NONCOMMUTATIVE ALGEBRAS II & III: GOLDIE'S THEOREM

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4. Semisimple rings

In this section, we will generalize the Wedderburn-Artin theorem. The rings we look at will be "quotient rings," taking the place of fields (or finite direct products thereof) in the commutative case.

Definition 4.1. An ideal is *semiprime* if it is an intersection of prime ideals. A ring is semiprime if 0 is a semiprime ideal.

Proposition 4.2. *Let A be an ideal of R. Then the following are equivalent:*

(1) *A is semiprime,*
(2) *if I is an ideal and $I^2 \subset A$, then $I \subset A$,*
(3) *if I is a right ideal and $I^2 \subset A$, then $I \subset A$,*
(4) *if I is a left ideal and $I^2 \subset A$, then $I \subset A$,*
(5) *if $b \in R$ and $bRb \subset A$, then $b \in A$.*

(Check.) □

As in the commutative case, we will connect semiprime artinian rings to prime artinian rings.

The main tool is

Lemma 4.3 (Chinese Remainder Theorem)**.** *Let R be a ring and let $A_1, \ldots, A_n$ be proper distinct ideals such that $A_i + A_j = R$ when $i \neq j$. Then $R/(\cap_{i=1}^{j} A_i) \cong \prod_{i=1}^{j} R/A_i$.*

Proof. The proof is as in the commutative case. □

Lemma 4.4. *Let R be right artinian. Then any prime ideal is maximal and R has finitely many prime ideals P_i.*

Proof. Let P be prime. Then R/P is prime right artinian, hence simple by Corollary 3.4. So P is maximal.

Let $P_1, P_2, \ldots$ be all distinct primes. Then $P_1 \supset P_1 \cap P_2 \supset P_1 \cap P_2 \cap P_3 \supset \ldots$. Since R is right artinian, this chain stops for some n. If P is any prime, then $P \supset P_1 \cap \cdots \cap P_n$, and so $P \supset P_i$ for some i. Since P_i is maximal, $P_i = P$. So there are only finitely many distinct primes. □

Theorem 4.5. *Let R be semiprime right artinian. Then*

$$R \cong \prod_{i=1}^{j} M_{n_i}(D_i)$$

for some n_i and division rings D_i. So R is artinian and noetherian.

Proof. Since R is right artinian, there are only finitely many primes P_i, and they are each maximal, so $P_i + P_j = R$ for $i \neq j$. Since R is semiprime, $0 = \cap_{i=1}^{n} P_i$. So by the Chinese Remainder Theorem, $R \cong \prod_{i=1}^{n} R/P_i$.

The conclusion then follows from Corollary 3.4. □

We now turn to a different way of looking at semiprime artinian rings.

Definition 4.6. Let M be an R-module. Then the *socle* of M is the sum of the simple submodules of M, denoted $\mathrm{soc}(M)$. If M has no simple submodules, then $\mathrm{soc}(M) = 0$.

A module is *semisimple* if $\mathrm{soc}(M) = M$. A ring is (right) semisimple if R_R is semisimple.

Proposition 4.7. *A ring R is semiprime artinian if and only if R is semisimple.*

Proof. If R is semiprime artinian, then we know from Theorem 4.5 that $R \cong \prod_{i=1}^{j} M_{n_i}(D_i)$. It can be checked directly that such a ring is the sum of its minimal right ideals.

On the other hand, if R is semisimple, then let $R = \sum L_i$, with the L_i minimal right ideals. Then $1 = a_{i_1} + \cdots + a_{i_n}$ for some $a_j \in L_j, a_j \neq 0$. So $R = a_{i_1}R + \ldots a_{i_n}R = L_{i_1} + \cdots + L_{i_n}$. So R is a finite sum of L_i. Renumber those L_{i_j} as L_j. Then $R \supset \sum_{i>1} L_i \supset \sum i > 2L_i \supset \ldots$ is a composition series for R. Hence R is right artinian.

Now since each L_i is simple, $P_i = \mathrm{r.ann}(L_i)$ is primitive (because L_i is a simple faithful R/P_i-module). Since R is right artinian, these are prime ideals. Since $0 = \mathrm{r.ann}(1) = \cap\, \mathrm{r.ann}(L_i)$, we have that 0 is semiprime. □

We shall return to semisimple rings soon.

