

# **Advanced School and Conference on Non-commutative Geometry**

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## **Homological algebra, abelian and derived categories**

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These are preliminary lecture notes, intended only for distribution to participants

### General categories: settheory

The theory of categories fits somewhat uncomfortably inside set theory. Luckily one can usually ignore set theory.

#### Possible foundation

Godel-Bernays axioms: “sets” and “classes”.

There is no set of all sets, but there is a class of all sets.

Mild extension of classical set theory (Zermelo Fraenkel axioms).

#### Alternative

“Universes”: more flexible, but requires a more serious extension of set theory.

### General categories: axioms

A category  $\mathcal{C}$  consist of the following data:

- A class of “objects”  $\text{Ob}(\mathcal{C})$ .
- For every  $X, Y \in \text{Ob}(\mathcal{C})$  a set of “maps”:  $X \rightarrow Y$  denoted by  $\text{Hom}_{\mathcal{C}}(X, Y)$ .
- For every  $X, Y, Z \in \text{Ob}(\mathcal{C})$  a “composition”

$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z): (f, g) \mapsto g \circ f$$

with the following axioms:

- Composition is associative.
- For every  $X \in \text{Ob}(\mathcal{C})$  there is an element  $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$  behaving as a left and right identity for composition.

**Remark** It follows that  $\text{id}_X$  is unique.

## Examples

**Set** : The category of (all) sets.

**Grp** : The category of groups.

**Ab** : The category of abelian groups.

**Rng** : The category of rings (with unit).

**Top** : The category of topological spaces with continuous maps.

etc. . .

**Note** Any set can be viewed as a category with no (non-identity) arrows.

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### Size matters

- A category is *small* if  $\text{Ob}(\mathcal{C})$  is a set.
- A category is *essentially small* if the class of isomorphism classes of objects forms a set.
- A “big” category is a category without the restriction that  $\text{Hom}_{\mathcal{C}}(-, -)$  is a set.

Size is often implied by context.

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## Properties

Standard properties of maps in these concrete categories can be mimicked in abstract categories.

**Example** A map  $f : X \rightarrow Y$  is an *isomorphism* if it has an *inverse* i.e. there is a map  $g : Y \rightarrow X$  such that  $fg = \text{id}_Y$  and  $gf = \text{id}_X$ .

Two objects are *isomorphic* if there is an isomorphism between them.

We will see a systematic way of doing this below using representable functors.

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## Functors

**Definition** Let  $\mathcal{C}, \mathcal{D}$  be categories. A *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of

- A map  $F : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ .
- For all  $X, Y \in \text{Ob}(\mathcal{C})$  maps  $F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ .

such that

- For all  $X \in \text{Ob}(\mathcal{C})$  one has  $F(\text{id}_X) = \text{id}_{F(X)}$ .
- For all maps  $f : X \rightarrow Y, g : Y \rightarrow Z$  in  $\mathcal{C}$  one has  $F(g \circ f) = F(f) \circ F(g)$ .

### Notation

$\text{Fun}(\mathcal{C}, \mathcal{D})$  : functors  $\mathcal{C} \rightarrow \mathcal{D}$  (a class in general, a set if  $\mathcal{C}, \mathcal{D}$  are small).

**Cat** : the category of *small* categories.

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## Examples

Very common functors are *Forgetful functors* (which forget part of a structure).

$$\mathbf{Ab} \rightarrow \mathbf{Set}$$

$$\mathbf{Rng} \rightarrow \mathbf{Ab}$$

$$\mathbf{Top} \rightarrow \mathbf{Set}$$

Examples of non-forgetful functors:

- $U : \mathbf{Rng} \rightarrow \mathbf{Grp} : R \mapsto R^*$  where  $R^* = \{x \in R \mid \exists y \in R : yx = xy = 1\}$
- $\mathbf{Set} \rightarrow \mathbf{Ab} : S \mapsto \mathbb{Z}S$  where  $\mathbb{Z}S$  is the free abelian group with basis  $S$ .

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## Contravariant functors

A functor as defined above is often called a *covariant* functor. We also use *contravariant* functors which invert the direction of arrows.

**Definition** If  $\mathcal{C}$  is a category then  $\mathcal{C}^\circ$  is the category with

- $\text{Ob}(\mathcal{C}^\circ) = \text{Ob}(\mathcal{C})$ .
- For all  $X, Y \in \text{Ob}(\mathcal{C}) : \text{Hom}_{\mathcal{C}^\circ}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ .

**Definition** Let  $\mathcal{C}, \mathcal{D}$  be categories. A *contravariant* functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor  $F : \mathcal{C}^\circ \rightarrow \mathcal{D}$ .

**Remark** A functor  $F : \mathcal{C}^\circ \rightarrow \mathcal{D}$  is the same thing as a functor  $\mathcal{C} \rightarrow \mathcal{D}^\circ$ .

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## Standard properties of functors

**Definition** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.  $F$  is faithful (resp. full, resp. fully faithful) if for all  $X, Y \in \text{Ob}(\mathcal{C})$  the map

$$F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective (resp. surjective, resp. bijective).

**Example** The forgetful functors on the previous transparency are all faithful, but not full.

**Definition** A full subcategory of a category  $\mathcal{D}$  is a category  $\mathcal{C}$  such that  $\text{Ob}(\mathcal{C}) \subset \text{Ob}(\mathcal{D})$  and such that for all  $X, Y \in \text{Ob}(\mathcal{C})$  we have  $\text{Hom}_{\mathcal{C}}(X, Y) = \text{Hom}_{\mathcal{D}}(X, Y)$

**Note** A full subcategory is uniquely determined by its set of objects. We say that  $\mathcal{C}$  is *spanned* by its set of objects.

**Note** If  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$  then the obvious inclusion functor  $I : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful.

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## Natural transformations

**Definition** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *natural transformation*  $\theta : F \rightarrow G$  consists of, for all  $X \in \text{Ob}(\mathcal{C})$ , a map  $\theta(X) : F(X) \rightarrow G(X)$  in  $\mathcal{D}$  such that for every map  $f : X \rightarrow Y$  in  $\mathcal{C}$  the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta(X)} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\theta(Y)} & G(Y) \end{array}$$

**Notation**  $\text{Hom}(F, G) : \text{the natural transformations } F \rightarrow G$ .

**Note** If  $\mathcal{C}, \mathcal{D}$  are categories then  $\text{Fun}(\mathcal{C}, \mathcal{D})$  is itself a category (the maps are the natural transformations).

Thus in particular the  $\text{Hom}$ -sets in  $\mathbf{Cat}$  are categories.

$\mathbf{Cat}$  is more than just a category. It is a so-called 2-category.

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## Isomorphism and equivalence

$F : \mathcal{C} \rightarrow \mathcal{D}$  a functor.

**Definition** A  $F$  is an isomorphism if it has an *inverse* for  $F$  i.e. a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $GF = \text{id}_{\mathcal{C}}$ ,  $FG = \text{id}_{\mathcal{D}}$ .

Isomorphisms between categories are quite rare.

**Definition**  $F$  is an equivalence if it has a quasi-inverse, i.e. a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $GF \cong \text{id}_{\mathcal{C}}$  in  $\text{Fun}(\mathcal{C}, \mathcal{C})$ ,  $FG \cong \text{id}_{\mathcal{D}}$  in  $\text{Fun}(\mathcal{D}, \mathcal{D})$ .

### Definition

- The essential image of  $F$  consist of the objects in  $\text{Ob}(\mathcal{D})$  which are isomorphic to objects in the image of  $F$ .
- $F$  is essentially surjective if its essential image is  $\text{Ob}(\mathcal{D})$ .

**Theorem**  $F$  is an equivalence if and only if it is fully faithful and essentially surjective.

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## Representable functors

Let  $\mathcal{C}$  be a category and  $X \in \text{Ob}(\mathcal{C})$ . We define functors

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\circ} &\rightarrow \mathbf{Set} : Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X) \\ \text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} &\rightarrow \mathbf{Set} : Z \mapsto \text{Hom}_{\mathcal{C}}(X, Z) \end{aligned}$$

**Theorem** (Yoneda) Let  $F : \mathcal{C}^{\circ} \rightarrow \mathbf{Set}$  be a contravariant functor. Then the map

$$\text{Hom}_{\text{Fun}(\mathcal{C}^{\circ}, \mathbf{Set})}(\text{Hom}_{\mathcal{C}}(-, X), F) \rightarrow F(X) : \theta \rightarrow \theta(\text{id}_X)$$

is a bijection.

**Theorem** (dual version) Let  $G : \mathcal{C} \rightarrow \mathbf{Set}$  be a covariant functor. Then the map

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathbf{Set})}(\text{Hom}_{\mathcal{C}}(X, -), G) \rightarrow G(X) : \theta \rightarrow \theta(\text{id}_X)$$

is a bijection.

**Corollary** The functors (the “Yoneda embeddings”)

$$\begin{aligned} \mathcal{C} &\rightarrow \text{Fun}(\mathcal{C}^{\circ}, \mathbf{Set}) : X \mapsto \text{Hom}_{\mathcal{C}}(-, X) \\ \mathcal{C}^{\circ} &\rightarrow \text{Fun}(\mathcal{C}, \mathbf{Set}) : X \mapsto \text{Hom}_{\mathcal{C}}(X, -) \end{aligned}$$

are fully faithful.

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## Isomorphism and equivalence II

**Example** Let  $\mathbf{FinSet}$  be the full subcategory of  $\mathbf{Set}$  spanned by the finite sets and let  $I$  be the full subcategory of  $\mathbf{Set}$  spanned by  $\emptyset$  and the intervals  $\{1, \dots, n\}$  for  $n = 1, 2, 3, \dots$

Then  $I$  and  $\mathbf{FinSet}$  are equivalent.

Indeed the obvious map

$$F : I \rightarrow \mathbf{FinSet}$$

is clearly fully faithful and essentially surjective.

**Note**  $F$  has no canonical quasi-inverse!

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## Representable functors II

**Definition** A contravariant functor  $F : \mathcal{C}^{\circ} \rightarrow \mathbf{Set}$  is *representable* if

$$F \cong \text{Hom}_{\mathcal{C}}(-, X)$$

for some  $X \in \text{Ob}(\mathcal{C})$ .

**In that case** : The object  $X$ , together with the isomorphism  $F \cong \text{Hom}_{\mathcal{C}}(-, X)$  is called a *representing object* for  $F$ .

**Analogously** : a covariant functor  $G : \mathcal{C} \rightarrow \mathbf{Set}$  is representable if

$$G \cong \text{Hom}_{\mathcal{C}}(Y, -)$$

for  $Y \in \text{Ob}(\mathcal{C})$ .

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## Representable functors III

**Note** By Yoneda's theorem: a representing object is unique, up to unique isomorphism.

Given natural isomorphisms

$$\theta : \text{Hom}_{\mathcal{C}}(-, X) \rightarrow F$$

and

$$\theta' : \text{Hom}_{\mathcal{C}}(-, X') \rightarrow F$$

there is a unique isomorphism  $f : X \rightarrow X'$  in  $\mathcal{C}$  such that the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(-, X) & \xrightarrow{\theta} & F \\ \text{Hom}_{\mathcal{C}}(-, f) \downarrow & & \parallel \\ \text{Hom}_{\mathcal{C}}(-, X') & \xrightarrow{\theta'} & F \end{array}$$

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## Monomorphisms and epimorphisms

**General principle** Use Yoneda embeddings to define properties of objects and maps.

**Definition**  $f : X \rightarrow Y$  is a *monomorphism* if

$$\text{Hom}_{\mathcal{C}}(Z, f) : \text{Hom}_{\mathcal{C}}(Z, X) \rightarrow \text{Hom}_{\mathcal{C}}(Z, Y)$$

is an injection for all  $Z \in \text{Ob}(\mathcal{C})$ .

**Traditional (equivalent) definition**

A map  $f : X \rightarrow Y$  is a *monomorphism* if for all diagrams

$$\begin{array}{ccccc} & & p & & \\ & & \searrow & & \\ Z & \xrightarrow{\quad} & X & \xrightarrow{f} & Y \\ & & q & & \end{array}$$

such that  $fp = fq$  one has  $p = q$ .

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## Monomorphisms and epimorphisms II

**Definition**  $f : X \rightarrow Y$  is an *epimorphism* if

$$\text{Hom}_{\mathcal{C}}(f, Z) : \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$

is an injection for all  $Z \in \text{Ob}(\mathcal{C})$ .

**Traditional definition**

A map  $f : X \rightarrow Y$  is an *epimorphism* if for all diagrams

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{p} & Z \\ & & & \searrow q & \end{array}$$

such that  $pf = qf$  one has  $p = q$ .

**Remark** An isomorphism is both a mono- and an epimorphism but the converse is not generally true. Counter examples: **Rng** and **Top**.

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## Split maps

If we have a commutative diagram

$$\begin{array}{ccccc} & & \text{id}_X & & \\ & & \searrow & & \nearrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & X \end{array}$$

then  $f$  is mono and  $g$  is epi.

**We say**

- A monomorphism  $f$  is *split* if  $g$  exists as in the diagram.
- An epimorphism  $g$  is *split* if  $f$  exists as in the diagram.

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## Generators

**Definition** An object  $G$  in a category  $\mathcal{C}$  is a *generator* if

$$\text{Hom}_{\mathcal{C}}(G, -) : \mathcal{C} \rightarrow \mathbf{Set}$$

is faithful (i.e. is injective on Hom-sets).

**Traditional definition:**

$G$  is a generator if for all pairs with  $p \neq q$

$$X \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} Y$$

there is a map  $f : G \rightarrow X$  such that  $pf \neq qf$ .

**Dual notion : cogenerator.**

**Examples**

- The singleton is a generator for **Set** and **Top**.
- The two element set is a cogenerator for **Set**.
- $\mathbb{Z}$  is a generator for **Ab** and **Grp**.
- $\mathbb{Z}[X]$  is a generator for **Rng**.

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## Limits exist in Set

Let  $N : I \rightarrow \mathbf{Set}$  be a functor. Then

$$\lim N$$

will stand for the following *concrete* construction.

It is the set of all

$$(a_i)_i \in \prod_{i \in \text{Ob}(I)} N(i)$$

subject to the condition :

$$\forall p : i \rightarrow j \text{ in } I : N(p)(a_i) = a_j.$$

The  $p_i$  are the restrictions of the projection maps

$$\prod_{i \in \text{Ob}(I)} N(i) \rightarrow N(i)$$

Construction works in other “concrete” categories (e.g. **Ab**, **Rng**).

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## Limits

Let  $I$  be a small category and  $N : I \rightarrow \mathcal{C}$  a functor.

A *cone* over  $N$  is an object  $X$  in  $\mathcal{C}$  together with maps for all  $i \in \text{Ob}(I)$

$$a_i : X \rightarrow N(i)$$

such that for all  $q : i \rightarrow j$  in  $I$  there is a commutative diagram

$$\begin{array}{ccc} N(i) & \xrightarrow{N(q)} & N(j) \\ a_i \uparrow & \nearrow a_j & \\ X & & \end{array}$$

$\lim N$  is a universal cone over  $N$  (unique if existing).

$$\begin{array}{ccccc} & & N(i) & \xrightarrow{N(p)} & N(j) \\ & \nearrow p_i & \uparrow p_j & \nearrow & \\ \lim N & \xleftarrow{\exists!} & X & & \end{array}$$

The  $p_i$  are sometimes called the “projection maps”.

**Alternative (ambiguous) notation**

$$\lim_{i \in I} N(i)$$

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## Definition using representable functors

We work again in a category  $\mathcal{C}$ . The universal property may be restated as:

$$\text{Hom}_{\mathcal{C}}(-, \lim_{i \in I} N(i)) \cong \lim_{i \in I} \text{Hom}_{\mathcal{C}}(-, N(i))$$

(limit in **Set**)

( $\cong$  as contravariant functors).

**Usual formulation :** there are isomorphisms

$$\text{Hom}_{\mathcal{C}}(X, \lim_{i \in I} N(i)) \cong \lim_{i \in I} \text{Hom}_{\mathcal{C}}(X, N(i))$$

“natural in  $X$ ”

**Notation** If

$$f : X \rightarrow \lim_{i \in I} N(i)$$

is a map then we write  $f_k$  for the composition

$$X \xrightarrow{f} \lim_{i \in I} N(i) \xrightarrow{p_k} N(k)$$

It is the projection of  $f$  on  $\text{Hom}_{\mathcal{C}}(X, N(k))$ .

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## Special limits

**Products** If  $I$  is a set (no arrows) then a functor

$$N : I \rightarrow \mathcal{C}$$

is the same as a set of objects  $N(i) \in \text{Ob}(\mathcal{C})$ .

The limit is denoted by

$$\prod_{i \in I} N(i)$$

and is called the *product* of the  $N(i)$ .

Finite products (i.e. if  $I = \{1, \dots, n\}$ ) are written as:

$$N(1) \times \dots \times N(n)$$

### Limit (or product) over the empty set

The limit over (the unique) functor  $N : \emptyset \rightarrow \mathcal{C}$  is a *final* object in  $\mathcal{C}$ , i.e. an object  $F$  such that for all  $X \in \mathcal{C}$ :

$$|\text{Hom}_{\mathcal{C}}(X, F)| = 1$$

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## Functors and diagrams

**Principle** Diagrams can be viewed as functors.

**Example** Let  $I$  be the category.

$$p \xrightarrow{\alpha} q$$

- two objects  $p, q$ ;
- one (non-identity) arrow  $\alpha : p \rightarrow q$

A functor  $M : I \rightarrow \mathcal{C}$  is determined by

- Objects  $X = M(p), Y = M(q)$ .
- A map  $f : X \rightarrow Y$ , given by  $f = M(\alpha)$ .

I.e. a functor  $M : I \rightarrow \mathcal{C}$  is the same as a (very small) diagram

$$X \xrightarrow{f} Y$$

in  $\mathcal{C}$ .

**Notation :**  $\text{Maps}(\mathcal{C}) = \text{Fun}(\bullet \rightarrow \bullet, \mathcal{C})$ .

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## Functors and diagrams II

**Example** Let  $J$  be the category.

$$\begin{array}{ccc} p & \xrightarrow{\alpha} & q \\ \beta \downarrow & & \downarrow \delta \\ r & \xrightarrow{\gamma} & s \end{array}$$

with "relation"  $\delta\alpha = \gamma\beta$ .

(i.e. there are 4 objects  $p, q, r, s$  and 5 non-identity arrows  $\alpha, \beta, \gamma, \delta, \delta\alpha = \gamma\beta$ ).

A functor  $J \rightarrow \mathcal{C}$  is a *commutative diagram* in  $\mathcal{C}$ .

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## Special limits II

**Equalizers** The limit of a pair of arrows

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

is called the *equalizer* of  $f, g$ .

**Pullbacks (fiber products)** The limit of a diagram

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Y & \xrightarrow{g} & Z \end{array}$$

is called the *pullback* (or fiber product) of  $(f, g)$ .

**Notation :**  $X \times_Z Y$

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## Completeness

A category  $\mathcal{C}$  is complete if it has all limits.

**Equivalent with :** All products and equalizers exist.

**Proof** Assume products and equalizers exist.

If  $N : I \rightarrow \mathcal{C}$  is a functor then  $\lim N$  is the equalizer of

$$\prod_{i \in I} N(i) \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{s} \end{array} \prod_{\phi: i \rightarrow j \text{ in } I} N(j)$$

where

$$r_{\phi: i \rightarrow j} = p_j$$

$$s_{\phi: i \rightarrow j} = \phi \circ p_i$$

**Definition** A functor is continuous if it commutes with all (existing) limits.

**Example** The representable functors  $\text{Hom}_{\mathcal{C}}(X, -)$  and  $\text{Hom}_{\mathcal{C}}(-, X)$  are continuous.

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## Chapter II

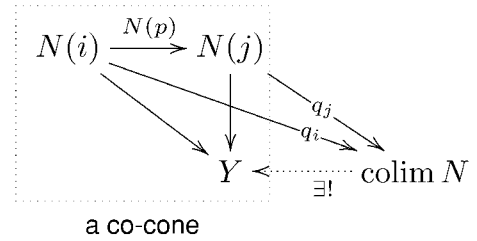
(Pre-)additive and abelian categories.

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## Colimits

The dual notion of a limit is a colimit.

