NONCOMMUTATIVE ALGEBRAS I: WEDDERBURN-ARTIN THEOREM

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1. Introduction

Convention 1.1. All rings R have a multiplicative unity 1. If $\phi : R \to S$ is a ring homomorphism, then $\phi(1_R) = 1_S$. In particular, if S is a subring of R, then $1_R = 1_S$ (that is, the multiplicative unities of R and S are equal).

Ideal means "two-sided ideal." Our one-sided ideals tend to be right ideals. Our modules tend to be right modules. Modules are unital in that $m1 = m$ for all $m \in M$.

Definitions for ideals in R

(1) $I + J = \{i + j : i \in I, j \in J\}$
(2) $IJ = \{i_1 j_1 + i_2 j_2 + \cdots + i_n j_n : i_k \in I, j_k \in J\}$
(3) An ideal J is a *maximal* ideal if whenever K is an ideal and $J \subsetneq K \subset R$, then $K = R$.
(4) An ideal P is a *prime* ideal if for all ideals I, J, if $IJ \subset P$, then $I \subset P$ or $J \subset P$.
(5) An ideal P is *completely prime* if for all $a, b \in R$, if $ab \in P$, then $a \in P$ or $b \in P$.

Warning: For noncommutative rings, completely prime is stronger than prime. However:

Proposition 1.2. *Let P be an ideal in R. Then the following are equivalent:*

(1) *P is prime,*
(2) *for all right ideals I, J, if $IJ \subset P$, then $I \subset P$ or $J \subset P$,*
(3) *for all left ideals I, J, if $IJ \subset P$, then $I \subset P$ or $J \subset P$,*
(4) *for all $a, b \in R$, if $aRb \subset P$, then $a \in P$ or $b \in P$, where $aRb = \{arb : r \in R\}$.*

Definitions for (right) modules

(1) R_R means R is thought of as a right module over itself,
(2) ${}_RR$ means R is thought of as a left module over itself,
(3) If X is a subset of a module M, then $\operatorname{r.ann} X = \{r \in R : Xr = 0\}$. Check that $\operatorname{r.ann} X$ is a right ideal, and when X is a submodule of M, that $\operatorname{r.ann} X$ is an ideal,
(4) a module M is *simple* if $M \neq 0$ and the only submodules of M are 0 and M,

(5) a module M is *faithful* if $\operatorname{r.ann} M = 0$,
(6) $M^{(n)} = M \oplus \cdots \oplus M$ (n times).

Definitions for rings

(1) a ring R is *simple* if 0 is a maximal ideal,
(2) a ring R is *primitive* if R has a faithful simple module,
(3) a ring R is *prime* if 0 is a prime ideal,
(4) a ring R is a *domain* if 0 is completely prime, i.e., $ab = 0$ implies $a = 0$ or $b = 0$,
(5) a ring R is a *division ring* if every non-zero element of R is invertible.

Proposition 1.3. *In general, simple ring $\implies$ primitive ring $\implies$ prime ring*

Proof. (simple $\implies$ primitive): By Zorn's Lemma, R has a maximal right ideal I. Then R/I is simple. Since $\operatorname{r.ann}(R/I) \neq R$ (since $1 \in R$), we must have $\operatorname{r.ann}(R/I) = 0$. So R/I is a faithful simple module.

(primitive $\implies$ prime) Let M be a faithful simple module and suppose I, J are ideals with $IJ = 0$. Then $MIJ = 0$. Since M is simple, we have $MI = 0$ or $MI = M$. If $MI = 0$, then $I = 0$ since M is faithful. If $MI = M$, then $MIJ = MJ = 0$, so $J = 0$ since M is faithful. Since $I = 0$ or $J = 0$, we have that R is a prime ring. □

The theorems we'll present are trivial in the commutative case. If R is commutative and primitive, then R is simple, and hence a field.

Proof. Let M be a faithful simple R-module, with non-zero element x. Then $xR = M$ since M is simple. Since R is commutative, $\operatorname{r.ann}(x) = \operatorname{r.ann}(xR) = 0$. But $\operatorname{r.ann}(x)$ is the kernel of the natural R-module epimorphism $R \to M$ given by $r \mapsto xr$. So $R \cong M$ as R-modules. So R is a simple R-module, and hence has only ideals 0 and R. Hence R is also simple as a ring, and is a field. □

This is why "primitive rings" aren't a topic in commutative algebra.

Note: for noncommutative rings, saying R is simple as a right module over itself (i.e., R_R is simple), is stronger than saying R is a simple ring. For example, $R = M_2(\mathbb{C})$ is a simple ring, but it has non-zero proper right submodules (its submodule of top row vectors for instance). In fact, if R_R is simple, then R is a division ring. (Check.)

2. Primitive rings and Jacobson Density

Proposition 2.1. *Let D be a division ring and ${}_DM$ a* left *D-module (also known as a left vector space). Let $R = \operatorname{End}({}_DM)$. (That is R is the* ring *of all left D-module endomorphisms $M \to M$.) Then M_R is a faithful simple right R-module, so R is a primitive ring.* □

Example: if ${}_DM$ is infinite dimensional, then $R = \mathrm{End}({}_DM)$ is not simple. This is because $I = \{f \in R : f(m)$ is finite dimensional for all $m \in M\}$ is an ideal. (Check.) So R is a primitive, non-simple ring.