5. Quotient rings and the Ore condition

In commutative algebra, the procedure for localization is relatively simple. Given any commutive ring R and multiplicatively closed subset S, we can form the localization RS^{-1}. For noncommutative rings, the procedure is not so simple. It is quite common to have multiplicatively closed subsets which do not yield a localization.

Let us be more specific about what we want.

Definition 5.1. Let R be a ring and let S be a multiplicatively closed subset ($1 \in S$ and if $a, b \in S$, then $ab \in S$). Then a ring Q is a *(right) ring of fractions* of R with respect to S if

(1) There is a ring homomorphism $\nu : R \to Q$ such that $\nu(s)$ is invertible for any $s \in S$, with $\operatorname{Ker} \nu = \{r \in R : rs = 0, \text{ some } s \in S\}$.
(2) every element of Q has the form $(\nu(r))(\nu(s))^{-1} = rs^{-1}$ (abusing notation) for some $r \in R$ and some $s \in S$.

We write $Q = RS^{-1}$.

The main possible problem with a given S is how to write $s^{-1}r = r'(s')^{-1}$ for some $r' \in R, s' \in S$. We would need that:
(i) Given $r \in R, s \in S$, there exists $r' \in R, s' \in S$ such that $rs' = sr'$

Another issue is that if $sr = 0$, we must make sure that $\nu(r) = 0$. That is becase, we need $s^{-1}0 = s^{-1}sr = r = 0 \in Q$. Given the structure of $\operatorname{Ker} \nu$, we need that there is $s' \in S$ with $rs' = 0$. So
(ii) If $sr = 0$ for $r \in R, s \in S$, then there is $s' \in S$ with $rs' = 0$.

Definition 5.2. A multiplicatively closed set S which satisfies (i) and (ii) is called a right denominator set.

We won't prove it here, but it turns out that R has a right ring of fractions with respect to S if and only if S is a right denominator set.

We will focus on a special multiplicatively closed set S, the set of all *regular elements*. An element s is regular if $rs \neq 0$ and $sr \neq 0$ for all $r \in R, r \neq 0$. It is easy to see that the set of regular elements is mutiplicatively closed, and automatically satisfies (ii). For this special case, (i) is called the *Ore condition* and a ring R with S, the set of regular elements, satisfying (i) is called *right Ore*.

If R is right Ore, we say that R has a *right quotient ring* $Q = RS^{-1}$. The ring R is a *right order* in Q if Q is a right quotient ring of R. It is a fact that Q is unique up to isomorphism. Also, since $\operatorname{Ker} \nu = 0$, we can think of $R \subset Q$.

There are many domains which are not right Ore, and hence have no right quotient rings. For instance, if k is a field, and $R = k\langle x, y\rangle$, non-commutative polynomials in k, then R is a domain and x, y are regular. But there are no non-zero polynomials f, g such that $xf = yg$, so one could not write $x^{-1}y$ in the form gf^{-1}.

6. Orders in semisimple rings

We will show that a large class of rings are right orders in semisimple rings. These are known as the semiprime right *Goldie* rings. Let us examine some of the properties they must have.

We need the following definitions

Definition 6.1. Let M be an R-module.

(1) A submodule N of M is *essential* (or *large*) in M if $N \cap N' \neq 0$ for all submodules $N' \neq 0$ of M. This is written $N \leq_e M$.
(2) A submodule N is *uniform* if $N' \cap N'' \neq 0$ for all non-zero submodules N', N''. That is $N' \leq_e N$ for every submodule N.

Essentialness (essentiality?) is connected to semisimplicity as follows.

Proposition 6.2. *Let M be an R-module. Then M is semisimple if and only if M is the only essential submodule of M.*

Proof. Suppose M is semisimple. Since $\operatorname{soc}(M) = M$, if $N \subsetneq M$, there must be a simple submodule S of M which is not contained in N. Thus $N \cap S = 0$, so N is not essential in M.

The other direction would require more lemmas and propositions, which we won't present. □

Definition 6.3. A ring R which has no infinite direct sum of right ideals and has ACC on ideals of the form $\operatorname{r.ann}(A), A \subset R$ (called right annhilators) is called right *Goldie*.