**Universal property**



**Equivalent condition :** There are isomorphisms

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{i \in I} N(i), Y) \cong \lim_{i \in I} \text{Hom}_{\mathcal{C}}(N(i), Y)$$

natural in  $Y$ .

**Note** We only refer to *limits* in **Set** (not colimits).

**Dual notions :** Coproduct ( $\coprod$ ), initial object, coequalizer, pushout, cocomplete, cocontinuity.

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## Pre-additive categories

**Definition** A *pre-additive* category is a category where the  $\text{Hom}$ -sets have the additional structure of an abelian group and compositions are bilinear.

**Example**  $\mathbf{Ab}$  is pre-additive.

**Example** If  $R$  is a ring then  $\text{Mod}(R)$ , the category of left  $R$ -modules is pre-additive. Note:  $\mathbf{Ab} = \text{Mod}(\mathbb{Z})$ .

**Example** Assume  $\mathcal{C}$  pre-additive and  $|\text{Ob}(\mathcal{C})| = 1$  e.g.  $\text{Ob}(\mathcal{C}) = \{*\}$ . Then  $\mathcal{C}$  is determined by  $R = \text{End}_{\mathcal{C}}(*)$ . It is easy to see that  $R$  is a ring (always with unit).

A pre-additive category with one object is the same as a ring!

**Alternative name for a pre-additive category (Mitchell)**

“A ring with many objects.”

**Excercise** If  $\mathcal{C}$  is pre-additive and  $I$  is small then  $\text{Fun}(I, \mathcal{C})$  is pre-additive (in a natural way) as well.

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## Additive functors

**Definition** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between pre-additive categories is *additive* if the maps

$$F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are linear.

**Notation**  $\text{Add}(\mathcal{C}, \mathcal{D})$ : additive functors  $\mathcal{C} \rightarrow \mathcal{D}$ .

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## Special properties

$\mathcal{C}$  pre-additive category.

For  $X \in \text{Ob}(\mathcal{C})$  put

$1_X$  : the identity  $\text{id}_X$  in  $\text{Hom}_{\mathcal{C}}(X, X)$

$0_X$  : the zero map in  $\text{Hom}_{\mathcal{C}}(X, X)$

**Definition** A *zero object* in  $\mathcal{C}$  is an object  $X$  such that

$$1_X = 0_X$$

**Proposition** The properties of being an initial, final or zero object are equivalent.

**Notation** A zero object is denoted by ... 0.

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## Basic example

- Let  $R$  be a ring, considered as pre-additive category with one object  $*$ .
- Let  $F : R \rightarrow \mathbf{Ab}$  be an additive functor.

$F$  is determined by

- An abelian group  $M = F(*)$ .
- A map of abelian groups:

$$F : R \rightarrow \text{Hom}_{\mathbf{Ab}}(M, M)$$

compatible with composition. I.e. it should be a ring map.

Putting for  $r \in R, m \in M: r \cdot m = F(r)m$  defines a left  $R$ -module structure on  $M$ .

This construction yields an *isomorphism* between  $\text{Add}(R, \mathbf{Ab})$  and  $\text{Mod}(R)$

For a *small* pre-additive category  $\mathcal{C}$  we put

$$\text{Mod}(\mathcal{C}) = \text{Add}(\mathcal{C}, \mathbf{Ab})$$

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## Special properties continued

Let  $(M_i)_{i \in I}$  be objects in  $\mathcal{C}$  and assume  $\prod_{i \in I} M_i$  exists.

As for general categories we have “projection maps”

$$p_k : \prod_i M_i \rightarrow M_k$$

For  $f : X \rightarrow \prod_i M_i$  put  $f_k = p_k f$ . Under the isomorphism

$$\text{Hom}_{\mathcal{C}}(X, \prod_i M_i) \cong \prod_i \text{Hom}_{\mathcal{C}}(X, M_i)$$

$f_k$  is the image of  $f$  under the projection on  $\text{Hom}_{\mathcal{C}}(X, M_k)$

We now also have “inclusion maps”

$$q_j : M_j \rightarrow \prod_i M_i$$

which are defined by

$$(q_j)_k = \begin{cases} 1_{M_j} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

**Note** Here we notice the importance of having a *canonical* element 0 in every  $\text{Hom}_{\mathcal{C}}(X, Y)$ .

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## Special properties continued

The coproduct in a pre-additive category is usually denoted by  $\oplus$ .

Assume  $\oplus_i M_i$  exists.

We again have canonical maps

$$M_j \xrightarrow{q_j} \oplus_i M_i \xrightarrow{p_j} M_j$$

There is also a canonical map

$$c : \oplus_i M_i \rightarrow \prod_i M_i$$

Defined as follows

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\oplus_i M_i, \prod_j M_j) &\cong \prod_j \text{Hom}_{\mathcal{C}}(\oplus_i M_i, M_j) \\ &\cong \prod_j \prod_i \text{Hom}_{\mathcal{C}}(M_i, M_j) \end{aligned}$$

Denote the projection on  $\text{Hom}_{\mathcal{C}}(M_i, M_j)$  by  $(-)_ij$ .

Then

$$c_{ij} = \begin{cases} 1_{M_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

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## Special properties continued

Using biproducts one obtains.

**Theorem** Let  $M_1, \dots, M_n$  be objects in  $\mathcal{C}$ . Then  $M_1 \times \dots \times M_n$  exists if and only if  $M_1 \oplus \dots \oplus M_n$  exists. In that case the canonical map

$$c : M_1 \oplus \dots \oplus M_n \rightarrow M_1 \times \dots \times M_n$$

is an isomorphism.

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## Biproducts

The similarity between products and coproducts in pre-additive categories leads to the notion of a *biproduct*.

Fix a *finite* set of objects  $M_1, \dots, M_n$ .

**Definition** A *biproduct* of  $M_1, \dots, M_n$  is

- an object  $N$
- maps  $q_i : M_i \rightarrow N, p_i : N \rightarrow M_i$

satisfying

$$\sum_i q_i p_i = 1_N$$

$$p_i q_j = \begin{cases} 1_{M_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

One easily proves.

- Biproducts are unique up to unique isomorphism.
- The coproduct and the product of the  $M_i$  (if they exist) are biproducts.
- A biproduct is both a product and a coproduct.

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## Functors

Since biproducts are defined by equations, the following is clear.

**Theorem** An additive functor between pre-additive categories preserves biproducts (and hence finite products and coproducts).

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## Additive categories

**Definition** A pre-additive category is *additive* if

- it has a zero object and
- finite products (or equivalently coproducts or biproducts) exist.

**Remark** It is of course sufficient that *binary* products exist.

**Remark** A zero object is a final object so it is a product over the empty set.

**Example** A (non-zero) ring viewed as a pre-additive category is not additive. In fact it has no zero object.

**Example** If  $R$  is a ring then the category  $\text{Mod}(R)$  is additive.

**Example** More generally if  $\mathcal{C}$  is a small pre-additive category then  $\text{Mod}(\mathcal{C})$  is additive.

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## Kernels II

**Definition using representable functors**

$\ker f \rightarrow M$  is the kernel of  $f : M \rightarrow N$  if for all  $X \in \text{Ob}(\mathcal{C})$  the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(X, \ker f) \rightarrow \text{Hom}_{\mathcal{C}}(X, M) \rightarrow \text{Hom}_{\mathcal{C}}(X, N)$$

is exact

**In particular**

- $\ker f \rightarrow M$  is a monomorphism.
- $f$  is a monomorphism if and only if  $\ker f = 0$ .

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## Kernels

$\mathcal{C}$  an additive category,  $f : M \rightarrow N$  a map in  $\mathcal{C}$ .

**Definition** The *kernel*  $\ker f$  of  $f$  is the pullback of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & & \uparrow \\ & & 0 \end{array}$$

(need not exist)

**Universal property**

$$\begin{array}{ccccc} \ker f & \longrightarrow & M & \xrightarrow{f} & N \\ & \swarrow \exists! & \uparrow & \searrow 0 & \\ & & X & & \end{array}$$

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## Cokernels

$\mathcal{C}$  an additive category,  $f : M \rightarrow N$  a map in  $\mathcal{C}$ .

**Definition** The *cokernel*  $\text{coker } f$  of  $f$  is the pushout of the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

(need not exist)

**Universal property**

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \longrightarrow & \text{coker } f \\ & \searrow 0 & \downarrow & \swarrow \exists! & \\ & & X & & \end{array}$$

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## Abelian categories

Assume that  $f : M \rightarrow N$  has both a kernel and a cokernel.

Consider the following commutative diagram.

$$\begin{array}{ccccc}
 \ker f & \xrightarrow{i} & M & \xrightarrow{f} & N & \xrightarrow{j} & \text{coker } f \\
 & & \searrow & & \nearrow & & \\
 & & & & \text{coker } i & \xrightarrow{\exists!} & \ker j
 \end{array}$$

The universal properties for kernel and cokernel imply that the dotted arrow exists and is unique.

**Definition** An additive category is *abelian* if

- Every map  $f$  has a kernel and a cokernel.
- The canonical map  $\text{coker } \ker f \rightarrow \ker \text{coker } f$  is an isomorphism.

We call  $\text{coker } \ker f \cong \ker \text{coker } f$  the *image* of  $f$  and denote it by  $\text{im } f$ .

## Cokernels II

**Definition using representable functors**

$N \rightarrow \text{coker } f$  is the cokernel of  $f : M \rightarrow N$  if for all  $X \in \text{Ob}(\mathcal{C})$  the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(\text{coker } f, X) \rightarrow \text{Hom}_{\mathcal{C}}(N, X) \rightarrow \text{Hom}_{\mathcal{C}}(M, X)$$

is exact.

**In particular**

- $N \rightarrow \text{coker } f$  is an epimorphism.
- $f$  is an epimorphism if and only if  $\text{coker } f = 0$ .

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## Examples

Here are some examples of abelian categories.

- If  $\mathcal{A}$  is abelian then so is  $\mathcal{A}^\circ$  (ker and coker are exchanged).
- If  $R$  is a ring then  $\text{Mod}(R)$  is an abelian category.
- Assume  $R$  is a  $(\mathbb{Z}\text{-})$ graded ring. I.e.  $R$  comes with a decomposition

$$R = \bigoplus_{n \in \mathbb{Z}} R_n$$

such that  $R_m R_n \subset R_{m+n}$ . A graded  $R$ -module

is an  $R$ -module with a (given) decomposition

$$M = \bigoplus_n M_n \text{ such that } R_n M_m \subset M_{n+m}.$$

The category  $\text{Gr}(R)$  of graded  $R$ -modules is abelian.

- (For those who know) If  $X$  is a topological space then

$\text{Pre}(X)$  : Presheaves on  $X$

$\text{Sh}(X)$  : Sheaves on  $X$

are abelian categories.

## A non-example

Let  $\mathcal{F}$  be the category of torsion free abelian groups (as a full subcategory of  $\mathbf{Ab}$ ).

- $\mathcal{F}$  has arbitrary products and coproducts (computed as in  $\mathbf{Ab}$ ).
- $\mathcal{F}$  clearly has kernels (computed as in  $\mathbf{Ab}$ ).

**Less obvious** :  $\mathcal{F}$  also has cokernels.

$$\text{coker}_{\mathcal{F}} f = \text{coker}_{\mathbf{Ab}} f / \{\text{torsion}\}$$

**However** the identity  $\text{coker } \ker = \ker \text{coker}$  does not hold.

**Example**  $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ .

$$\begin{array}{ccccccc}
 0 & \xrightarrow{i} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{j} & 0 \\
 & & \searrow \cong & & \nearrow \cong & & \\
 & & & & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}
 \end{array}$$

The dotted arrow is *not* an isomorphism.

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## Epi-mono factorization

We have a commutative diagram

$$\begin{array}{ccccc} \ker f & \xrightarrow{i} & M & \xrightarrow{f} & N & \xrightarrow{j} & \operatorname{coker} f \\ & & \searrow p & & \nearrow q & & \\ & & & \operatorname{im} f & & & \end{array}$$

**Note**

- $p$  is epi (being a cokernel).
- $q$  is mono (being a kernel).

**We say :**  $f = pq$  is the epi-mono factorization of  $f$ .

## A non-example II

**Note**  $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$  has zero kernel and cokernel. So it is both a monomorphism and an epimorphism. However it is clearly not an isomorphism.

**Remark** Conversely in an abelian category, the identity  $\operatorname{coker} \ker = \ker \operatorname{coker}$  implies that a map which is both a monomorphism and an epimorphism is an isomorphism.

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## Exact sequences

We work in an abelian category  $\mathcal{A}$ .

Consider a diagram

$$M \xrightarrow{f} N \xrightarrow{g} P \quad (*)$$

with  $gf = 0$ .

We obtain a commutative diagram

$$\begin{array}{ccccc} \ker g & \longrightarrow & N & \xrightarrow{g} & P \\ & \swarrow q' & \uparrow q & & \\ M & \xrightarrow{p} & \operatorname{im} f & & \end{array}$$

(since  $gqp = 0$  and  $p$  is epi we obtain  $gq = 0$ ).

**Definition** The diagram (\*) is exact (at  $N$ ) if the canonical map

$$\operatorname{im} f \rightarrow \ker g$$

is an isomorphism.

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## Generalization

**Definition** A sequence of maps

$$P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} P_n \xrightarrow{d_n} P_{n+1}$$

is *exact* if

$$d_{i+1}d_i = 0$$

for all  $i$  and

$$\operatorname{im} d_0 = \ker d_1$$

$$\operatorname{im} d_1 = \ker d_2$$

$\vdots$

$$\operatorname{im} d_{n-1} = \ker d_n$$

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## Special cases

$$0 \rightarrow M \xrightarrow{f} N$$

is exact iff  $f$  is a monomorphism.

$$N \xrightarrow{g} P \rightarrow 0$$

is exact iff  $g$  is an epimorphism.

$$0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} P$$

is exact iff  $gf = 0$  and the canonical map  $M \rightarrow \ker g$  is an isomorphism. We call this a *(short) left exact sequence*.

$$M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$$

is exact iff  $gf = 0$  and the canonical map  $\operatorname{coker} f \rightarrow P$  is an isomorphism. We call this a *(short) right exact sequence*.

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## Split exact sequences

Assume that we have a short exact sequence in an abelian category

$$0 \longrightarrow A \xrightarrow{q} B \xrightarrow{p} C \longrightarrow 0$$

with  $p$  split by  $q'$ . I.e.  $pq' = \operatorname{id}_C$ .

**One proves :** There is a (unique) left splitting  $p'$  of  $q$  such that

$$A \begin{array}{c} \xrightarrow{q} \\ \xleftarrow{p'} \end{array} B \begin{array}{c} \xleftarrow{q'} \\ \xrightarrow{p} \end{array} C$$

is a biproduct of  $A$  and  $C$ .

**Proposition** A short exact sequence is split on the left if and only if it is split on the right. In that case the middle object is a biproduct of the outer objects.

**Corollary** Split exact sequences remain exact under application of any additive functor.

**Note :** Any biproduct yields a split short exact sequence.

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## Short exact sequences

A diagram of the form

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

which is exact, is called a *short exact sequence*.

Below we frequently encounter the category  $\operatorname{Ex}(\mathcal{A})$  of short exact sequences in  $\mathcal{A}$ . The morphisms in  $\operatorname{Ex}(\mathcal{A})$  are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

$\operatorname{Ex}(\mathcal{A})$  is an additive category in a natural way.

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## Left and right exact functors

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor between abelian categories.

**We know :**  $F$  preserves finite (co)products.

**Definition**

- $F$  is *left exact* if it preserves left exact sequences (or equivalently: kernels).
- $F$  is *right exact* if it preserves right exact sequences (or equivalently: cokernels).
- $F$  is *exact* if it preserves short exact sequences (or equivalently, if it is both left and right exact).

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## The Freyd-Mitchell embedding theorem

**Theorem** Let  $\mathcal{A}$  be an essentially small abelian category. Then there exist a fully faithful exact functor

$$\mathcal{A} \rightarrow \text{Mod}(R)$$

where  $R$  is some ring.

This is the basis for the technique of *diagram chasing*. I.e. to prove theorems in an abelian category we may assume that we are dealing with objects in a module category.

**Warning** We need to be careful applying this principle. For example a product of exact sequences is exact in a module category but not in a general abelian category.

**Reason:** the Freyd-Mitchell embedding functor does not need to preserve products.

**However:** being additive the F-M functor preserves *finite* (co)products.

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## The five lemma

Start with a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

**Theorem** If

- $\beta, \delta$  are isomorphisms.
- $\epsilon$  is mono.
- $\alpha$  is epi.

then  $\gamma$  is an isomorphism.

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## The snake lemma

$$\begin{array}{ccccccc} \ker f & \longrightarrow & \ker g & \longrightarrow & \ker h & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \searrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{coker } f & \longrightarrow & \text{coker } g & \longrightarrow & \text{coker } h & \longrightarrow & \dots \end{array}$$

(exact rows)

The snake lemma asserts that

- The dotted arrow exists (in a canonical way).
- We obtain an exact sequence

$$\ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \text{coker } f \rightarrow \text{coker } g \rightarrow \text{coker } h$$

**Additional info**

- If  $A \rightarrow B$  is mono then so is  $\ker f \rightarrow \ker g$ .
- If  $B' \rightarrow C'$  is epi then so is  $\text{coker } g \rightarrow \text{coker } h$ .

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## Projective objects

$\mathcal{A}$  abelian category.

**Definition** An object  $P \in \text{Ob}(\mathcal{A})$  is projective if  $\text{Hom}_{\mathcal{A}}(P, -)$  is an exact functor.

Since  $\text{Hom}_{\mathcal{A}}(P, -)$  is always left exact this is equivalent to

- An object  $P \in \text{Ob}(\mathcal{A})$  is projective if and only if  $\text{Hom}_{\mathcal{A}}(P, -)$  sends epimorphisms to surjections.

This leads to the usual definition in terms of commutative diagrams:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & 0 \\ & \nearrow & \uparrow & & \\ & & P & & \end{array}$$

$\exists$

**Other characterization :**  $P$  is projective if and only if any epimorphism

$$A \longrightarrow P \longrightarrow 0$$

splits.

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**Projective objects: properties**

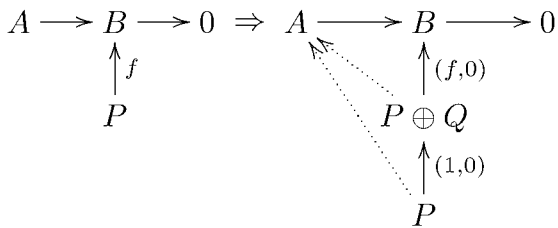
**Proposition** If  $(P_i)_{i \in I}$  are projective objects then so is  $\bigoplus_{i \in I} P_i$  (if the latter exists).

**Follows from**

$$\text{Hom}_{i \in I}(\bigoplus_i P_i, -) = \prod_i \text{Hom}_i(P_i, -)$$

which is exact.

**Proposition** If  $P \oplus Q$  is projective then so are  $P$  and  $Q$ .



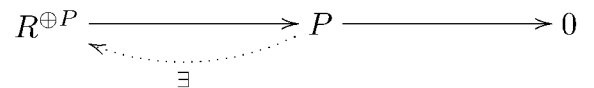
**Definition** If  $X \cong P \oplus Q$  then  $P, Q$  are said to be *summands* of  $X$ .

**Modules**

Let  $R$  be a ring.

**Definition** A free  $R$  module is one which is of the form  $R^{\oplus I}$  for some set  $I$ .

**Proposition** The projective modules are precisely the direct summands of free modules.

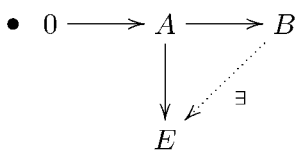


**Injective objects**

**Short definition** : The injective objects in  $\mathcal{A}$  are the projective objects in  $\mathcal{A}^\circ$ .

**Proposition** The following are equivalent for  $E \in \text{Ob}(\mathcal{A})$ .

- $E$  is injective.
- $\text{Hom}_{\mathcal{A}}(-, E)$  is exact.
- $\text{Hom}_{\mathcal{A}}(-, E)$  sends monomorphisms to surjections.



- Any monomorphism

$$0 \longrightarrow E \longrightarrow A$$

splits.

**Example**  $\mathbb{Q}$  and  $\mathbb{Q}/\mathbb{Z}$  are injective objects in **Ab**.

**In general** : Injective objects are quite complicated.

**Chapter III**

Grothendieck's AB properties.

