proves.

Proposition Assume that $\mathcal{A}$ has enough projectives. Then the following numbers are the same.

- The maximum of the projective dimensions of the objects in $\mathcal{A}$.
- The mininum number n such that $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for all $i > n$ and all $A, B \in \mathrm{Ob}(\mathcal{A})$.

There is a dual result for injective dimensions.

Definition We define the *global dimension* $\mathrm{gldim}\,\mathcal{A}$ of $\mathcal{A}$ as one of the following numbers (whenever they are defined):

- The maximum of the projective dimensions of the objects in $\mathcal{A}$.
- The mininum number n such that $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for all $i > n$ and all $A, B \in \mathrm{Ob}(\mathcal{A})$.
- The maximum of the injective dimensions of the objects in $\mathcal{A}$.

Global dimension of rings

R a ring. $\mathrm{Mod}(R)$ has both enough projectives and injectives. We put

$$\mathrm{r.\,gldim}\, R = \mathrm{gldim}\, \mathrm{Mod}(R^\circ)$$

(right global dimension)

$$\mathrm{l.\,gldim}\, R = \mathrm{gldim}\, \mathrm{Mod}(R)$$

(left global dimension)

We have

$$\mathrm{w.\,dim}\, R \leq \mathrm{r.\,gldim}\, R$$
$$\mathrm{w.\,dim}\, R \leq \mathrm{l.\,gldim}\, R$$

Special results for rings

It is well-known that $E \in \mathrm{Mod}(R^\circ)$ is injective if we have the following lifting property for all right ideals I.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[dl, dotted, "\exists"] \\ & E & \end{tikzcd}$$

This leads to

Proposition The following are equivalent for $E \in \mathrm{Mod}(R^\circ)$.

(1) E is injective.

(2) $\mathrm{Ext}^1_R(R/I, E) = 0$ for all right ideals $I \subset R$.

(3) E is acyclic for all functors $\mathrm{Hom}_R(R/I, -)$.

This leads to:

Proposition $\mathrm{r.\,gldim}\, R$ is also equal to

$$\sup_I \mathrm{pd}(R/I)$$

with I a right ideal in R.

Noetherian rings

Reminder A ring R is right noetherian if any right ideal is finitely generated.

Equivalent : Every submodule of a finitely generated right module is finitely generated.

Terminology : A ring is noetherian if it is both left and right noetherian.

Note A finitely generated right module over a right noetherian ring has a resolution consisting of finitely generated projective modules.

Fact If R is right noetherian then every finitely generated flat right module is projective.

For a right noetherian ring this leads to

$$\mathrm{w.\,dim}\, R = \mathrm{r.\,gldim}\, R$$

and hence for any noetherian ring

$$\mathrm{l.\,gldim}\, R = \mathrm{r.\,gldim}\, R$$

Commutative rings

R a commutative ring.

One has

$$\mathrm{gl\,dim}\, R = \sup_m R_m$$

where m runs through the maximal ideals of R and R is the localization of R at m (i.e. invert $R - m$).

This reduces the problem to *local rings* (i.e. rings with a unique maximal ideal).

Theorem Let R be a commutative noetherian ring with maximal ideal m. The following are equivalent.

- $\mathrm{gl\,dim}\, R = n$.
- $\mathrm{pd}(R/m) = n$.
- R is a *regular local ring* of dimension n. I.e. $\dim_{R/m} m/m^2 = n$.

Graded rings

If A is a graded ring then we put

$$\mathrm{r.\,gr.\,gldim}\,A = \mathrm{gl\,dim\,Gr}(A^\circ)$$

In a similar way we define *graded weak, graded flat, graded projective and graded injective dimension.*

Let k be a field.

Definition A is *connected graded* if

- $A_i = 0$ for $i < 0$.
- $A_0 = k$.
- A_i is finite dimensional over k if $i > 0$.

Notation We write simply k for the graded A-bimodule $A/(A_{>0})$.

Theorem Assume that A is connected graded. One has

$$\begin{aligned} &\mathrm{r.\,gldim}\,A = \mathrm{r.\,gr.\,gldim}\,A = \mathrm{gr.\,pd}(k_A) \\ &= \mathrm{gr.\,pd}({}_A k) = \mathrm{l.\,gr.\,gldim}\,A = \mathrm{l.\,gldim}\,A \\ &= \mathrm{w.\,dim}\,A = \mathrm{gr.\,w.\,dim}\,A \end{aligned}$$

Rings of low global dimension

The following result is classical.

Artin-Wedderburn Theorem Let R be a ring. Then $\mathrm{r.\,gldim}\,R = 0$ (or equivalently $\mathrm{l.\,gldim}\,R = 0$) if and only if

$$R = \prod_{i=1}^n M_{m_i}(D_i)$$

with the D_i being skew fields.

Examples of global dimension one

The free algebra

$$k\langle X_1, \ldots, X_n\rangle$$

Upper triangular matrices

$$\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

So-called “hereditary orders”

$$\begin{pmatrix} k[x] & k[x] \\ xk[x] & k[x] \end{pmatrix}$$

Polynomial rings

General result :

$$\mathrm{gl\,dim}\,R[x] = 1 + \mathrm{gl\,dim}\,R$$

Hence in particular

$$\mathrm{gl\,dim}\,k[x_1, \ldots, x_n] = n$$

An example of global dimension three

“Non-commutative projective planes”

$$A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

$$\begin{aligned} f_1 &= ayz + bzy + cx^2 \\ f_2 &= azx + bxz + cy^2 \\ f_3 &= axy + byx + cz^2 \end{aligned}$$

$(a, b, c) \in \mathbb{P}^2 - \{$finite “bad” set$\}$

A is connected graded with $\deg x = \deg y = \deg z = 1$.

Resolution of ${}_A k$.

$$0 \to A \xrightarrow{\cdot(\,x\ y\ z\,)} A^3 \xrightarrow{\cdot\left(\begin{smallmatrix} cx & bz & ay \\ az & cy & bx \\ by & ax & cz \end{smallmatrix}\right)} A^3 \xrightarrow{\cdot\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)} A \to 0$$

Sheaf cohomology

Let $\mathrm{Sh}(X)$ be the category of sheaves on a topological space X. Recall that $\mathrm{Sh}(X)$ has enough injectives (being a Grothendieck category).

Define the (left exact) *global section* functor as

$$\Gamma(X,-):\mathrm{Sh}(X)\to\mathbf{Ab}:\mathcal{F}\mapsto\mathcal{F}(X)$$

For a sheaf $\mathcal{F}$ we put

$$H^i(X,\mathcal{F})=R^i\Gamma(X,\mathcal{F})$$

Chapter VII

Derived categories and triangulated categories

(Universal) localization of categories

$\mathcal{C}$ category, $S\subset\mathrm{Maps}(\mathcal{C})$.

Definition An S*-inverting map* is a functor $\mathcal{C}\to\mathcal{B}$ such that the elements of S are mapped to isomorphisms.

