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Advanced School and Conference on Non-commutative Geometry

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Algebraic geometry

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Commutative geometry
Trieste summer school

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Aug 12, 2004

Schedule

1. Introduction

2. More introduction

3. Schemes, **Spec**, and **Proj**

4. **Spec** and **Proj** for modules

5. Projective space and line bundles

6. Vector bundles

7. Elliptic curves

8. Vector bundles and stability

First lecture

Algebraic geometry

Algebraic geometry:

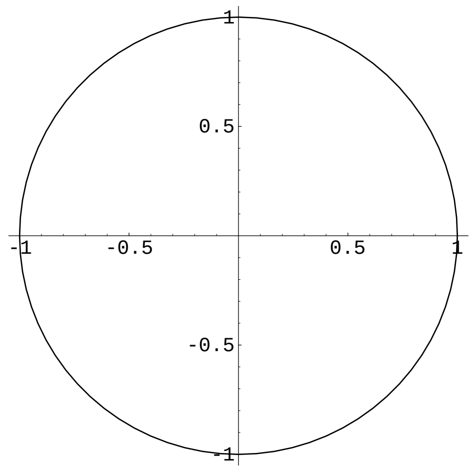
Part of general geometry; analytic, differential, projective geometry, topology.

Algebraic geometry:

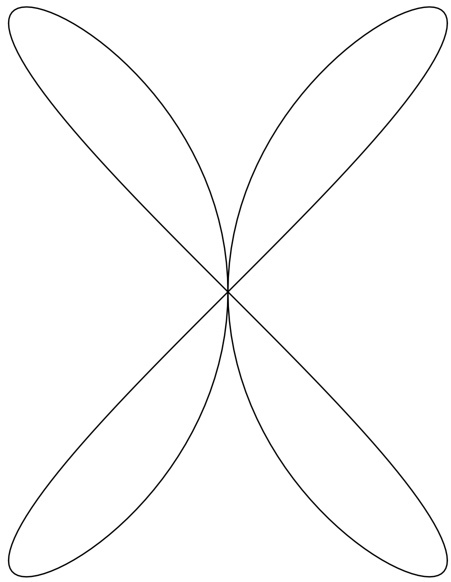
- "Controlled" nice geometry (vs. e.g. fractal geometry).
- (Can use) purely algebraic methods.
- $\Rightarrow$ Applications to arithmetic, non-commutative algebraic geometry. . .

Geometry $\longleftrightarrow$ Algebra

$x^2 + y^2 = 1$:



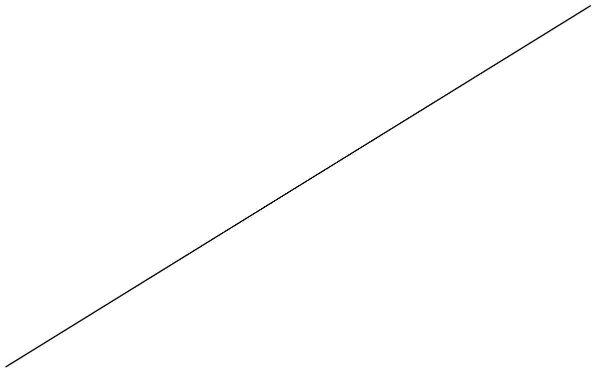
$(x^2 - y^2)x^2 + y^6 = 0$:



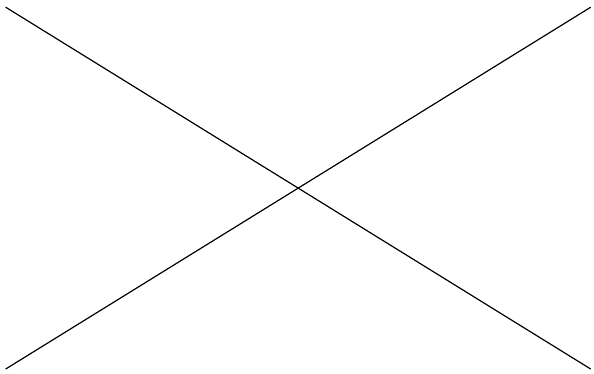
How to tell shape from equation?

Linear algebra

- Line: $ax + by = c$

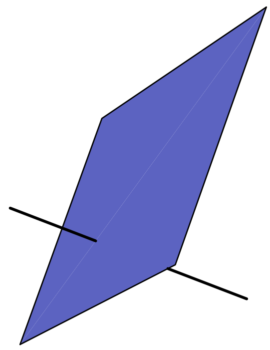


- Lines: $\left\{ \begin{array}{rcl} ax + by & = & c \\ a'x + b'y & = & c' \end{array} \right.$

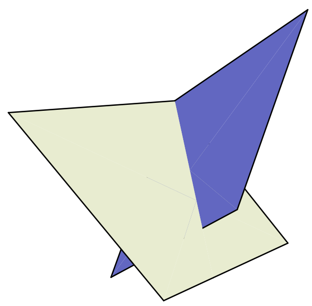


These equations can easily be solved algebraically.

- Line-plane: $\begin{cases} \begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases} \\ ax+by+cz=d \end{cases}$



- Plane-plane: $\begin{cases} a_1x+b_1y+c_1z = d_1 \\ a_2x+b_2y+c_2z = d_2 \end{cases}$



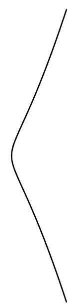
Drawing curves, real case

Real solutions: $y^2 = f(x)$

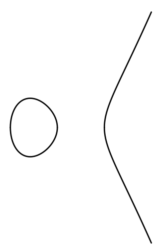
Solutions: $y = \begin{cases} 0 & \text{if } f(x) = 0 \\ \pm\sqrt{f(x)} & \text{if } f(x) > 0 \\ - & \text{if } f(x) < 0 \end{cases}$

- $x^2 + y^2 = 1 \iff$
  $y^2 = 1 - x^2 = (1-x)(1+x)$,
  $x \in [-1, 1]$.

- $y^2 = x^3 - 1 = (x-1)((x+1/2)^2 + 3/4)$,
  $x \in [1, \infty[$.



- $y^2 = x^3 - x = (x-1)x(x+1)$,
  $x \in [-1, 0] \cup [1, \infty[$.



Drawing curves, complex case

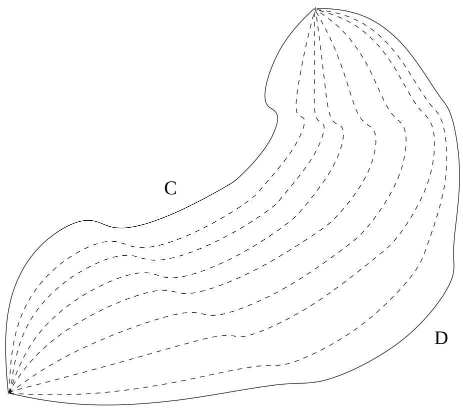
$y^2 = x$, $x, y \in \mathbf{C}$ *always* solvable.

Furthermore:

Lemma 0.1 *Let* $f\colon U \to \mathbf{C}^*$ *be a continuous from an open subset of* $\mathbf{C}$ *and* $\gamma\colon [a, b] \to U$ *also continuous. Then for any* $c \in \mathbf{C}$ *with* $c^2 = f(\gamma(a))$ *there is a unique continuous* $\rho\colon [a, b] \to \mathbf{C}^*$ *with*

- $\rho^2(t) = f(\gamma(t))$ *for all* $t \in [a, b]$*,*
- $\rho(a) = c$*.*

End point homotopic curves C and D:



Lemma 0.2 *If f as in Lemma 0.1, $\gamma, \gamma': [a,b] \to U$ are* end point homotopic *and $\rho^2 = \gamma$, $\rho'^2 = \gamma'$ with $\rho(a) = \rho'(a)$ then $\rho(b) = \rho'(b)$.*

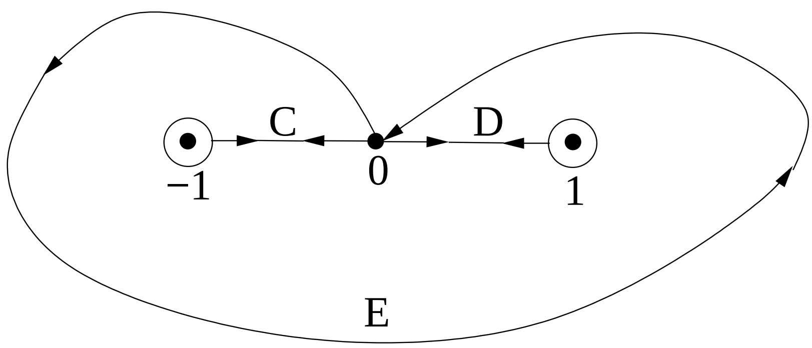
$y^2 = 1 - x^2$, $x, y \in \mathbf{C}$:

$1 - x^2 \colon \mathbf{C} \setminus \{\pm 1\} \to \mathbf{C}^*$

For basepoint choose 0.

$y \colon \mathbf{C} \setminus \{\pm 1\} \to \mathbf{C}^*$ well defined once starting value ± 1 and a homotopy class of paths from 1 to x has been chosen.

$\pm 1 \to \mp 1 \Rightarrow y \to -y$



Choose $y(0) = 1$.

$y(D)$:

Around 1, $x - 1 = e^{i\theta}$, $\theta \in [0, 2\pi]$, so $\sqrt{x-1} = e^{i\theta/2}$.

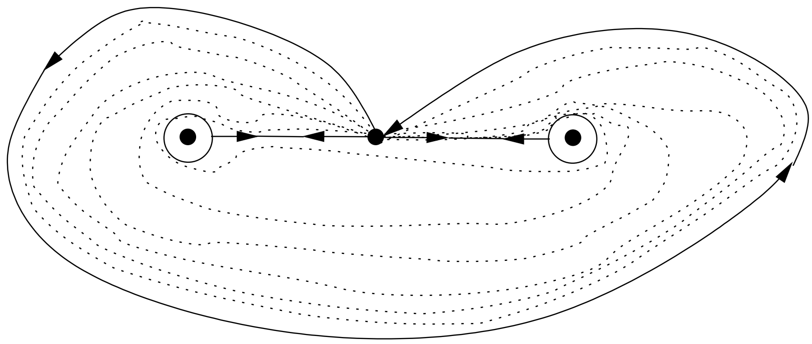
$-x - 1 = (\sqrt{-x-1})^2$

Once around gives $e^{i2\pi/2} = -1$ and hence the square root changes sign.

Back and forth gives same result so that $\boxed{y(D) = -1}$

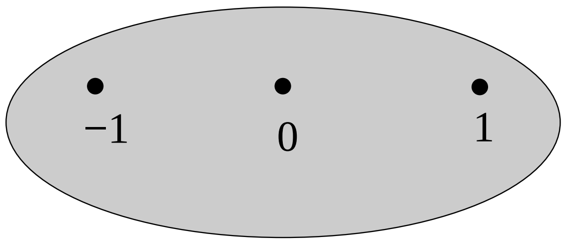
Similarly $\boxed{y(C) = -1}$.

Now E is homotopic to the composite, $D \circ C$, of C and D.



Hence $y(E) = 1$.

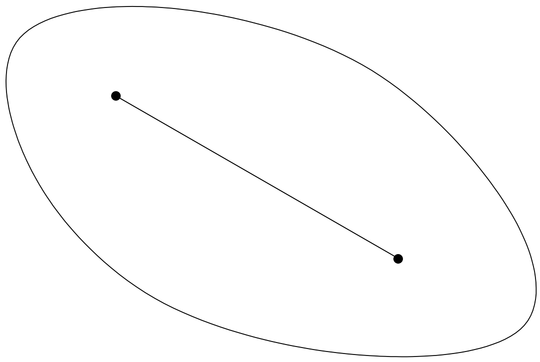
This gives that outside of the cutout, $|z| \geq 2$ say,



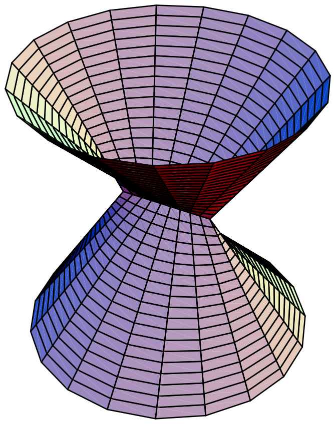
$x^2 - 1$ has two continuous square roots.

Therefore, the subset of the curve $\{(x, y) \in \mathbf{C}^2 \mid x^2 + y^2 = 1\}$ for which $|x| \geq 2$ consists of two copies of $|z| \geq 2$.

What about the inside $|z| \leq 2$? Cut up the disc along $[-1, 1]$. Again the square roots are well-defined away from the cut but are interchanged when passing through the cut.