## Grothendieck's list

Grothendieck made up a list of possible good properties of abelian category  $\mathcal{A}$ .

The relevant properties are (AB3-5) and their duals (AB3\* -5\*).

(AB3)  $\mathcal{A}$  is cocomplete.

(AB4)  $\mathcal{A}$  satisfies (AB3) and coproducts are exact.

i.e. if we have a family of exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$$

indexed by a set  $I$  then

$$0 \rightarrow \bigoplus_i A_i \rightarrow \bigoplus_i B_i \rightarrow \bigoplus_i C_i \rightarrow 0$$

is also exact.

## Note about limits and colimits

An abelian category has all finite limits and colimits.

This follows from the fact that (co)equalizers and finite (co)products exist.

**Example :** The equalizer of

$$\begin{array}{ccc} & f & \\ A & \xrightarrow{\quad} & B \\ & g & \end{array}$$

is the kernel of  $f - g$ .

An abelian category is (co) complete if and only if it has all (co) products.

## Filtered partially ordered sets

**Definition** Let  $(I, \leq)$  be a partially ordered set. We say that  $I$  is *filtered* if for all  $i, j \in I$  there exists  $k \in I$  such that  $i \leq k, j \leq k$ .

**Construction** A partially ordered set may be viewed as a category as follows.

$$\text{Hom}_I(i, j) = \begin{cases} \{*\} & \text{if } i \leq j \\ \emptyset & \text{otherwise} \end{cases}$$

( $\{*\}$  is a fixed singleton). The compositions are defined by  $* \circ * = *$ .

## Colimits over filtered posets

**Fact** Let  $I \rightarrow \text{Mod}(R)$  be a functor with  $I$  a filtered partially ordered set.

$$\text{colim}_{i \in I} M(i) = \coprod_{i \in I} M(i) / \sim$$

where

$$(m \in M(i)) \sim (n \in M(j)) \Leftrightarrow \exists k \in I, i, j \leq k, M(i \rightarrow k)(m) = M(j \rightarrow k)(n)$$

**Terminology** If  $\mathcal{C}$  is a category then a set of objects  $(M_i)_{i \in I}$  indexed by a partially ordered set is a functor  $M : I \rightarrow \mathcal{C}$  with  $M(i) = M_i$ .

## The (AB5) axiom

(AB5)  $\mathcal{A}$  satisfies (AB3) and filtered colimits are exact.

I.e. if we have a family of exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0$$

indexed by a partially ordered set  $I$  then

$$0 \rightarrow \operatorname{colim}_{i \in I} A_i \rightarrow \operatorname{colim}_{i \in I} B_i \rightarrow \operatorname{colim}_{i \in I} C_i \rightarrow 0$$

is also exact.

**Fact**  $\operatorname{Mod}(R)$  satisfies (AB5) and (AB4\*). Typical categories in algebraic geometry (i.e. sheaves) satisfy (AB5) and (AB3\*).

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## Grothendieck categories

**Definition** A *Grothendieck category* is an abelian category which has a generator and which satisfies (AB5).

Grothendieck categories have some highly non-obvious properties.

**Proposition** A Grothendieck category satisfies (AB3\*).

**Proposition** A Grothendieck category has enough injectives. I.e. any object has a monomorphism to an injective object.

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## Note about generators

### Proposition

Let  $\mathcal{A}$  be a cocomplete abelian category. Then  $G \in \operatorname{Ob}(\mathcal{A})$  is a generator if and only if for any  $A \in \operatorname{Ob}(\mathcal{A})$  there is an epimorphism

$$G^{\oplus I} \rightarrow A$$

for some  $I$ .

**Proof** Exercise.

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## The Gabriel-Popescu theorem

The deepest result about Grothendieck categories is the *Gabriel-Popescu theorem*.

**Proposition** Let  $\mathcal{A}$  be a Grothendieck category and let  $\mathcal{G}$  be a generator of  $\mathcal{A}$ . Put  $S = \operatorname{End}_{\mathcal{A}}(\mathcal{G})$ . Then the functor

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, -) : \mathcal{A} \rightarrow \operatorname{Mod}(S^{\circ})$$

is fully faithful (and has an exact left adjoint).

**Note** Let  $A \in \operatorname{Ob}(\mathcal{A})$ . The right  $S$  module structure on  $\operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, A)$  is obtained from the composition

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, A) \times \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{G}) \rightarrow \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}, A) : (f, g) \mapsto fg$$

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## The embedding theorem

For  $\mathcal{A}$  an essentially small abelian category put

$$\text{Lex}(\mathcal{A}) = \{\text{left exact additive functors } \mathcal{A}^\circ \rightarrow \mathbf{Ab}\}$$

**Theorem**  $\text{Lex}(\mathcal{A})$  is a Grothendieck category and the functor

$$\mathcal{A} \mapsto \text{Lex}(\mathcal{A}) : A \mapsto \text{Hom}_{\mathcal{A}}(-, A)$$

is fully faithful (Yoneda!) and exact.

**Remark** One may show that  $\text{Lex}(\mathcal{A})$  is in a certain sense the formal closure  $\mathcal{A}$  under filtered colimits.

**Alternative notation**

$$\text{Ind}(\mathcal{A})$$

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## General Morita theory

Let  $\mathcal{A}$  be a cocomplete abelian category.

**Definition**  $A \in \text{Ob}(\mathcal{A})$  *small* if for any family of objects  $(B_i)_{i \in I}$  the canonical map

$$\bigoplus_i \text{Hom}_{\mathcal{A}}(A, B_i) \rightarrow \text{Hom}_{\mathcal{A}}(A, \bigoplus_i B_i)$$

is an isomorphism.

**One proves** : A projective object  $P \in \text{Mod}(R)$  is small if and only if it is finitely generated.

**Definition** A small projective generator in  $\mathcal{A}$  is called a *progenerator*.

**Theorem** Assume that  $P \in \text{Ob}(\mathcal{A})$  is a progenerator. Put  $S = \text{End}_{\mathcal{A}}(P)$ . Then the functor

$$\text{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \rightarrow \text{Mod}(S^\circ)$$

is an equivalence of categories.

**Note** This result may be deduced from the Gabriel-Popescu theorem but here the proof is much easier.

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## Chapter IV

### Morita theory

## Proof

We have to prove that  $\text{Hom}_{\mathcal{A}}(P, -)$  is fully faithful and essentially surjective.

**Full faithfulness**

We have to prove that the natural map

$$\text{Hom}_{\mathcal{A}}(M, N) \rightarrow \text{Hom}_S(\text{Hom}_{\mathcal{A}}(P, M), \text{Hom}_{\mathcal{A}}(P, N)) :$$

$$f \mapsto \text{Hom}_{\mathcal{A}}(P, f)$$

is an isomorphism for all  $M, N \in \mathcal{A}$ .

We use the fact that the functor

$$F : \text{Hom}_S(\text{Hom}_{\mathcal{A}}(P, -), \text{Hom}_{\mathcal{A}}(P, N)) : \mathcal{A}^\circ \rightarrow \mathbf{Ab}$$

is left exact (since  $P$  is projective), and sends sums to products (since  $P$  is small).

Furthermore we have

$$\begin{aligned} F(P) &= \text{Hom}_S(\underbrace{\text{Hom}_{\mathcal{A}}(P, P)}_S, \text{Hom}_{\mathcal{A}}(P, N)) \\ &= \text{Hom}_{\mathcal{A}}(R, N) \end{aligned}$$

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## Proof cont'd

### Proof cont'd

Since  $P$  is a generator we may construct a right exact sequence

$$P^{\oplus J} \rightarrow P^{\oplus I} \rightarrow M \rightarrow 0$$

for sets  $I, J$ . This yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_{\mathcal{A}}(M, N) & \rightarrow & \text{Hom}_{\mathcal{A}}(P, N)^{\times I} & \rightarrow & \text{Hom}_{\mathcal{A}}(P, N)^{\times J} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & F(M) & \rightarrow & F(P)^{\times I} & \rightarrow & F(P)^{\times J} \end{array}$$

Since  $F(P) = \text{Hom}_{\mathcal{A}}(P, N)$  the two rightmost vertical maps are iso's. Hence so is the leftmost one.

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### A converse result

If there is any equivalence

$$H : \mathcal{A} \rightarrow \text{Mod}(S^{\circ})$$

between a cocomplete abelian category and a module category then  $S \cong \text{End}_{\mathcal{A}}(P)$  for a progenerator  $P$  in  $\mathcal{A}$ , and furthermore the equivalence is of the form  $\text{Hom}_{\mathcal{A}}(P, -)$ .

**Hint :** Take a (quasi-)inverse  $H^{-1}$  for  $H$  and let  $P = H^{-1}(S)$ .

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### Essential surjectivity

Let  $Z \in \text{Mod}(S^{\circ})$ . We have to write it (up to isomorphism) as  $\text{Hom}_{\mathcal{A}}(P, X)$ .

**Idea :** We can do this if  $Z = S^{\oplus I}$ . Take  $X = P^{\oplus I}$ .

For general  $Z$  construct a short exact sequence

$$S^{\oplus J} \xrightarrow{g} S^{\oplus I} \rightarrow Z \rightarrow 0$$

Since  $\text{Hom}_{\mathcal{A}}(P, -)$  is fully faithful there is some

$$f : P^{\oplus J} \rightarrow P^{\oplus I}$$

such that

$$g = \text{Hom}_{\mathcal{A}}(P, f)$$

It now suffices to take

$$X = \text{coker } f$$

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### Morita equivalent rings

**Definition** Rings  $R, S$  are *Morita equivalent* if  $\text{Mod}(R^{\circ}) \cong \text{Mod}(S^{\circ})$ .

Applying the above theory with  $\mathcal{A} = \text{Mod}(R^{\circ})$  we obtain a second equivalent definition.

**Definition** Rings  $R, S$  are *Morita equivalent* if  $S \cong \text{End}_R(P)$  for a progenerator  $P \in \text{Mod}(R^{\circ})$ .

**Example** If  $P = R^n$  then  $S = M_n(R)$  ( $n \times n$ -matrices over  $R$ ).

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## Notes on duality

If  $M \in \text{Mod}(R)$  then we define the *dual*  $M^* \in \text{Mod}(R^\circ)$  of  $M$  as:

$$M^* = \text{Hom}_R(M, R)$$

As usual there is a canonical map

$$\text{ev}_M : M \rightarrow M^{**} : m \mapsto (\phi \mapsto \phi(m))$$

### One proves

- If  $P$  is finitely generated projective then  $\text{ev}_P$  is an isomorphism.
- $(-)^*$  defines a (contravariant) equivalence between the categories of finitely generated left and right projective  $R$ -modules.

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## Morita equivalence for left modules

- Assume  $\text{Mod}(R^\circ) \cong \text{Mod}(S^\circ)$ .
- This equivalence corresponds to a progenerator  $P \in \text{Mod}(R^\circ)$  with  $S = \text{End}_R(P)$ .
- Then  $P^* = \text{Hom}_R(P, R)$  is a progenerator of  $\text{Mod}(R)$ .
- Hence

$$\text{Mod}(R) \cong \text{Mod}(T^\circ)$$

for  $T = \text{End}_R(P^*)$ .

- But one has

$$T = \text{End}_R(P^*) \cong \text{End}_R(P)^\circ = S^\circ$$

- Thus we obtain

$$\text{Mod}(R) \cong \text{Mod}(S^{\circ\circ}) = \text{Mod}(S)$$

**Conclusion :** One has  $\text{Mod}(R^\circ) \cong \text{Mod}(S^\circ)$

if and only if  $\text{Mod}(R) \cong \text{Mod}(S)$ .

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## Duality for progenerators

**Note** If  $P$  is a finitely generated projective in  $\text{Mod}(R)$  then  $P$  is a progenerator if and only if  $R$  is a summand of some  $P^{\oplus n}$ .

**We obtain :**  $P$  is a progenerator if and only if this is the case for  $P^*$ .

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## Chapter V

Presheaves and sheaves.

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## Presheaves

Throughout let  $X$  be a topological space.

Let  $\text{Open}(X)$  be the set of all open subsets of  $X$ . We view  $\text{Open}(X)$  as a partially ordered set (ordered by inclusion), and hence as a category.

We define the category of *presheaves of abelian groups* on  $X$  as

$$\text{Pre}(X) = \text{Fun}(\text{Open}(X)^\circ, \mathbf{Ab})$$

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## Sheaves

**Definition** A sheaf is a presheaf  $\mathcal{F}$  such that

- for every open  $U \subset X$
- and every open covering  $\bigcup_{i \in I} U_i = U$
- and every family of sections  $s_i \in \mathcal{F}(U_i)_i$
- such that  $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
- there exists a unique section  $s \in \mathcal{F}(U)$
- such that  $s|_{U_i} = s_i$  for all  $i$ .

For a presheaf  $\mathcal{F}$  the “sheaffication”  $a\mathcal{F}$  is defined by the following universal property for any sheaf  $\mathcal{G}$ :

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & a\mathcal{F} \\ \downarrow & \searrow \exists! & \\ \mathcal{G} & & \end{array}$$

**Roughly speaking** :  $a\mathcal{F}$  is constructed by first dividing out the sections which are locally zero and then by adjoining new sections which are defined on a covering.

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## Alternative definition

A pre-sheaf  $\mathcal{F}$  (of abelian groups) consists of

- For every open  $U \subset X$  an abelian group  $\mathcal{F}(U)$ .
- For every inclusion of opens  $U \subset V$ : *restriction maps*  $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ .

such that for every inclusion of opens  $U \subset V \subset W$  we have equality

$$\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$$

**Notation** : For  $x \in \mathcal{F}(V)$  we write  $x|_U = \rho_{V,U}(x)$ .

**Terminology** The elements of the  $\mathcal{F}(U)$  are called *sections* of  $\mathcal{F}$ . The elements of  $\mathcal{F}$  are called *global sections*.

**Easy** :  $\text{Pre}(X)$  is an abelian category. Kernels and cokernels can be computed on each open set.

$$\ker(\mathcal{F} \rightarrow \mathcal{G})(U) = \ker(\mathcal{F}(U) \rightarrow \mathcal{G}(U))$$

$$\text{coker}(\mathcal{F} \rightarrow \mathcal{G})(U) = \text{coker}(\mathcal{F}(U) \rightarrow \mathcal{G}(U))$$

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## Abelian structure

**Notation** :  $\text{Sh}(X)$  is the full subcategory of  $\text{Pre}(X)$  whose objects are sheaves.

**Fact** :  $\text{Sh}(X)$  is an abelian category.

**However** : The inclusion

$$\text{Sh}(X) \subset \text{Pre}(X)$$

is left exact (it preserves kernels), but *not right exact* (it does not preserve cokernels).

**Principle** : The non-exactness of this inclusion functor is the basis for the theory of *sheaf cohomology*.

**Formula** :

$$\text{coker}_{\text{Sh}(X)}(\mathcal{F} \rightarrow \mathcal{G}) = a(\text{coker}_{\text{Pre}(X)}(\mathcal{F} \rightarrow \mathcal{G}))$$

**Fact** : The sheaffication functor

$$a : \text{Pre}(X) \rightarrow \text{Sh}(X)$$

is exact.

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## Stalks

Let  $\mathcal{F}$  be a presheaf on  $X$  and  $x \in X$ . Let

$$\text{Open}(X, x) = \{U \in \text{Open}(X) \mid x \in U\}$$

(viewed as poset and as category)

The *stalk* of  $\mathcal{F}$  at  $x$  is defined as

$$\mathcal{F}_x = \text{colim}_{U \in \text{Open}(X, x)^\circ} \mathcal{F}(U)$$

**One proves :**  $(\mathcal{F})_x = (a\mathcal{F})_x$ .

Since  $\text{Open}(X, x)^\circ$  is filtered(!) one also sees that  $(-)_x$  is an exact functor on  $\text{Pre}(X)$  and on  $\text{Sh}(X)$ .

**One proves :** A diagram in  $\text{Sh}(X)$

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact at  $\mathcal{F}$  if and only if for all  $y \in X$

$$\mathcal{F}_y \xrightarrow{f_y} \mathcal{G}_y \xrightarrow{g_y} \mathcal{H}_y$$

is exact at  $\mathcal{G}_y$ .

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## Example: the exponential sequence

$X$  : manifold.

$\mathcal{O}$  : the sheaf of complex valued continuous functions on  $X$  (with the additive abelian group structure).

$\mathcal{O}^*$  : functions which are everywhere non-zero (with the multiplicative group structure).

$\underline{\mathbb{Z}}^p$  : the *constant presheaf* with values in  $\mathbb{Z}$ .

$\underline{\mathbb{Z}} = a(\underline{\mathbb{Z}}^p)$  (the constant *sheaf* with values in  $\mathbb{Z}$ ).

**Fact :** There is an exact sequence of sheaves.

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

(look at stalks).

**However** this sequence is usually not exact as presheaves.

**Example :** Let  $X = \mathbb{C}^*$  and let  $f \in \mathcal{O}^*(X)$  be the non-zero function  $z \mapsto z$ .  $f$  is not of the form  $\exp(g)$ , as  $\log f$  cannot be made continuous on  $\mathbb{C}^*$ .

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## Grothendieck categories

**Fact :** Both  $\text{Pre}(X)$  and  $\text{Sh}(X)$  are Grothendieck categories. Hence they have enough injectives.

**Formula :**

$$\text{colim}_{i, \text{Sh}(X)} \mathcal{F}_i = a(\text{colim}_{i, \text{Pre}(X)} \mathcal{F}_i)$$

**Generators for  $\text{Pre}(X)$**

$U \subset X$  open.

$$\mathbb{Z}_U^p(V) = \begin{cases} \mathbb{Z} & \text{if } V \subset U \\ 0 & \text{otherwise} \end{cases}$$

**Formula :**

$$\text{Hom}_{\text{Pre}(X)}(\mathbb{Z}_U^p, \mathcal{F}) = \mathcal{F}(U)$$

**Fact :** The  $\mathbb{Z}_U^p$  are generators for  $\text{Pre}(X)$ .

**Generators for  $\text{Sh}(X)$**

$$\mathbb{Z}_U = a(\mathbb{Z}_U^p)$$

**Fact :** The  $\mathbb{Z}_U$  are generators for  $\text{Sh}(X)$ .

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## Special properties of $\text{Sh}(X)$

- $\text{Pre}(X)$  has enough projectives, but  $\text{Sh}(X)$  usually has not.
- $\text{Pre}(X)$  satisfies (AB4\*) but  $\text{Sh}(X)$  only satisfies (AB34\*).

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## Chapter VI

### Classical homological algebra.

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### Graded objects

If  $\mathcal{D}$  is any category then the category  $\text{Gr}(\mathcal{D})$  of *graded objects* over  $\mathcal{D}$  is the category of sequences of objects in  $\mathcal{D}$

$$(A_n)_{n \in \mathbb{Z}}$$

with Hom-sets.

$$\text{Hom}_{\text{Gr}(\mathcal{D})}((A_n)_n, (B_n)_n) = \prod_n \text{Hom}_{\mathcal{D}}(A_n, B_n)$$

#### Shift functor

$$s((A_n)_n) = (A_{n+1})_n$$

(shift to left).

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## Complexes

**Definition** A *graded category* is a category  $\mathcal{C}$  with an automorphism  $s : \mathcal{C} \rightarrow \mathcal{C}$  (the shift functor).

**Convention :** We write

$$A[n] = s^n A$$

and

$$\text{Hom}_{\mathcal{C}}^n(A, B) = \text{Hom}_{\mathcal{C}}(A, B[n])$$

**Terminology :**  $\text{Hom}_{\mathcal{C}}^n(A, B)$  are the maps of *degree*  $n$ .

**Notations :**  $|f| = \deg f = n$ .

We obtain compositions

$$\begin{aligned} \text{Hom}_{\mathcal{C}}^m(B, C) \times \text{Hom}_{\mathcal{C}}^n(A, B) &\rightarrow \text{Hom}^{m+n}(A, C): \\ (f, g) &\rightarrow s^n(f)g \end{aligned}$$

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### Complexes II

If  $\mathcal{A}$  is abelian then so is  $\text{Gr}(\mathcal{A})$  and  $\ker$ ,  $\text{coker}$  may be computed componentwise.

**Definition** A complex over  $\mathcal{A}$  is

- An object  $A$  in  $\text{Gr}(\mathcal{A})$ .
- A map  $d \in \text{Hom}_{\text{Gr}(\mathcal{A})}^1(A, A)$  with  $d^2 = 0$ .