Proposition There is a "universal" S-inverting functor

$$Q:\mathcal{C}\to S^{-1}\mathcal{C}$$

defined by the following universal property

$$\begin{tikzcd} & \mathcal{B} & \\ \mathcal{C} \arrow[ur, "S\text{ inverting}"] \arrow[rr, "Q"'] & & S^{-1}\mathcal{C} \arrow[ul, dotted, "\exists!"'] \end{tikzcd}$$

Sketch of construction

- $\mathrm{Ob}(S^{-1}\mathcal{C})=\mathrm{Ob}(\mathcal{C})$.
- Morphisms in $S^{-1}\mathcal{C}$ are (composable) formal paths

  $$f_1\cdot s_1^{-1}\cdot f_2\cdot s_2^{-1}\cdots s_{n-1}^{-1}\cdot f_n$$

  with f_i in $\mathcal{C}$ and s_i in S, modulo a suitable equivence relation.

Simplifying conditions

Definition S is *multiplicatively closed* if it contains all the identy maps and is closed under composition.

Defintion A multiplicatively closed set is *saturated* if for maps s,t in $\mathcal{C}$ with $s,t\in S$ we have

$$s\in S\Leftrightarrow t\in S$$

Fact We can replace a set of maps always by its *multiplicative closure* or *saturation* without changing the corresponding localization.

Öre sets

Principle Localization becomes a lot easier if the *Öre conditions hold.*

Definition Let S be a multiplicatively closed set of maps. S is Öre if the following conditions hold.

(ORE1) For all $s : Z \to Y$ in S, $f : X \to Y$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{tikzcd} W \arrow[r, dotted, "f'"] \arrow[d, dotted, "s' \in S"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(think: $s^{-1}f = f's'^{-1}$).

(ORE1') Dual condition.

(ORE2) For $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{C}$ the following are equivalent

- $\exists s \in S$ such that $sf = sg$.
- $\exists t \in S$ such that $ft = gt$.

Localization for Öre sets

Let $S \subset \mathrm{Maps}(\mathcal{C})$ be an Öre set.

New construction of $S^{-1}\mathcal{C}$

- $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
- Morphisms are equivalence classes of diagrams

$$\begin{tikzcd} & X_1 \arrow[dl, "s \in S"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

(think fs^{-1}).

Equivalence relation

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & \sim & & X_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y & & X & & Y \end{tikzcd}$$

if there is a commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & X_3 \arrow[l, dotted, "s_3 \in S"] \arrow[r, dotted, "f_3"'] \arrow[u, dotted, "u_1"'] \arrow[d, dotted, "u_2"] & Y \\ & X_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

Note : Not required that $u_1, u_2 \in S$. Automatic if S is saturated.

Composition of diagrams

$$\begin{tikzcd} & & X_2 \arrow[dl, dotted, "t'"'] \arrow[dr, dotted, "f'"] \arrow[ddll, dotted, bend right] \arrow[ddrr, dotted, bend left] & & \\ & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \text{(ORE1)} & Y_1 \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

Technical problem

- A priori "Hom"-sets in $S^{-1}\mathcal{C}$ might not be sets.
- Will be solved later in concrete cases.

Alternative diagrams

Principle We may also define $S^{-1}\mathcal{C}$ via equivalence classes of diagrams of the following type.

$$\begin{tikzcd}
& Y_1 & \\
X \arrow[ur, "g"] & & Y \arrow[ul, "t\in S"']
\end{tikzcd}$$

(think: $t^{-1}g$). The result is the same!

We go from the old diagrams to the new diagrams (and back) via the Öre conditions.

$$\begin{tikzcd}
& Y_1 & \\
X \arrow[ur, dotted, "g"] & \text{(ORE1)} & Y \arrow[ul, dotted, "t"'] \\
& X_1 \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Öre localization preserves pre-additivity

$$\begin{tikzcd}
& X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & & & X_2 \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & X & & Y
\end{tikzcd}
\quad + $$

(ORE1'):
$$\begin{tikzcd}
X_1 \arrow[d, "s"'] & X_3 \arrow[l, dotted, "t'\in S"'] \arrow[d, dotted, "p"] \arrow[dl, dotted, "u\in S"] \\
X & X_2 \arrow[l, "t"]
\end{tikzcd}$$

(think: u is a common denominator for s, t).

$$\text{Sum} \overset{\text{def}}{=} \begin{tikzcd}
& X_1 \arrow[dl, "u"'] \arrow[dr, "ft'+gp"] & \\
X & & Y
\end{tikzcd}$$

The derived category of an abelian category

$\mathcal{A}$ abelian category.

Definition $A \xrightarrow{f} B$ in $\mathrm{C}(\mathcal{A})$ is a *quasi-isomorphism* if $H(f)$ is an isomorphism.

$$S_{\text{q.i.}} = \{\text{quasi-isomorphisms in } \mathrm{C}(\mathcal{A})\}$$

Properties

- $S_{\text{q.i.}}$ is a saturated multiplicatively closed set containing all isomorphisms.
- $S_{\text{q.i.}}$ is not an Öre set.

Definition $D(\mathcal{A}) = S_{\text{q.i.}}^{-1}\,\mathrm{C}(\mathcal{A})$.

Elementary properties

Commutative diagram

$$\begin{tikzcd} \mathrm{C}(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "H"'] & & S_{\text{q.i.}}^{-1}\,\mathrm{C}(\mathcal{A}) = D(\mathcal{A}) \arrow[dl, dotted, "\exists! H"] \\ & \mathrm{Gr}(\mathcal{A}) & \end{tikzcd}$$

Similary The shift functor $-[1]$ on $\mathrm{C}(\mathcal{A})$ descends to a shift functor $-[1]$ on $D(\mathcal{A})$.

Define

$$i : \mathcal{A} \to D(\mathcal{A}) : A \mapsto Q(0 \to \underbrace{A}_{(\deg 0)} \to 0)$$

Commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\mathrm{id}_{\mathcal{A}}"'] & D(A) \arrow[d, "H^0"] \\ & \mathcal{A} \end{tikzcd}$$

Hence i is fully faithful.

Convention : We view i as an inclusion.

Application

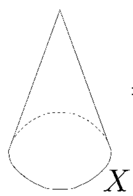
New definition of Ext without assuming enough projectives or injectives.

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[n])$$

We will show : this coincides with our earlier definitions.

The cone of a morphism: topological inspiration

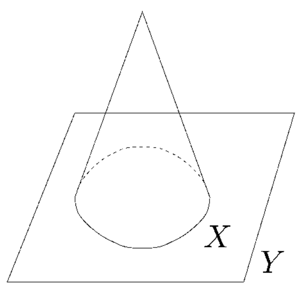
$f : X \to Y$ continous map.

$\mathrm{cone}(X) =$ 

$= X \times I/((x, 1) \sim (x', 1))$

$$\mathrm{cone}(f) = \frac{\mathrm{cone}(X) \coprod Y}{(x, 0) \sim f(x)}$$

Schematically

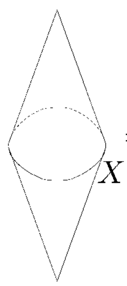


The cone of a morphism: topological inspiration II

There are maps

$$X \xrightarrow{f} Y \to \mathrm{cone}(f) \to SX \qquad (*)$$

where

$SX =$ 

$= X \times I \Big/ \begin{pmatrix} (x,0) \sim (x',0) \\ (x,1) \sim (x',1) \end{pmatrix}$

(the unreduced suspension of X).

Note The composition of two consecutive maps in (*) is homotopic to a constant map.

Remark The functor SX is a kind of shift functor on topological spaces.

The cone of a morphism between complexes

$f : A \to B$ in $\mathrm{C}(\mathcal{A})$.