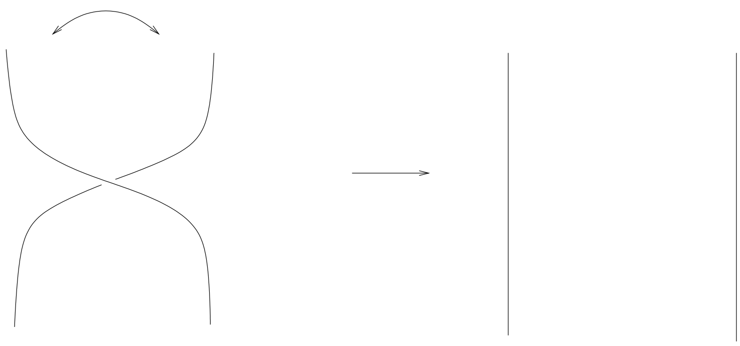
giving as inverse image something like



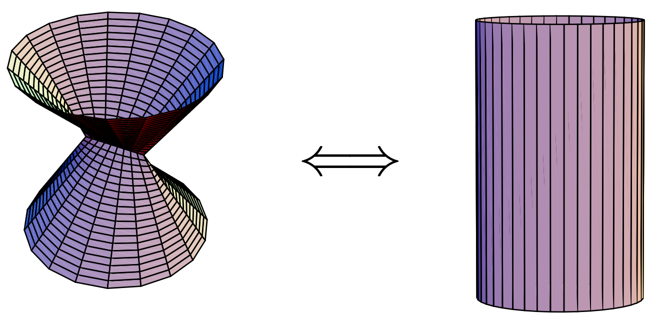
Note that the seeming intersection in the middle is no intersection.

The top part may be pushed through itself.

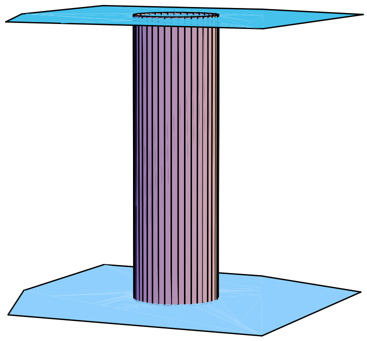
Cross section:



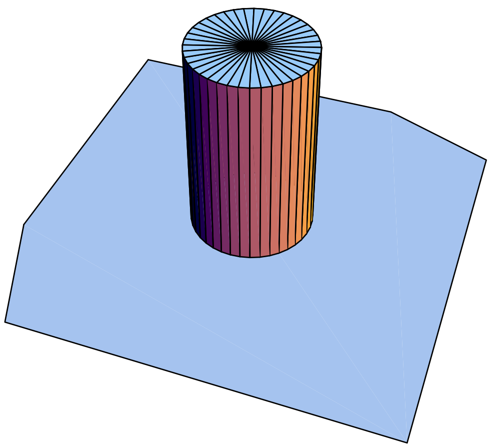
resulting in a cylinder:



Putting all together gives



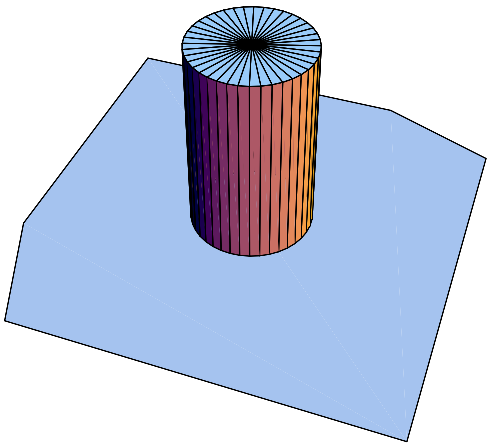
and “folding” the top plane gives a cap of the cylinder



and pushing the cylinder down gives a *punctured* plane (centre missing).

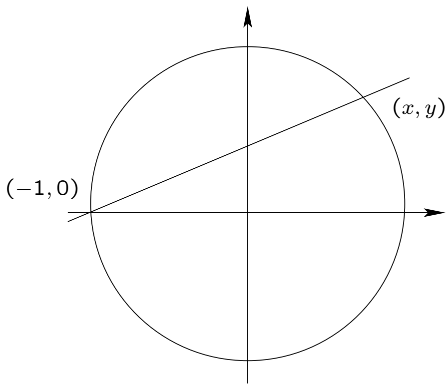
Second lecture

Shape of the curve
$\{(x, y) \in \mathbf{C} \mid x^2 + y^2 = 1\}$:



"Topologically" the punctured plane,
$\mathbf{C} \setminus \{0\}$.
This is true also algebraically:

$$\begin{cases} x^2 + y^2 = 1 \\ y = k(x+1) \end{cases}$$



Solution:

$$k \mapsto \left(\frac{1-k^2}{1+k^2}, \frac{2k}{1+k^2} \right)$$

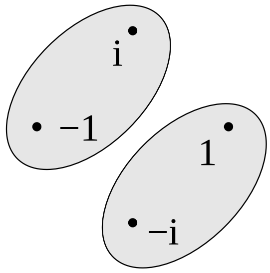
$$(x, y) \mapsto \frac{y}{x+1}$$

Missing points!

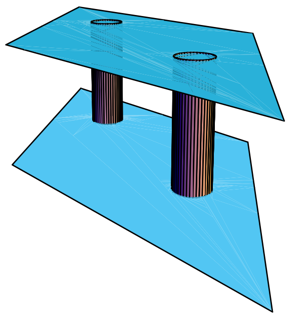
$\left\{ {k=\pm i \atop (x,y)=(-1,0)} \right. \rightsquigarrow$ **Projective geometry**

$$y^2 = x^4 - 1 = (x \pm 1)(x \pm i)$$

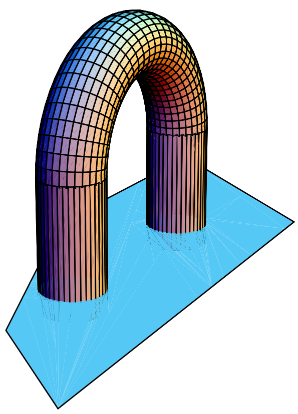
Two cutouts:



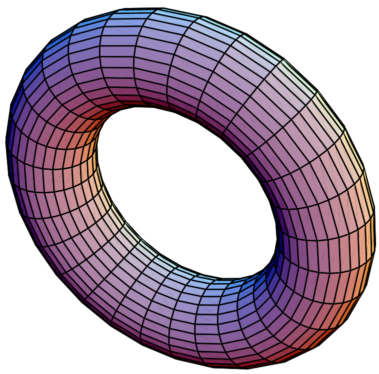
Hence



Folding top plane:

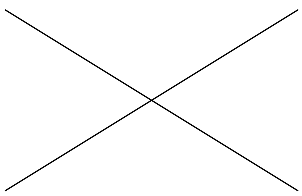


Folding also bottom plane gives,

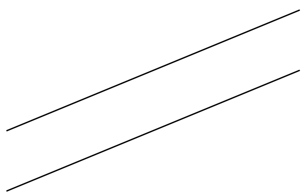


i.e., a torus *with two points missing*.

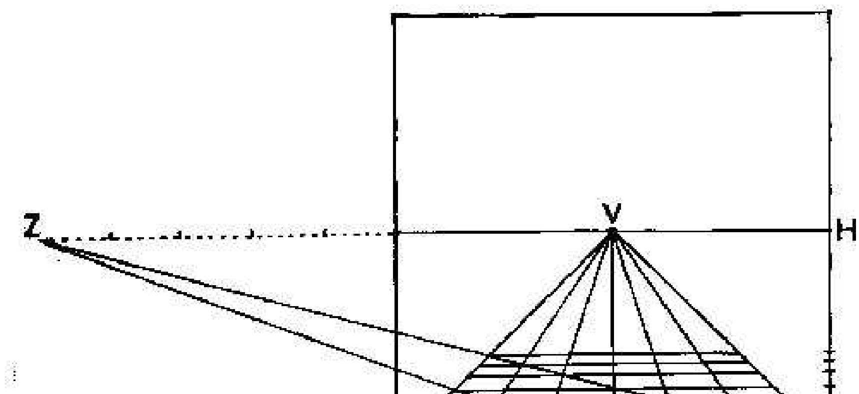
The story of the missing points



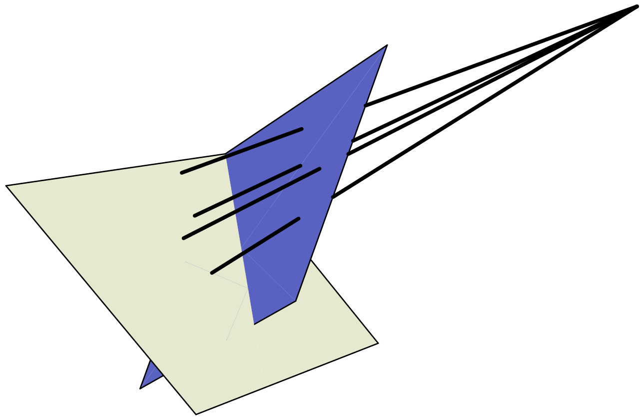
vs.



Laws of perspective (Alberti, 1404–1472).



Central projection:



Parametrising a plane in K^3 (K a field):

$$\begin{array}{ccc} K^2 & \longrightarrow & K^3 \\ (s,t) & \mapsto & s\mathbf{v_1} + t\mathbf{v_2} + \mathbf{v_3} \end{array}$$

General form for central projection between parametrised planes:

$$(s,t) \mapsto \left(\frac{a_{11}s + a_{12}t + a_{13}}{a_{31}s + a_{32}t + a_{33}}, \frac{a_{21}s + a_{22}t + a_{23}}{a_{31}s + a_{32}t + a_{33}} \right)$$

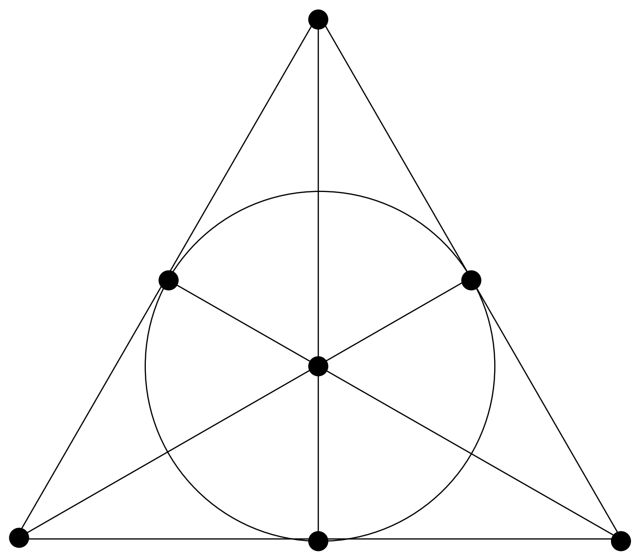
Not always defined!

For natural reasons!

Definition 0.3 *i) The* n-dimensional projective space *over a field* K*,* $\mathbf{P}^n(K)$*, is the set of* 1*-dimensional linear subspaces of* K^{n+1}*.*

ii) A k-dimensional projective subspace *of* $\mathbf{P}^n(K)$ *consists of the* 1*-dimensional linear subspaces contained in a linear subspace of dimension* $k+1$*.*

$\mathbf{P}^2(\mathbf{F}_2)$:



Homogeneous coordinates

$L \subseteq K^{n+1}$, $\dim L = 1 \Rightarrow$
$L = Kv$, $0 \neq v = (a_0, a_1, \ldots, a_n)$

$Kv = Kv' \iff$
$(a'_0, a'_1, \ldots, a'_n) = \lambda(a_0, a_1, \ldots, a_n)$, $\lambda \neq 0$

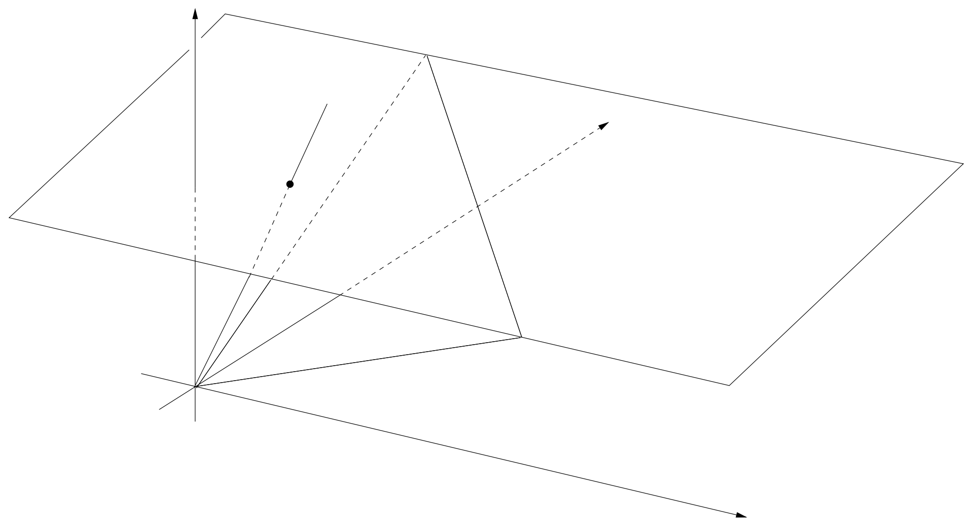
$(a_0 : a_1 : \ldots : a_n) :=$
equivalence class of $(a_0, a_1, \ldots, a_n)$

$a_0 \neq 0$:

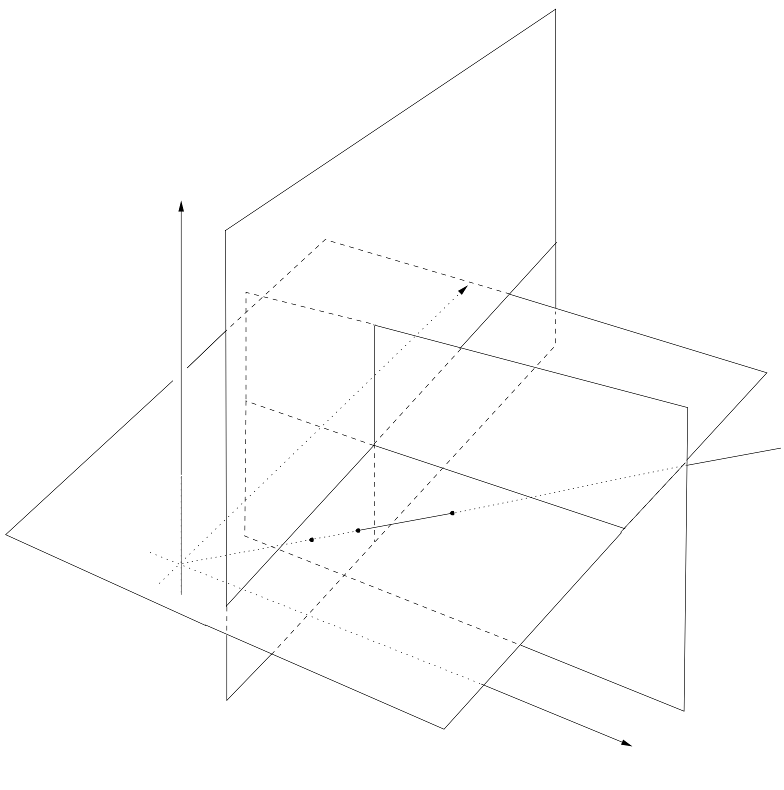
$$(a_0 : a_1 : \ldots : a_n) = (1 : a_1/a_0 : \ldots : a_n/a_0)$$

Unique! $\{(a) \in \mathbf{P}^n(K) \mid a_0 \neq 0\} = K^n$.

$z \neq 0$:



Standard affine patches:



$$(x:y:1) = (x/y:1:1/y) = (1:y/x:1/x)$$

Projective completion

Polynomials not welldefined on $\mathbf{P}^n(K)$.
$\{\mathbf{v} \in \mathbf{P}^n(K) \mid f((v)) = 0\}$ is welldefined if f is *homogeneous*, i.e.,

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n).$$

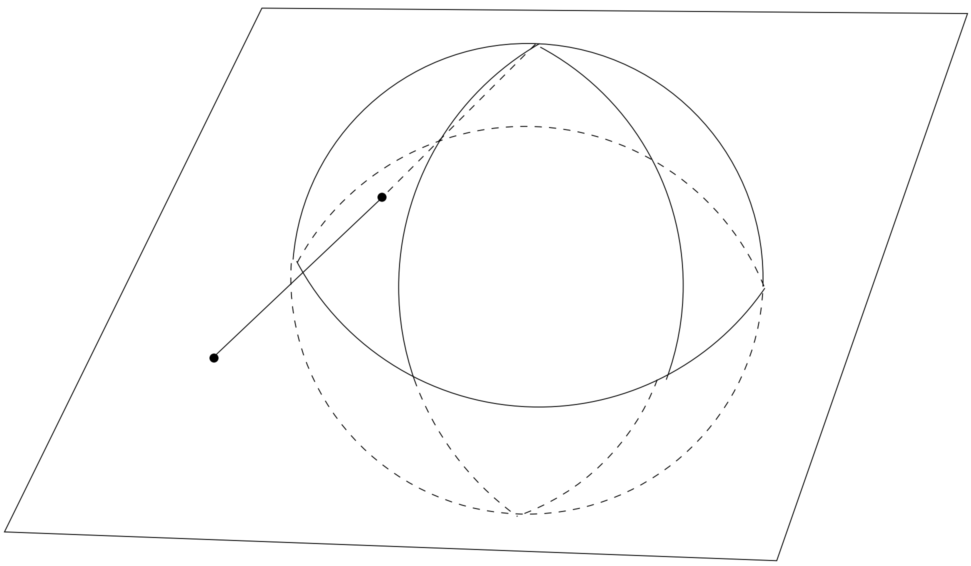
$f(x,y,z) = y^2 + x^2 - z^2$: Affine patch $z \neq 0$
$0 = f(x,y,1) = y^2 + x^2 - 1 \iff$
$y^2 + x^2 = 1$

Points at infinity:
$0 = f(x,y,0) = x^2 + y^2$, $(\pm i : 1 : 0)$
Missing points!

$(x : y : z) = (x/z : y/z : 1)$
$f(x,y,z) = z^2 f(x/z, y/z, 1)$
Adding points at ∞

The projective line

$$\mathbf{P}^1(K) = \{(x:1) \mid x \in K\} \cup \{\infty := (1:0)\}$$



$$\mathbf{P}^1(K) = \{x \mid x \in K\} \cup \{y \mid y \in K\},$$
$$y = 1/x,\ x = 1/y$$

The circle revisited

Put

$$Q := \{(x:y:z) \in \mathbf{P}^1(K) \mid x^2 + y^2 = z^2\}$$

Inverse maps:

$$\begin{array}{ccc} \mathbf{P}^1(K) & \to & Q \\ (k:\ell) & \mapsto & (k^2 - \ell^2 : 2k\ell : k^2 + \ell^2) \end{array}$$

and

$$\begin{array}{ccc} Q & \to & \mathbf{P}^1(K) \\ (x:y:z) & \mapsto & (y : x + z) \end{array}$$

NB: $(-1:0:1) \mapsto (0:0)$, not defined!
For $K = \mathbf{C}$ $(-1:0:1) \mapsto (1:0)$ gives a continuous extension.

Third lecture

Spec and all that

Affine n-dimensional space:

$$\mathbf{A}^n(K) := K^n$$

$$\begin{array}{c}\{\varphi\colon K[x_1,\ldots,x_n] \to K \mid \varphi\ K\text{-alg. hom.}\}\\ \updownarrow \\ \mathbf{A}^n(K)\end{array}$$

$\varphi \leftrightarrow (\varphi(x_1),\ldots,\varphi(x_n))$
$f \in K[x_1,\ldots,x_n]$

$f(a_1,\ldots,a_n) \iff \varphi(f) = 0 \iff \varphi$ vanishes on the ideal generated by f.

System of equations $\{f_i\} \leftrightarrow K[x_1,\ldots,x_n]/(f_i)$

Pitfalls

$x^2 = 0 \leftrightarrow K[x]/(x^2)$
$x = 0 \leftrightarrow K[x]/(x)$
$K[x]/(x^2) \ncong K[x]/(x)$

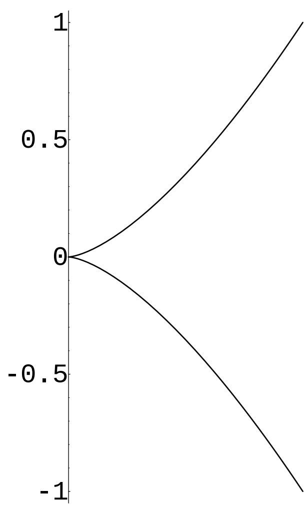
Bijection ($t = x/y$):

$$\begin{array}{ccc} \mathbf{A}^1(K) & \to & \{(x,y) \in K^2 \mid y^2 = x^3\} \\ t & \mapsto & (t^2, t^3) \end{array}$$

$K[t] \ncong K[x,y]/(y^2 - x^3)$

The "ring side" is in the right!

$y^2 = x^3$:



Schemes

> ...to hard-core pornography. I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description; and perhaps I could never succeed in intelligibly doing so. But I know it when I see it,...
>
> Justice Stewart, US Supreme Court

Affine K-schemes:
The opposite category of the category of K-algebras.
$\mathbf{Spec}\, R$ the affine K-scheme of R.

$\mathbf{A}^n$, the *affine n-space*, is $\mathbf{Spec}\, K[x_1, \ldots, x_n]$.
$\mathbf{Spec}\, K$, or generally, $\mathbf{Spec}\, L$, L a K-extension field, are points.

Definition 0.4 *If L is an extension field of K, then an L-*point *of a K-scheme X is a map $\mathbf{Spec} L \to X$. The set of L-points of X is denoted $X(L)$.*

$\mathbf{A}^n(L) = \mathbf{A}^n(L)$!!

Unqualified "scheme": The opposite category of commutative rings.

$\mathbf{Spec}\mathbf{Z}(L) \leftrightarrow \mathbf{Z} \to L$, exactly one for each L.

$\mathbf{Spec}\mathbf{Z}[x](L) \leftrightarrow \mathbf{Z}[x] \to L$:

- $\operatorname{char} L = 0$, $\mathbf{Q}[x] \to L \leftrightarrow$ $\mathbf{Q}[x]/(f(x)) \hookrightarrow L$, $f = 0$ or f irred.

- $\operatorname{char} L = p > 0$, $\mathbf{Z}/p\mathbf{Z}[x] \to L \leftrightarrow$ $\mathbf{Z}/p\mathbf{Z}[x]/(f(x)) \hookrightarrow L$, $f = 0$ or f irred.

General schemes

Remark: The actual gluing maps that can be used to construct schemes are more general than those we shall consider here.

Requirements:

- An affine scheme, $\mathbf{Spec}\, R$, is a scheme.
- An intersection of "open" affine schemes shall be affine.
- A scheme is a union of open affine schemes.

$$X = \bigcup_i U_i, \ U_i = \mathbf{Spec}\, R_i$$

$$\begin{array}{ccccc} & & \dot\cup_{ij} U_{ij} & & \\ & \swarrow & & \searrow & \\ \dot\cup_i U_i & & & & \dot\cup_j U_j \\ & \searrow & & \swarrow & \\ & & X & & \end{array} \qquad U_{ij} := U_i \cap U_j$$

$$U_{ij} = \mathbf{Spec}\, R_{ij}, \ U_{ij} \hookrightarrow U_i \leftrightarrow R_i \to R_{ij}$$

When is $U_{ij} \hookrightarrow U_i$ open?
Example: $R_{ij} = R_i[r_{ij}^{-1}]$, $r_{ij} \in R_i$.

Following seems reasonable:

$$\begin{array}{ccccc} & & \mathbf{Spec}\, K[x, x^{-1}] & & \\ & \swarrow & & \searrow & \\ \mathbf{Spec}\, K[x] & & & & \mathbf{Spec}\, K[x^{-1}] \\ & \searrow & & \swarrow & \\ & & \mathbf{P}^1 & & \end{array}$$

More precisely: Isomorphisms $\psi_{ij}\colon R_i[r_{ij}^{-1}] \to R_j[r_{ji}^{-1}]$ with compatibility condition.

$\mathbf{P}^1$:

$$\begin{array}{ccc} K[x][x^{-1}] & \to & K[y][y^{-1}] \\ x & \mapsto & y^{-1} \end{array}$$

Remark: Could have chosen $x \mapsto y$. That would identify K^* with itself but getting two zeroes. This is a *non-separated scheme*. Topologically (over $\mathbf{R}$ or $\mathbf{C}$) for instance $(1/n)$ converges to *both* zeroes.

Projective space

$U_i := \mathbf{Spec}\, K[x_0/x_i, \ldots, x_n/x_i]$,
$K[x_0/x_i, \ldots, x_n/x_i] \subset K(x_0, \ldots, x_n)$

$$U_{ij} = \mathbf{Spec}\, K[x_0/x_i, \ldots, x_n/x_i][(x_j/x_i)^{-1}]$$

ψ_{ij} given by

$$K[x_0/x_i, \ldots, x_n/x_i][(x_j/x_i)^{-1}] = K[x_0/x_j, \ldots, x_n/x_j][(x_i/x_j)^{-1}]$$

Separating the rings more
$U_i := \mathbf{Spec}\, K[x_{i0}, \ldots, \widehat{x_{ii}}, \ldots, x_{in}]$
$U_{ij} = \mathbf{Spec}\, K[x_{i0}, \ldots \widehat{x_{ii}}, \ldots, x_{in}][x_{ij}^{-1}]$
$\psi_{ij}(x_{ik}) = x_{jk}/x_{ji}$

Remark: Even more natural is to take
$U_i = \mathbf{Spec}\, K[x_{i0}, \ldots, x_{in}]/(x_{ii} - 1)$.

A K-point $\mathbf{Spec}\, K \to \mathbf{P}^n$ lands in one U_i (covering!).
$\text{Hom}(\mathbf{Spec}\, K, U_i) = \{(a_0, \ldots, \widehat{a_i}, \ldots, a_n)\}$.
Different i: $\text{Hom}(\mathbf{Spec}\, K, U_{ij}) \overset{\psi_{ij}}{=} \text{Hom}(\mathbf{Spec}\, K, U_{ji}) \Rightarrow (a_0 : \ldots : 1 : \ldots : a_n)$ welldefined.

Projective spectrum

Let $R = \bigoplus_i R_i$ be a graded K-algebra. Assume (for simplicity) that R is generated by a finite set $\{f_i\}$ of positively graded elements, $d_i := \deg f_i$.
Then $R[f_i^{-1}]$ is a graded ring, let $R[f_i^{-1}]_0$ be the subring of degree 0 elements.