#### Standard view

$$\cdots \rightarrow A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} A_{n+2} \rightarrow \cdots$$

with  $d_{n+1}d_n = 0$ .

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### Complexes III

The category  $C(\mathcal{A})$  has as objects the complexes over  $\mathcal{A}$  and Hom-sets:

$$\text{Hom}_{C(\mathcal{A})}((A, d), (A', d')) = \{f \in \text{Hom}_{\text{Gr}(\mathcal{A})}((A, d), (A', d')) \mid d'f = fd\}$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{d_n} & A_{n+1} & \xrightarrow{d_{n+1}} & A_{n+2} & \longrightarrow & \cdots \\ & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} & & \\ \cdots & \longrightarrow & A'_n & \xrightarrow{d'_n} & A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_{n+2} & \longrightarrow & \cdots \end{array}$$

$C(\mathcal{A})$  is also abelian and  $\ker, \text{coker}$  may be computed termwise.

**Grading**  $(A, d_A)[1] = (A[1], -d_A)$ .

### Homology

$$(A, d) \in C(\mathcal{A}).$$

$$H(A) \stackrel{\text{def}}{=} \ker d / \text{im } d$$

**Homology functor**

$$H : C(\mathcal{A}) \rightarrow \text{Gr}(\mathcal{A})$$

**Notation :**  $H^n(A) = H(A)_n$ .

$$\cdots \rightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \rightarrow \cdots$$

$$H^n(A) = \ker d_n / \text{im } d_{n-1}$$

### The long exact sequence for homology

Let

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

Then there exists a *connecting morphism*

$$\delta : H(C) \rightarrow H(A)[1]$$

such that there is a long exact sequence

$$\begin{array}{ccccccc} & & & & \cdots & \xrightarrow{H^{i-1}(\beta)} & H^{i-1}(C) & \xrightarrow{\delta^{i-1}} & \cdots \\ & & & & \nearrow & & & & \\ \cdots & \xrightarrow{H^i(\alpha)} & H^i(A) & \xrightarrow{H^i(\beta)} & H^i(B) & \xrightarrow{H^i(\beta)} & H^i(C) & \xrightarrow{\delta^i} & \cdots \\ & & \nearrow & & & & & & \\ \cdots & \xrightarrow{H^{i+1}(\alpha)} & H^{i+1}(A) & \longrightarrow & \cdots & & & & \end{array}$$

### The long exact sequence II

**Remarks**

- The result is proved using diagram chasing.
- The statement must be completed by saying that any map between short exact sequences yields a map between the corresponding long exact sequences (viewed as complexes). The only problem is the connecting morphism which requires some diagram chasing.
- Will give a better proof later using derived categories.

## Homotopy

For  $A, B \in \text{Gr}(\mathcal{A})$  we define

$$\underline{\text{Hom}}_{\text{Gr}(\mathcal{A})}(A, B) = (\text{Hom}_{\text{Gr}(\mathcal{A})}^n(A, B))_n \in \text{Gr}(\mathbf{Ab})$$

If  $A, B \in \text{C}(\mathcal{A})$  then  $\underline{\text{Hom}}_{\text{Gr}(\mathcal{A})}(A, B)$  becomes an element of  $\text{C}(\mathcal{A})$  by defining

$$d(f) = d_B f - (-1)^n f d_A$$

for  $f \in \text{Hom}_{\text{Gr}(\mathcal{A})}^n(A, B)$ .

We denote this complex by  $\underline{\text{Hom}}_{\text{C}(\mathcal{A})}(A, B)$

**Note :** If  $|f| = 0$  then  $f \in \text{Hom}_{\text{C}(\mathcal{A})}(A, B)$  if and only if  $d(f) = 0$ .

**Definition**  $f, g \in \text{Hom}_{\text{C}(\mathcal{A})}(A, B)$  are *homotopic* if  $f - g = dh$  for  $h \in \text{Hom}_{\text{C}(\mathcal{A})}^{-1}(A, B)$ . i.e. if

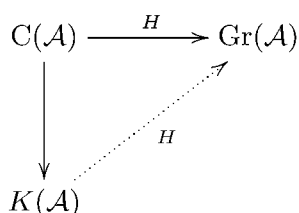
$$f - g = d_B h + h d_A$$

**Notation :**  $f \sim g$  ( $h$  is called the *homotopy* connecting  $f$  and  $g$ ).

**Note :** Two homotopic maps induces the same map on homology (since in homology,  $d_A, d_B$  become zero).

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## Fundamental diagram



**General principle** Homological algebra takes place in the homotopy category (and later : in the derived category).

**Note :** Any additive

$$F : \mathcal{A} \rightarrow \mathcal{B}$$

functor can be lifted to a functor

$$C(F) : C(\mathcal{A}) \rightarrow C(\mathcal{B})$$

(evaluating termwise). This yields a well defined functor

$$K(F) : K(\mathcal{A}) \rightarrow K(\mathcal{B})$$

The above diagram is compatible with these functors.

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## The homotopy category.

We define the *homotopy category*  $K(\mathcal{A})$  of  $\mathcal{A}$  as follows:

- $\text{Ob}(K(\mathcal{A})) = \text{Ob}(C(\mathcal{A}))$ .
- For  $A, B \in \text{Ob}(K(\mathcal{A}))$

$$\begin{aligned} \text{Hom}_{K(\mathcal{A})}(A, B) &= H^0(\underline{\text{Hom}}_{\text{C}(\mathcal{A})}(A, B)) \\ &= \text{Hom}_{\text{C}(\mathcal{A})}(A, B) / \sim \end{aligned}$$

The composition in  $\text{Hom}_{\text{C}(\mathcal{A})}(A, B)$  induces a composition in  $K(\mathcal{A})$  which is well defined because of the following identity:

**Identity :** For arrows of degrees  $m, n$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we have

$$d(gf) = d(g)f + (-1)^m g d(f)$$

**Note :** The category  $K(\mathcal{A})$  is (almost never) abelian (in contrast to  $C(\mathcal{A})$ ).

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## Analogy with topology

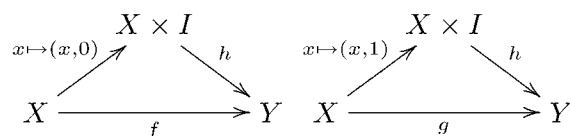
**Principle** The notion of homotopy equivalence for complexes is analogous to the notion of homotopy in algebraic topology.

Put  $I = [0, 1]$ . Let  $X, Y \in \mathbf{Top}$ .

**Definition**  $f, g : X \rightarrow Y$  are *homotopy equivalent* if there is a map

$$h : X \times I \rightarrow Y$$

such that there are commutative diagrams



In algebraic topology one constructs a functor

$$C : \mathbf{Top} \rightarrow C(\mathbf{Ab})$$

(the singular chain complex) such that if  $f, g$  are homotopy equivalent then so are  $C(f), C(g)$ .

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## Projective resolutions

**Definition**  $\mathcal{A}$  has enough projectives if for any  $A \in \text{Ob}(\mathcal{A})$  there is an epimorphism:

$$P \longrightarrow A \longrightarrow 0$$

with  $P$  projective.

**Definition** A projective resolution of  $A \in \text{Ob}(\mathcal{A})$  is a complex of projective objects

$$\cdots \rightarrow P_{-2} \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \rightarrow 0$$

together with a map  $P_0 \rightarrow A$  such that

$$\cdots \rightarrow P_{-2} \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \rightarrow A \rightarrow 0$$

is exact.

**Note :** If  $\mathcal{A}$  has enough projectives then a projective resolution always exists.

$$P_1 \longrightarrow \ker p \hookrightarrow P_0 \xrightarrow{p} A$$

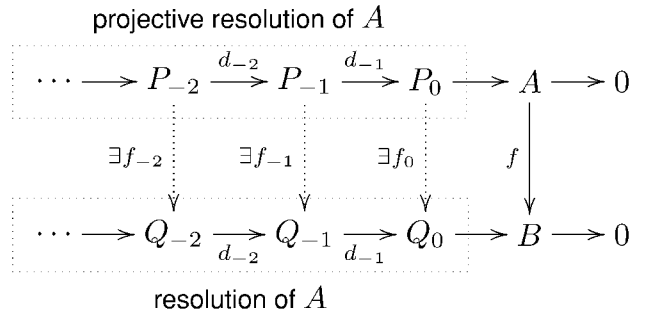
**Note :** If we drop the requirement that the  $P_i$  are projective then we speak of a (left) resolution of  $A$ .

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## Uniqueness of projective resolutions

### Principle

- Maps between objects lift to maps between projective resolutions.
- Such a lifted map is far from unique but it is unique up to homotopy.



The lifted map  $f : P \rightarrow Q$  is unique in  $K(\mathcal{A})$ .

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## Uniqueness II

Apply lifting of identity to two resolutions an object.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_{-2} & \xrightarrow{d_{-2}} & P_{-1} & \xrightarrow{d_{-1}} & P_0 \longrightarrow A \longrightarrow 0 \\ & & \downarrow f_{-2} & & \downarrow f_{-1} & & \downarrow f_0 \\ \cdots & \longrightarrow & Q_{-2} & \xrightarrow{d_{-2}} & Q_{-1} & \xrightarrow{d_{-1}} & Q_0 \longrightarrow A \longrightarrow 0 \\ & & \downarrow g_{-2} & & \downarrow g_{-1} & & \downarrow g_0 \\ \cdots & \longrightarrow & P_{-2} & \xrightarrow{d_{-2}} & P_{-1} & \xrightarrow{d_{-1}} & P_0 \longrightarrow A \longrightarrow 0 \end{array}$$

$\text{id}_A \downarrow$

Now

$$gf : P \rightarrow P$$

is a lifting of the identity on  $A$ . Hence  $gf \sim \text{id}_P$ .

Likewise  $fg \sim \text{id}_Q$ .

**Conclusion** Projective resolutions are unique up to (unique) isomorphism in  $K(\mathcal{A})$ .

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## Uniqueness III

Assume that  $\mathcal{A}$  has enough projectives. Pick for any  $A \in \text{Ob}(\mathcal{A})$  a projective resolution  $P(A)$  in  $C(\mathcal{A})$ . We obtain a commutative diagram of functors.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{P} & K(\mathcal{A}) \\ & \searrow \cong & \downarrow H^0 \\ & & \mathcal{A} \end{array}$$

The diagonal arrow is naturally isomorphic to the identity functor.

**Note :** Different choices of projective resolutions yield naturally isomorphic functors  $\mathcal{A} \rightarrow K(\mathcal{A})$ .

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## The horseshoe lemma

**Principle :** Assume we have an exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

and projective resolutions

$$\cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow P_0 \rightarrow A \rightarrow 0$$

$$\cdots \rightarrow Q_{-2} \rightarrow Q_{-1} \rightarrow Q_0 \rightarrow C \rightarrow 0$$

Then we can construct a commutative diagram of projective resolutions

$$\cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow P_0 \rightarrow A \rightarrow 0$$

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightarrow & R_{-2} & \rightarrow & R_{-1} & \rightarrow & R_0 \rightarrow B \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightarrow & Q_{-2} & \rightarrow & Q_{-1} & \rightarrow & Q_0 \rightarrow C \rightarrow 0 \end{array}$$

such that

$$0 \rightarrow P \rightarrow R \rightarrow Q \rightarrow 0$$

is exact (necessarily split in  $\text{Gr}(\mathcal{A})$ ).

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## Left derived functors

Assume

- $F : \mathcal{A} \rightarrow \mathcal{B}$  is a right exact functor.
- $\mathcal{A}$  has enough projectives.

For  $A \in \text{Ob}(\mathcal{A})$  fix a projective resolution

$$\cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow P_0 \rightarrow A \rightarrow 0$$

We apply  $F$  to this projective resolution.

$$\cdots \rightarrow F(P_{-2}) \rightarrow F(P_{-1}) \rightarrow F(P_0) \rightarrow F(A) \rightarrow 0$$

If  $F$  is not exact then this will in general not be an exact sequence.

We define

$$L_i F(A) = H^{-i}(C(F)(P))$$

**Note :**  $L_0 F(A) \cong A$  (canonically).

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## Construction

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \ker p & \longrightarrow & P_0 & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \downarrow (1,0) & & \downarrow (1,\phi) & & \downarrow \\ \ker(1,\phi) & \longrightarrow & P_0 \oplus Q_0 & \xrightarrow{(1,\phi)} & B & \longrightarrow & 0 \\ & & \downarrow (0,1) & \nearrow \exists \phi & \downarrow & & \downarrow \\ \ker p' & \longrightarrow & Q_0 & \xrightarrow{p'} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The snake lemma (applied to the dotted rectangle) implies that

$$0 \rightarrow \ker p \rightarrow \ker(1, \phi) \rightarrow \ker p' \rightarrow 0$$

is exact.

**Hence** We may repeat to obtain the desired resolutions.

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## Well definedness and functoriality

- We may also write

$$L_i F(A) = H^{-i}(K(F)(P))$$

(since homology may be computed in the homotopy category).

- Two projective resolutions  $P, Q$  of  $A$  are canonically isomorphic in  $K(\mathcal{A})$ .
- Thus  $K(F)(P)$  and  $K(F)(Q)$  are canonically isomorphic in  $K(\mathcal{B})$ .
- Hence  $H^{-i}(K(F)(P))$  and  $H^{-i}(K(F)(Q))$  are canonically isomorphic. We identify them and write them as  $L_{-i} F(A)$ .

By lifting maps in  $\mathcal{A}$  to projective resolutions (unique in the homotopy category!) we obtain functors

$$L_i F : \mathcal{A} \rightarrow \mathcal{B}$$

The sequence of functors  $(L_i F)_i$  is called the left derived functor of  $F$ .

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### Example: Tor-functors

Let  $R$  be a ring and let  $M$  be a right  $R$ -module. Then we have a right exact functor.

$$M \otimes_R - : \text{Mod}(R) \rightarrow \mathbf{Ab}$$

The derived functors are written as

$$\text{Tor}_i^R(M, -)$$

### The long exact sequence

$F : \mathcal{A}, \mathcal{B}$  right exact,  $\mathcal{A}$  enough projectives.

**Theorem** Assume that we have a long exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

Then there are "connecting maps"

$$\delta_i : L_i F(C) \rightarrow L_{i-1} F(A)$$

such that there is a long exact sequence

$$\begin{array}{ccccccc} & & & & L_{i+1} F(C) & \xrightarrow{\delta_{i+1}} & \\ & & & & \uparrow & & \\ & & & & L_{i+1} F(C) & \xrightarrow{L_{i+1} F(\beta)} & \\ & & & & \uparrow & & \\ & & & & L_i F(C) & \xrightarrow{L_i F(\beta)} & \\ & & & & \uparrow & & \\ & & & & L_i F(B) & \xrightarrow{L_i F(\alpha)} & \\ & & & & \uparrow & & \\ & & & & L_i F(B) & \xrightarrow{L_i F(\alpha)} & \\ & & & & \uparrow & & \\ & & & & L_{i-1} F(A) & \xrightarrow{L_{i-1} F(\alpha)} & \\ & & & & \uparrow & & \\ & & & & L_{i-1} F(A) & \xrightarrow{L_{i-1} F(\alpha)} & \\ & & & & \vdots & & \\ & & & & F(A) & \xrightarrow{F(\alpha)} & \\ \dots & \longrightarrow & F(A) & \xrightarrow{F(\alpha)} & F(B) & \xrightarrow{F(\beta)} & F(C) \longrightarrow 0 \end{array}$$

### Example

Assume  $R = \mathbb{Z}$ . We will compute  $\text{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$ .

**Projective resolution**

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

**Tensoring...**

$$M \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{(\times p) \otimes_{\mathbb{Z}} \mathbb{Z}} N \otimes_{\mathbb{Z}} \mathbb{Z}$$

But  $M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M$ , etc... Thus  $\text{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$  is the homology of the complex

$$M \xrightarrow{\times p} M$$

**Hence**

$$\text{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z}) = \begin{cases} M/pM & \text{if } i = 0 \\ \ker(M \xrightarrow{\times p} M) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

explaining the notation "Tor(sion)".

### Proof of theorem (sketch)

We start by using the horseshoe lemma to construct a commutative diagram of projective resolutions

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_{-2} & \longrightarrow & P_{-1} & \longrightarrow & P_0 \longrightarrow A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \dots & \longrightarrow & R_{-2} & \longrightarrow & R_{-1} & \longrightarrow & R_0 \longrightarrow B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \dots & \longrightarrow & Q_{-2} & \longrightarrow & Q_{-1} & \longrightarrow & Q_0 \longrightarrow C \longrightarrow 0 \end{array}$$

such that

$$0 \rightarrow P \rightarrow R \rightarrow Q \rightarrow 0$$

is exact in  $C(\mathcal{A})$

Since this sequence is split in  $\text{Gr}(\mathcal{A})$  we obtain an exact sequence

$$0 \rightarrow C(F)(P) \rightarrow C(F)(R) \rightarrow C(F)(Q) \rightarrow 0$$

in  $\text{Gr}(\mathcal{B})$ . This sequence is then also exact in  $C(\mathcal{B})$ .

It now suffices to apply the long exact sequence for homology.

### Proof cont'd

The proof is incomplete as the constructed connecting maps may depend on the chosen projective resolutions. Proving that this is not the case is slightly tricky.

**Main point :** Suppose we have a commutative diagram of complexes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

and we have constructed projective resolutions of the top and the bottom row using the horseshoe lemma

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & R & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ \\ 0 & \longrightarrow & P' & \longrightarrow & R' & \longrightarrow & Q' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

Then these resolutions may be assembled in one big commutative diagram. 117

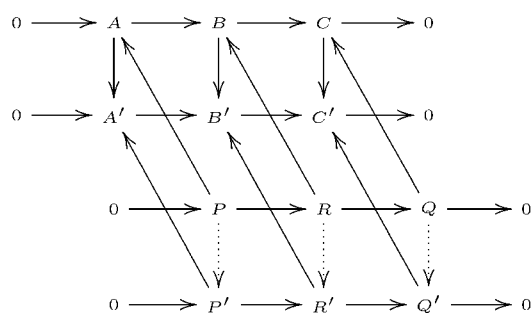
### Another property

$F, \mathcal{A}, \mathcal{B}$  as above.

**Proposition** The functors  $L_i F$  for  $i > 0$  are zero on projective objects.

**Proof** This is trivial since a projective object is its own projective resolution.

### Proof cont'd



If we take the two exact sequences equal then the long exact sequence associated to the two projective resolutions yields that the resulting connecting maps are the same.

The proof also yields.

**Proposition** A map between short exact sequences give a map between the corresponding long exact sequences.

### $\delta$ -functors

**Definition** Let  $(F^i)_{i \in \mathbb{Z}}$  be a series of functors between abelian categories  $\mathcal{A}, \mathcal{B}$  such that for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

there is given a connecting morphism

$$\delta^i : F^i(C) \rightarrow F^{i+1}(A)$$

which fit in a long exact sequence

$$\begin{array}{ccccccc} & & & & F^{i-1}(\beta) & & \delta^{i-1} \\ & & & & \longrightarrow & F^{i-1}(C) & \cdots \\ & & & & & & \nearrow \\ \cdots & \longrightarrow & F^i(A) & \xrightarrow{F^i(\alpha)} & F^i(B) & \xrightarrow{F^i(\beta)} & F^i(C) \cdots \\ & & & & & & \searrow \\ \cdots & \longrightarrow & F^{i+1}(A) & \xrightarrow{F^{i+1}(\alpha)} & \cdots & & \end{array}$$

Assume furthermore that any map between short exact sequences gives rise to a map between long exact sequences (in a functorial way). Then we say that the  $(F^i)_i$  (together with the  $\delta^i$ ) form a  $\delta$ -functor  $\mathcal{A} \rightarrow \mathcal{B}$ .

## Examples

- Let  $\mathcal{A}, \mathcal{B}$  be abelian categories. Then the functors  $(H^i)_i : C(\mathcal{A}) \rightarrow \mathcal{B}$  form a  $\delta$ -functor.
- Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor between an abelian categories, such that  $\mathcal{A}$  has enough projectives. Then the functors  $(L_{-i}F)_i$  form a  $\delta$ -functor.