Informally

$$\begin{tikzcd} A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_{n+2} \arrow[d, "f_{n+2}"] \\ B_n \arrow[r, "d_n"'] & B_{n+1} \arrow[r, "d_{n+1}"'] & B_{n+2} \end{tikzcd}$$

Construction

$$\begin{tikzcd} A[1]: & A_{n+1} \arrow[r, "-d"] \arrow[dr, "f_{n+1}"] & A_{n+2} \arrow[r, "-d"] \arrow[dr, "f_{n+2}"] & A_{n+3} \\ & \oplus & \oplus & \oplus \\ B & B_n \arrow[r, "d"'] & B_{n+1} \arrow[r, "d"'] & B_{n+2} \end{tikzcd}$$

Matrix convention

Assume

$A, B, C, \ldots \quad \in \quad$ abelian (or additive) category

$$\begin{pmatrix} A \\ B \\ C \\ \vdots \end{pmatrix} \overset{\text{not.}}{=} A \oplus B \oplus C \oplus \cdots$$

Maps

$$A \oplus B \oplus C \oplus \cdots \to A' \oplus B' \oplus C' \oplus \cdots$$

will be written as matrices.

The cone of a morphism between complexes II

$f : A \to B$ map in $\mathrm{C}(\mathcal{A})$.

$\mathrm{cone}(f) = A[1] \oplus B$ as graded objects

$$d_{\mathrm{cone}(f)} = \begin{pmatrix} d_{A[1]} & 0 \\ f & d_B \end{pmatrix}$$

Helices

$$A \xrightarrow{f} B \xrightarrow{\binom{0}{1}} \mathrm{cone}(f) \xrightarrow{(1\ 0)} A[1]$$

We may extend this to an infinite sequence

$$\cdots \xrightarrow{\binom{0}{1}} \mathrm{cone}(f)[-1] \overset{(1\ 0)}{\cdots\cdots>} A \xrightarrow{f} B \xrightarrow{\binom{0}{1}} \mathrm{cone}(f) \overset{(1\ 0)}{\cdots\cdots>} A[1] \xrightarrow{f[1]} \cdots$$

Terminology We will call this the *helix* of f.

Convenient summary

$$\begin{tikzcd} & \mathrm{cone}(f) \arrow[dl, "(1)"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul] \end{tikzcd}$$

Note Composition of any two maps is zero in $K(\mathcal{A})$.

Long exact sequences

Proposition

- $H^0(\text{helix})$ is exact.
- Let C be a complex. Then

$$\operatorname{Hom}_{K(\mathcal{A})}(C, \text{helix})$$

and

$$\operatorname{Hom}_{K(\mathcal{A})}(\text{helix}, C)$$

are exact.

Proof (sketch) We have an exact sequence in $\mathrm{C}(\mathcal{A})$

$$0 \to B \xrightarrow{\binom{0}{1}} \operatorname{cone}(f) \xrightarrow{(1\,0)} A[1] \to 0$$

split in $\operatorname{Gr}(\mathcal{A})$. The long exact sequence for homology for this exact sequence of complexes turns out to be precisely $H^0(\text{helix})$.

Proof cont'd

Since the sequence is split in $\operatorname{Gr}(\mathcal{A})$ applying $\underline{\operatorname{Hom}}_{\mathrm{C}(\mathcal{A})}(C, -) = \underline{\operatorname{Hom}}_{\operatorname{Gr}(\mathcal{A})}(C, -)$ yields a short exact sequence

$$0 \to \underline{\operatorname{Hom}}_{\mathrm{C}(\mathcal{A})}(C, B) \xrightarrow{\binom{0}{1}} \underline{\operatorname{Hom}}_{\mathrm{C}(\mathcal{A})}(C, \operatorname{cone}(f)) \xrightarrow{(1\,0)} \underline{\operatorname{Hom}}_{\mathrm{C}(\mathcal{A})}(C, A[1]) \to 0$$

$$\operatorname{Hom}_{K(\mathcal{A})}(C, \text{helix})$$

is the long exact sequence for homology associated to this short exact sequence.

Note on the functoriality of cones

Clearly $\operatorname{cone}(-)$ defines a functor

$$\operatorname{Maps}(\mathrm{C}(\mathcal{A})) \to \mathrm{C}(\mathcal{A})$$

The world would be much nicer if $\operatorname{cone}(-)$ would define a functor on maps in the homotopy category but this is not the case.

Functoriality cont'd

More precisely Suppose we have a diagram in $\mathrm{C}(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

commutative in $K(\mathcal{A})$ then we can construct a corresponding commutative diagram of helices in $K(\mathcal{A})$

$$\begin{tikzcd}
 & \operatorname{cone}(f) \arrow[dl, "(1)"'] \arrow[dd, bend left, "r"] & \\
A \arrow[rr, "f"] \arrow[dd, "p"'] & & B \arrow[ul] \arrow[dd, "q"] \\
 & \operatorname{cone}(g) \arrow[dl, "(1)"'] & \\
C \arrow[rr, "g"'] & & D \arrow[ul]
\end{tikzcd}$$

Functoriality cont'd

However The construction of r turns out to depend on the chosen homotopy between qf and gp. There is no natural choice!

Weak result If p, q are isomorphims then for any homotopy the constructed r will be an isomorphism.

The derived category via the homotopy category

The homology functor is defined on $K(\mathcal{A})$ so we can speak about quasi-isomorphims in $K(\mathcal{A})$.

One may prove : The natural functor

$$S_{\text{q.i.}}^{-1}\,\mathrm{C}(\mathcal{A}) \to S_{\text{q.i.}}^{-1}K(\mathcal{A})$$

is an isomorphism of categories.

It requires checking that $S_{\text{q.i.}}^{-1}K(\mathcal{A})$ satisfies the required universal property.

The Öre condition

The interpretation

$$D(\mathcal{A}) = S_{\text{q.i.}}^{-1}K(\mathcal{A})$$

simplifies life considerably since we have the following fundamental result.

Proposition $S_{\text{q.i.}}$ is an Öre set in $K(\mathcal{A})$.

Example (ORE1) We must complete in $K(\mathcal{A})$ for $t \in S_{\text{q.i.}}$, g arbitrary.

$$\begin{tikzcd}
D \arrow[r, dotted, "f"] \arrow[d, dotted, "s\in S_{\text{q.i.}}"'] & C \arrow[d, "t"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

We define D, s, f by the following helix

$$\begin{tikzcd}
 & \operatorname{cone}(-g,t) \arrow[r, equal, "\text{def}"] \arrow[dl, "\binom{s}{f}"', "(1)"] & D[1] \\
A \oplus C \arrow[rr, "(-g,t)"'] & & B \arrow[ul]
\end{tikzcd}$$

Example cont'd

Since the composition of any two maps is zero in a helix we have

$$-gs + tf = 0$$

Furthermore : The long exact sequence for homology yields that s is a quasi-isomorphism.

Triangles

Principle The cone of a map plays the role of kernel *and* cokernel in the homotopy category. We need a substitute for the axiom $\ker \operatorname{coker} = \operatorname{coker} \ker$ in an abelian category.

Triangles : Let $\mathcal{D}$ be graded a category.

A *triangle* (X, Y, Z, u, v, w) in $\mathcal{D}$ is a sequence of objects and maps

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

also written as

$$\begin{tikzcd} & Z \arrow[dl, "w", "(1)"'] & \\ X \arrow[rr, "u"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

Triangles II

A map between triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Notation : $\Delta(\mathcal{D})$: the category of triangles in $\mathcal{D}$.