Proposition 6.4. *Let R be a right order in a semisimple ring Q. Then R is semiprime right Goldie.*

Proof. Suppose N is an ideal with $N^2 = 0$. Then by Zorn's Lemma, there exists a right ideal N' with $N \cap N' = 0$ and $L = N + N' \leq_e R_R$. If J is a non-zero right ideal of Q, then $rR \subset J$ for some non-zero $r \in R$ (because $rs^{-1} \in J$ implies $r \in J$). Since $L \cap rR \neq 0$, we have $LQ \cap J \neq 0$. So $LQ \leq_e Q$. Thus by the previous proposition, $LQ = Q$. So $1 = \sum l_i s_i^{-1}$ and $\prod_i s_i = \sum_i (l_i \prod_{j \neq i} s_i) \in L$. So L contains a regular element. Now $NL = N^2 + NN' = 0$. Since L contains a regular element, we have $N = 0$.

We leave it to the reader to check that if $\oplus L_i$ is a direct sum of ideals of R, then $\oplus L_i Q$ is a direct sum of ideals of Q. So the direct sum cannot be infinite.

Now suppose A_i, A_{i+1} are such that $\operatorname{r.ann}_R(A_i) \subsetneq \operatorname{r.ann}_R(A_{i+1})$ where A_i are subsets of R. (Here the subscript R tells us that these annhilators are in R.) Let $B_j = \{q \in Q | q \operatorname{r.ann}_R(A_j) = 0\}$. Then $B_i \supset B_{i+1}$. Since $\operatorname{r.ann}_R(A_i) \subsetneq \operatorname{r.ann}_R(A_{i+1})$, there exists $a_i \in A_i$ such that $a_i \operatorname{r.ann}_R(A_{i+1}) \neq 0$. Then $a_i \in B_i \setminus B_{i+1}$. Since Q is left artinian, we have DCC on the B_i. So we must have ACC on the right annhilators in R. □

Of special note is that right noetherian rings are right Goldie, as are commutative domains.

7. Semiprime right Goldie rings have quotient rings

For this section, R is always a semiprime right Goldie ring. Most of our material comes from [1]. (Draft available online at

`https://www.dcc.ufrj.br/ collier/goldie.pdf`.
Get the published version if you can, but the only real mathematical problem I know of is in the last line of page 11 of the draft: it should say $a^n x = 0$, not $a^n = 0$.)

Our main goal will be to show

Claim 7.1. *A right ideal I of R is essential if and only if I contains a regular element.*

Given the claim, we have

Theorem 7.2 (Goldie's Theorem)**.** *Let R be semiprime right Goldie. Then R has a ring of quotients.*

Proof. Let $a, s \in R$, with s regular. Let $E = \{x \in R : ax \in sR\}$.

Now suppose I is a non-zero right ideal of R. If $aI = 0$, then $I \subset E$. If $aI \neq 0$, then $aI \cap cR \neq 0$ since $cR \leq_e R_R$. So $I \cap E \neq 0$. So $E \leq_e R_R$. Thus E contains a regular element s'.

Then there exists $b \in R$ such that $as' = sb$. So the right Ore condition holds. □

First, we go about showing that if s is regular, then sR is essential. We need a lemma.

Lemma 7.3. *Every right ideal contains a uniform right ideal.*

Proof. Suppose not. Let I be a counterexample. Then there are non-zero $I_1, I_1' \subset I$ such that $I_1 \cap I_1' = 0$. Similarly, there are non-zero $I_2, I_2' \subset I_1$ such that $I_2 \cap I_2' = 0$ (and further $(I_2 + I_2') \cap I_1' = 0$. So $I_1' + I_2'$ is direct. Continuing in this manner, we get an infinite direct sum $I_1' \oplus I_2' \oplus I_3' \oplus \ldots$. This contradicts the righ Goldie condition. □

So any semiprime right Goldie ring contains uniform ideals. It turns out that the maximal length n of a direct sum of uniform right ideals is an invariant of R, called the *Goldie rank* or *uniform rank* of R. The proof that this is an invariant is omitted.

Proposition 7.4. *Let s be a regular element. Then $sR \leq_e R_R$.*

Proof. Let R have Goldie rank n and let $U_1 \oplus \cdots \oplus U_n$ be a maximal direct sum of uniform right ideals. Since s is regular, $sU_i \neq 0$ and sU_i is uniform (check that right ideals in sU_i are of the form sI with I a right ideal in U_i, and that pairs of such non-zero ideals have non-zero intersection). Then $sU_1 \oplus \cdots \oplus sU_n$ is also a maximal direct sum of uniform right ideals.