$U_i := \mathbf{Spec}\, R[f_i^{-1}]_0$
$U_{ij} := \mathbf{Spec}\, R[f_i^{-1}]_0[(f_j^{d_i}/f_i^{d_j})^{-1}]$
$\psi_{ij}\colon R[f_i^{-1}]_0[(f_j^{d_i}/f_i^{d_j})^{-1}] = R[(f_i f_j)^{-1}]_0 =$

$$R[f_j^{-1}]_0[(f_i^{d_j}/f_j^{d_i})^{-1}]$$

The *projective spectrum*:

$$\mathbf{Proj}\, R := \cup_i U_i$$

Examples

- $R = K[t]$, $\deg t = 1$.
  $\{t\}$ set of generators. $R[t^{-1}] = K[t, t^{-1}]$.
  $R[t^{-1}]_0 = K$, $\boxed{\mathbf{Proj}\, R = \mathbf{Spec}\, K}$.

- $R = K[x_0, \ldots, x_n]$, $\deg x_i = 1$.
  $\{x_i\}$ generators.
  $R[x_i^{-1}]_0 = K[x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i]$.
  $\boxed{\mathbf{Proj}\, R = \mathbf{P}^n}$.

- $R = K[x, y, z]$, $\deg(x, y, z) = (2, 3, 1)$.

  - $R[x^{-1}]_0 = K[z^2/x, y^2/x^3, yz/x^2] = K[r, s, t]/(rs - t^2)$,
    not a polynomial ring.

  - $R[y^{-1}]_0 = K[z^3/y, x^3/y^2, xz/y] = K[a, b, c]/(ab - c^3)$,
    not a polynomial ring.

  - $R[z^{-1}]_0 = K[x/z^2, y/z^3] = K[u, v]$.

  $\boxed{\mathbf{Proj}\, R =: \mathbf{P}(2, 3, 1)}$, a *weighted projective space*

Points of $\mathbb{P}(2,3,1)$

- $U_x(K) = \{(r,s,t) \in K^3 \mid rs = t^2\}$.

- $U_y(K) = \{(a,b,c) \in K^3 \mid ab = c^3\}$.

- $U_z(K) = \{(u,v) \in K^2\}$

Gluing:

- $U_x(K) \cap U_y(K)$: $bs = 1$,
  $r = c^2/b, t = c/b, c = t/s, a = rt/s$

- $U_x(K) \cap U_z(K)$: $ru = 1$,
  $v = t/r^2, s = v^2/u^3, t = v/u^2$

- $U_y(K) \cap U_z(K)$: $av = 1$,
  $u = c/a, b = u^3/v^2, c = u/v$

Weighted homogeneous coordinates:

$(x : y : z) = (\lambda^2 x : \lambda^3 y : \lambda z)$, $\lambda \in K^*$.

- $x \neq 0$:
  $(x : y : z) \mapsto (z^2/x, y^2/x^3, yz/x^2) \in U_x(K)$

- $y \neq 0$:
  $(x : y : z) \mapsto (z^3/y, x^3/y^2, xz/y) \in U_y(K)$

- $z \neq 0$:
  $(x : y : z) \mapsto (x/z^2, y/z^3) \in U_z(K)$

This gives a bijection,
homogeneous coord's $\leftrightarrow$ $\mathbf{P}(2,3,1)(K)$ *if*
K is algebraically closed.

General homogeneous coordinates

Definition 0.5 *R as before, $X := \mathbf{Proj}\, R$,*
A graded K-algebra homomorphism $R \to K[t]$, $\deg t = 1$*, is* irrelevant *if it is zero on positive elements.*

$R \to K[t]$ not irrelevant $\Rightarrow$ some $f_i \not\mapsto 0$
$\rightsquigarrow R[f_i^{-1}] \to K[t, t^{-1}] \rightsquigarrow R[f_i^{-1}]_0 \to K$
$\rightsquigarrow x \in U_i(K)$

"Independent" of $i \rightsquigarrow x \in X(K)$.

$\lambda \in K^*$: $\varphi_\lambda \colon K[t] \to K[t]$, $\varphi_\lambda(t) = \lambda t$.
$f \colon R \to K[t]$ and $\varphi_\lambda \circ f$ give same x.

Proposition 0.6 *If K is algebraically closed then this gives a bijection between $X(K)$ and non-irrelevant $R \to K[t]$ modulo the action of the φ_λ.*

Fourth lecture

Quasi-coherent sheaves

Definition 0.7 *i) Let R be a commutative ring. A* quasi-coherent sheaf *on $X := \mathbf{Spec}R$ is an R-module M and similarly for morphisms between them. A* global section *of M is a module homomorphism $R \to M$. The set of global sections is denoted $M(X)$.*

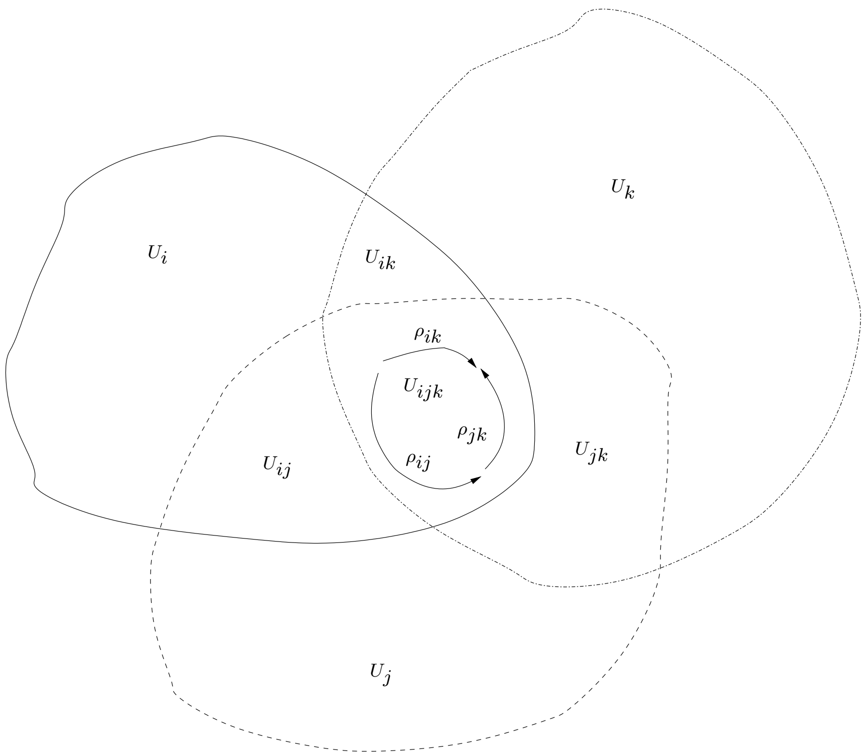
ii) If $f\colon R \to S$ is a ring homomorphism (of commutative rings) and M a quasi-coherent sheaf on $\mathbf{Spec}R$, then $f^*M := S \bigotimes_R M$.

Note: $R \to M \leftrightarrow m \in M$ by $1 \mapsto m$.

Definition 0.8 *Let* $X = (\{U_i\}, \{U_{ij}\}, \psi_{ij})$*,* $U_i := \mathbf{Spec}\, R_i$*,* $U_{ij} := \mathbf{Spec}\, R_{ij}$*,* $\psi_{ij}\colon U_{ij} \cong U_{ji}$*, be a scheme. A* quasi-coherent sheaf *on* X *consists of the choice of a quasi-coherent sheaf* M_i *on* U_i *and the following data*

- *isomorphisms* $\rho_{ij}\colon M_{ij} \to \psi_{ij}^* M_{ji}$*, where* $M_{ij} := s^* M_i$ *with* $s\colon U_{ij} \to U_i$ *the "inclusions",*

- *a compatability* $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$ *on* $U_{ijk} := \mathbf{Spec}\, R_{ij} \bigotimes_{R_j} R_{jk}$*.*



Examples

- $M_i = R_i$. This is the *structure sheaf* denoted $\mathcal{O}_X$.

- All constructions of modules that "commute with pullbacks" (i.e., with f^*). They include *direct sum*, $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \bigoplus \mathcal{N}$ and *tensor products*, $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \bigotimes_{\mathcal{O}_X} \mathcal{N}$. These are constructed requiring for instance $(\mathcal{M} \bigoplus \mathcal{N})_i = \mathcal{M}_i \bigoplus \mathcal{N}_i$.

- Let $x \in X(L)$. We define $\mathbf{k}_x$ as follows: If $x \in U_i(L)$ then $(\mathbf{k}_x)_i := R_i/m_i$, where m_i is the kernel of $R_i \to L$. If $x \notin U_i(L)$, $(k_x)_i = 0$. Similarly $m_x \subseteq \mathcal{O}_X$ is defined by $(m_x)_i = m_i$ resp. R_i.

Examples, cont'd

The category of quasi-coherent sheaves have most of the properties of the category of R-modules:

- Abelian category:
  $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$
  is exact precisely when
  $0 \to \mathcal{M}_i \to \mathcal{N}_i \to \mathcal{P}_i \to 0$ is.

- Enough injective objects.

- Exact directed limits.

- *Not* enough projective objects (in general).

Global sections:
$\Gamma(X, \mathcal{M}) = \mathcal{M}(X) := \mathrm{Hom}(\mathcal{O}_X, \mathcal{M})$.
In general *not* exact:
$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ exact $\Rightarrow$
$0 \to \mathcal{M}(X) \to \mathcal{N}(X) \to \mathcal{P}(X)$ exact.

The projective line

$U_1 := \mathbf{Spec}\, K[x]$, $U_2 := \mathbf{Spec}\, K[x^{-1}]$.

$\mathcal{M}$: M_1 $K[x]$-module, M_2 $K[x^{-1}]$-module,
ρ: $M_1[x^{-1}] \cong M_2[x]$.

- $x = (0:1)$.
  $(\mathbf{k}_x)_1 = K[x]/(x)$, $(\mathbf{k}_x)_2 = 0$.

- $x = (1:0)$.
  $(\mathbf{k}_x)_1 = 0$, $(\mathbf{k}_x)_2 = K[x^{-1}]/(x^{-1})$.

- $x = (\lambda:1)$, $\lambda \neq 0$.
  $(\mathbf{k}_x)_1 = K[x]/(x - \lambda)$,
  $(\mathbf{k}_x)_2 = K[x^{-1}]/(x^{-1} - \lambda^{-1})$,
  $\rho = \overline{id}$.

$\mathcal{M} = \mathcal{O}_{\mathbf{P}^1}(n)$:
$\mathcal{M}_1 = K[x]$, $\mathcal{M}_2 = K[x^{-1}]$, $\rho(1) = x^{-n}$.

$s\colon \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(n)$ global section.
$s(1) =: p(x) \in M_1$, $s(1) =: q(x^{-1}) \in M_2$,
$q(x^{-1}) = \rho(p(x)) = x^{-n}p(x)$.

- $n < 0$: $x^{-n}p(x) \in xK[x]$,
  $q(x^{-1}) \in K[x^{-1}]$,
  $p = q = 0$.

- $n \geq 0$: $x^{-n}p(x) \in x^{-n}K[x]$,
  $q(x^{-1}) \in K[x^{-1}]$,
  $\deg q \leq n$.

- $n = 0$: $q = 1$ gives isomorphism
  $\mathcal{O}_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(0)$.

$\mathbf{Proj}$ of modules

R graded K-algebra (as before) and M a *graded* R-module.

$X := \mathbf{Proj}\, R, \mathcal{M} = \mathbf{Proj}\, M$:
$\mathcal{M}_i := M[f_i^{-1}]_0$,
$\rho_{ij}\colon M[f_i^{-1}]_0[(f_j^{d_i}/f_i^{d_j})^{-1}] = M[(f_i f_j)^{-1}]_0$
$M[f_j^{-1}]_0[(f_i^{d_j}/f_j^{d_i})^{-1}]$.

Examples for $\mathbf{P}^1$:

- Shift of M, $M(n)\colon M(n)_i = M_{n+i}$,
  $R = K[x, y]$, $M = K[x, y]$,
  $\mathbf{Proj}\, M(n) = \mathcal{O}_{\mathbf{P}^1}(n)$.

- $R = K[x, y]$, $M = K[t]$,
  $x \cdot 1 = at$, $y \cdot 1 = bt$, $(a, b) \neq (0, 0)$,
  $\mathbf{Proj}\, M = \mathbf{k}_x$, $x = (a\colon b)$.

$m \in M_k \rightsquigarrow m/1 \in \Gamma(X, \mathbf{Proj}\, M(k))$.

Examples for $\mathbf{P}^n$ ($R = K[x_0, \ldots, x_n]$):

- $M = K[x_0, \ldots, x_n]$,
  $\mathcal{O}_{\mathbf{P}^n}(k) = \mathbf{Proj}\, M(k)$,
  $K[x_0, \ldots, x_n]_k \to \Gamma(\mathbf{P}^n, \mathcal{O}(k))$,
  $K[x_0, \ldots, x_n]_k$ homogeneous polynomials of degree k.

- M graded R-module:
  $\mathbf{Proj}\, M(k) = (\mathbf{Proj}\, M) \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(k)$.