Pretriangulated categories

Definition A pretriangulated category is a graded pre-additive category $\mathcal{D}$ together with a full subcategory

$$\Delta(\mathcal{D})^{\text{dist}} \subset \Delta(\mathcal{D})$$

of *distinguished triangles* satisfying the following axioms.

(TR1a)

$$\begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "\mathrm{id}_A"'] & & A \arrow[ul] \end{tikzcd}$$

is distinguished.

(TR1b) A triangle isomorphic to a distinguished one is distinguished.

(TR1c) For all $u : A \to B \in \operatorname{Maps}(\mathcal{D})$ there is a distinguished triangle

$$\begin{tikzcd} & C \arrow[dl, "(1)"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul] \end{tikzcd}$$

Pretriangulated categories II

(TR2) (the *rotation axiom*) A triangle

$$\begin{tikzcd} & C \arrow[dl, "w"', "(1)"] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$

is distinguished if and only if the following triangle is distinguished.

$$\begin{tikzcd} & A[1] \arrow[dl, "-u[1]"', "(1)"] & \\ B \arrow[rr, "v"'] & & C \arrow[ul, "w"'] \end{tikzcd}$$

Pretriangulated categories III

(TR3) (the double helix axiom) A commutative diagram with rows which are distinguished triangles may be completed to a map between distinguished triangles

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dotted, "\exists\ h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1]
\end{tikzcd}$$

Note on axioms

- An abelian category is an additive category satisfying *extra axioms*.
- A pretriangulated category has *extra structure* besides extra axioms.

Note also The axioms of a pretriangulated category only assert the existence of certain objects. These objects are *in no way unique or functorial.*

Contrast with : $\ker, \operatorname{coker}$ in an abelian category are functorial.

Elementary properties

$\mathcal{D}$ pretriangulated.

Proposition Let

$$\begin{tikzcd}
& C \arrow[dl, "w"', "(1)"] & \\
A \arrow[rr, "u"'] & & B \arrow[ul, "v"']
\end{tikzcd}$$

be a distinguished triagle.

- The composition of any two arrows is zero.
- For any $D \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(D, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, D)$ applied to the corresponding helix yield long exact sequences.

Elementary properties II

Consider (TR3).

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dotted, "\exists\ h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1]
\end{tikzcd}$$

Proposition If f, g are isomorphisms then so is h.

Proof It follows from the long exact sequence and the five lemma that $\mathrm{Hom}_{\mathcal{D}}(D, h)$ is an isomorphism for any D.

Corollary A distinguished triangle is up to isomorphism determined by its base.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "v"] \arrow[d, "\mathrm{id}_B"'] & C \arrow[r, "w"] \arrow[d, dotted, "\cong"'] & A[1] \arrow[d, "\mathrm{id}_A[1]"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & A'[1]
\end{tikzcd}$$

Notation for top of triangle with base u : $\mathrm{cone}(u)$ (determined up to iso).

Additivity

Proposition A pretriangulated category is additive.

Proof Consider a distinguished triangle

$$\begin{tikzcd} & C \arrow[dl, "w"'] \arrow[d, phantom, "(1)"] & \\ A \arrow[rr, "0"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$

One proves

- v is split mono, w is split epi.
- $C \cong B \oplus A[1]$

Definition A distinguished triangle where one of the arrows is zero is called a *split* triangle.

Coproducts

Proposition If there are distinguished triangles indexed by $i \in I$

$$A_i \xrightarrow{u_i} B_i \xrightarrow{v_i} C_i \xrightarrow{w_i} A_i[1]$$

such that

$$\oplus_i A_i, \qquad \oplus_i B_i, \qquad \oplus_i C_i$$

exist (e.g. if I is finite), then

$$\oplus_i A_i \xrightarrow{\oplus_i u_i} \oplus_i B_i \xrightarrow{\oplus_i v_i} \oplus_i C_i \xrightarrow{\oplus_i w_i} \oplus_i A_i[1]$$

is distinguished.

Localizing Öre set

$\mathcal{D}$ pretriangulated.

$S \subset \mathrm{Maps}(\mathcal{D})$: Öre set.

Definition S is *localizing* if the following holds:

(LOC1) $s \in S \Leftrightarrow s[1] \in S$.

(LOC2) If in (TR3) $f, g \in S$ then h may be chosen in S as well.

Localization of pretriangulated categories

$S \subset \mathrm{Maps}(\mathcal{D})$ localizing Öre set.

$S^{-1}\mathcal{D}$ is a graded category.

$$\left(\begin{tikzcd} & X_1 & \\ X \arrow[ur, "s \in S"] & & Y \arrow[ul, "f"'] \end{tikzcd}\right)[1] = \begin{tikzcd} & X_1[1] & \\ X[1] \arrow[ur, "{s[1]}"] & & Y[1] \arrow[ul, "{f[1]}"'] \end{tikzcd}$$

Definition A triangle in $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic (in $S^{-1}\mathcal{D}$) to the image of a distinguished triangle in $\mathcal{D}$.

Proposition $S^{-1}\mathcal{D}$ is pretriangulated.

Note on functors

Definition An additive functor $F : \mathcal{D} \to \mathcal{E}$ between pretriangulated categories is *exact* if it sends distinguished triangles to distinguished triangles.

Example If S is localizing in $\mathcal{D}$ then the functor

$$Q : \mathcal{D} \to S^{-1}\mathcal{D}$$

is exact.

The homotopy category is pretriangulated

Definition A triangle is $K(\mathcal{A})$ is distinguished if it is isomorphic to a *standard triangle* of the form

$$A \xrightarrow{f} B \xrightarrow{\binom{0}{1}} \mathrm{cone}(f) \xrightarrow{(1\ 0)} A[1]$$

Proposition With this choice of distinguished triangles $K(\mathcal{A})$ is pretriangulated.

The derived category is pretriangulated

Proposition $S_{\text{q.i.}} \subset K(\mathcal{A})$ is localizing.

Corollary The derived category

$$D(\mathcal{A}) = S_{q.i.}^{-1} K(\mathcal{A})$$

is pretriangulated.

Exact sequence of complexes

Theorem Assume that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "u"] & B \arrow[l, dotted, shift left, "p"] \arrow[r, shift left, "v"] & C \arrow[l, dotted, shift left, "q"] \arrow[r] & 0 \end{tikzcd}$$

is an exact sequence in $\mathrm{C}(\mathcal{A})$, split in $\mathrm{Gr}(\mathcal{A})$.

$$\begin{aligned} vq &= 1_C & vu &= 0 \\ pu &= 1_A & pq &= 0 \\ up + qv &= 1_B \end{aligned}$$

Then

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{-p\,d_B q} A[1]$$

is a distinguished triangle in $K(\mathcal{A})$.

Exact sequences of complexes II

Theorem Assume that

$$0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$$

is an exact sequence in $\mathrm{C}(\mathcal{A})$.

There is a (canonical) map

$$C \xrightarrow{w} A[1] \qquad \text{in } D(\mathcal{A})$$

such that

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is a distinguished triangle in $D(\mathcal{A})$.