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## Universal $\delta$ -functors

**Definition** A  $\delta$ -functor  $(F^i)_i$  is a *universal homological  $\delta$ -functor* if

- $F^i = 0$  for  $i > 0$ .
- For any  $\delta$ -functor  $(G^i)_i$  and any map  $\theta^0 : G^0 \rightarrow F^0$  there is a *unique* extension of  $\theta^0$  to a map of  $\delta$ -functors  $\theta : (G^i)_i \rightarrow (F^i)_i$ .

**Note :** Being given by a universal property, a universal homological  $\delta$ -functors  $(F^i)_i$  is determined, up to unique isomorphism, by  $F^0$ .

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## Morphisms between $\delta$ -functors

**Definition** A morphism  $(F^i)_i \rightarrow (G^i)_i$  between  $\delta$ -functors is a sequence of natural transformation  $\theta^i : F^i \rightarrow G^i$  compatible with the connecting maps. I.e. for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

there is a commutative diagram

$$\begin{array}{ccc} F^i(C) & \xrightarrow{\delta^i} & F^{i+1}(A) \\ \theta^i(C) \downarrow & & \downarrow \theta^{i+1}(A) \\ G^i(C) & \xrightarrow{\delta^i} & G^{i+1}(A) \end{array}$$

**Example** If  $\theta : F \rightarrow G$  is a natural transformation then we obtain a corresponding morphism  $L\theta : (L_{-i}F)_i \rightarrow (L_{-i}G)_i$  of  $\delta$ -functors.

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## Characterization of universal homological $\delta$ -functors

**Definition** An additive functor  $H : \mathcal{A} \rightarrow \mathcal{B}$  is *coeffaceable* if for any  $A \in \text{Ob}(\mathcal{A})$  there exists an epimorphism  $u : P \rightarrow A$  such that  $H(u) = 0$ .

**Theorem** Let  $(F^i)_i$  be a  $\delta$ -functor  $\mathcal{A} \rightarrow \mathcal{B}$ . Assume that

- $F^i = 0$  for  $i > 0$
- $F^i$  is coeffaceable for  $i < 0$ .

Then  $(F^i)_i$  is a universal homological  $\delta$ -functor.

**Example** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor and assume that  $\mathcal{A}$  has enough projectives. Then the  $\delta$ -functor  $(L_{-i}F)_i$  satisfies the hypotheses of the theorem and hence is universal.

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## Proof

Assume  $(F^i)_i$  satisfies the conditions of the theorem, let  $(G^i)_i$  be an arbitrary  $\delta$ -functor  $\mathcal{A} \rightarrow \mathcal{B}$  and let  $\theta^0$  be a map  $G^0 \rightarrow F^0$ .

We need to construct (for  $i \geq 0$ )

$$\theta^{-i} : G^{-i} \rightarrow F^{-i}$$

compatible with the connecting morphisms, i.e. for any exact sequence

$$0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$$

we should have a commutative diagram

$$\begin{array}{ccc} G^{-i-1}(A) & \xrightarrow{\delta^{-i-1}} & F^{-i}(B) \\ \theta^{-i-1}(A) \downarrow & & \downarrow \theta^{-i}(B) \\ F^{-i-1}(A) & \xrightarrow{\delta^{-i-1}} & F^{-i}(B) \end{array}$$

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## Proof cont'd

We proceed by induction.  $\theta^0$  is already given.

Assume that we have constructed  $(\theta^{-i})_{i \leq n}$ . Let  $A \in \text{Ob}(\mathcal{A})$  and construct a short exact sequence

$$0 \rightarrow B \rightarrow P \xrightarrow{u} A \rightarrow 0$$

such  $F^{-n-1}(u) = 0$ .

If  $\theta^{-n-1}(A)$  exists then it should be equal to  $\theta^{-n-1}(A, u)$ , defined by the dotted arrow in the following diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & G^{-n-1}(A) & \xrightarrow{\delta^{-n-1}} & G^{-n}(B) & \longrightarrow & G^{-n}(P) \\ & & \theta^{-n-1}(A, u) \downarrow \text{dotted} & & \theta^{-n}(B) \downarrow & & \theta^{-n}(P) \downarrow \\ F^{-n-1}(P) & \xrightarrow[\text{F}^{-n-1}(u)]{0!} & F^{-n-1}(A) & \xrightarrow[\delta^{-n-1}]{} & F^{-n}(B) & \longrightarrow & F^{-n}(P) \end{array}$$

We want

- $\theta^{-n-1}(A, u)$  is independent of  $u$ .
- If we put  $\theta^{-n}(A) = \theta^{-n}(A, u)$  then  $\theta^{-n}(A)$  is a natural transformation compatible with the connecting maps.

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## Proof cont'd

Consider first a commutative diagram of the form.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & P & \xrightarrow{u} & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & B' & \longrightarrow & P' & \xrightarrow{u'} & A' \longrightarrow 0 \end{array}$$

with  $F^{-n-1}(u) = F^{-n-1}(u') = 0$ .

Then we get a diagram

$$\begin{array}{ccccc} G^{-n-1}(A) & \xrightarrow{G^{-n-1}(\alpha)} & G^{-n-1}(A') & & \\ \downarrow \theta^{-n-1}(A, u) & \searrow & \swarrow & \downarrow \theta^{-n-1}(A', u') & \\ & G^{-n}(B) & \longrightarrow & G^{-n}(B') & \\ & \downarrow & & \downarrow & \\ & F^{-n}(B) & \longrightarrow & F^{-n}(B') & \\ \downarrow \theta^{-n-1}(A) & \swarrow & \searrow & \downarrow \theta^{-n-1}(A') & \\ F^{-n-1}(A) & \xrightarrow{F^{-n-1}(\alpha)} & F^{-n-1}(A') & & \end{array}$$

where the trapezoids and the middle square are commutative. Since the lower diagonal arrows are monomorphisms this implies that the outer square is commutative as well.

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## Proof cont'd

Thus we obtain a commutative diagram

$$\begin{array}{ccc} G^{-n-1}(A) & \xrightarrow{G^{-n-1}(\alpha)} & G^{-n-1}(A') \\ \theta^{-n-1}(A, u) \downarrow & & \downarrow \theta^{-n-1}(A', u') \\ F^{-n-1}(A) & \xrightarrow{F^{-n-1}(\alpha)} & F^{-n-1}(A') \end{array}$$

In particular :

- If  $A = A'$  then  $\theta^{-n-1}(A, u) = \theta^{-n-1}(A, u')$ . Since two cofacings of  $F^{-n-1}$  at  $A$  can be dominated by a third we obtain that  $\theta^{-n-1}(A, u)$  is independent of  $u$ .
- Dropping the  $u$ 's from the diagram we see that  $\theta^{-n-1}$  is a natural transformation.

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### Proof cont'd

Now we prove that  $\theta^{-n-1}$  is compatible with the connecting maps. We start with

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B'' & \longrightarrow & P & \xrightarrow{u} & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow v & & \parallel \\
 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A \longrightarrow 0 \\
 & & \text{(given exact sequence)} & & & & 
 \end{array}$$

with  $F^{-n-1}(v) = 0$ . It follows that  $F^{-n-1}(u) = 0$  as well.

This gives a commutative diagram

$$\begin{array}{ccc}
 G^{-n-1}(A) & \longrightarrow & G^{-n}(B'') \\
 \parallel & & \downarrow \\
 G^{-n-1}(A) & \xrightarrow{\delta^{-n-1}} & G^{-n}(B)
 \end{array}$$

and a similar one for  $(F^i)_i$ .

### Proof cont'd

The proof now ends with a final commutative diagram

$$\begin{array}{ccccc}
 & & \delta^{-n-1} & & \\
 & & \curvearrowright & & \\
 G^{-n-1}(A) & \longrightarrow & F^{-n-1}(B'') & \longrightarrow & G^{-n}(B) \\
 \downarrow \theta^{-n-1}(A) & & \downarrow \theta^{-n}(B'') & & \downarrow \theta^{-n}(B) \\
 F^{-n-1}(A) & \longrightarrow & F^{-n-1}(B'') & \longrightarrow & F^{-n}(B) \\
 & & \delta^{-n-1} & & 
 \end{array}$$

The left square is commutative by the construction of  $\theta^{-n-1}$  (and its independence of the coefficient of  $F^{-n-1}$  at  $A$ ). The right square is commutative since  $\theta^{-n}$  is a natural transformation. Therefore the outer rectangle is commutative, finishing the proof.

**Note :** This proof illustrates the technique of *degree (or dimension) shifting*. By construction exact sequences with suitable middle term one reduces things to lower degree.

## Tor as a bi-functor

Write temporarily

$${}^{II} \text{Tor}_i^R(M, -) = \text{Tor}_i^R(M, -)$$

Let  $N$  be a *left*  $R$ -module and let  $\text{Mod}(R^\circ)$  be the category of right  $R$ -modules. Then we have a right exact functor

$$- \otimes_R N : \text{Mod}(R^\circ) \rightarrow \text{Mod}(R^\circ)$$

Denote the left derived functors by

$${}^I \text{Tor}_i^R(-, N)$$

**Theorem** There are isomorphisms

$${}^I \text{Tor}_i^R(M, N) \cong {}^{II} \text{Tor}_i^R(M, N)$$

natural in  $M, N$ .

### Proof

We first show that

$${}^I \text{Tor}_i^R(-, N) \cong {}^{II} \text{Tor}_i^R(-, N)$$

Since both functors have the same value for  $i = 0$ , it is sufficient to show that  ${}^{II} \text{Tor}_i^R(-, N)$  is a universal homological  $\delta$ -functor. I.e. it is sufficient that

- ${}^{II} \text{Tor}_i^R(-, N)$  is a  $\delta$ -functor.
- ${}^{II} \text{Tor}_i^R(Q, N)$  is zero for  $i > 0$  if  $Q$  is projective.

## First assertion

**We prove**  ${}^{II} \text{Tor}_i^R(-, N)$  is a  $\delta$ -functor.

Take a projective resolution of  $N$

$$\cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow P_0 \rightarrow N \rightarrow 0$$

and an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

Tensoring with a projective  $R$ -module is exact so we obtain a commutative diagram with exact columns

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M' \otimes_R P_2 & \longrightarrow & M' \otimes_R P_1 & \longrightarrow & M' \otimes_R P_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & M \otimes_R P_2 & \longrightarrow & M \otimes_R P_1 & \longrightarrow & M \otimes_R P_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & M'' \otimes_R P_2 & \longrightarrow & M'' \otimes_R P_1 & \longrightarrow & M'' \otimes_R P_0 \end{array}$$

The long exact sequence for homology yields the connecting maps.

**Remark** The connecting maps are independent of the chosen resolution of  $N$  since this resolution is unique up to homotopy.

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## Second assertion

**We prove**  ${}^{II} \text{Tor}_i^R(Q, N) = 0$  for  $i > 0$ ,  $Q$  projective (or flat).

Let the resolution of  $N$  be as above.

$$\cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow P_0 \rightarrow N \rightarrow 0$$

Tensoring with  $Q$  is exact. So we obtain an exact sequence

$$Q \otimes_R P_{-2} \rightarrow Q \otimes_R P_{-1} \rightarrow Q \otimes_R P_0 \rightarrow Q \otimes_R N \rightarrow 0$$

So we have indeed  $\text{Tor}_i^R(Q, N) = 0$  for  $i > 0$ .

To finish the proof we need that the isomorphism  ${}^I \text{Tor}_i^R(-, N) \cong {}^{II} \text{Tor}_i^R(-, N)$  is natural in  $N$ . This follows easily from the fact that they are universal homological  $\delta$ -functors.

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## Acyclic objects

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor. Assume that  $\mathcal{A}$  has enough projectives.

**Definition** An object  $A \in \text{Ob}(\mathcal{A})$  is *acyclic* for  $F$  if  $L_i F(A) = 0$  for  $i < 0$ .

**Note :** Any projective object is acyclic but there are usually others.

**Proposition** Let  $M \in \text{Ob}(\mathcal{A})$  and assume there is a resolution by acyclic objects

$$\cdots \rightarrow A_{-2} \rightarrow A_{-1} \rightarrow A_0 \rightarrow M \rightarrow 0$$

Then  $L_i F(M) \cong H^{-i}(C(F)(A))$ .

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## Homological characterization of flatness

**Proposition** The following are equivalent for  $M \in \text{Mod}(R^\circ)$ .

- (1)  $M$  is flat (i.e.  $M \otimes_R -$  is exact).
- (2)  $\text{Tor}_1^R(M, N) = 0$  for all  $N \in \text{Mod}(R)$ .
- (3)  $M$  is acyclic for all functors  $- \otimes_R N$ .

**Proof** (1)  $\Rightarrow$  (3) has already been shown. (3)  $\Rightarrow$  (2) is trivial. (2)  $\Rightarrow$  (1) follows from the long exact sequence for  $M \otimes_R -$ .

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## Flat dimension

**Definition** Let  $M \in \text{Mod}(R^\circ)$ . The *flat dimension*

$$\text{fd } M$$

of  $M$  is the minimal length of a resolution of  $M$  by right flat objects (by convention it is infinite if such a resolution does not exist).

**Proposition** The following are equivalent for  $M \in \text{Mod}(R^\circ)$ .

- $\text{fd } M \leq n$ .
- For any resolution

$$0 \rightarrow M_{-n} \rightarrow F_{-n+1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \quad (*)$$

with  $F_0, \dots, F_{-n+1}$  flat we have that  $M_{-n}$  is flat.

- $\text{Tor}_{n+1}^R(M, N) = 0$  for all  $N \in \text{Mod}(R)$ .
- $\text{Tor}_i^R(M, N) = 0$  for all  $N \in \text{Mod}(R)$  and all  $i > n$ .

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### Proof cont'd

(1)  $\Rightarrow$  (4) We now have a resolution as in (\*) with  $M_{-n}$  flat. It follows from (\*\*) that  $\text{Tor}_i^R(M, N) = 0$  for  $i > n$ .

(4)  $\Rightarrow$  (3) This is trivial.

(3)  $\Rightarrow$  (2) It follows from (\*\*) that  $\text{Tor}_j^R(M_{-n}, N) = 0$  for  $j > 0$  for all  $N$ . Hence  $M_{-n}$  is flat.

(2)  $\Rightarrow$  (1) A resolution of length  $n$  as in (\*) always exists. For example take the  $F_i$  projective.

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## Proof

We start with a preliminary computation. Assume that we have a resolution as in (\*). If we break up this resolution in short exact sequences

$$0 \rightarrow M_{-1} \rightarrow F_0 \rightarrow M \rightarrow 0$$

$$0 \rightarrow M_{-2} \rightarrow F_{-1} \rightarrow M_{-1} \rightarrow 0$$

...

$$0 \rightarrow M_{-n} \rightarrow F_{-n+1} \rightarrow M_{-n+1} \rightarrow 0$$

then from the long exact sequence for  $- \otimes_R N$  we obtain for  $j > 0$ .

$$\begin{aligned} \text{Tor}_j^R(M_{-n}, N) &\cong \text{Tor}_{j+1}^R(M_{-n+1}, N) \\ &\cong \dots \\ &\cong \text{Tor}_{j+n}^R(M, N) \quad (**) \end{aligned}$$

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## Weak dimension

**Definition** The *right weak dimension*

$$\text{r. w. dim } R$$

of  $R$  is the maximum of the flat dimensions of all right  $R$ -modules. The *left weak dimension* is defined similarly.

**Definition** The *Tor-dimension*

$$\text{Tdim } R$$

of  $R$  is the minimal number  $n$  such that  $\text{Tor}_i^R(M, N) = 0$  for  $i \geq n + 1$  and all  $M \in \text{Mod}(R^\circ)$ ,  $N \in \text{Mod}(R)$  (infinite if such a number does not exist).

**Theorem** There is equality

$$\text{l. w. dim } R = \text{Tdim } R = \text{r. w. dim } R$$

**Proof** The second equality follows from the proposition on flat dimension. The first equality follows by symmetry.

**Below we write :**  $\text{w. dim } R = \text{r. w. dim } R$ .

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## Dual notions

$\mathcal{A}$  abelian.

- We say that  $\mathcal{A}$  has enough injectives if for any  $A$  there is a monomorphism  $A \hookrightarrow E$  with  $E$  injective (example:  $\text{Mod}(R)$  has enough injectives).

- If  $\mathcal{A}$  has enough in injectives then we may construct *injective resolutions*

$$0 \rightarrow A \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$$

- Such resolutions have the usual functoriality and uniqueness properties in the homotopy category.
- If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a left exact functor then we define the right-derived functors of  $F$  as  $R^i F(A) = H^i(C(F)(E))$ . This is well-defined and functorial in the usual sense.
- An exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

yields a long exact sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \xrightarrow{\delta^0} R^1 F(A) \rightarrow R^1 F(B) \rightarrow \dots$$

with the usual naturality properties.

## Dual notions II

- A  $\delta$ -functor  $(F^i)_i$  is a *universal cohomological  $\delta$ -functor* if  $F^i = 0$  for  $i < 0$  and for any  $\delta$ -functor  $(G^i)_i$  and any map  $\theta^0 : F^0 \rightarrow G^0$  there is a *unique* extension of  $\theta^0$  to a map of  $\delta$ -functors  $\theta : (F^i)_i \rightarrow (G^i)_i$ .
- An additive functor  $H : \mathcal{A} \rightarrow \mathcal{B}$  is *effaceable* if for any  $A \in \text{Ob}(\mathcal{A})$  there exists a monomorphism  $u : A \rightarrow E$  such that  $H(u) = 0$ .
- If  $(F^i)_i$  is a  $\delta$ -functor such that  $F^i = 0$  for  $i < 0$  and  $F^i$  is effaceable for  $i > 0$ . then it is a universal cohomological  $\delta$ -functor.
- If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is left exact then  $(R^i F(A))_i$  is a universal cohomological  $\delta$ -functor.

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## Ext-functors

Let  $\mathcal{A}$  be an abelian category.

**Definition** Assume that  $\mathcal{A}$  has enough injectives and let  $A \in \text{Ob}(\mathcal{A})$ . Then we define

$${}^{II} \text{Ext}_{\mathcal{A}}^i(A, -)_i$$

as the right derived functor of

$$\text{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \rightarrow \mathbf{Ab}$$

**Definition** Assume that  $\mathcal{A}$  has enough projectives and let  $B \in \text{Ob}(\mathcal{A})$ . Then we define

$${}^I \text{Ext}_{\mathcal{A}}^i(-, B)_i$$

as the right derived functors of

$$\text{Hom}_{\mathcal{A}}(-, B) : \mathcal{A}^\circ \rightarrow \mathbf{Ab}$$

**Note**  $\mathcal{A}$  has enough projectives if and only if  $\mathcal{A}^\circ$  has enough injectives.

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## Ext-functors II

One proves as for the Tor-functors.

**Theorem** If  $\mathcal{A}$  has both enough injectives and projectives then there are isomorphisms

$${}^I \text{Ext}_{\mathcal{A}}^i(A, B) \cong {}^{II} \text{Ext}_{\mathcal{A}}^i(A, B)$$

natural in  $A, B$ .

Below we write  $\text{Ext}_{\mathcal{A}}^i(A, B)$  for both  ${}^I \text{Ext}_{\mathcal{A}}^i(A, B)$  and  ${}^{II} \text{Ext}_{\mathcal{A}}^i(A, B)$  whenever these are defined.

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## Homological characterizations of projectives and injectives

### Not enough projectives or injectives

$\mathcal{A}$  is an essentially small abelian category then we may define

$$\text{Ext}_{\mathcal{A}}^i(A, B) = \text{Ext}_{\text{Ind}(\mathcal{A})}^i(A, B)$$

**One may show :** This coincides with the earlier definitions if there are enough injectives or projectives.

**Principle** The results proved for  $\text{Tor}$  have analogs for  $\text{Ext}$ . For example:

**Proposition** Assume that  $\mathcal{A}$  has enough projectives. Then the following are equivalent. for  $P \in \text{Ob}(\mathcal{A})$ .

- (1)  $P$  is projective.
- (2)  $\text{Ext}_{\mathcal{A}}^1(P, B) = 0$  for all  $B \in \text{Ob}(\mathcal{A})$ .
- (3)  $P$  is acyclic for all functors  $\text{Hom}_{\mathcal{A}}(-, B) = 0$ .

There is of course a dual result for injective objects.