It turns out that any primitive ring R with faithful simple module M is closely related to $\mathrm{End}({}_DM)$ for some division ring D. Our path begins with

Lemma 2.2 (Schur's Lemma)**.** *Let M_R be a simple module. Then $D = \mathrm{End}(M_R)$ is a division ring.*

Proof. Let $f \in D, f \neq 0$. Then since M is simple, $\mathrm{Im}\, f = M$ and $\mathrm{Ker}\, f = 0$. So f is onto and 1-1, hence invertible. □

Definition 2.3. Let D be a division ring and let R be a subring of $\mathrm{End}({}_DM)$. Then R is *dense* in $\mathrm{End}({}_DM)$ if for every $n \in \mathbb{N}$ and every D-independent set $\{x_1, \ldots, x_n\}$ in M, and any $y_1, \ldots, y_n \in M$, there exists $r \in R$ such that $x_i r = y_i$ for $i = 1, \ldots, n$.

Theorem 2.4 (Jacobson's density theorem)**.** *Suppose R has a faithful simple module M_R and $D = \mathrm{End}\, M_R$. Then R is dense in $\mathrm{End}({}_DM)$.*

Proof. First, since M is faithful, R is (isomorphic to) a subring of $\mathrm{End}({}_DM)$.

Let $\{x_1, \ldots, x_n\}$ be D-independent on M, and let $y_1, \ldots, y_n \in M$. We induct on n to show there is r with $x_i r = y_i$. If $n = 1$, then since M is simple, $x_1 R = M$, so there is r with $x_1 r = y_1$.

Now suppose we've proven the claim for $n-1$. Suppose for contradiction that there does not exist $r \in R$ with $x_n r \neq 0$ and $x_i r = 0$ for $i \neq n$. Then there is a right R-module homorphism $f : M^{(n-1)} \to M$ given by $f(x_1, \ldots, x_{n-1}) = x_n$. It is well-defined since if $x_i r = 0$ for $i \neq n$, then $x_n r = 0$ as well. And f is defined on all of $M^{(n-1)}$ by the induction hypothesis. But $f \in \mathrm{Hom}_R(M^{(n-1)}, M \cong (\mathrm{Hom}_R(M, M))^{(n-1)} = D^{(n-1)}$. So there are $d_i \in D$ such that $\sum_{i=1}^{n-1} d_i x_i = f(x_1, \ldots, x_{n-1}) = x_n$. This contradicts the D-independence of the x_i. So there is r such that $x_n r \neq 0$ and $x_i r = 0$ for $i \neq n$.

The above argument would have been true for any indice, not just n. So now choose r_j such that $x_j r_j \neq 0$ and $x_i r_j = 0$ for all $i \neq j$. Choose r'_j such that $(x_j r_j) r'_j = y_j$. Let $r = \sum r_j r'_j$. Then $x_j r = y_j$ as desired. □

3. The Wedderburn-Artin Theorem

Recall the chain conditions:

Definition 3.1. A module M has ACC (the ascending chain condition) on submodules if for any chain of submodules $M_0 \subset M_1 \subset \ldots$, there exists i such that $M_i = M_j$ for all $j \geq i$. Such a module is also called *noetherian*. DCC (the descending chain condition) and *artinian* are defined similarly.

A ring R is right noetherian if R_R is a noetherian module. Left noetherian and right/left artinian are defined similarly. A ring R is noetherian if R is both right and left noetherian (and similarly for artinian).

Theorem 3.2 (Wedderburn-Artin)**.** *Let R be primitive and right artinian. Then $R \cong M_n(D)$ for some n and division ring D. Hence R is simple artinian (and noetherian).*

Proof. Let $D = \mathrm{End}(M_R)$, which is a division ring by Schur's Lemma. Then R is dense in $\mathrm{End}({}_DM)$.

Let $\{x_i : i \in I\}$ be a base for M over D. Let $L_i = \mathrm{r.ann}\{x_j : j \leq i\}$. Then $L_1 \supset L_2 \supset \dots$. Those inclusions are strict, since by density, there exists $r \in R$ such that $x_j r = 0, j < i$ and $x_i r \neq 0$. This leads to a contradiction of DCC, unless I is finite.

So M is finite dimensional over D, say of dimension n. Thus by linear algebra, $\mathrm{End}({}_DM) \cong M_n(D)$. Again by linear algebra, for R to be dense, the injection $R \to M_n(D)$ must be onto, and hence an isomorphism. □

The above is just one possible statement of Wedderburn-Artin. We know of course that a prime artinian commutative ring is just a field. This generalizes:

Proposition 3.3. *Let R be a prime ring with a minimal right ideal L. Then R is primitive.*

Proof. Since L is minimal, L is a simple right R-module. Now we have $L(\mathrm{r.ann}\, L) = 0$. Since R is prime and $L \neq 0$, we have $\mathrm{r.ann}\, L = 0$. So L is faithful. Hence R is primitive. □

Corollary 3.4. *Let R be a prime right artinian ring. Then R is simple artinian and hence $R \cong M_n(D)$ for some n and division ring D.* □

References

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