Proof Use the functoriality of $\mathrm{cone}(-)$ in $\mathrm{C}(\mathcal{A})$.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d] & B \arrow[r] \arrow[d, "v"] & \mathrm{cone}(u) \arrow[r, "p"] \arrow[d, dotted, "w'"] & A[1] \arrow[d] \\
0 \arrow[r] & C \arrow[r, "\mathrm{id}_C"'] & \underset{(\mathrm{cone}(0))}{C} \arrow[r] & 0
\end{tikzcd}$$

The long exact sequences for homology yields that $w' \in S_{\text{q.i.}}$.

Take $w = p \circ (w')^{-1}$

The octahedral axiom: motivation

Assume we have monomorphisms in $\mathrm{C}(\mathcal{A})$.

$$A \hookrightarrow B \hookrightarrow C$$

split in $\mathrm{Gr}(\mathcal{A})$.

This yields 4 triangles in $K(\mathcal{A})$.

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \arrow[rr] \arrow[d] & & B/A \arrow[r] \arrow[ddl] & {} \\
& C \arrow[dr] \arrow[dd] & & & \\
& & C/A \arrow[dl] \arrow[dr] & & \\
& C/B \arrow[dl] \arrow[d] & & {} & \\
{} & {} & & &
\end{tikzcd}$$

More symmetric representation: the octahedron

Top

$$\begin{tikzcd}
C/B \arrow[dddd, "(1)"'] \arrow[ddrr, "(1)"] & & & & C \arrow[llll] \\
& & d & & \\
& + & B \arrow[uurr] \arrow[ddll] & + & \\
& & d & & \\
B/A \arrow[rrrr, "(1)"'] & & & & A \arrow[uull] \arrow[uuuu]
\end{tikzcd}$$

Bottom

$$\begin{tikzcd}
C/B \arrow[dddd, "(1)"'] & & & & C \arrow[llll] \arrow[ddll] \\
& & + & & \\
& d & C/A \arrow[uull] \arrow[ddrr, "(1)"] & d & \\
& & + & & \\
B/A \arrow[uurr] \arrow[rrrr, "(1)"'] & & & & A \arrow[uuuu]
\end{tikzcd}$$

$+ =$ commutative, $d =$ distinguished.

Octahedra in pretriangulated categories

$\mathcal{D}$ pretriangulated.

Definition A octahedron in $\mathcal{D}$ is a diagram of the form

Top

$$\begin{tikzcd}
A' \arrow[dddd, "(1)"'] \arrow[ddrr, "(1)"] & & & & C \arrow[llll] \\
& & d & & \\
& + & B \arrow[uurr] \arrow[ddll] & + & \\
& & d & & \\
C' \arrow[rrrr, "(1)"'] & & & & A \arrow[uull] \arrow[uuuu]
\end{tikzcd}$$

Bottom

$$\begin{tikzcd}
A' \arrow[dddd, "(1)"'] & & & & C \arrow[llll] \arrow[ddll] \\
& & + & & \\
& d & B' \arrow[uull] \arrow[ddrr, "(1)"] & d & \\
& & + & & \\
C' \arrow[uurr] \arrow[rrrr, "(1)"'] & & & & A \arrow[uuuu]
\end{tikzcd}$$

Triangulated categories

Definition A pretriangulated category $\mathcal{D}$ is *triangulated* if the following axiom holds

(TR4) Any two consecutive maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

may be completed to an octahedrom.

Remark The *top* of the octahedron is determined up to isomorphism by f, g.

Remark The most useful part of the octahedron remains

$$\begin{tikzcd}
A \arrow[rr] \arrow[drr] & & B \arrow[rrrr] \arrow[d] & & & & C' \arrow[r] \arrow[ddlll] & {} \\
 & & C \arrow[dr] \arrow[dd] & & & & & \\
 & & & B' \arrow[dl] \arrow[dr] & & & & \\
 & & A' \arrow[dl] \arrow[d] & & {} & & & \\
 & {} & {} & & & & &
\end{tikzcd}$$

Localization of pretriangulated categories

Theorem If $\mathcal{D}$ is triangulated and $S \subset \mathrm{Maps}(\mathcal{D})$ is localizing then so is $S^{-1}\mathcal{D}$.

Homotopy category and the derived category

Theorem The homotopy category is triangulated.

Proof is an easy, but tedious verification.

Corollary The derived category is triangulated.

Subcategories of the derived category

$\mathcal{A}$ abelian category, $\mathcal{A}' \subset \mathcal{A}$ a full abelian (i.e. closed under $\ker, \mathrm{coker}$) subcategory of $\mathcal{A}$.

The following full subcategories of $D(\mathcal{A})$ inherit its triangulated structure.

$$D^+(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } i \ll 0\}$$

$$D^-(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } i \gg 0\}$$

$$D^b(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } |i| \gg 0\}$$

and for $* =, +, -, b$.

$$D^*_{\mathcal{A}'}(\mathcal{A}) = \{A \in D^*(\mathcal{A}) \mid \forall i : H^i(A) \in \mathcal{A}'\}$$

Remark There is an obvious exact functor

$$D^*(\mathcal{A}') \to D^*_{\mathcal{A}'}(\mathcal{A})$$

In general this functor is neither full nor faithful.

Truncation functors

$$\tau_{\leq n} : \mathrm{C}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})$$

Definition

$$\tau_{\leq n}(\cdots \to A_{n-1} \to A_n \xrightarrow{d_n} A_{n+1} \to \cdots)$$
$$= \cdots \to A_{n-1} \to \ker d_n \to 0 \to 0 \to \cdots$$

One has

$$H^i(\tau_{\leq n} A) = \begin{cases} H^i(A) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Hence : $\tau_{\leq n}$ preserves quasi-isomorphisms. So there is a corresponding functor.

$$\tau_{\leq n} : D(\mathcal{A}) \to D(\mathcal{A})$$

Dual truncation functor

$$\tau_{\geq n}(\cdots \to A_{n-1} \xrightarrow{d_{n-1}} A_n \to A_{n+1} \to \cdots)$$
$$= \cdots 0 \to 0 \to \operatorname{coker} d_{n-1} \to A_{n+1} \to \cdots$$

Naive truncation functors

$$\tau_{\leq n} : \mathrm{C}(\mathcal{A}) \to \mathrm{C}(\mathcal{A})$$

Definition

$$\sigma_{\leq n}(\cdots \to A_{n-1} \to A_n \to A_{n+1} \to \cdots)$$
$$= \cdots \to A_{n-1} \to A_n \to 0 \to 0 \to \cdots$$

Note $\sigma_{\leq n}$ does preserves quasi-isomorphisms and hence does not define a functor on the derived category.

Left closed objects

$S \subset \mathrm{Maps}(\mathcal{C})$: Öre.

Definition

(1) $A \in \mathrm{Ob}(\mathcal{C})$ is *left closed* if $\mathrm{Hom}_{\mathcal{C}}(A, -)$ sends elements of S to isomorphisms.

(2) $\mathcal{C}$ has enough left closed objects if

$$\forall A \in \mathcal{C} : \exists s : A' \to A \in S$$

such that A' is left closed.

Theorem Assume that $\mathcal{C}$ has enough left closed objects. Let $\mathcal{C}^{\text{l.c.}}$ be the corresponding category. Then the inclusion

$$\mathcal{C}^{\text{l.c.}} \to \mathcal{C}$$

induces an equivalence

$$Q \mid \mathcal{C}^{\text{l.c}} : \mathcal{C}^{\text{l.c.}} \to S^{-1}\mathcal{C}$$

In particular : $S^{-1}\mathcal{C}$ may be identified with a full subcategory of $\mathcal{C}$.