- $\boxed{\mathcal{O}_{\mathbf{P}^n}(k) \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(\ell) \cong \mathcal{O}_{\mathbf{P}^n}(k + \ell)}$.

- $\boxed{\mathcal{O}_{\mathbf{P}^n} \cong \mathcal{O}_{\mathbf{P}^n}(0)}$.

General properties

$X := \mathbf{Proj}R$, R graded (as before).

- $M \mapsto \mathbf{Proj}M$ is *exact*.
- $\mathbf{Proj}M = 0 \iff$
  $\forall m \in M_k, f \in R_+ \exists n\colon f^n m = 0$.
- If M is f.g. then $M_k \to \Gamma(X, \mathbf{Proj}M(k))$ isomorphism for $k \gg 0$.
- Every quasi-coherent sheaf on X is of the form $\mathbf{Proj}M$.

Kähler differentials.

R K-algebra. R-module $\Omega_{R/K}$:
Generators $df, f \in R$. Relations:

- $d(f+g) = df + dg$.
- $d(fg) = fdg + gdf$.
- $d\lambda = 0, \lambda \in K$.

$f \in R$: $\Omega_{R[f^{-1}]/K} = \Omega_{R/K}[f^{-1}] \rightsquigarrow \Omega_{X/K}$,
X K-scheme
$(d(g/f^n) = f^{-n}dg - ngf^{-n-1}df)$.

The projective line

$R = K[x]$: $d(p(x)) = p'(x)dx$, $\Omega_{K[x]/K}$ free on dx.

$\Omega_{\mathbf{P}^1/K}$: $K[x]dx$, $K[x^{-1}]d(x^{-1})$,
$d(1/x) = x^{-2}dx$,
$\rho(dx) = x^2 d(1/x) \rightsquigarrow \Omega_{\mathbf{P}^1/K} \cong \mathcal{O}_{\mathbf{P}^1}(-2)$.

Fifth lecture

Definition 0.9 *i) A scheme is* noetherian *if it can be given by noetherian gluing data* (U, V, s, t)*, i.e.,* $U = \mathbf{Spec}\, R$ *for a noetherian ring* R*.*

ii) A quasi-coherent sheaf on X *is* coherent *if it is the form* (M, ρ) *with* M *a finitely generated* R*-module for some noetherian gluing data.*

Assume $U = \dot\cup_i U_i$:
noetherian $\iff$ finite indexset and R_i noetherian
coherence $\iff$ M_i finitely generated.

From now on we shall assume that all schemes are noetherian.

Line bundles

Definition 0.10 *A* line bundle *or* invertible sheaf *on a scheme X is a quasi-coherent sheaf $\mathcal{L}$ such that there is another quasi-coherent sheaf $\mathcal{L}^*$ so that $\mathcal{L}^* \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.*

Recall: $\mathcal{O}_{\mathbf{P}^n}(k) \otimes_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{O}_{\mathbf{P}^n}(\ell) \cong \mathcal{O}_{\mathbf{P}^n}(k+\ell)$, $k := -\ell$

R *Dedekind ring*: $\mathfrak{n}$ fractionary ideal $\mathfrak{n}^{-1} := \{r \in \operatorname{Frac} R \mid r\mathfrak{n} \subseteq R\}$

$$\begin{array}{ccc} \mathfrak{n}^{-1} \otimes_R \mathfrak{n} & \to & \mathfrak{n}^{-1}\mathfrak{n} \\ r \otimes s & \mapsto & rs \end{array}$$

is an isomorphism $\Rightarrow$ $\mathfrak{n}$ line bundle ($\mathfrak{n}^{-1}\mathfrak{n} = R \iff R$ is Dedekind)

$\operatorname{Pic}(X) :=$ iso classes of line bundles: Group under $\otimes_{\mathcal{O}_X}$

Proposition 0.11 *Line bundles are coherent.*

Line bundles on Dedekind ring R:
Fractionary ideals $\rightsquigarrow$
$\mathrm{Pic}(\mathbf{Spec}R) = \mathrm{C}\ell(R)$

Line bundles on $\mathbf{A}^1$: Only $\mathcal{O}_{\mathbf{A}^1}$ (PID).

Line bundles on $\mathbf{P}^1$:
Restriction to patches trivial
ρ: $K[x, x^{-1}] \cong K[x, x^{-1}], \rho(1) = \lambda x^n$
Mult. by λ on first patch $\rightsquigarrow \rho(1) = x^n \rightsquigarrow$
$\mathcal{O}_{\mathbf{P}^1}(n) \Rightarrow \mathrm{Pic}(\mathbf{P}^1) = \mathbf{Z}$

Čech cocycles

Assume gluing data:
$(\dot\cup_i U_i, \dot\cup_{ij} U_{ij}, s, t)$, $s\colon U_{ij} \to U_i$, $t\colon U_{ij} \to U_j$.

Assume $\mathcal{L}$ line bundle *and*
$\mathcal{L}_{|U_u i} \cong R_i e_i$, $U_i = \mathbf{Spec}\, R_i$,
$\rho_{ij}(e_i) = u_{ij} e_j, e_{ij} \in R_{ij}^*, U_{ij} = \mathbf{Spec}\, R_{ij}$,
Cocycle condition:
$u_{ik} = u_{jk} u_{ij} \in R_{ijk} := R_{ij} \bigotimes_{R_j} R_{jk}$.

$e_i' = u_i e_i \rightsquigarrow u_{ij}' = u_j u_i^{-1} u_{ij}$
u_{ij} and u_{ij}' define same line bundle

Projective space

$M = K[x_0, \ldots, x_n](m)$:
$M[x_i^{-1}]_0$: generator x_i^{-m}
$\rho_{ij}(x_i^{-m}) = (x_i/x_j)^m x_i^{-m}$

$\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathcal{O}_{\mathbf{P}^n}}(m))$:

$s \in \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathcal{O}_{\mathbf{P}^n}}(m)) \Rightarrow s_i = p_i x_i^{-m}$,
$p_i \in K[x_0/x_i, \ldots, x_n/x_i]$
$s_j = \rho_{ij}(s_i) \Rightarrow$
$p_j = (x_i/x_j)^m p_i \Rightarrow x_j^m p_j = x_i^m p_i$
$\deg p_j \leq m$
$K[x_0, \ldots, x_n]_m = M_0 \to \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathcal{O}_{\mathbf{P}^n}}(m)) \Rightarrow$

$\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathcal{O}_{\mathbf{P}^n}}(m)) = K[x_0, \ldots, x_n]_m$

Line bundle $\mathcal{L}$ on $\mathbf{P}^n$:

Line bundle on $\mathbf{A}^n$ trivial:
$K[x_0, \ldots, x_n]$ UFD.

$\Rightarrow \mathcal{L}$ determined by
$u_{ij} \in K[x_0/x_i, \ldots, x_n/x_i][(x_j/x_i)^{-1}]$
$u_{ij} = \lambda_{ij}(x_i/x_j)^{n_{ij}}, \lambda_{ij} \in K^*$
Cocycle condition $\leadsto$
$\lambda_{ij} = \lambda_j/\lambda_i \leadsto u_{ij}(x_i/x_j)^{n_{ij}}$
Cocycle condition $\iff$
$(x_i/x_k)^{n_{ik}} = (x_j/x_k)^{n_{jk}}(x_i/x_j)^{n_{ij}} \iff$

$$\begin{cases} n_{ik} & = & n_{ij} \\ n_{ik} & = & n_{jk} \\ n_{jk} & = & -n_{ij} \end{cases} \iff n_{ij} = m, i < j$$

Conclusion $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^n}(m)$

Maps to projective space

$V \subset K^{n+1}$, *hyperplane* (i.e., $\dim V = n$),
$V \mapsto V^\perp := \{ u \in K^{n+1} \mid \forall v \in V \colon \langle u, v \rangle = 0 \}$
lines $\leftrightarrow$ hyperplanes

X K-scheme,
$\mathcal{L}$ line bundle on X, $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L}) \rightsquigarrow$
$\varphi \colon \mathcal{O}^{n+1} \xrightarrow{(s_0 \ldots, s_n)} \mathcal{L}$

Definition 0.12 *The* s_i generate $\mathcal{L}$ *if* φ *is surjective.*

Idea: $x \in X(K), \mathcal{L}_x := x^*(\mathcal{L})$ 1-dimensional,
$x^*(\varphi) \colon K^{n+1} \to \mathcal{L}_x$ surjective $\Rightarrow$
kernel hyperplane $H_x \rightsquigarrow$

$$\begin{array}{ccc} X(K) & \to & \mathbf{P}^n(K) \\ x & \mapsto & H_x^\perp \end{array}$$

should come from $X \to \mathbf{P}^n$

Remark: For this reason sometimes
$\mathbf{P}^n(K) := \{\text{hyperplanes} \subset K^{n+1}\}$

More concretely:
$s_i^x \in \mathcal{L}_x, \dim \mathcal{L}_x = 1 \Rightarrow$
$\exists i_0\colon s_{i_0}^x \neq 0 \Rightarrow \mathcal{L}_x = K s_{i_0}^x \Rightarrow s_i^x = \lambda_i s_{i_0}^x$:
$z \mapsto (\lambda_0 \colon \ldots \colon \lambda_n)$

X_i: open subscheme where
$s_i\colon \mathcal{O}_X \to \mathcal{L}$ is iso.

$X = \mathbf{Spec}\, R, \mathcal{L} = \mathcal{O}_X \Rightarrow$
$s_i \in R \Rightarrow X_i = \mathbf{Spec}\, R[s_i^{-1}]$

In general exists gluing data
$(\dot\cup_k U_k, \dot\cup_{k\ell} U_{k\ell})$ such that
$\mathcal{L}_{|U_k} \cong \mathcal{O}_{U_k} \Rightarrow X_i \cap U_k = \mathbf{Spec}\, R_k[s_i^{-1}]$

On X_i $\mathcal{L} = \mathcal{O} s_i \Rightarrow s_j = f_j^i s_i, f_j^i \in \Gamma(X_i, \mathcal{O}_{X_i})$

$f \in \Gamma(Y, \mathcal{O}_Y) \leftrightarrow f\colon Y \to \mathbf{A}^1$:
$f \leftrightarrow K[t] \to R \leftrightarrow \mathbf{Spec}\, R \to A^1$

Gives $f^i\colon X_i \xrightarrow{(f_0^i,\ldots,f_n^i)} \mathbf{A}^n \subset \mathbf{P}^n$

symbolically: $x \mapsto (f_0^i(x), \ldots, f_n^i(x))$
lies in i'th affine patch ($f_i^i = 1$)

On $X_i \cap X_j$ $f^i = f^j \rightsquigarrow f\colon X \to \mathbf{P}^n$.

Properties:

- $f^*\mathcal{O}_{\mathbf{P}^n}(1) = \mathcal{L}$.

- $x_i \in \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, $s_i = f^*x_i$.

- This characterises f

Examples

- $X = \mathbf{P}^1$, $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(n)$, $x_0^n, x_0^{n-1}x_1, \ldots, x_1^n$
  $(s : t) \mapsto (s^n : s^{n-1}t : \ldots : t^n)$

- $x = (a_0 : a_1 : \ldots : a_n) \in \mathbf{P}^n(\mathbf{Q})$
  $a_i \in L := \mathbf{Z}a_0 + \cdots + \mathbf{Z}a_n$
  set of generators $\rightsquigarrow$
  $\mathbf{Spec}\,\mathbf{Z} \to \mathbf{P}^n$ s.t. composite
  $\mathbf{Spec}\,\mathbf{Q} \to \mathbf{Spec}\,\mathbf{Z} \to \mathbf{P}^n$ equals x
  $\rightsquigarrow \mathbf{P}^n(\mathbf{Q}) = \mathbf{P}^n(\mathbf{Z})$.

  $g :=$ gcd$(a_0, a_1, \ldots, a_n), L = \mathbf{Z}g \Rightarrow$
  $x = (b_0 : b_1 : \ldots : b_n), b_i := a_i/g,$
  $b_i \in \mathbf{Z},$ gcd$(b_0, \ldots, b_n) = 1 \rightsquigarrow$
  $\mathbf{Spec}\,\mathbf{Z} =$
  $\{(a_0, \ldots, a_n) \in \mathbf{Z}^{n+1}|$
  gcd$(a_0, \ldots, a_n) = 1\}/\{\pm 1\}$

Examples, cont'd

- $X = \mathbf{Spec}\, R$, R Dedekind ring,
  $K :=$ Frac R
  $x = (a_0 : a_1 : \ldots : a_n) \in \mathbf{P}^n(K)$
  $a_i \in L := Ra_0 + \cdots + Ra_n$
  set of generators $\rightsquigarrow$
  $\mathbf{Spec}\, R \to \mathbf{P}^n$
  NB: L may *not* be a trivial line bundle, e.g.,
  $R = \mathbf{Z}[\sqrt{-5}], x = (2 : 1 + \sqrt{-5})$

- $X = \mathbf{P}^m \times \mathbf{P}^n$, $\mathcal{L}_1 := \pi_1^* \mathcal{O}_{\mathbf{P}^m}(1)$,
  $\mathcal{L}_2 := \pi_2^* \mathcal{O}_{\mathbf{P}^n}(1)$, $\mathcal{L} := \mathcal{L}_1 \otimes \mathcal{L}_2$,
  $x_{ij} := \pi_1^* x_i \otimes \pi_2^* x_j$,
  $((x_0 : x_1 : \ldots : x_m), (y_0 : y_1 : \ldots : y_m)) \mapsto$
  $(x_0 y_0 : \ldots : x_0 y_n : \ldots : x_m y_n)$
  This, the *Segre embedding*, identifies $\mathbf{P}^m \times \mathbf{P}^n$ with a closed subscheme of $\mathbf{P}^{mn+m+n}$.