### Projective and injective dimension

**Definition** Let  $\mathcal{A}$  be an abelian category with enough projectives.

- Assume that  $\mathcal{A}$  has enough projectives and  $A \in \text{Ob}(\mathcal{A})$ . The *projective dimension*

$$\text{pd}(A)$$

of  $A$  is the minimal length of a finite projective resolution of  $A$  (as usual infinite if such a finite resolution does not exist).

- Assume that  $\mathcal{A}$  has enough injectives and  $B \in \text{Ob}(\mathcal{A})$ . The *injective dimension*

$$\text{id}(B)$$

of  $B$  is the minimal length of a finite injective resolution of  $B$

### Global dimension

As for  $\text{Tor}$  one proves.

**Proposition** Assume that  $\mathcal{A}$  has enough projectives. Then the following numbers are the same.

- The maximum of the projective dimensions of the objects in  $\mathcal{A}$ .
- The minimum number  $n$  such that  $\text{Ext}_{\mathcal{A}}^i(A, B) = 0$  for all  $i > n$  and all  $A, B \in \text{Ob}(\mathcal{A})$ .

There is a dual result for injective dimensions.

**Definition** We define the *global dimension*  $\text{gldim } \mathcal{A}$  of  $\mathcal{A}$  as one of the following numbers (whenever they are defined):

- The maximum of the projective dimensions of the objects in  $\mathcal{A}$ .
- The minimum number  $n$  such that  $\text{Ext}_{\mathcal{A}}^i(A, B) = 0$  for all  $i > n$  and all  $A, B \in \text{Ob}(\mathcal{A})$ .
- The maximum of the injective dimensions of the objects in  $\mathcal{A}$ .

## Special results for rings

It is well-known that  $E \in \text{Mod}(R^\circ)$  is injective if we have the following lifting property for all right ideals  $I$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \longrightarrow & R \\ & & \downarrow & \nearrow \exists & \\ & & E & & \end{array}$$

This leads to

**Proposition** The following are equivalent for  $E \in \text{Mod}(R^\circ)$ .

- (1)  $E$  is injective.
- (2)  $\text{Ext}_R^1(R/I, E) = 0$  for all right ideals  $I \subset R$ .
- (3)  $E$  is acyclic for all functors  $\text{Hom}_R(R/I, -)$ .

This leads to:

**Proposition**  $\text{r. gldim } R$  is also equal to

$$\sup_I \text{pd}(R/I)$$

with  $I$  a right ideal in  $R$ .

## Global dimension of rings

$R$  a ring.  $\text{Mod}(R)$  has both enough projectives and injectives. We put

$$\text{r. gldim } R = \text{gldim Mod}(R^\circ)$$

(right global dimension)

$$\text{l. gldim } R = \text{gldim Mod}(R)$$

(left global dimension)

We have

$$\text{w. dim } R \leq \text{r. gldim } R$$

$$\text{w. dim } R \leq \text{l. gldim } R$$

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## Noetherian rings

**Reminder** A ring  $R$  is right noetherian if any right ideal is finitely generated.

**Equivalent :** Every submodule of a finitely generated right module is finitely generated.

**Terminology :** A ring is noetherian if it is both left and right noetherian.

**Note** A finitely generated right module over a right noetherian ring has a resolution consisting of finitely generated projective modules.

**Fact** If  $R$  is right noetherian then every finitely generated flat right module is projective.

For a right noetherian ring this leads to

$$\text{w. dim } R = \text{r. gldim } R$$

and hence for any noetherian ring

$$\text{l. gldim } R = \text{r. gldim } R$$

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## Commutative rings

$R$  a commutative ring.

**One has**

$$\text{gl dim } R = \sup_m \text{gl dim } R_m$$

where  $m$  runs through the maximal ideals of  $R$  and  $R_m$  is the localization of  $R$  at  $m$  (i.e. invert  $R - m$ ).

This reduces the problem to *local rings* (i.e. rings with a unique maximal ideal).

**Theorem** Let  $R$  be a commutative noetherian ring with maximal ideal  $m$ . The following are equivalent.

- $\text{gl dim } R = n$ .
- $\text{pd}(R/m) = n$ .
- $R$  is a *regular local ring* of dimension  $n$ . I.e.  $\dim_{R/m} m/m^2 = n$ .

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## Graded rings

If  $A$  is a graded ring then we put

$$\text{r. gr. gldim } A = \text{gl dim Gr}(A^\circ)$$

In a similar way we define *graded weak*, *graded flat*, *graded projective* and *graded injective dimension*.

Let  $k$  be a field.

**Definition**  $A$  is *connected graded* if

- $A_i = 0$  for  $i < 0$ .
- $A_0 = k$ .
- $A_i$  is finite dimensional over  $k$  if  $i > 0$ .

**Notation** We write simply  $k$  for the graded  $A$ -bimodule  $A/(A_{>0})$ .

**Theorem** Assume that  $A$  is connected graded. One has

$$\begin{aligned} \text{r. gldim } A &= \text{r. gr. gldim } A = \text{gr. pd}(k_A) \\ &= \text{gr. pd}({}_A k) = \text{l. gr. gldim } A = \text{l. gldim } A \\ &= \text{w. dim } A = \text{gr. w. dim } A \end{aligned}$$

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## Polynomial rings

**General result :**

$$\text{gl dim } R[x] = 1 + \text{gl dim } R$$

Hence in particular

$$\text{gl dim } k[x_1, \dots, x_n] = n$$

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## Rings of low global dimension

The following result is classical.

**Artin-Wedderburn Theorem** Let  $R$  be a ring. Then  $\text{r. gldim } R = 0$  (or equivalently  $\text{l. gldim } R = 0$ ) if and only if

$$R = \prod_{i=1}^n M_{m_i}(D_i)$$

with the  $D_i$  being skew fields.

**Examples of global dimension one**

The free algebra

$$k\langle X_1, \dots, X_n \rangle$$

Upper triangular matrices

$$\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$$

So-called "hereditary orders"

$$\begin{pmatrix} k[x] & k[x] \\ xk[x] & k[x] \end{pmatrix}$$

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## An example of global dimension three

"Non-commutative projective planes"

$$A = k\langle x, y, z \rangle / (f_1, f_2, f_3)$$

$$f_1 = ayz + bzy + cx^2$$

$$f_2 = azx + bxz + cy^2$$

$$f_3 = axy + byx + cz^2$$

$$(a, b, c) \in \mathbb{P}^2 - \{\text{finite "bad" set}\}$$

$A$  is connected graded with  $\text{deg } x = \text{deg } y = \text{deg } z = 1$ .

Resolution of  ${}_A k$ .

$$0 \rightarrow A \xrightarrow{\cdot \begin{pmatrix} x & y & z \end{pmatrix}} A^3 \xrightarrow{\cdot \begin{pmatrix} cx & bz & ay \\ az & cy & bx \\ by & ax & cz \end{pmatrix}} A^3 \xrightarrow{\cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \rightarrow 0$$

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## Sheaf cohomology

Let  $\text{Sh}(X)$  be the category of sheaves on a topological space  $X$ . Recall that  $\text{Sh}(X)$  has enough injectives (being a Grothendieck category).

Define the (left exact) *global section* functor as

$$\Gamma(X, -) : \text{Sh}(X) \rightarrow \mathbf{Ab} : \mathcal{F} \mapsto \mathcal{F}(X)$$

For a sheaf  $\mathcal{F}$  we put

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$

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## Chapter VII

### Derived categories and triangulated categories

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### (Universal) localization of categories

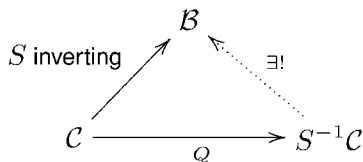
$\mathcal{C}$  category,  $S \subset \text{Maps}(\mathcal{C})$ .

**Definition** An  $S$ -inverting map is a functor  $\mathcal{C} \rightarrow \mathcal{B}$  such that the elements of  $S$  are mapped to isomorphisms.

**Proposition** There is a "universal"  $S$ -inverting functor

$$Q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$$

defined by the following universal property



**Sketch of construction**

- $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$ .
- Morphisms in  $S^{-1}\mathcal{C}$  are (composable) formal paths

$$f_1 \cdot s_1^{-1} \cdot f_2 \cdot s_2^{-1} \cdots s_{n-1}^{-1} \cdot f_n$$

with  $f_i$  in  $\mathcal{C}$  and  $s_i$  in  $S$ , modulo a suitable equivalence relation.

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### Simplifying conditions

**Definition**  $S$  is *multiplicatively closed* if it contains all the identity maps and is closed under composition.

**Definition** A multiplicatively closed set is *saturated* if for maps  $s, t$  in  $\mathcal{C}$  with  $s, t \in S$  we have

$$s \in S \Leftrightarrow t \in S$$

**Fact** We can replace a set of maps always by its *multiplicative closure* or *saturation* without changing the corresponding localization.

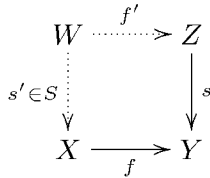
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## Öre sets

**Principle** Localization becomes a lot easier if the *Öre conditions hold*.

**Definition** Let  $S$  be a multiplicatively closed set of maps.  $S$  is *Öre* if the following conditions hold.

(ORE1) For all  $s : Z \rightarrow Y$  in  $S$ ,  $f : X \rightarrow Y$  in  $\mathcal{C}$  there is a commutative diagram



(think:  $s^{-1}f = f's'^{-1}$ ).

(ORE1') Dual condition.

(ORE2) For  $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$  in  $\mathcal{C}$  the following are equivalent

- $\exists s \in S$  such that  $sf = sg$ .
- $\exists t \in S$  such that  $ft = gt$ .

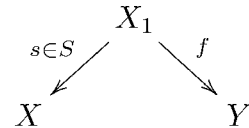
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## Localization for Öre sets

Let  $S \subset \text{Maps}(\mathcal{C})$  be an Öre set.

**New construction of  $S^{-1}\mathcal{C}$**

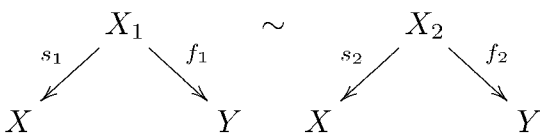
- $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$ .
- Morphisms are equivalence classes of diagrams



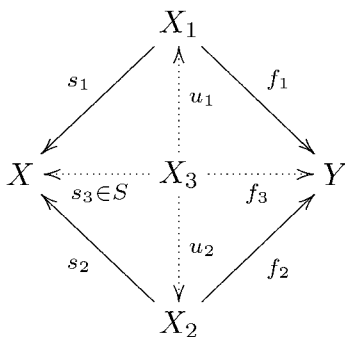
(think  $fs^{-1}$ ).

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## Equivalence relation



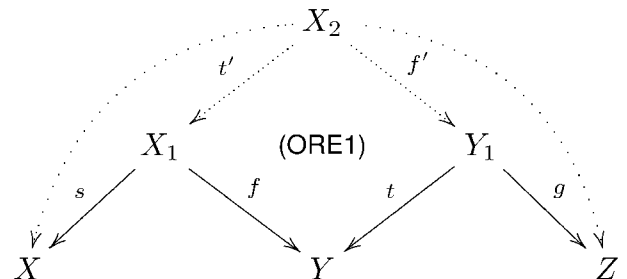
if there is a commutative diagram



**Note :** Not required that  $u_1, u_2 \in S$ . Automatic if  $S$  is saturated.

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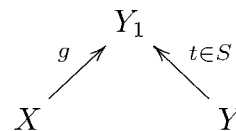
## Composition of diagrams



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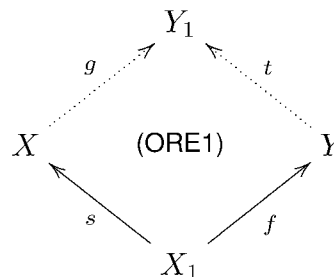
## Alternative diagrams

**Principle** We may also define  $S^{-1}\mathcal{C}$  via equivalence classes of diagrams of the following type.



(think:  $t^{-1}g$ ). The result is the same!

We go from the old diagrams to the new diagrams (and back) via the Ore conditions.



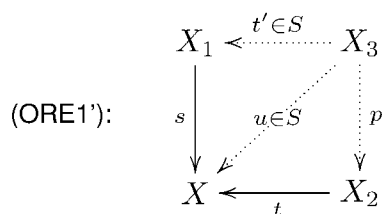
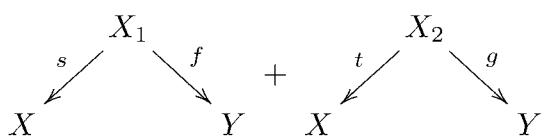
## Technical problem

- A priori “Hom”-sets in  $S^{-1}\mathcal{C}$  might not be sets.
- Will be solved later in concrete cases.

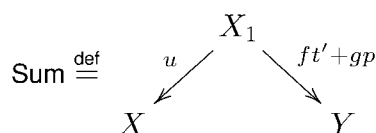
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## Ore localization preserves pre-additivity



(think:  $u$  is a common denominator for  $s, t$ ).



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## The derived category of an abelian category

$\mathcal{A}$  abelian category.

**Definition**  $A \xrightarrow{f} B$  in  $C(\mathcal{A})$  is a *quasi-isomorphism* if  $H(f)$  is an isomorphism.

$$S_{\text{q.i.}} = \{\text{quasi-isomorphisms in } C(\mathcal{A})\}$$

### Properties

- $S_{\text{q.i.}}$  is a saturated multiplicatively closed set containing all isomorphisms.
- $S_{\text{q.i.}}$  is not an Ore set.

**Definition**  $D(\mathcal{A}) = S_{\text{q.i.}}^{-1}C(\mathcal{A})$ .

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## Elementary properties

### Commutative diagram

$$\begin{array}{ccc}
 C(\mathcal{A}) & \xrightarrow{Q} & S_{\text{qi}}^{-1} C(\mathcal{A}) = D(\mathcal{A}) \\
 \searrow H & & \swarrow \exists! H \\
 & & \text{Gr}(\mathcal{A})
 \end{array}$$

**Similary** The shift functor  $-[1]$  on  $C(\mathcal{A})$  descends to a shift functor  $-[1]$  on  $D(\mathcal{A})$ .

### Define

$$i : \mathcal{A} \rightarrow D(\mathcal{A}) : A \mapsto Q(0 \rightarrow \underbrace{A}_{(\text{deg}0)} \rightarrow 0)$$

### Commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{i} & D(A) \\
 \searrow \text{id}_{\mathcal{A}} & & \downarrow H^0 \\
 & & \mathcal{A}
 \end{array}$$

Hence  $i$  is fully faithful.

**Convention :** We view  $i$  as an inclusion.

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## Application

New definition of  $\text{Ext}$  without assuming enough projectives or injectives.

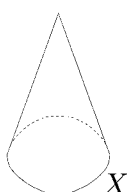
$$\text{Ext}_{\mathcal{A}}^i(A, B) = \text{Hom}_{D(\mathcal{A})}(A, B[n])$$

**We will show :** this coincides with our earlier definitions.

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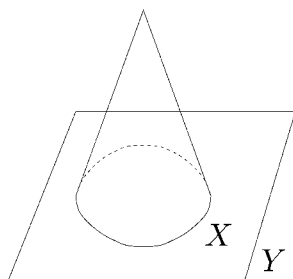
## The cone of a morphism: topological inspiration

$f : X \rightarrow Y$  continous map.

$$\text{cone}(X) = \text{cone}(X) = X \times I / ((x, 1) \sim (x', 1))$$


$$\text{cone}(f) = \frac{\text{cone}(X) \amalg Y}{(x, 0) \sim f(x)}$$

### Schematically



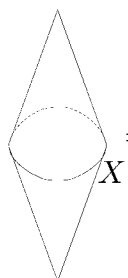
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## The cone of a morphism: topological inspiration II

**There are maps**

$$X \xrightarrow{f} Y \rightarrow \text{cone}(f) \rightarrow SX \quad (*)$$

where

$$SX = \text{cone}(SX) = X \times I / \left( \begin{array}{l} (x, 0) \sim (x', 0) \\ (x, 1) \sim (x', 1) \end{array} \right)$$


(the unreduced suspension of  $X$ ).

**Note** The composition of two consecutive maps in  $(*)$  is homotopic to a constant map.

**Remark** The functor  $SX$  is a kind of shift functor on topological spaces.

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## Long exact sequences

### Proposition

- $H^0(\text{helix})$  is exact.
- Let  $C$  be a complex. Then

$$\text{Hom}_{K(\mathcal{A})}(C, \text{helix})$$

and

$$\text{Hom}_{K(\mathcal{A})}(\text{helix}, C)$$

are exact.

**Proof (sketch)** We have an exact sequence in  $C(\mathcal{A})$

$$0 \rightarrow B \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \text{cone}(f) \xrightarrow{(1 \ 0)} A[1] \rightarrow 0$$

split in  $\text{Gr}(\mathcal{A})$ . The long exact sequence for homology for this exact sequence of complexes turns out to be precisely  $H^0(\text{helix})$ .

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### Proof cont'd

Since the sequence is split in  $\text{Gr}(\mathcal{A})$  applying  $\underline{\text{Hom}}_{C(\mathcal{A})}(C, -) = \underline{\text{Hom}}_{\text{Gr}(\mathcal{A})}(C, -)$  yields a short exact sequence

$$0 \rightarrow \underline{\text{Hom}}_{C(\mathcal{A})}(C, B) \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \underline{\text{Hom}}_{C(\mathcal{A})}(C, \text{cone}(f)) \xrightarrow{(1 \ 0)} \underline{\text{Hom}}_{C(\mathcal{A})}(C, A[1]) \rightarrow 0$$

$$\text{Hom}_{K(\mathcal{A})}(C, \text{helix})$$

is the long exact sequence for homology associated to this short exact sequence.

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### Note on the functoriality of cones

Clearly  $\text{cone}(-)$  defines a functor

$$\text{Maps}(C(\mathcal{A})) \rightarrow C(\mathcal{A})$$

The world would be much nicer if  $\text{cone}(-)$  would define a functor on maps in the homotopy category but this is not the case.

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### Functoriality cont'd

**More precisely** Suppose we have a diagram in  $C(\mathcal{A})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p \downarrow & & \downarrow q \\ C & \xrightarrow{g} & D \end{array}$$

commutative in  $K(\mathcal{A})$  then we can construct a corresponding commutative diagram of helices in  $K(\mathcal{A})$

$$\begin{array}{ccccc} & & \text{cone}(f) & & \\ & \swarrow (1) & \uparrow & \searrow & \\ A & \xrightarrow{f} & B & & \\ p \downarrow & & \downarrow q & & \\ & \swarrow (1) & \uparrow & \searrow & \\ C & \xrightarrow{g} & D & & \end{array}$$

$\text{cone}(g)$

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### Functoriality cont'd

**However** The construction of  $r$  turns out to depend on the chosen homotopy between  $qf$  and  $gp$ . There is no natural choice!

**Weak result** If  $p, q$  are isomorphisms then for any homotopy the constructed  $r$  will be an isomorphism.

### The derived category via the homotopy category

The homology functor is defined on  $K(\mathcal{A})$  so we can speak about quasi-isomorphisms in  $K(\mathcal{A})$ .

**One may prove :** The natural functor

$$S_{q.i.}^{-1} C(\mathcal{A}) \rightarrow S_{q.i.}^{-1} K(\mathcal{A})$$

is an isomorphism of categories.

It requires checking that  $S_{q.i.}^{-1} K(\mathcal{A})$  satisfies the required universal property.

### The Öre condition

The interpretation

$$D(\mathcal{A}) = S_{q.i.}^{-1} K(\mathcal{A})$$

simplifies life considerably since we have the following fundamental result.

**Proposition**  $S_{q.i.}$  is an Öre set in  $K(\mathcal{A})$ .

**Example (ORE1)** We must complete in  $K(\mathcal{A})$  for  $t \in S_{q.i.}, g$  arbitrary.

$$\begin{array}{ccc} D & \xrightarrow{\quad f \quad} & C \\ s \in S_{q.i.} \downarrow & & \downarrow t \\ A & \xrightarrow{\quad g \quad} & B \end{array}$$

We define  $D, s, f$  by the following helix

$$\begin{array}{ccc} & \text{cone}(-g, t) \xrightarrow{\text{def}} D[1] & \\ \begin{pmatrix} s \\ f \end{pmatrix} \swarrow & & \searrow \\ A \oplus C & \xrightarrow{(-g, t)} & B \end{array}$$

(1)

### Example cont'd

Since the composition of any two maps is zero in a helix we have

$$-gs + tf = 0$$

**Furthermore :** The long exact sequence for homology yields that  $s$  is a quasi-isomorphism.

## Triangles

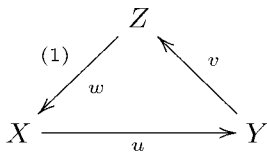
**Principle** The cone of a map plays the role of kernel and cokernel in the homotopy category. We need a substitute for the axiom  $\ker \operatorname{coker} = \operatorname{coker} \ker$  in an abelian category.