Hence : "Hom"-sets in $S^{-1}\mathcal{C}$ are actually sets.

Left closed objects II

Assume that $\mathcal{C}$ has enough left closed sets.

Fact (A', s) as in (2) is unique up to unique isomorphism.

Pick a representant

$$LA \xrightarrow{s_A} A$$

(we call this a (left) "resolution" of A).

Fact II L can be made functorial using the following commutative diagram.

$$\begin{array}{ccc} LA & \xrightarrow{s_A} & A \\ {\scriptstyle \exists!\, Lf}\downarrow & & \downarrow{\scriptstyle f} \\ LB & \xrightarrow[s_B]{} & B \end{array}$$

Formula : $\mathrm{Hom}_{S^{-1}\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(LA, B)$.

Principle : Hom's in localized categories may be computed using resolutions.

Left closed objects III

One proves : The functor

$$L : \mathcal{C} \to \mathcal{C}^{\text{l.c}}$$

sends the elements of S to isomorphisms.

The associated functor

$$S^{-1}\mathcal{C} \to \mathcal{C}^{\text{l.c}}$$

is an equivalence, and a quasi-inverse to the earlier functor

$$Q \mid \mathcal{C}^{\text{l.c}} : \mathcal{C}^{\text{l.c}} \to S^{-1}\mathcal{C}$$

Left closed objects in $K(\mathcal{A})$

Definition $A \in K(\mathcal{A})$ is *acyclic* if $H(A) = 0$.

Excercise A map is a quasi-isomorphism if and only if its cone is acyclic.

Excercise $P \in K(\mathcal{A})$ is left closed is and only if, for all A acyclic we have

$$\mathrm{Hom}_{K(\mathcal{A})}(P, A) = 0$$

Definition A left closed object in $K(\mathcal{A})$ is called *homotopically projective*. Category : $K(\mathcal{A})^{\text{h.p.}}$.

Define $\mathrm{C}^-(\mathcal{A})$: complexes which are zero in high degree (short: *right bounded complexes*). $K^-(\mathcal{A})$: corresponding homotopy category.

Proposition A right bounded complex consisting of projectives is homotopically projective.

Left closed objects in $K^-(\mathcal{A})$

One proves (using truncation functors)

$$D^-(\mathcal{A}) = S_{\text{q.i.}}^{-1} K^-(\mathcal{A})$$

Proposition Assume that $\mathcal{A}$ has enough projective. Then for any $A \in \mathrm{C}^-(\mathcal{A})$ there exists a quasi-isomorphism

$$P \to A$$

such that P is right bounded complex consisting of projectives

Hence $K^-(\mathcal{A})$ has enough left closed objects.

Notation $K^-(P(\mathcal{A}))$: full subcategory of $K^-(\mathcal{A})$ of complexes consisting of projectives.

Corollary $D^-(\mathcal{A}) \cong K^-(P(\mathcal{A}))$.

Unbounded complexes

R ring.

Notation

$$K(R) = K(\mathrm{Mod}(R))$$
$$D(R) = D(\mathrm{Mod}(R))$$

Proposition $K(R)$ has enough homotopically projective (=left closed) objects.

Corollary

$$D(R) \cong K(R)^{\text{h.p}}$$

Warning

Not every complex consisting of projectives is homotopically projective.

Example Consider (for $R = \mathbb{Z}$)

$$A: \qquad \cdots \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \cdots$$

This complex is *acyclic.*

If it where homotopically projective then

$$\mathrm{Hom}_{K(R)}(A, A) = 0$$

I.e. $A = 0$ in $K(\mathcal{A})$.

But then (excercise) : $A \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$ in $K(R)$

$$A \otimes \mathbb{Z}/2\mathbb{Z}: \qquad \cdots \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \cdots$$

Not acyclic and hence not zero.

Dual notions

- Terminology and notation: right closed, homotopically injective, $K(\mathcal{A})^{\text{h.i.}}$, $\mathrm{C}^+(\mathcal{A})$, $K^+(\mathcal{A})\ldots$.
- Left bounded complexes of injectives are homotopically injective.
- If $\mathcal{A}$ has enough injectives then $D^+(\mathcal{A}) \cong K^+(I(\mathcal{A}))$ where $K^+(I(\mathcal{A}))$ is the homotopy category of left bounded injective complexes.

Grothendieck categories

Theorem Assume that $\mathcal{C}$ is a Grothendieck category. Then $K(\mathcal{C})$ has enough homotopically injective (=right closed) objects.

Corollary $D(\mathcal{C}) \cong K(\mathcal{C})^{\text{h.i.}}$.

Very useful theorem.

Proof is difficult !

Note about Ext

$\mathcal{A}$ abelian category with enough injectives.

Inclusion (fully faithful).

$$\mathcal{A} \to D^+(\mathcal{A})$$

Recall principle Hom's in localized categories may be computed using resolutions.

$A, B \in \mathcal{A}$. Pick an injective resolution.

$$0 \to B \to E_0 \to E_1 \to E_2 \to \cdots$$

We compute

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) &= \mathrm{Hom}_{K(\mathcal{A})}(A, E[n]) \\ &= H^n(\underline{\mathrm{Hom}}_{\mathrm{C}(\mathcal{A})}(A, E)) \\ &= \mathrm{Ext}^n_{\mathcal{A}}(A, B) \end{aligned}$$

Note about essentially small categories

$\mathcal{A}$ essentially small, abelian.

Recall We have two definitions for Ext in $\mathcal{A}$.

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[n])$$

and

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Ext}^n_{\mathrm{Ind}(\mathcal{A})}(A,B)$$

Fact These definitions are equivalent.

We know : $\mathrm{Ind}(\mathcal{A})$ is a Grothendieck category, hence has enough injectives.

One may show that the natural functor

$$D^b(\mathcal{A}) \to D^b_{\mathcal{A}}(\mathrm{Ind}(\mathcal{A}))$$

is fully faithful.

Thus for $A, B \in \mathcal{A}$ we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) &= \mathrm{Hom}_{D(\mathrm{Ind}(\mathcal{A}))}(A, B[n]) \\ &= \mathrm{Ext}^n_{\mathrm{Ind}(\mathcal{A})}(A,B)\end{aligned}$$

Derived functors: introduction

$F : \mathcal{A} \to \mathcal{B}$: additive functor between abelian categories.

Commutative diagram

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q"'] & K(\mathcal{B}) \arrow[d, "Q"] \\ D(\mathcal{A}) \arrow[r, dotted, "?"'] & D(\mathcal{B}) \end{tikzcd}$$

Fact If F is not exact there will be *no* $\dashrightarrow$ making the diagram commutative...

Sometimes there is a best approximation (theory of Kan extensions).

Abstract left and right derived functors

Consider

$$\begin{tikzcd} & S^{-1}\mathcal{C} \arrow[dr, dotted, "?"] & \\ \mathcal{C} \arrow[ur, "Q"] \arrow[rr, "F"'] & & \mathcal{E} \end{tikzcd}$$

The right and left derived functors RF and LF of F are determined by the conditions that there should be isomorphisms

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(F, G\circ Q) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(RF, G)$$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(G\circ Q, F) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(G, LF)$$

natural in G.

Note Since RF, LF are representing objects for certain functors, they are unique up to unique isomorphism.