Sixth lecture

Definition 0.13 *i) A module over a commutative ring is* projective *if it is a direct factor of a free module.*

ii) A quasi-coherent sheaf (M, ρ) *with respect to gluing data* $(\mathbf{Spec} R, \mathbf{Spec} S, s, t)$ *on a scheme* X *is a* vector bundle *or a* locally free sheaf *if* M *is a finitely generated projective* R*-module.*

Examples:

- Line bundles are vector bundles.
- Direct sums of vector bundles are vector bundles.
- More generally, extensions of vector bundles are vector bundles.
- $\Omega^1_{\mathbf{P}^n/K}$ is a vector bundle: Its restriction to an affine patch is $\Omega^1_{\mathbf{A}^n/K}$ and $\Omega^1_{K[x_1,\ldots,x_n]/K}$ has $dx_1, dx_2, \ldots, dx_n$ as basis.

- The tensor product of vector bundles is a vector bundle.

- M R-module, $\Lambda^n M :=$
  $M^{\otimes n}/\{a_1 \otimes \cdots \otimes a_i \otimes a_i \otimes \cdots \otimes a_{n-1}\}$,
  exterior power
  $\mathcal{E}$ vector bundle $\Rightarrow \Lambda^n \mathcal{E}$ v.b.

- *Symmetric powers* of vector bundles are vector bundles.

The projective line

Finitely generated modules over PID R:
Exact sequence

$$R^m \xrightarrow{A} R^n \to M \to 0$$

$m \times n$-matrix A
Elementary row and column operations $\rightsquigarrow$
$A = NDM, M \in \mathsf{SL}_m(R), N \in \mathsf{SL}_n(R)$,
D diagonal $\rightsquigarrow$
M is a sum of cyclic modules (i.e.,
$\cong R/(f)$)
Torsion free $\Rightarrow$ all f's zero $\Rightarrow$ M free.

$\mathcal{M}$ vector bundle over $\mathbf{P}^1 \leftrightarrow$
M_1 projective $K[x]$-module,
M_2 projective $K[x^{-1}]$-module,
ρ: $M_1[x^{-1}] \cong M_2[x]$.

$M_1 \cong K[x]^n, M_2 \cong K[x^{-1}]^n$
$\rho = R \in \mathsf{GL}_n(K[x, x^{-1}])$
$R \sim NRM, M \in \mathsf{GL}_n(K[x]), N \in \mathsf{GL}_n(K[x^{-1}])$

Elementary row ($/K[x^{-1}]$) and column ($/K[x]$) ops $\rightsquigarrow$:
$R = NDM, M \in \mathsf{GL}_n(K[x])$,
$N \in \mathsf{GL}_n(K[x^{-1}])$,
D diagonal w. entries $x^{k_i} \Rightarrow \mathcal{M} \cong \bigoplus_i \mathcal{O}_{\mathbf{P}^1}(k_i)$,
$k_1 \leq k_2 \leq \ldots \leq k_n$

The k_i determined:

$\mathcal{M} \bigotimes \mathcal{O}(m) \cong \bigoplus_i \mathcal{O}_{\mathbf{P}^1}(m + k_i)$
$\dim \Gamma(\mathbf{P}^1, \mathcal{M} \bigotimes \mathcal{O}(m)) = \sum_{m+k_i \geq 0} m + k_i$

Remark: Krull-Schmidt also does the trick

$x, y \in \Gamma(\mathbf{P}^1, \mathcal{O}(1)) \rightsquigarrow$

$$0 \to \mathcal{L} \longrightarrow \mathcal{O} \bigoplus \mathcal{O} \longrightarrow \mathcal{O}(1) \to 0$$

$\mathcal{L} \cong ?$

$\operatorname{rk} \mathcal{E}, \det \mathcal{E} := \Lambda^{\operatorname{rk} \mathcal{E}} \mathcal{E},$
$0 \to \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \to 0 \Rightarrow$
$\det \mathcal{F} \cong \det \mathcal{E} \bigotimes \det \mathcal{G} \Rightarrow$
$\mathcal{O}_{\mathbf{P}^1} \cong \mathcal{L} \bigotimes \mathcal{O}_{\mathbf{P}^1}(1) \Rightarrow \mathcal{L} \cong \mathcal{O}(-1)$

$q_1, q_2, q_3 \in \Gamma(\mathbf{P}^1, \mathcal{O}(2))$ generating $\mathcal{O}(2)$:

$$0 \to \mathcal{E} \longrightarrow \mathcal{O}^3 \longrightarrow \mathcal{O}(2) \to 0$$

$\mathcal{E} \cong \mathcal{O}(k_1) \bigoplus \mathcal{O}(k_2) \Rightarrow k_1 + k_2 = -2$

$$0 \to \Gamma(\mathbf{P}^1, \mathcal{E}) \to K^3 \xrightarrow{(q_1, q_2, q_3)} K[x, y]_2$$

$(q_1, q_2, q_3) = (x^2, xy, y^2) \Rightarrow$
$\Gamma(\mathbf{P}^1, \mathcal{E}) = 0 \Rightarrow$
$k_1, k_2 < 0 \Rightarrow k_1 = k_2 = -1$

$(q_1, q_2, q_3) = (x^2, 0, y^2) \Rightarrow$
$\Gamma(\mathbf{P}^1, \mathcal{E}) = K \Rightarrow (k_1, k_2) = (-2, 0)$

The Kähler differentials

$R = K[x_0, \ldots, x_n]$

$\partial/\partial x_i \colon \Omega^1_{R/K} \to K(-1)$, $df \mapsto \partial f/\partial x_i \rightsquigarrow$
$\Omega^1_{R/K} \cong R(-1)^{n+1}$
$\mathbf{Proj} \rightsquigarrow \Omega^1_{\mathbf{P}^n/K} \to \mathcal{O}(-1)^{n+1}$

Euler's theorem: $Ef := \sum_i x_i \partial f/\partial x_i =$ $\deg f \cdot f \Rightarrow f \in K(x_0, \ldots, x_n)_0 \Rightarrow Ef = 0 \rightsquigarrow$ Complex:

$$0 \to \Omega^1_{\mathbf{P}^1/K} \to \mathcal{O}_{\mathbf{P}^n}(-1)^{n+1} \xrightarrow{(x_0,\ldots,x_n)} \mathcal{O}_{\mathbf{P}^n} \to 0$$

An exact sequence.
$\Omega^1_{\mathbf{P}^1 n/K}$ indecomposable

Seventh lecture

Some definitions

- *Dedekind scheme*:
  Gluing data $(\mathbf{Spec}R, \mathbf{Spec}S, s, t)$,
  R Dedekind

- *finite type K-scheme*:
  Gluing data $(\mathbf{Spec}R, \mathbf{Spec}S, s, t)$,
  R finitely generated K-algebra

- *smooth curve*:
  Dedekind K-scheme, finite type

- *projective K-scheme*:
  $X = \mathbf{Proj}R$

- *connected scheme*:
  not a non-trivial disojoint union

Asssume $K = \overline{K}$

C proj. smooth connected curve

Proposition 0.14
$\exists!$*grp hom* deg: $\mathrm{Pic}(C) \to \mathbf{Z}$ *s.t.*
$\mathcal{L} \subseteq \mathcal{M} \Rightarrow \deg(\mathcal{M}) - \deg(\mathcal{L}) = \mathrm{length}\, \mathcal{M}/\mathcal{L}$

Special case:
$s\colon \mathcal{O}_C \to \mathcal{L}$, $\mathrm{length}\, \mathcal{M}/\mathcal{O} = \#$ of zeroes
$\deg(\mathcal{M}) - \deg(\mathcal{O}) = deg(\mathcal{M})$

$\mathcal{F}$ coherent sheaf $/C \Rightarrow h^0(\mathcal{F}) :=$
$\dim H^0(C, \mathcal{F}) < \infty$

Riemann's theorem:

$$\exists g \geq 0\colon \mathcal{L} \in \mathrm{Pic}(C) \Rightarrow$$

$$\boxed{h^0(\mathcal{L}) \geq \deg \mathcal{L} + 1 - g}$$

$\Omega^1_{C/K} \in \mathrm{Pic}(C), g = h^0(\Omega^1_{C/K})$

Proposition 0.15

$H^0(C, \mathcal{O}_C) = K$

PROOF:

$\mathcal{O}_C$ coherent $\Rightarrow$

$T = H^0(C, \mathcal{O}_C)$ fin. alg.

Non-trivial idemp.: C non-conn.

$T \subset R$, Dedekind $\Rightarrow$ no nilp.

$K = \overline{K} \Rightarrow T = K$ $\square$

Consequence:

$\mathcal{L} \in \mathrm{Pic}(C) \Rightarrow H^0(C, \mathcal{L}) = K$

Curves of Genus 0

$g = 0$:

$x \in C(K), \mathcal{O}(x) := m_x^{-1}, \rightsquigarrow$
$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}(x) \longrightarrow \mathbf{k}_x \to 0 \Rightarrow$
deg: $(x) = 1 \Rightarrow$
$h^0(\mathcal{O}(x)) \geq 1 + 1 - 0 = 2 \Rightarrow$
$\exists t \in H^0(C, \mathcal{O}(x))$: $t_x \neq 0 \Rightarrow$
s, t span $\mathcal{O}(x) \rightsquigarrow$
$C \to \mathbf{P}^1$,
$\deg \mathcal{O}(x) = 1 \Rightarrow$ is isom.

Argument gives:
$\exists \mathcal{L} \in \mathrm{Pic}(C)$: $h^0(\mathcal{L}) \geq 2 \Rightarrow C \cong \mathbf{P}^1$

Elliptic curves

char $K \neq 2$

$E := \mathbf{Proj}\, K[x, y, z]/(y^2 z - F(x, z))$
$F(x, z) = z^3 f(x/z), f$ homog of deg 3,
f no mult. roots

$z \neq 0$ patch: $y^2 = f(x)$

Complex case:
Going around ∞ permutes roots $\rightsquigarrow$
Torus w. 1 missing pt.

$z = 0$: $F(x, 0) = 0 \Rightarrow (0:1:0)$
Missing point!

Genus of elliptic curve

$dx/y \in \Gamma(E, \Omega^1_{E/K})$:

$y^2 = f(x) \Rightarrow$ $\boxed{2ydy = f'(x)dx}$

- $z \neq 0$:
  - $y \neq 0$: $\boxed{dx/y}$,
    Also $f'(x) \neq 0 \Rightarrow dx/y = 2dy/f'(x)$
  - $f'(x) \neq 0$: $\boxed{2dy/f'(x)}$
  - $y = 0, f'(x) = 0 \Rightarrow f(x) = 0 \Rightarrow$
    $\boxed{\text{Double root}}$

- $y \neq 0$: $(x' : 1 : z') = (x'/z' : 1/z' : 1) \Rightarrow$
  $dx/y = (d(x'/z'))/(1/z') = dx' - x'/z'dz'$
  $g(x', z')z' = x'^3, g(0,0) \neq 0 \Rightarrow$
  May assume $h = g^{-1}$ exists $\Rightarrow$
  $z' = gx'^3 \Rightarrow \frac{x'}{z'}dz' = x'hdg + 3dx'$

- $y = 0, z = 0 \Rightarrow 0 = F(x, 0) \Rightarrow x = 0$

dx/y **generator** $\Rightarrow$ $\boxed{g = 1}$

Picard group

$\infty := (0:1:0)$,
$x \in E(K)$: $\mathcal{O}(x) := m_x^{-1}$

$\mathrm{Pic}^0(E) := \ker \deg$

Theorem 0.16 *The map*

$$\begin{array}{ccc} E(K) & \to & \mathrm{Pic}^0(E) \\ x & \mapsto & \mathcal{O}(x) \bigotimes \mathcal{O}(\infty)^{-1} \end{array}$$

is a bijection.

PROOF:
Injectivity:
$\mathcal{O}(x) \bigotimes \mathcal{O}(\infty)^{-1} \cong \mathcal{O}(y) \bigotimes \mathcal{O}(\infty)^{-1} \iff$
$\mathcal{O}(x) \cong \mathcal{O}(y)$:
$s^x \in \Gamma(E, \mathcal{O}(x)), s^y \in \Gamma(E, \mathcal{O}(y))$
$g \neq 0 \Rightarrow h^0(\mathcal{O}(x)) \leq 1 \Rightarrow x = y$

Surjectivity:
$\mathcal{L} \in \mathrm{Pic}^0(E) \Rightarrow$
$h^0(\mathcal{L} \bigotimes \mathcal{O}(\infty)) \geq 1 \Rightarrow$
$0 \to \mathcal{O}_E \longrightarrow \mathcal{L} \bigotimes \mathcal{O}(\infty) \longrightarrow x \to 0 \rightsquigarrow$
$\mathcal{L} \bigotimes \mathcal{O}(\infty) \cong \mathcal{O}(x)$ □

Group structure

$x, y \in E(K) \Rightarrow \exists! z \in E(K)$:
$\mathcal{O}(x) \otimes \mathcal{O}(y) \cong \mathcal{O}(z) \otimes \mathcal{O}(\infty)$

Which is it?