**Triangles :** Let  $\mathcal{D}$  be graded a category.

A *triangle*  $(X, Y, Z, u, v, w)$  in  $\mathcal{D}$  is a sequence of objects and maps

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

also written as



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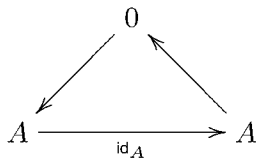
## Pretriangulated categories

**Definition** A pretriangulated category is a graded pre-additive category  $\mathcal{D}$  together with a full subcategory

$$\Delta(\mathcal{D})^{\text{dist}} \subset \Delta(\mathcal{D})$$

of *distinguished triangles* satisfying the following axioms.

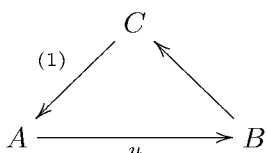
(TR1a)



is distinguished.

(TR1b) A triangle isomorphic to a distinguished one is distinguished.

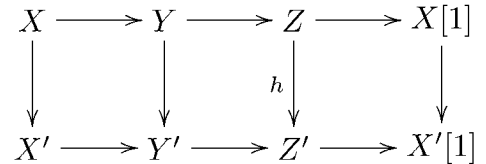
(TR1c) For all  $u : A \rightarrow B \in \operatorname{Maps}(\mathcal{D})$  there is a distinguished triangle



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## Triangles II

A map between triangles is a commutative diagram

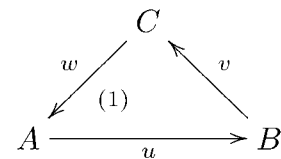


**Notation :**  $\Delta(\mathcal{D})$  : the category of triangles in  $\mathcal{D}$ .

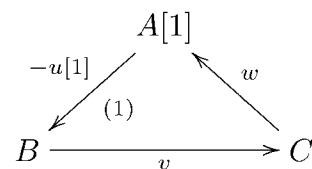
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## Pretriangulated categories II

(TR2) (the *rotation axiom*) A triangle



is distinguished if and only if the following triangle is distinguished.



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### Pretriangulated categories III

(TR3) (the double helix axiom) A commutative diagram with rows which are distinguished triangles may be completed to a map between distinguished triangles

$$\begin{array}{ccccccc}
 A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
 f \downarrow & & g \downarrow & & \exists h \downarrow & & \downarrow f[1] \\
 A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1]
 \end{array}$$

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### Elementary properties

$\mathcal{D}$  pretriangulated.

**Proposition** Let

$$\begin{array}{ccc}
 & C & \\
 w \swarrow & & \nwarrow v \\
 A & \xrightarrow{u} & B
 \end{array}
 \quad (1)$$

be a distinguished triangle.

- The composition of any two arrows is zero.
- For any  $D \in \text{Ob}(\mathcal{D})$ ,  $\text{Hom}_{\mathcal{D}}(D, -)$  and  $\text{Hom}_{\mathcal{D}}(-, D)$  applied to the corresponding helix yield long exact sequences.

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### Note on axioms

- An abelian category is an additive category satisfying *extra axioms*.
- A pretriangulated category has *extra structure* besides extra axioms.

**Note also** The axioms of a pretriangulated category only assert the existence of certain objects. These objects are *in no way unique or functorial*.

**Contrast with :** ker, coker in an abelian category are functorial.

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### Elementary properties II

Consider (TR3).

$$\begin{array}{ccccccc}
 A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
 f \downarrow & & g \downarrow & & \exists h \downarrow & & \downarrow f[1] \\
 A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1]
 \end{array}$$

**Proposition** If  $f, g$  are isomorphisms then so is  $h$ .

**Proof** It follows from the long exact sequence and the five lemma that  $\text{Hom}_{\mathcal{D}}(D, h)$  is an isomorphism for any  $D$ .

**Corollary** A distinguished triangle is up to isomorphism determined by its base.

$$\begin{array}{ccccccc}
 A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
 \text{id}_A \downarrow & & \text{id}_B \downarrow & & \cong \downarrow & & \downarrow \text{id}_{A[1]} \\
 A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1]
 \end{array}$$

**Notation** for top of triangle with base  $u$  :  $\text{cone}(u)$  (determined up to iso).

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## Additivity

**Proposition** A pretriangulated category is additive.

**Proof** Consider a distinguished triangle

$$\begin{array}{ccc}
 & C & \\
 w \swarrow & & \searrow v \\
 A & \xrightarrow{0} & B
 \end{array}
 \quad (1)$$

One proves

- $v$  is split mono,  $w$  is split epi.
- $C \cong B \oplus A[1]$

**Definition** A distinguished triangle where one of the arrows is zero is called a *split* triangle.

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## Coproducts

**Proposition** If there are distinguished triangles indexed by  $i \in I$

$$A_i \xrightarrow{u_i} B_i \xrightarrow{v_i} C_i \xrightarrow{w_i} A_i[1]$$

such that

$$\bigoplus_i A_i, \quad \bigoplus_i B_i, \quad \bigoplus_i C_i$$

exist (e.g. if  $I$  is finite), then

$$\bigoplus_i A_i \xrightarrow{\bigoplus_i u_i} \bigoplus_i B_i \xrightarrow{\bigoplus_i v_i} \bigoplus_i C_i \xrightarrow{\bigoplus_i w_i} \bigoplus_i A_i[1]$$

is distinguished.

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## Localizing Öre set

$\mathcal{D}$  pretriangulated.

$S \subset \text{Maps}(\mathcal{D})$ : Öre set.

**Definition**  $S$  is *localizing* if the following holds:

(LOC1)  $s \in S \Leftrightarrow s[1] \in S$ .

(LOC2) If in (TR3)  $f, g \in S$  then  $h$  may be chosen in  $S$  as well.

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## Localization of pretriangulated categories

$S \subset \text{Maps}(\mathcal{D})$  localizing Öre set.

$S^{-1}\mathcal{D}$  is a graded category.

$$\left( \begin{array}{ccc}
 & X_1 & \\
 s \in S \nearrow & & \nwarrow f \\
 X & & Y
 \end{array} \right) [1] = \begin{array}{ccc}
 & X_1[1] & \\
 s[1] \nearrow & & \nwarrow f[1] \\
 X[1] & & Y[1]
 \end{array}$$

**Definition** A triangle in  $S^{-1}\mathcal{D}$  is distinguished if it is isomorphic (in  $S^{-1}\mathcal{D}$ ) to the image of a distinguished triangle in  $\mathcal{D}$ .

**Proposition**  $S^{-1}\mathcal{D}$  is pretriangulated.

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### Note on functors

**Definition** An additive functor  $F : \mathcal{D} \rightarrow \mathcal{E}$  between pretriangulated categories is *exact* if it sends distinguished triangles to distinguished triangles.

**Example** If  $S$  is localizing in  $\mathcal{D}$  then the functor

$$Q : \mathcal{D} \rightarrow S^{-1}\mathcal{D}$$

is exact.

### The derived category is pretriangulated

**Proposition**  $S_{q.i.} \subset K(\mathcal{A})$  is localizing.

**Corollary** The derived category

$$D(\mathcal{A}) = S_{q.i.}^{-1}K(\mathcal{A})$$

is pretriangulated.

### The homotopy category is pretriangulated

**Definition** A triangle in  $K(\mathcal{A})$  is distinguished if it is isomorphic to a *standard triangle* of the form

$$A \xrightarrow{f} B \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \text{cone}(f) \xrightarrow{(1 \ 0)} A[1]$$

**Proposition** With this choice of distinguished triangles  $K(\mathcal{A})$  is pretriangulated.

### Exact sequence of complexes

**Theorem** Assume that

$$0 \longrightarrow A \begin{array}{c} \xrightarrow{u} \\ \leftarrow \text{---} p \end{array} B \begin{array}{c} \xrightarrow{v} \\ \leftarrow \text{---} q \end{array} C \longrightarrow 0$$

is an exact sequence in  $C(\mathcal{A})$ , split in  $\text{Gr}(\mathcal{A})$ .

$$vq = 1_C \quad vu = 0$$

$$pu = 1_A \quad pq = 0$$

$$up + qv = 1_B$$

Then

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{-p d_B q} A[1]$$

is a distinguished triangle in  $K(\mathcal{A})$ .

## Exact sequences of complexes II

**Theorem** Assume that

$$0 \rightarrow A \xrightarrow{u} B \xrightarrow{v} C \rightarrow 0$$

is an exact sequence in  $C(\mathcal{A})$ .

There is a (canonical) map

$$C \xrightarrow{w} A[1] \quad \text{in } D(\mathcal{A})$$

such that

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is a distinguished triangle in  $D(\mathcal{A})$ .

**Proof** Use the functoriality of  $\text{cone}(-)$  in  $C(\mathcal{A})$ .

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \longrightarrow & \text{cone}(u) & \xrightarrow{p} & A[1] \\ \downarrow & & \downarrow v & & \downarrow w' & & \downarrow \\ 0 & \longrightarrow & C & \xrightarrow{\text{id}_C} & C & \longrightarrow & 0 \\ & & & & (\text{cone}(0)) & & \end{array}$$

The long exact sequences for homology yields that  $w' \in S_{q.i.}$

**Take**  $w = p \circ (w')^{-1}$  201

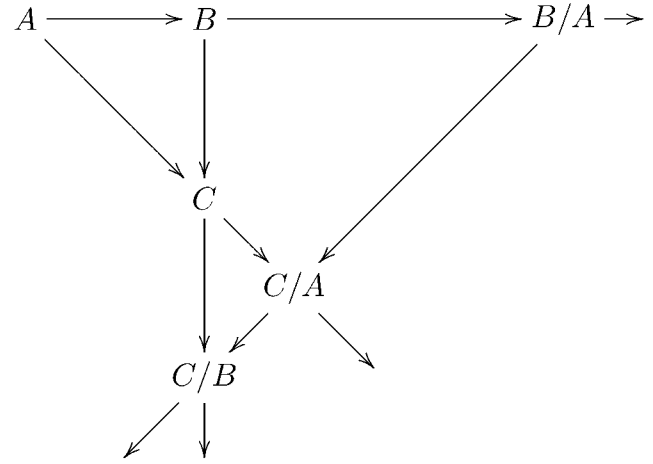
## The octahedral axiom: motivation

Assume we have monomorphisms in  $C(\mathcal{A})$ .

$$A \hookrightarrow B \hookrightarrow C$$

split in  $\text{Gr}(\mathcal{A})$ .

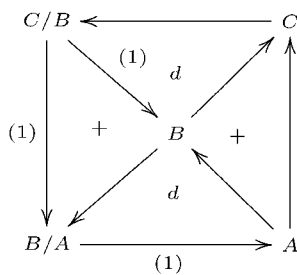
This yields 4 triangles in  $K(\mathcal{A})$ .



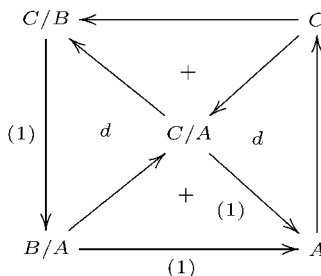
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## More symmetric representation: the octahedron

**Top**



**Bottom**



$+$  = commutative,  $d$  = distinguished.

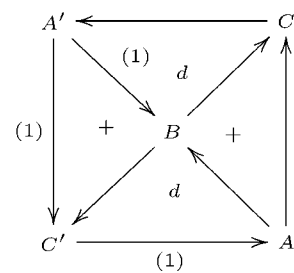
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## Octahedra in pretriangulated categories

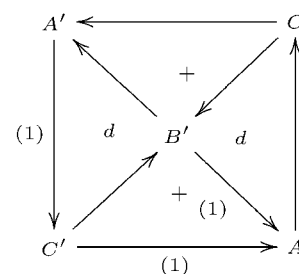
$\mathcal{D}$  pretriangulated.

**Definition** An octahedron in  $\mathcal{D}$  is a diagram of the form

**Top**



**Bottom**



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## Triangulated categories

**Definition** A pretriangulated category  $\mathcal{D}$  is *triangulated* if the following axiom holds

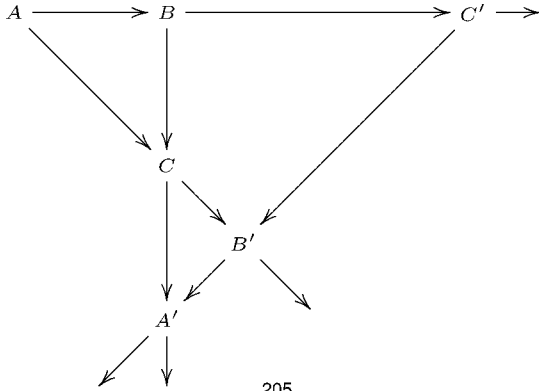
(TR4) Any two consecutive maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

may be completed to an octahedron.

**Remark** The *top* of the octahedron is determined up to isomorphism by  $f, g$ .

**Remark** The most useful part of the octahedron remains



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## Homotopy category and the derived category

**Theorem** The homotopy category is triangulated.

Proof is an easy, but tedious verification.

**Corollary** The derived category is triangulated.

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## Localization of pretriangulated categories

**Theorem** If  $\mathcal{D}$  is triangulated and  $S \subset \text{Maps}(\mathcal{D})$  is localizing then so is  $S^{-1}\mathcal{D}$ .

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## Subcategories of the derived category

$\mathcal{A}$  abelian category,  $\mathcal{A}' \subset \mathcal{A}$  a full abelian (i.e. closed under  $\ker, \text{coker}$ ) subcategory of  $\mathcal{A}$ .

The following full subcategories of  $D(\mathcal{A})$  inherit its triangulated structure.

$$D^+(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } i \ll 0\}$$

$$D^-(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } i \gg 0\}$$

$$D^b(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ for } |i| \gg 0\}$$

and for  $*$   $=, +, -, b$ .

$$D_{\mathcal{A}'}^*(\mathcal{A}) = \{A \in D^*(\mathcal{A}) \mid \forall i : H^i(A) \in \mathcal{A}'\}$$

**Remark** There is an obvious exact functor

$$D^*(\mathcal{A}') \rightarrow D_{\mathcal{A}'}^*(\mathcal{A})$$

In general this functor is neither full nor faithful.

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## Truncation functors

$$\tau_{\leq n} : C(\mathcal{A}) \rightarrow C(\mathcal{A})$$

### Definition

$$\begin{aligned} \tau_{\leq n}(\cdots \rightarrow A_{n-1} \rightarrow A_n \xrightarrow{d_n} A_{n+1} \rightarrow \cdots) \\ = \cdots \rightarrow A_{n-1} \rightarrow \ker d_n \rightarrow 0 \rightarrow 0 \rightarrow \cdots \end{aligned}$$

### One has

$$H^i(\tau_{\leq n} A) = \begin{cases} H^i(A) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

**Hence :**  $\tau_{\leq n}$  preserves quasi-isomorphisms. So there is a corresponding functor.

$$\tau_{\leq n} : D(\mathcal{A}) \rightarrow D(\mathcal{A})$$

### Dual truncation functor

$$\begin{aligned} \tau_{\geq n}(\cdots \rightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \rightarrow A_{n+1} \rightarrow \cdots) \\ = \cdots 0 \rightarrow 0 \rightarrow \operatorname{coker} d_{n-1} \rightarrow A_{n+1} \rightarrow \cdots \end{aligned}$$

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## Left closed objects

$S \subset \operatorname{Maps}(\mathcal{C}) : \text{Öre.}$

### Definition

- (1)  $A \in \operatorname{Ob}(\mathcal{C})$  is *left closed* if  $\operatorname{Hom}_{\mathcal{C}}(A, -)$  sends elements of  $S$  to isomorphisms.
- (2)  $\mathcal{C}$  has enough left closed objects if

$$\forall A \in \mathcal{C} : \exists s : A' \rightarrow A \in S$$

such that  $A'$  is left closed.

**Theorem** Assume that  $\mathcal{C}$  has enough left closed objects. Let  $\mathcal{C}^{\text{l.c.}}$  be the corresponding category. Then the inclusion

$$\mathcal{C}^{\text{l.c.}} \rightarrow \mathcal{C}$$

induces an equivalence

$$Q | \mathcal{C}^{\text{l.c.}} : \mathcal{C}^{\text{l.c.}} \rightarrow S^{-1}\mathcal{C}$$

**In particular :**  $S^{-1}\mathcal{C}$  may be identified with a full subcategory of  $\mathcal{C}$ .

**Hence :** “Hom”-sets in  $S^{-1}\mathcal{C}$  are actually sets.

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## Naive truncation functors

$$\tau_{\leq n} : C(\mathcal{A}) \rightarrow C(\mathcal{A})$$

### Definition

$$\begin{aligned} \sigma_{\leq n}(\cdots \rightarrow A_{n-1} \rightarrow A_n \rightarrow A_{n+1} \rightarrow \cdots) \\ = \cdots \rightarrow A_{n-1} \rightarrow A_n \rightarrow 0 \rightarrow 0 \rightarrow \cdots \end{aligned}$$

**Note**  $\sigma_{\leq n}$  does not preserve quasi-isomorphisms and hence does not define a functor on the derived category.

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## Left closed objects II

Assume that  $\mathcal{C}$  has enough left closed sets.

**Fact**  $(A', s)$  as in (2) is unique up to unique isomorphism.

Pick a representant

$$LA \xrightarrow{s_A} A$$

(we call this a (left) “resolution” of  $A$ ).

**Fact II**  $L$  can be made functorial using the following commutative diagram.

$$\begin{array}{ccc} LA & \xrightarrow{s_A} & A \\ \exists! \downarrow Lf & & \downarrow f \\ LB & \xrightarrow{s_B} & B \end{array}$$

**Formula :**  $\operatorname{Hom}_{S^{-1}\mathcal{C}}(A, B) = \operatorname{Hom}_{\mathcal{C}}(LA, B)$ .

**Principle :** Hom’s in localized categories may be computed using resolutions.

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### Left closed objects III

**One proves :** The functor

$$L : \mathcal{C} \rightarrow \mathcal{C}^{\text{l.c}}$$

sends the elements of  $S$  to isomorphisms.

The associated functor

$$S^{-1}\mathcal{C} \rightarrow \mathcal{C}^{\text{l.c}}$$

is an equivalence, and a quasi-inverse to the earlier functor

$$Q | \mathcal{C}^{\text{l.c}} : \mathcal{C}^{\text{l.c}} \rightarrow S^{-1}\mathcal{C}$$

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### Left closed objects in $K^-(\mathcal{A})$

**One proves** (using truncation functors)

$$D^-(\mathcal{A}) = S_{\text{q.i.}}^{-1}K^-(\mathcal{A})$$

**Proposition** Assume that  $\mathcal{A}$  has enough projective. Then for any  $A \in C^-(\mathcal{A})$  there exists a quasi-isomorphism

$$P \rightarrow A$$

such that  $P$  is right bounded complex consisting of projectives

**Hence**  $K^-(\mathcal{A})$  has enough left closed objects.

**Notation**  $K^-(P(\mathcal{A}))$  : full subcategory of  $K^-(\mathcal{A})$  of complexes consisting of projectives.

**Corollary**  $D^-(\mathcal{A}) \cong K^-(P(\mathcal{A}))$ .

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### Left closed objects in $K(\mathcal{A})$

**Definition**  $A \in K(\mathcal{A})$  is *acyclic* if  $H(A) = 0$ .

**Exercise** A map is a quasi-isomorphism if and only if its cone is acyclic.

**Exercise**  $P \in K(\mathcal{A})$  is left closed is and only if, for all  $A$  acyclic we have

$$\text{Hom}_{K(\mathcal{A})}(P, A) = 0$$

**Definition** A left closed object in  $K(\mathcal{A})$  is called *homotopically projective*. Category :  $K(\mathcal{A})^{\text{h.p.}}$ .