Putting $G = RF$ (LF) and considering id_{RF} (id_{LF}) we obtain associated maps

$$\begin{aligned}\eta_F &: F \to RF \circ Q \\ \eta_F &: LF \circ Q \to F\end{aligned}$$

Existence of right derived functors

$F : \mathcal{C} \to \mathcal{E}$: functor.

$S \subset \mathrm{Maps}(\mathcal{C})$: Öre.

Proposition Assume that $\mathcal{C}$ has enough right closed objects. Then RF exists and is determined by the following commutative diagram.

$$\begin{tikzcd} S^{-1}\mathcal{C} \arrow[d, bend right=60, "R"'] \arrow[ddrr, "RF"] & & \\ \mathcal{C}^{\mathrm{r.c}} \arrow[u, "Q|\mathcal{C}^{\mathrm{r.c}}"', "\cong"] \arrow[d, hook] \arrow[drr, "F|\mathcal{C}^{\mathrm{r.c}}"] & & \\ \mathcal{C} \arrow[rr, "F"'] & & \mathcal{E} \end{tikzcd}$$

I.e. $RF = (F \mid \mathcal{C}^{\mathrm{r.c}}) \circ R$

Principle : Derived functors may be computed using appropriate resolutions.

Note : Construction may be done in triangulated setting and yields exact functors.

Derived functors in $D^*(\mathcal{A})$

$F : \mathcal{A} \to \mathcal{B}$ functor between abelian categories.

$* = \emptyset, +, -, b$.

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q"'] \arrow[dr, "Q\circ K(F)"] & K(\mathcal{B}) \arrow[d, "Q"] \\ D^*(\mathcal{A}) \arrow[r, dotted, "?"'] & D(\mathcal{B}) \end{tikzcd}$$

Look for

$$R(Q \circ K(F)) \qquad L(Q \circ K(F))$$

Notation (in case of existence).

$$RF \qquad LF$$

Terminology : Left and right derived functors of F.

Classical derived functors

$$R^iF = H^i(RF(-)) : \mathcal{A} \to \mathcal{B}$$
$$L_iF = H^{-i}(LF(-)) : \mathcal{A} \to \mathcal{B}$$

Example

$F : \mathcal{A} \to \mathcal{B}$, $*$ as above.

One obtains : RF exists in the following cases.

- If $* = +$ and $\mathcal{A}$ has enough injectives.
- If $* = \emptyset$ and $\mathcal{A}$ is a Grothendieck category.

Note on classical derived functors

$F : \mathcal{A} \to \mathcal{B}$ functor between abelian categories. Assume $\mathcal{A}$ as enough injectives.

Recall principle Derived functors are computed by resolutions.

Pick $A \in \mathcal{A}$ together with an injective resolution

$$0 \to A \to E_0 \to E_1 \to E_2 \to \cdots$$

Then

$$\begin{aligned} R^iF(A) &= H^i(R(Q \circ K(F)(A) \\ &= H^i((Q \circ K(F))(E)) \\ &= H^i(K(F)(E)) \end{aligned}$$

So the new definition of R^iF coincides with the old one.

Standard derived functors : RHom

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : K(\mathcal{A})^\circ \times K(\mathcal{A}) \to K(\mathbf{Ab}) :$$
$$(A, B) \mapsto \underline{\mathrm{Hom}}_{\mathrm{C}(\mathcal{A})}(A, B)$$

Assume : $\mathcal{A}$ has enough injectives.

Pick $A \in K(\mathcal{A})$. Then

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(A, -) : D^+(\mathcal{A}) \to D(\mathcal{A})$$

exists (II = second factor).

We obtain : a bifunctor:

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(-,-) : K(\mathcal{A})^\circ \times D^+(\mathcal{A}) \to D(\mathcal{A})$$

One shows : for $B \in D(\mathcal{A})$

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(-, B) : K(\mathcal{A})^\circ \to D(\mathcal{A})$$

preserves standard triangles and quasi-isomorphisms.

We obtain : a bifunctor:

$$R_I R_{II}\mathrm{Hom}_{\mathcal{A}}(-,-) : D(\mathcal{A})^\circ \times D^+(\mathcal{A}) \to D(\mathcal{A})$$

RHom cont'd

Similarly : If $\mathcal{A}$ has enough projectives then we obtain a bifunctor

$$R_{II}R_I\mathrm{Hom}_{\mathcal{A}}(-,-) : D^-(\mathcal{A})^\circ \times D(\mathcal{A}) \to D(\mathcal{A})$$

If $\mathcal{A}$ has both enough injectives and projectives then

$R_I R_{II}\mathrm{Hom}_{\mathcal{A}}(-,-)$ and $R_{II}R_I\mathrm{Hom}_{\mathcal{A}}(-,-)$

coincide when restricted to $D^-(\mathcal{A})^\circ \times D^+(\mathcal{A})$.

Notation

$$\begin{aligned}\mathrm{RHom}_{\mathcal{A}}(-,-) &= R_I R_{II}\mathrm{Hom}_{\mathcal{A}}(-,-)\\ &= R_{II}R_I\mathrm{Hom}_{\mathcal{A}}(-,-)\end{aligned}$$

whenever defined.

Formula : For $A, B \in \mathcal{A}$ we have

$$\mathrm{Ext}^i_{\mathcal{A}}(A,B) = H^i(\mathrm{RHom}_{\mathcal{A}}(A,B))$$

RHom variation

If $\mathcal{A}$ is a Grothendieck category then $K(\mathcal{A})$ has enough homotopically injective (=right closed) objects.

Hence We may define $\mathrm{RHom}_{\mathcal{A}}(-,-)$ as a bifunctor.

$$D(\mathcal{A})^\circ \times D(\mathcal{A}) \to D(\mathcal{A})$$

without any restriction.

Tensor product of complexes

$A \in \mathrm{C}(R^\circ)$, $B \in \mathrm{C}(R)$.

$$(A \otimes_R B)_n = \oplus_i A_i \otimes_R B_{n-i}$$

Differential :

$$d(a \otimes b) = da \otimes b + a \otimes (-1)^{|a|} a\, db$$

Fact : As for RHom, we can derive $- \otimes_R -$ in both arguments to obtain a bifunctor.

$$- \overset{L}{\otimes}_R - : D(R^\circ) \times D(R) \to D(\mathbf{Ab})$$

Formula : For A a right and B a left R-module we have

$$\mathrm{Tor}_i^R(A,B) = H^{-i}(A \overset{L}{\otimes}_R B)$$

t-structures: motivation

$\mathcal{A}$ abelian category.

Define

$$D^{\le n}(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ if } i > n\}$$

$$D^{\ge n}(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ if } i < n\}$$

We have

$$\mathcal{A} = D^{\le 0}(\mathcal{A}) \cap D^{\ge 0}(\mathcal{A})$$

Idea : Make this abstract for general triangulated categories in order to recognize abelian subcategories.

t-structures: definition

$\mathcal{D}$ triangulated category.

Definition A t-structure on $\mathcal{D}$ is a pair of full additive subcategories $\mathcal{D}^{\leq 0}$, $\mathcal{D}^{\geq 0}$ closed under isomorphism such that (putting $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$)

- If $X \in \mathcal{D}^{\leq 0}$, $Y \in \mathcal{D}^{\geq 1}$ then
$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$$
- $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.
- For every $X \in \mathcal{D}$ there is a distiguished triangle
$$A \to X \to B \to$$
with $A \in \mathcal{D}^{\leq 0}$, $B \in \mathcal{D}^{\geq 1}$.

t-structures: the heart

$\mathcal{D}$ triangulated category with t-structure.