$q \in \Gamma(\mathbf{P}^2, \mathcal{O}(n))$:
Restriction $q \colon \mathcal{O}_E \to \mathcal{O}(n)_{|E}$ has zeroes $x_1, \ldots, x_m$ with multiplicity!

- $m = 3n$ (Bezout's theorem)
- $\mathcal{O}(n)_{|E} \cong \mathcal{O}_E(x_1) \otimes \cdots \mathcal{O}_E(x_m)$
  Factorisation into prime ideals!

Consequence:
$\mathcal{O}_E(x_1) \otimes \cdots \mathcal{O}_E(x_m)$ indep. of q

$n = 1, q = z$:

$z = 0 \wedge y^2 z = F(x, z) \Rightarrow (x : y : z) = \infty \Rightarrow$

$x_1 = x_2 = x_3 = \infty$

General $\ell \in \Gamma(\mathbf{P}^2, \mathcal{O}(1))$:

Zeroes x_1, x_2, x_3

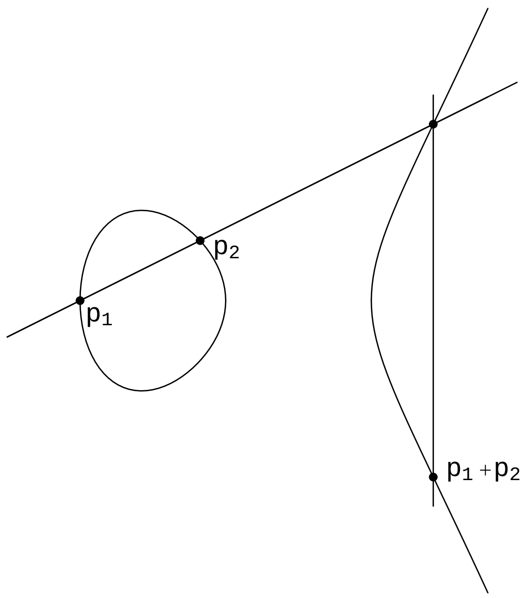
$\mathcal{O}_E(x_1) \otimes \mathcal{O}_E(x_2) \otimes \cdots \mathcal{O}_E(x_3) \cong$

$\mathcal{O}_E(\infty) \otimes \mathcal{O}_E(\infty) \otimes \cdots \mathcal{O}_E(\infty)$

Symbolically: $\boxed{x_1 + x_2 + x_3 \equiv 3\infty}$

$\ell = x - \lambda z \Rightarrow$

$(\lambda : \mu : 1) + (\lambda : -\mu : 1) + \infty \equiv 3\infty$



$P_1 + P_2 + Q \equiv 3\infty \wedge$

$Q + (P_1 + P_2) + \infty \equiv 3\infty \Rightarrow$

$P_1 + P_2 \equiv (P_1 + P_2) + \infty$

$\infty \leftrightarrow 0$: $(P_1, P_2) \mapsto P_1 + P_2$

Theorem 0.17 *There is a (unique) commutative group operation* $+\colon E(K)\times E(K)\to E(K)$ *s.t.* $\mathcal{O}(x)\otimes\mathcal{O}(y)\cong\mathcal{O}(x+y)\otimes\mathcal{O}(\infty)$

$(x_1 : y_1 : 1) + (x_1 : y_1 : 1) = (x_3 : y_3 : 1)$:

$$\begin{cases} y_3^2 = f(x_3) \\ y_3 = \frac{y_2-y_1}{x_2-x_1}(x_3-x_1)+y_1 \end{cases} \quad k := \frac{y_2-y_1}{x_2-x_1}$$

$k^2(x_3-x_1)^2 = x_3^3 + bx_3^2 + cx_3 + d$
$x_1 + x_2 + x_3 = b - k^2 \Rightarrow$
$x_3 = b - k^2 - x_1 - x_2,$
$y_3 = k(x_3 - x_1) + y_1 \rightsquigarrow$

Algebraic group scheme:

$$\boxed{+\colon E\times E\to E}$$

Remark:
i) Associativity non-obvious!
Clear from previous results

ii) *Projective* group scheme

Putting things together

$f(x) \in \mathbf{Z}[x]$:
E $\mathbf{Z}$-scheme (=scheme) $\mathbf{P}^2(\mathbf{Q}) = \mathbf{P}^2(\mathbf{Z}) \Rightarrow$
$E(\mathbf{Q}) = E(\mathbf{Z})$
$\mathbf{Z} \to \mathbf{Z}/\mathbf{Z}p \rightsquigarrow E(\mathbf{Q}) \to E(\mathbf{Z}/\mathbf{Z}p)$

$f(x) = x^3 + 4x - 1$
$(1/8 : 1/4 : 1) \in E(\mathbf{Q})$:
$(1/8)^2 = (1/4)^3 + 4 \cdot 1/4 - 1 = 0$
$(1:2:8) \in E(\mathbf{Z}) \rightsquigarrow$
$(1 : -1 : -1) = (-1 : 1 : 1) \in E(\mathbf{Z}/\mathbf{Z}5)$

$E(\mathbf{Q}) \to E(\mathbf{Z}/\mathbf{Z}p)$ *group homomorphism*:

Lines reduce to lines *or*
$f\colon X \to Y \Rightarrow \mathcal{L} \rightsquigarrow f^*\mathcal{L}$
$f^*(\mathcal{L} \bigotimes \mathcal{M}) = f^*\mathcal{L} \bigotimes f^*\mathcal{M}$

Classification for $g = 1$

$g(C) = 1$:
Pick $p \in C(K), 0 \neq z \in \Gamma(C, \mathcal{O}(p))$,
$x^2 := x \otimes x \in \Gamma(C, \mathcal{O}(2p))$,
$\mathcal{O}(2p) := \mathcal{O}(p) \bigotimes \mathcal{O}(p)$
Riemann: $h^0(\mathcal{O}(2p)) \geq 2 + 1 - 1 = 2 \Rightarrow$
$x \in \Gamma(C, \mathcal{O}(2p))$, x^2 and y lin. indep.

$x_p \neq 0$:
$x_p = 0 \Rightarrow x = zt, t \in \Gamma(C, \mathcal{O}(p))$,
$\Gamma(C, \mathcal{O}(p)) = Kx$ as $\deg \mathcal{O}(p) = 1 \Rightarrow$
$h^0(\mathcal{O}(p)) \leq 1 \Rightarrow x \in Kzt = Kz^2$

Conclusion: x and z^2 generate $\mathcal{O}(2p)$.

$h^0(\mathcal{O}(3p)) \geq 3 + 1 - 1 = 3 \Rightarrow$
$y \in \Gamma(C, \mathcal{O}(3p)), y \notin Kxz + Kz^3$
$y^2, yxz, yz^3, x^3, x^2z^2, xz^4, z^6 \in \Gamma(C, \mathcal{O}(6p))$

Classification, cont'd

Riemann-Roch
$\Rightarrow h^0(\mathcal{O}(6p)) = 6 + 1 - 1 = 6 \Rightarrow$
$y^2, yxz, yz^2, x^3, x^2z, xz, z^3$
linearly *dependent*: $\rightsquigarrow$
$y^2 + a_1yxz + a_3yz^3 =$
$x^3 + a_2x^2z + a_4xz^4 + a_6z^6$

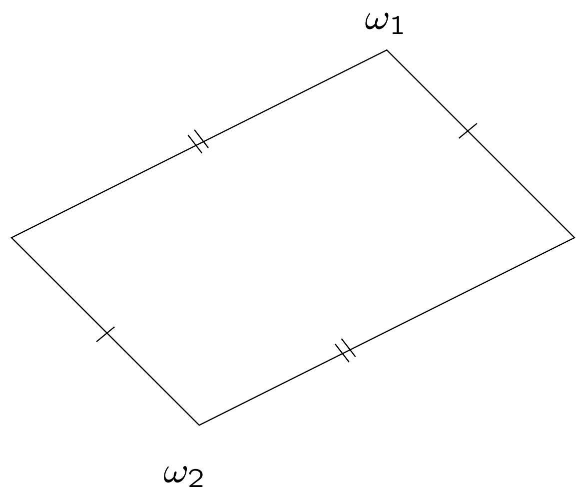
x, y not determined.
$$\begin{cases} y \mapsto uy + rxz + sz^3 \\ x \mapsto vx + tz^2 \end{cases}$$
char $K \neq 2 \Rightarrow$
$y^2 = x^3 + a_2x^2z + a_4xz^4 + a_6z^6$

Homogeneous if
$\deg x = 2, \deg y = 3, \deg z = 1$.
weighted projective space!
$z \neq 0$: $(x : y : 1) \Rightarrow y^2 = f(x)$
$z = 0$: $y^2 = x^3 \rightsquigarrow (1 : 1 : 0)$

Conclusion: Completes by one point.

Highlights

- Over $\mathbf{C}$ torus, $E = \mathbf{C}/\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$.



- Construct interesting *non-commutative* algebras (*Sklyanin algebras*).

- $f(x) \in \mathbf{Q}[x]$, $E(\mathbf{Q})$ finitely generated group.

- $f(x) \in \mathbf{F}_q$, $|\mathbf{F}_q| = q$, $|\#E(\mathbf{F}_q) - q - 1| \leq 2\sqrt{q}$.

- Ramanujam's conjecture:
  $q\prod_i(1-q^n)^{24}=\sum_n \tau(n)q^n \Rightarrow$
  $|\tau(n)| \le \sigma_0(n)n^{11/2}$
  $\sigma_0(n)=\sum d|n1$

- String theory

- "Monstrous moonshine"

- Arithmetic of torsion points:
  $\{x \in E(\overline{\mathbf{Q}}) \mid nx = 0\}$

- Elliptic curve cryptography

Eighth lecture

Vector bundles on curves

C smooth, proper, connected curve

$\mathcal{F} \subseteq \mathcal{E}$ two vector bundles on C.
$\mathcal{F}$ is a *subbundle* if
$\mathcal{E}/\mathcal{F}$ is a vector bundle.
$\mathcal{F} \subseteq \mathcal{E}$ subbundle $\iff$
$\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{E} \wedge \operatorname{rk} \mathcal{F} = \operatorname{rk} \mathcal{F}' \Rightarrow \mathcal{F} = \mathcal{F}'$

$\mathcal{L} \subset \mathcal{M}$, line bundles,
$\mathcal{L}$ subbundle $\iff$ $\mathcal{L} = \mathcal{M}$.

$\mathcal{E}$ vector bundle, *slope*,
$s(\mathcal{E}) := \deg \mathcal{E} / \operatorname{rk} \mathcal{E}, \deg \mathcal{E} := \deg \det \mathcal{E}$

Definition 0.18 *A vector bundle $\mathcal{E}$ on C is* stable *resp.* semi-stable *if for every subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $s(\mathcal{F}) < s(\mathcal{E})$ resp. $s(\mathcal{F}) \leq s(\mathcal{E})$.*

$\mathcal{S}_s :=$ category of semi-stable vector bundles of fixed slope s.

Lemma 0.19 $b, d > 0, \frac{a}{b} \leq \frac{c}{d} \Rightarrow$
$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$

Consequence:
$0 \to \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \to 0$ exact $\Rightarrow$
$\deg \mathcal{F} = \deg \mathcal{E} + \deg \mathcal{G}, \operatorname{rk} \mathcal{F} = \operatorname{rk} \mathcal{E} + \operatorname{rk} \mathcal{G} \Rightarrow$
$t := \max(s(\mathcal{E}), s(\mathcal{G})), r := \min(s(\mathcal{E}), (\mathcal{G}))$
$\Rightarrow \boxed{r \leq s(\mathcal{F}) \leq t}$

Big surprise:

Proposition 0.20 *$\mathcal{S}_s$ is an abelian subcategory of the category of quasi-coherent sheaves. Every object has finite length and the simple objects are the
stable sheaves.*

Proof:
Let $f : \mathcal{E} \to \mathcal{F}$, $\mathcal{E}, \mathcal{F} \in \mathcal{S}_s$
$\mathcal{G} := \ker f, \mathcal{H} := \mathrm{I}m\, f,$
Semistability of $\mathcal{E} \Rightarrow$
$s(\mathcal{G}) \leq s(\mathcal{E}) \Rightarrow s(\mathcal{H}) \geq s(\mathcal{E}) = s$
Semistability of $\mathcal{F} \Rightarrow$
$s(\mathcal{H}) \leq s(\mathcal{F}) = s$
Together: $s(\mathcal{G}) = s = s(\mathcal{H})$

Proof, cont'd

$s(\text{ker } f) = s(\text{I}m\, f) = s(\text{coker } f) = s$
Assume
$\mathcal{E}, \mathcal{F} \in \mathcal{S}_s, \mathcal{E} \hookrightarrow \mathcal{F}, \text{rk}\,\mathcal{E} = m = \text{rk}\,\mathcal{F}$
$\Rightarrow \deg \mathcal{E} = ms = \deg \mathcal{F}$
$\mathcal{E} \to \mathcal{F}$ iso $\iff \Lambda^m \mathcal{E} \to \Lambda^m \mathcal{F}$ iso
$\Leftarrow$ (det(A) invert. $\iff$ A invert.)
$\mathcal{E} \hookrightarrow \mathcal{F} \Rightarrow \det \mathcal{E} \hookrightarrow \det \mathcal{F}$
$0 = \deg \det \mathcal{E} - \det \mathcal{F} = \text{length} \det \mathcal{F} / \det \mathcal{E} \Rightarrow$

$\mathcal{E} \hookrightarrow \mathcal{F}$ iso

Conclusion: $\mathcal{E}, \mathcal{F} \in \mathcal{S}_s, \mathcal{E} \subsetneq \mathcal{F}$
$\Rightarrow \text{rk}\,\mathcal{E} < \text{rk}\,\mathcal{F} \Rightarrow$ finite length.