**Define**  $C^-(\mathcal{A})$ : complexes which are zero in high degree (short: *right bounded complexes*).  
 $K^-(\mathcal{A})$ : corresponding homotopy category.

**Proposition** A right bounded complex consisting of projectives is homotopically projective.

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### Unbounded complexes

$R$  ring.

**Notation**

$$K(R) = K(\text{Mod}(R))$$

$$D(R) = D(\text{Mod}(R))$$

**Proposition**  $K(R)$  has enough homotopically projective (=left closed) objects.

**Corollary**

$$D(R) \cong K(R)^{\text{h.p}}$$

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## Warning

Not every complex consisting of projectives is homotopically projective.

**Example** Consider (for  $R = \mathbb{Z}$ )

$$A : \quad \dots \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \dots$$

This complex is *acyclic*.

If it were homotopically projective then

$$\mathrm{Hom}_{K(R)}(A, A) = 0$$

i.e.  $A = 0$  in  $K(\mathcal{A})$ .

**But then** (exercise) :  $A \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$  in  $K(R)$

$$A \otimes \mathbb{Z}/2\mathbb{Z} : \quad \dots \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \dots$$

Not acyclic and hence not zero.

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## Grothendieck categories

**Theorem** Assume that  $\mathcal{C}$  is a Grothendieck category. Then  $K(\mathcal{C})$  has enough homotopically injective (=right closed) objects.

**Corollary**  $D(\mathcal{C}) \cong K(\mathcal{C})^{\mathrm{h.i.}}$ .

Very useful theorem.

Proof is difficult !

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## Dual notions

- Terminology and notation: right closed, homotopically injective,  $K(\mathcal{A})^{\mathrm{h.i.}}$ ,  $C^+(\mathcal{A})$ ,  $K^+(\mathcal{A}) \dots$
- Left bounded complexes of injectives are homotopically injective.
- If  $\mathcal{A}$  has enough injectives then  $D^+(\mathcal{A}) \cong K^+(I(\mathcal{A}))$  where  $K^+(I(\mathcal{A}))$  is the homotopy category of left bounded injective complexes.

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## Note about Ext

$\mathcal{A}$  abelian category with enough injectives.

**Inclusion** (fully faithful).

$$\mathcal{A} \rightarrow D^+(\mathcal{A})$$

**Recall principle** Hom's in localized categories may be computed using resolutions.

$A, B \in \mathcal{A}$ . Pick an injective resolution.

$$0 \rightarrow B \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$$

**We compute**

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) &= \mathrm{Hom}_{K(\mathcal{A})}(A, E[n]) \\ &= H^n(\underline{\mathrm{Hom}}_{C(\mathcal{A})}(A, E)) \\ &= \mathrm{Ext}_{\mathcal{A}}^n(A, B) \end{aligned}$$

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## Note about essentially small categories

$\mathcal{A}$  essentially small, abelian.

**Recall** We have two definitions for  $\text{Ext}$  in  $\mathcal{A}$ .

$$\text{Ext}_{\mathcal{A}}^n(A, B) = \text{Hom}_{D(\mathcal{A})}(A, B[n])$$

and

$$\text{Ext}_{\mathcal{A}}^n(A, B) = \text{Ext}_{\text{Ind}(\mathcal{A})}^n(A, B)$$

**Fact** These definitions are equivalent.

**We know :**  $\text{Ind}(\mathcal{A})$  is a Grothendieck category, hence has enough injectives.

**One may show** that the natural functor

$$D^b(\mathcal{A}) \rightarrow D_{\mathcal{A}}^b(\text{Ind}(\mathcal{A}))$$

is fully faithful.

Thus for  $A, B \in \mathcal{A}$  we have

$$\begin{aligned} \text{Hom}_{D(\mathcal{A})}(A, B[n]) &= \text{Hom}_{D(\text{Ind}(\mathcal{A}))}(A, B[n]) \\ &= \text{Ext}_{\text{Ind}(\mathcal{A})}^n(A, B) \end{aligned}$$

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## Derived functors: introduction

$F : \mathcal{A} \rightarrow \mathcal{B}$  : additive functor between abelian categories.

**Commutative diagram**

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ \downarrow Q & & \downarrow Q \\ D(\mathcal{A}) & \xrightarrow{\quad ? \quad} & D(\mathcal{B}) \end{array}$$

**Fact** If  $F$  is not exact there will be *no*  $\xrightarrow{\quad ? \quad}$  making the diagram commutative. . .

Sometimes there is a best approximation (theory of Kan extensions).

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## Abstract left and right derived functors

**Consider**

$$\begin{array}{ccc} & S^{-1}\mathcal{C} & \\ & \nearrow Q & \searrow ? \\ \mathcal{C} & \xrightarrow{F} & \mathcal{E} \end{array}$$

The right and left derived functors  $RF$  and  $LF$  of  $F$  are determined by the conditions that there should be isomorphisms

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{E})}(F, G \circ Q) \cong \text{Hom}_{\text{Fun}(\mathcal{D}, \mathcal{E})}(RF, G)$$

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{E})}(G \circ Q, F) \cong \text{Hom}_{\text{Fun}(\mathcal{D}, \mathcal{E})}(G, LF)$$

natural in  $G$ .

**Note** Since  $RF, LF$  are representing objects for certain functors, they are unique up to unique isomorphism.

Putting  $G = RF$  ( $LF$ ) and considering  $\text{id}_{RF}$  ( $\text{id}_{LF}$ ) we obtain associated maps

$$\eta_F : F \rightarrow RF \circ Q$$

$$\eta_F : LF \circ Q \rightarrow F$$

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## Existence of right derived functors

$F : \mathcal{C} \rightarrow \mathcal{E}$  : functor.

$S \subset \text{Maps}(\mathcal{C})$  : Öre.

**Proposition** Assume that  $\mathcal{C}$  has enough right closed objects. Then  $RF$  exists and is determined by the following commutative diagram.

$$\begin{array}{ccc} & S^{-1}\mathcal{C} & \\ & \uparrow Q|_{\mathcal{C}^{r.c}} \cong & \\ R & \curvearrowright & \mathcal{C}^{r.c} \\ & \downarrow & \downarrow F|_{\mathcal{C}^{r.c}} \\ & \mathcal{C} & \xrightarrow{F} \mathcal{E} \end{array}$$

**i.e.**  $RF = (F |_{\mathcal{C}^{r.c}}) \circ R$

**Principle :** Derived functors may be computed using appropriate resolutions.

**Note :** Construction may be done in triangulated setting and yields exact functors.

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## Derived functors in $D^*(\mathcal{A})$

$F : \mathcal{A} \rightarrow \mathcal{B}$  functor between abelian categories.

$* = \emptyset, +, -, b$ .

$$\begin{array}{ccc}
 K^*(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\
 \downarrow Q & \searrow Q \circ K(F) & \downarrow Q \\
 D^*(\mathcal{A}) & \xrightarrow{?} & D(\mathcal{B})
 \end{array}$$

Look for

$$R(Q \circ K(F)) \quad L(Q \circ K(F))$$

Notation (in case of existence).

$$RF \quad LF$$

Terminology : Left and right derived functors of  $F$ .

Classical derived functors

$$R^i F = H^i(RF(-)) : \mathcal{A} \rightarrow \mathcal{B}$$

$$L_i F = H^{-i}(LF(-)) : \mathcal{A} \rightarrow \mathcal{B}$$

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## Note on classical derived functors

$F : \mathcal{A} \rightarrow \mathcal{B}$  functor between abelian categories.

Assume  $\mathcal{A}$  as enough injectives.

**Recall principle** Derived functors are computed by resolutions.

Pick  $A \in \mathcal{A}$  together with an injective resolution

$$0 \rightarrow A \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$$

Then

$$\begin{aligned}
 R^i F(A) &= H^i(R(Q \circ K(F))(A)) \\
 &= H^i((Q \circ K(F))(E)) \\
 &= H^i(K(F)(E))
 \end{aligned}$$

So the new definition of  $R^i F$  coincides with the old one.

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## Example

$F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $*$  as above.

**One obtains :**  $RF$  exists in the following cases.

- If  $* = +$  and  $\mathcal{A}$  has enough injectives.
- If  $* = \emptyset$  and  $\mathcal{A}$  is a Grothendieck category.

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## Standard derived functors : $\mathbf{RHom}$

$$\mathrm{Hom}_{\mathcal{A}}(-, -) : K(\mathcal{A})^\circ \times K(\mathcal{A}) \rightarrow K(\mathbf{Ab}) :$$

$$(A, B) \mapsto \underline{\mathrm{Hom}}_{C(\mathcal{A})}(A, B)$$

**Assume :**  $\mathcal{A}$  has enough injectives.

Pick  $A \in K(\mathcal{A})$ . Then

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(A, -) : D^+(\mathcal{A}) \rightarrow D(\mathcal{A})$$

exists ( $II =$  second factor).

**We obtain :** a bifunctor:

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(-, -) : K(\mathcal{A})^\circ \times D^+(\mathcal{A}) \rightarrow D(\mathcal{A})$$

**One shows :** for  $B \in D(\mathcal{A})$

$$R_{II}\mathrm{Hom}_{\mathcal{A}}(-, B) : K(\mathcal{A})^\circ \rightarrow D(\mathcal{A})$$

preserves standard triangles and quasi-isomorphisms.

**We obtain :** a bifunctor:

$$R_I R_{II}\mathrm{Hom}_{\mathcal{A}}(-, -) : D(\mathcal{A})^\circ \times D^+(\mathcal{A}) \rightarrow D(\mathcal{A})$$

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## RHom cont'd

**Similarly :** If  $\mathcal{A}$  has enough projectives then we obtain a bifunctor

$$R_{II}R_I\mathrm{Hom}_{\mathcal{A}}(-, -) : D^-(\mathcal{A})^\circ \times D(\mathcal{A}) \rightarrow D(\mathcal{A})$$

If  $\mathcal{A}$  has both enough injectives and projectives then

$$R_I R_{II} \mathrm{Hom}_{\mathcal{A}}(-, -) \text{ and } R_{II} R_I \mathrm{Hom}_{\mathcal{A}}(-, -)$$

coincide when restricted to  $D^-(\mathcal{A})^\circ \times D^+(\mathcal{A})$ .

### Notation

$$\begin{aligned} \mathrm{RHom}_{\mathcal{A}}(-, -) &= R_I R_{II} \mathrm{Hom}_{\mathcal{A}}(-, -) \\ &= R_{II} R_I \mathrm{Hom}_{\mathcal{A}}(-, -) \end{aligned}$$

whenever defined.

**Formula :** For  $A, B \in \mathcal{A}$  we have

$$\mathrm{Ext}_{\mathcal{A}}^i(A, B) = H^i(\mathrm{RHom}_{\mathcal{A}}(A, B))$$

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## Tensor product of complexes

$$A \in C(R^\circ), B \in C(R).$$

$$(A \otimes_R B)_n = \bigoplus_i A_i \otimes_R B_{n-i}$$

**Differential :**

$$d(a \otimes b) = da \otimes b + a \otimes (-1)^{|a|} db$$

**Fact :** As for RHom, we can derive  $- \otimes_R -$  in both arguments to obtain a bifunctor.

$$- \overset{L}{\otimes}_R - : D(R^\circ) \times D(R) \rightarrow D(\mathbf{Ab})$$

**Formula :** For  $A$  a right and  $B$  a left  $R$ -module we have

$$\mathrm{Tor}_i^R(A, B) = H^{-i}(A \overset{L}{\otimes}_R B)$$

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## RHom variation

If  $\mathcal{A}$  is a Grothendieck category then  $K(\mathcal{A})$  has enough homotopically injective (=right closed) objects.

**Hence** We may define  $\mathrm{RHom}_{\mathcal{A}}(-, -)$  as a bifunctor.

$$D(\mathcal{A})^\circ \times D(\mathcal{A}) \rightarrow D(\mathcal{A})$$

without any restriction.

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## t-structures: motivation

$\mathcal{A}$  abelian category.

**Define**

$$D^{\leq n}(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ if } i > n\}$$

$$D^{\geq n}(\mathcal{A}) = \{A \in D(\mathcal{A}) \mid H^i(A) = 0 \text{ if } i < n\}$$

**We have**

$$\mathcal{A} = D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$$

**Idea :** Make this abstract for general triangulated categories in order to recognize abelian subcategories.

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## t-structures: definition

$\mathcal{D}$  triangulated category.

**Definition** A  $t$ -structure on  $\mathcal{D}$  is a pair of full additive subcategories  $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$  closed under isomorphism such that (putting  $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ ,  $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ )

- If  $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$  then

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$$

- $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}, \mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$ .
- For every  $X \in \mathcal{D}$  there is a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow$$

with  $A \in \mathcal{D}^{\leq 0}, B \in \mathcal{D}^{\geq 1}$ .

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## t-structures: the heart

$\mathcal{D}$  triangulated category with  $t$ -structure.

**Definition** The intersection

$$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$

is called the *heart* of the  $t$ -structure.

**Main theorem** The heart is an abelian category.

The proof uses truncation functors for arbitrary  $t$ -structures (see below).

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## t-structures: the derived category

The axioms hold for

$$\mathcal{D} = D(\mathcal{A})$$

with

$$\mathcal{D}^{\leq 0} = D^{\leq 0}(\mathcal{A}), \quad \mathcal{D}^{\geq 0} = D^{\geq 0}(\mathcal{A})$$

**Key points in proof**

- If  $A \in D^{\leq n}(\mathcal{A})$  one has

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, \tau_{\leq n} B)$$

- If  $B \in D^{\geq n}(\mathcal{A})$  one has

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(\tau_{\geq n} A, B)$$

- For  $X$  in  $C(\mathcal{A})$  the natural map

$$X/\tau_{\leq 0} X \rightarrow \tau_{\geq 1} X$$

is a quasi-isomorphism, so there is a distinguished triangle

$$\tau_{\leq 0} X \rightarrow X \rightarrow \tau_{\geq 1} X \rightarrow$$

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## t-structure: truncation functors

$\mathcal{D}$  triangulated category with  $t$ -structure.

**One proves** For  $X \in \mathcal{D}$  the triangle

$$A \rightarrow X \rightarrow B \rightarrow$$

with  $A \in \mathcal{D}^{\leq 0}, B \in \mathcal{D}^{\geq 1}$  is unique up to unique isomorphism.

**One puts**

$$\tau_{\leq 0} X \stackrel{\mathrm{def}}{=} A$$

$$\tau_{\geq 1} X \stackrel{\mathrm{def}}{=} B$$

and in general

$$\tau_{\leq n} X = (\tau_{\leq 0}(X[n]))[-n]$$

$$\tau_{\geq n} X = (\tau_{\geq 1}(X[n-1]))[-n+1]$$

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## t-structures: perverse cohomology

$\mathcal{D}$  triangulated category with t-structure and heart  $\mathcal{A}$ .

**Definition** The *perverse cohomology* of  $A \in \mathcal{D}$  is defined as

$$\begin{aligned} {}^p H^0(A) &= \tau_{\geq 0} \tau_{\leq 0} A \\ {}^p H^n(A) &= {}^p H^0(A[n]) \end{aligned}$$

**Note :**  $H^0(A)$  lies in  $\mathcal{A}$ .

**Theorem** If we have a distinguished triangle then  ${}^p H^0(-)$  applied to the corresponding helix yields a long exact sequence.

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## Example of tilting

$\mathcal{A}$  the abelian category of finitely generated abelian groups.

$\mathcal{D} = D(\mathcal{A})$  with its canonical t-structure.

**Define** a torsion theory on  $\mathcal{A}$  by

$$\begin{aligned} \mathcal{T} &= \{\text{torsion groups}\} \\ \mathcal{F} &= \{\text{torsion free groups}\} \end{aligned}$$

Then the heart of the tilted t-structure is represented by complexes of length 2

$$0 \rightarrow A_0 \xrightarrow{d} A_1 \rightarrow 0$$

such that  $\ker d \in \mathcal{F}$ ,  $\text{coker } d \in \mathcal{T}$ .

**One may show :** The heart of the tilted t-structure is equivalent to  $\mathcal{A}^\circ$ .

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## New t-structures from old: tilting

$\mathcal{D}$  triangulated category with t-structure and heart  $\mathcal{A}$ .

**Definition** A torsion theory in an abelian category  $\mathcal{B}$  is a pair of additive full subcategories  $(\mathcal{T}, \mathcal{F})$  such that

- For all  $T \in \mathcal{T}, F \in \mathcal{F}: \text{Hom}_{\mathcal{B}}(T, F) = 0$ .
- For all  $B \in \mathcal{B}$  there is a (necessarily unique) exact sequence

$$0 \rightarrow T \rightarrow B \rightarrow F \rightarrow 0$$

with  $T \in \mathcal{T}, F \in \mathcal{F}$ .

**Theorem** Let  $(\mathcal{T}, \mathcal{F})$  be a torsion theory in  $\mathcal{A}$ . Define

$$\begin{aligned} {}'\mathcal{D}^{\leq 0} &= \{D \in \mathcal{D}^{\leq 1} \mid {}^p H^1(D) \in \mathcal{T}\} \\ {}'\mathcal{D}^{\geq 0} &= \{D \in \mathcal{D}^{\geq 0} \mid {}^p H^0(D) \in \mathcal{F}\} \end{aligned}$$

Then  $({}'\mathcal{D}^{\leq 0}; {}'\mathcal{D}^{\geq 0})$  defines a new t-structure on  $\mathcal{D}$ .

**Terminology :** The new t-structure is called a *tilted* t-structure.

**Mental picture :** Walking around in  $\mathcal{D}$  by tilting produces new abelian categories from old.

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## Hereditary t-structures

$\mathcal{D}$  t-structure with heart  $\mathcal{A}$ .

**Assume** the following conditions on the t-structure.

- (*Non-degeneracy*) If  ${}^p H^n(D) = 0$  for all  $n$  then  $D = 0$ .
- (*Boundedness*) For every  $D$ , at most a finite number of  ${}^p H(D)$  are non-zero.
- (*Hereditary*) For every  $A, B \in \mathcal{A}$  we have

$$\text{Hom}_{\mathcal{D}}(A, B[n]) = 0 \text{ for } n \geq 2$$

**Note :**  $\text{Hom}_{\mathcal{D}}(A, B[n]) = 0$  for  $n < 0$ .

**Theorem** Under the above conditions for every  $D \in \mathcal{D}$  we have

$$D \cong \bigoplus_i H^i(D)[-i]$$

**Note :** The theorem applies if  $\mathcal{D} = D^b(\mathcal{A})$  where  $\mathcal{A}$  is a *hereditary abelian category*, i.e. an abelian category such that  $\text{Ext}_{\mathcal{A}}^i(-, -) \neq 0$  for  $i \neq 0, 1$ .

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**Sketch of proof**

We pretend that perverse homology behaves exactly like homology of complexes. This is true!

For  $D \in \mathcal{D}$  define  $l(D)$  as the number of non-zero homology objects. We will perform induction on  $l(D)$ . The key point is that if  ${}^p H^n(D)$  is the lowest cohomology group then there is a distinguished triangle

$${}^p H^n(D)[-n] \rightarrow D \rightarrow \tau_{\geq n+1} D \rightarrow$$

and  $l(\tau_{\geq n+1} D) = l(D) - 1$ .

**Step 1**

Let  $D \in \mathcal{D}^{\geq 2}$  and  $A \in \mathcal{A}$ . We claim that  $\text{Hom}_{\mathcal{D}}(D, A) = 0$

Apply to  $\text{Hom}_{\mathcal{D}}(-, A)$  to

$${}^p H^n(D)[-n] \rightarrow D \rightarrow \tau_{\geq n+1} D \rightarrow \quad (*)$$

where  $n \geq 2$ . The vanishing of  $\text{Hom}_{\mathcal{D}}^n(-, -)$  for  $n \geq 2$  implies that

$$\text{Hom}_{\mathcal{D}}(\tau_{\geq n+1} D, A) \rightarrow \text{Hom}_{\mathcal{D}}(D, A)$$

is epi. We finish by induction.

**Step 2**

Let  $D \in \mathcal{D}$  be arbitrary. It follows from Step 1 that in the triangle (\*) the map

$$\tau_{\geq n+1} D \rightarrow {}^p H^n(D)[-n][1]$$

is zero. Thus the triangle is split.

We obtain

$$D \cong {}^p H^n(D)[-n] \oplus \tau_{\geq n+1} D$$

We finish by induction.