Definition The intersection
$$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$
is called the *heart* of the t-structure.

Main theorem The heart is an abelian category.

The proof uses truncation functors for arbitrary t-structures (see below).

t-structures: the derived category

The axioms hold for
$$\mathcal{D} = D(\mathcal{A})$$
with
$$\mathcal{D}^{\leq 0} = D^{\leq 0}(\mathcal{A}), \qquad \mathcal{D}^{\geq 0} = D^{\geq 0}(\mathcal{A})$$

Key points in proof

- If $A \in D^{\leq n}(A)$ one has
$$\mathrm{Hom}_{D(\mathcal{A})}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, \tau_{\leq n} B)$$
- If $B \in D^{\geq n}(A)$ one has
$$\mathrm{Hom}_{D(\mathcal{A})}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(\tau_{\geq n} A, B)$$
- For X in $\mathrm{C}(\mathcal{A})$ the natural map
$$X/\tau_{\leq 0} X \to \tau_{\geq 1} X$$
is a quasi-isomorphism, so there is a distinguished triangle
$$\tau_{\leq 0} X \to X \to \tau_{\geq 1} X \to$$

t-structure: truncation functors

$\mathcal{D}$ triangulated category with t-structure.

One proves For $X \in \mathcal{D}$ the triangle
$$A \to X \to B \to$$
with $A \in \mathcal{D}^{\leq 0}$, $B \in \mathcal{D}^{\geq 1}$ is unique up to unique isomorphism.

One puts
$$\tau_{\leq 0} X \overset{\text{def}}{=} A$$
$$\tau_{\geq 1} X \overset{\text{def}}{=} B$$
and in general
$$\tau_{\leq n} X = (\tau_{\leq 0}(X[n]))[-n]$$
$$\tau_{\geq n} X = (\tau_{\geq 1}(X[n-1]))[-n+1]$$

t-structures: perverse cohomology

$\mathcal{D}$ triangulated category with t-structure and heart $\mathcal{A}$.

Definition The *perverse cohomology* of $A \in \mathcal{D}$ is defined as

$$
\begin{aligned}
{}^pH^0(A) &= \tau_{\geq 0}\tau_{\leq 0}A \\
{}^pH^n(A) &= {}^pH^0(A[n])
\end{aligned}
$$

Note : $H^0(A)$ lies in $\mathcal{A}$.

Theorem If we have a distinguished triangle then ${}^pH^0(-)$ applied to the corresponding helix yields a long exact sequence.

New t-structures from old: tilting

$\mathcal{D}$ triangulated category with t-structure and heart $\mathcal{A}$.

Definition A torsion theory in an abelian category $\mathcal{B}$ is a pair of additive full subcategories $(\mathcal{T}, \mathcal{F})$ such that

- For all $T \in \mathcal{T}$, $F \in \mathcal{F}$: $\mathrm{Hom}_{\mathcal{B}}(T, F) = 0$.
- For all $B \in \mathcal{B}$ there is a (necessarily unique) exact sequence $$0 \to T \to B \to F \to 0$$ with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

Theorem Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{A}$. Define

$$
\begin{aligned}
{}'\mathcal{D}^{\leq 0} &= \{D \in \mathcal{D}^{\leq 1} \mid {}^pH^1(D) \in \mathcal{T}\} \\
{}'\mathcal{D}^{\geq 0} &= \{D \in \mathcal{D}^{\geq 0} \mid {}^pH^0(D) \in \mathcal{F}\}
\end{aligned}
$$

Then $({}'\mathcal{D}^{\leq 0}, {}'\mathcal{D}^{\geq 0})$ defines a new t-structure on $\mathcal{D}$.

Terminology : The new t-structure is called a *tilted* t-structure.

Mental picture : Walking around in $\mathcal{D}$ by tilting produces new abelian categories from old.

Example of tilting

$\mathcal{A}$ the abelian category of finitely generated abelian groups.

$\mathcal{D} = D(\mathcal{A})$ with its canonical t-structure.

Define a torsion theory on $\mathcal{A}$ by

$$
\begin{aligned}
\mathcal{T} &= \{\text{torsion groups}\} \\
\mathcal{F} &= \{\text{torsion free groups}\}
\end{aligned}
$$

Then the heart of the tilted t-structure is represented by complexes of length 2

$$0 \to A_0 \xrightarrow{d} A_1 \to 0$$

such that $\ker d \in \mathcal{F}$, $\operatorname{coker} d \in \mathcal{T}$.

One may show : The heart of the tilted t-structure is equivalent to $\mathcal{A}^\circ$.

Hereditary t-structures

$\mathcal{D}$ t-structure with heart $\mathcal{A}$.

Assume the following conditions on the t-structure.

- *(Non-degeneracy)* If ${}^pH^n(D) = 0$ for all n then $D = 0$.
- *(Boundedness)* For every D, at most a finite number of ${}^pH(D)$ are non-zero.
- *(Hereditarity)* For every $A, B \in \mathcal{A}$ we have $$\mathrm{Hom}_{\mathcal{D}}(A, B[n]) = 0 \text{ for } n \geq 2$$

Note : $\mathrm{Hom}_{\mathcal{D}}(A, B[n]) = 0$ for $n < 0$.

Theorem Under the above conditions for every $D \in \mathcal{D}$ we have

$$D \cong \oplus_i H^i(D)[-i]$$

Note : The theorem applies if $\mathcal{D} = D^b(\mathcal{A})$ where $\mathcal{A}$ is a *hereditary abelian category*, i.e. an abelian category such that $\mathrm{Ext}^i_{\mathcal{A}}(-,-) \neq 0$ for $i \neq 0, 1$.

Sketch of proof

We pretend that perverse homology behaves exactly like homology of complexes. This is true!

For $D \in \mathcal{D}$ define $l(D)$ as the number of non-zero homology objects. We will perform induction on $l(D)$. The key point is that if ${}^pH^n(D)$ is the lowest cohomology group then there is a distinguished triangle

$${}^pH^n(D)[-n] \to D \to \tau_{\geq n+1}D \to$$

and $l(\tau_{\geq n+1}D) = l(D) - 1$.

Proof: cont'd

Step 1

Let $D \in \mathcal{D}^{\geq 2}$ and $A \in \mathcal{A}$. We claim that $\mathrm{Hom}_{\mathcal{D}}(D, A) = 0$

Appy to $\mathrm{Hom}_{\mathcal{D}}(-, A)$ to

$${}^pH^n(D)[-n] \to D \to \tau_{\geq n+1}D \to \qquad (*)$$

where $n \geq 2$. The vanishing of $\mathrm{Hom}^n_{\mathcal{D}}(-,-)$ for $n \geq 2$ implies that

$$\mathrm{Hom}_{\mathcal{D}}(\tau_{\geq n+1}D, A) \to \mathrm{Hom}_{\mathcal{D}}(D, A)$$

is epi. We finish by induction.

Step 2

Let $D \in \mathcal{D}$ be arbitrary. It follows from Step 1 that in the triangle (*) the map

$$\tau_{\geq n+1}D \to {}^pH^n(D)[-n][1]$$

is zero. Thus the triangle is is split.

We obtain

$$D \cong {}^pH^n(D)[-n] \oplus \tau_{\geq n+1}D$$

We finish by induction.