Clear: Stable $\iff$ simple

The Harder-Narasimhan filtration

$\mathcal{E}$ vector bundle on C
$s_{max}(\mathcal{E}) = s :=$ max $\{s(\mathcal{F}) \mid \mathcal{F} \subseteq \mathcal{E}\}$
s exists:
Riemann's theorem for vector bundles:
$h^0(\mathcal{G}) \geq \deg \mathcal{G} + \mathrm{rk}\, \mathcal{G}(1-g)$
$s(\mathcal{F})$ unbounded $\Rightarrow h^0(\mathcal{G})$ unbounded but
$h^0(\mathcal{F}) \leq h^0(\mathcal{E}) < \infty$

Let $\mathcal{F} \subseteq \mathcal{E}, s(\mathcal{F}) = s$, maximal rank
Assume $\mathcal{G} \subseteq \mathcal{E}, s(\mathcal{G}) = s$
$0 \to \mathcal{G} \cap \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G} \cap \mathcal{F} \to 0$
$s(\mathcal{G} \cap \mathcal{F}) \leq s \Rightarrow s(\mathcal{G}/\mathcal{G} \cap \mathcal{F}) \geq s$
$\mathcal{G}/\mathcal{G} \cap \mathcal{F} = \mathcal{G} + \mathcal{F}/\mathcal{F} \Rightarrow s(\mathcal{G} + \mathcal{F}) \geq s \Rightarrow$
$\mathrm{rk}\, \mathcal{G} + \mathcal{F} = \mathrm{rk}\, \mathcal{F} \Rightarrow \det \mathcal{F} \hookrightarrow \det(\mathcal{G} + \mathcal{F})$
$\deg \mathcal{F} = \det(\mathcal{G}+\mathcal{F}) \Rightarrow \mathcal{F} = \mathcal{G}+\mathcal{F} \Rightarrow \mathcal{G} \subseteq \mathcal{F}$

Conclusion

$\mathcal{E}_s := \mathcal{F}$ largest subsheaf with slope s.

- $\mathcal{E}_s$ subbundle: If not
  $\mathcal{E}_s \subsetneq \mathcal{F} \subseteq \mathcal{E}, \mathrm{rk}\, \mathcal{E}_s = \mathrm{rk}\, \mathcal{F} \Rightarrow$
  $s(\mathcal{F}) > s(\mathcal{E}_s)$

- $s_{max}(\mathcal{E}/\mathcal{E}_s) < s$: If not
  $\mathcal{E}_s \subsetneq \mathcal{F} \subseteq \mathcal{E}, s(\mathcal{F}/\mathcal{E}_s) \geq s \Rightarrow s(\mathcal{F}) \geq s$.

- $\mathcal{E}_s$ semi-stable: If not,
  $0 \neq \mathcal{F} \subsetneq \mathcal{E}_s$, $s(\mathcal{F}) > s$

Repeating:

Proposition 0.21 *$\mathcal{E}$ vector bundle $/C$*
Unique filtration by subbundles:
$0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \ldots \subsetneq \mathcal{E}_n = \mathcal{E}$
$s(\mathcal{E}_1) > s(\mathcal{E}_2/\mathcal{E}_1) > \ldots > s(\mathcal{E}/\mathcal{E}_{n-1})$
$\mathcal{E}_i/\mathcal{E}_{i-1}$ *semi-stable*

The projective line

$\mathcal{L} \in \mathrm{Pic}(C)$: $\mathcal{E}$ (semi-)stable $\iff \mathcal{E}\bigotimes\mathcal{L}$
$(-)\bigotimes\mathcal{L}$ exact with inverse $(-)\bigotimes\mathcal{L}^{-1}$
$\det(\mathcal{E}\bigotimes\mathcal{L}) = \det\mathcal{E}\bigotimes\mathcal{L}^{\otimes \operatorname{rk}\mathcal{E}} \Rightarrow$
$s(\mathcal{E}\bigotimes\mathcal{L}) = s(\mathcal{E}) + \deg\mathcal{L}$
Riemann:
$\frac{h^0(\mathcal{E})}{\operatorname{rk}\mathcal{E}} \geq s(\mathcal{E}) + 1 - g$

$C = \mathbf{P}^1$, $\mathcal{E}$ stable:

$\mathcal{E} \leftrightarrow \mathcal{E}(m)$: $-1 < s(\mathcal{E}) \leq 0$
$h^0(\mathcal{E})/\operatorname{rk}\mathcal{E} > -1 + 1 - 0 = 0 \Rightarrow$
$\mathcal{O}_{\mathbf{P}^1} \subseteq \mathcal{E}$, $s(\mathcal{O}) = 0 = s(\mathcal{E}) \Rightarrow \mathcal{E} = \mathcal{O}$

Semi-stable:
$\mathcal{E} = \mathcal{O}^n$ induction over rank
$\mathcal{F} \subseteq \mathcal{E}$, $\mathcal{F}$ stable of slope 0.
$\mathcal{E}/\mathcal{F} \cong \mathcal{O}^{n-1}$

General facts:
$0 \to \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \to 0 \leftrightarrow \alpha \in \mathrm{Ext}^1(\mathcal{G},\mathcal{E})$
$\mathcal{F} \cong \mathcal{E}\bigoplus\mathcal{G} \leftrightarrow 0$
$\mathrm{Ext}^1(\mathcal{G}\bigoplus\mathcal{G}',\mathcal{E}) = \mathrm{Ext}^1(\mathcal{G},\mathcal{E})\bigoplus\mathrm{Ext}^1(\mathcal{G}',\mathcal{E})$

Cont'd

Quasi-coherent sheaves on C:
$\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(C)$,
$\dim_K \mathrm{Ext}_{\mathcal{O}_C}(\mathcal{L}, \mathcal{M}) = h^0(\mathcal{L} \bigotimes \mathcal{M}^{-1} \bigotimes \Omega^1_{C/K})$

$\dim_K \mathrm{Ext}_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}, \mathcal{O}) =$
$h^0(\Omega^1_{\mathbf{P}^1/K}) = h^0(\mathcal{O}(-2)) = 0 \rightsquigarrow$
$\mathcal{E} \cong \mathcal{O}^n \Rightarrow \mathcal{E}$ semi-stable $\mathcal{E} \cong \mathcal{O}(m)^n, m =$
$s(\mathcal{E})$

$\mathcal{E}$ general:
$\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{O}(s_i)^{n_i}$, $s_1 < s_2 < \ldots < s_n$
$s < t \Rightarrow \dim \mathrm{Ext}_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(t), \mathcal{O}(s)) =$
$h^0(\mathcal{O}(-2 + s - t)) = 0$

Remark:
$0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}(1) \to 0$
non-split

Elliptic curves

E elliptic curve, $\mathcal{E}$ stable vector bundle
$\mathcal{E} \leftrightarrow \mathcal{E} \otimes \mathcal{L}$: May assume
$\deg \mathcal{L} = 0 \Rightarrow h^0(\mathcal{E} \otimes \mathcal{L}) = 0$
$\exists \mathcal{M} \in \mathrm{Pic}(E)$: $\deg \mathcal{L} \wedge h^0(\mathcal{E} \otimes \mathcal{M}) > 0$

Riemann: $0 = \frac{h^0(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} \geq s(\mathcal{E}) + 1 - 1 \Rightarrow$
$\boxed{s(\mathcal{E}) \leq 0}$
$\mathcal{O} \to \mathcal{E} \otimes \mathcal{M} \leftrightarrow \mathcal{M}^{-1} \to \mathcal{E} \Rightarrow$
$\boxed{-1 = s(\mathcal{M}^{-1}) < s(\mathcal{E})}$

Assume $\mathrm{rk}\,\mathcal{E} = 2$: $\Rightarrow$
$s(\mathcal{E}) = -1/2 \vee s(\mathcal{E}) = 0$

$s(\mathcal{E}) = -1/2$:
$\mathcal{M}^{-1} \subset \mathcal{E}$ subbundle,
if not $\mathcal{M}^{-1} \subsetneq \mathcal{L} \subset \mathcal{E} \Rightarrow$
$s(\mathcal{L}) > s(\mathcal{M}^{-1}) = -1$

Cont'd $s(\mathcal{E}) = -1/2$

$\mathcal{M} \cong \mathcal{O}(x), x \in E(K)$, $\mathcal{O}(-x) := \mathcal{O}(x)^{-1}$

$$0 \to \mathcal{O}(-x) \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0$$

$-1 = \deg \mathcal{E} = -1 + \deg \mathcal{L} \Rightarrow$
$\deg \mathcal{L} = 0$

$$0 \to \mathcal{O}(-x) \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0$$

Non-split!

Conversely, $\mathrm{Pic}(E) \ni \mathcal{M} \subset \mathcal{E}, \deg \mathcal{M} \geq 0$
$\mathcal{M} \to \mathcal{E} \to \mathcal{L}$, non-zero $\Rightarrow$ iso $\Rightarrow$ split $\Rightarrow$
$\mathcal{M} \to \mathcal{O}(-x)$, non-zero $\Rightarrow$
$-1 = \deg \mathcal{O}(-x) \geq \deg \mathcal{M} \geq 0$

Exists? $\dim \mathrm{Ext}_{\mathcal{O}_E}(\mathcal{L}, \mathcal{O}(-x)) =$
$h^0(\mathcal{O}(x) \otimes \mathcal{L} \otimes \Omega^1_{E/K}) = h^0(\mathcal{O}(x) \otimes \mathcal{L})$
$\deg \mathcal{O}(x) \otimes \mathcal{L} = 1 \Rightarrow h^0(\mathcal{O}(x) \otimes \mathcal{L}) = 1$
$\det \mathcal{E} = \mathcal{O}(-x) \otimes \mathcal{L} \cong \mathcal{O}(-y)$,
Depends only on $y \Rightarrow$

parameter $y \in E(K)$

$s(\mathcal{E}) = 0$:

Example:
$\dim \mathrm{Ext}_{\mathcal{O}_E}(\mathcal{O}_E, \mathcal{O}_E) = h^0(\Omega^1_{E/K}) = 1 \Rightarrow$
Non-split:

$$0 \to \mathcal{O}_E \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_E \to 0$$

Semi-stable

Stable $\mathcal{E}$:
$\mathcal{O}(-x) \hookrightarrow \mathcal{E}$, subbundle $\Rightarrow$

$$0 \to \mathcal{O}(-x) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(y) \to 0$$

$\rightsquigarrow \exists \mathcal{L} \in \mathrm{Pic}(E)$: $\mathcal{E} \cong \mathcal{F} \bigotimes \mathcal{L}$,
only semi-stable

Conclusion: Semi-stable bundles

Slope	Form
0	$\mathcal{O}(x - \infty) \bigoplus \mathcal{O}(y - \infty)$
0	$\mathcal{E}/\mathcal{O}(x - \infty) \cong \mathcal{O}(-x + \infty)$
$-1/2$	$\mathcal{E}/\mathcal{O}(-\infty) \cong \mathcal{O}(\infty - x)$

$\mathcal{O}(\infty - x) := \mathcal{O}(\infty) \bigotimes \mathcal{O}(x)^{-1}$

General curves

C smooth proper connected curve
$\mathcal{E}$ semi-stable, $n := \text{rk}\,\mathcal{E}, d := \deg \mathcal{E}$
Action Pic(C): $\mathcal{E} \mapsto \mathcal{E} \bigotimes \mathcal{L} \rightsquigarrow$
$0 \leq s(\mathcal{E}) < n$, $\det \mathcal{E}$ fixed

$\gcd(d, n) = 1 \Rightarrow \mathcal{E}$ stable

Results for stable bundles:

- $g = 1, (n, d) = (2, -1)$:
  $\det \mathcal{E}$ fixed $\Rightarrow \mathcal{E}$ unique (up to iso)

- $g = 1$: Generally $\mathcal{E}$ exists $\iff$ $\gcd(d, n)$ and then it is unique.

- $g > 1$: $\mathcal{E}$ exists for all (d, n).
  "Smooth parameter scheme" of dimension $(n^2 - 1)(g - 1)$

- $d = 0$: Over $\mathbf{C}$ $\mathcal{E} \leftrightarrow$ *irreducible* representations of the *fundamental group* of C in SU_n up to conjugation