If L is a right ideal, then by the previous lemma, it contains a uniform right ideal V. By the maximality of the length of the direct sum,

$$0 \neq (sU_1 \oplus \cdots \oplus sU_n) \cap V \subset sR \cap L.$$

Thus $sR \leq_e R_R$ □

Now we turn to showing that an essential right ideal contains a regular element. We need more lemmas.

Lemma 7.5. *Let I be a right ideal such that all elements are nilpotent. Then $I = 0$.*

Proof. Suppose there exists $a \in I, a \neq 0$. Consider the set of ideals

$$\mathbb{S} = \{\mathrm{r.ann}(za) : z \in R, za \neq 0\}.$$

Since R is right Goldie, this set $\mathbb{S}$ contains a maximal element, say $\mathrm{r.ann}(za)$.

Let $x \in R$. Then $axz \in I$, so there is n such that $(axz)^n = 0$. So

$$(xza)^{n+1} = xz(axz)^n a = 0.$$

Hence xza is nilpotent. Say $(xza)^k = 0$, but $(xza)^{k-1} \neq 0$. Then $\mathrm{r.ann}(za) \subset \mathrm{r.ann}((xza)^{k-1}) \neq R$. So $\mathrm{r.ann}(za) = \mathrm{r.ann}((xza)^{k-1})$. Thus $xza \in \mathrm{r.ann}(za)$. So $(za)x(za) = 0$. Since this is true for all $x \in R$, we have $(za)R(za) = 0$. Since R is semiprime, we have $za = 0$. This is a contradiction. □

Lemma 7.6. *Let $a \in R$. Then $a^n R \oplus \mathrm{r.ann}(a^n) \leq_e R_R$ for all n sufficiently large.*

Proof. Since R is right Goldie, there exists N such that $\mathrm{r.ann}(a^n) = \mathrm{r.ann}(a^{n+1})$ for all $n \geq N$. Let $n \geq N$. Choose $z \in a^n R \cap \mathrm{r.ann}(a^n)$. Then $z = a^n x \in \mathrm{r.ann}(a^n)$, so $(a^n)a^n x = a^{2n}x = 0$. Thus we have $x \in \mathrm{r.ann}(a^{2n}) = \mathrm{r.ann}(a^n)$, so $z = a^n x = 0$. Thus the sum $a^n R + \mathrm{r.ann}(a^n)$ is direct.

Now let I be a non-zero right ideal and suppose $(a^n R \oplus \mathrm{r.ann}(a^n)) \cap I = 0$. Since $I \not\subset \mathrm{r.ann}(a^n) = \mathrm{r.ann}(a^{kn})$, we have $a^{kn} I \neq 0$ for all $k \geq 0$.

We claim the sum

$$a^n I + a^{2n} I + \cdots + a^{kn} I$$

is direct for $k > 0$. This is trivially true for $k = 1$. Suppose by induction this is true for $k - 1$. Let $x \in a^n I \cap (a^{2n} I + \cdots + a^{kn} I)$. So $x = a^n y = a^{2n} z, y \in I, z \in R$. So $(y - a^n z) \in \mathrm{r.ann}(a^n)$. Thus $y \in I \cap (a^n R + \mathrm{r.ann}(a^n)) = 0$. But then $x = a^n y = 0$. So the sum of the $a^{kn} I$ is direct. This contradicts the right Goldie condition. □

We finally complete the Claim 7.1.

Proposition 7.7. *An essential right ideal contains a regular element.*

Proof. Let E be a non-zero right ideal. Since $E \neq 0$, it contains a non-nilpotent element x by Lemma 7.5. By the previous lemma, let n be such that $x^n R \cap \mathrm{r.ann}(x^n) = 0$. Set $a_1 = x^n$. If $\mathrm{r.ann}(a_1) \cap E = 0$, then stop. If $\mathrm{r.ann}(a_1) \cap E \neq 0$, repeat the argument, replacing E with $\mathrm{r.ann}(a_1) \cap E$. Then we have non-zero $a_2 \in \mathrm{r.ann}(a_1) \cap E$ such that $a_2 R \cap \mathrm{r.ann}(a_2) = 0$.

Check that $a_1R + a_2R + (\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap E)$ is a direct sum. If $(\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap E) \neq 0$, then repeat to get a direct sum

$$a_1R + a_2R + a_3R + (\mathrm{r.ann}(a_1) \cap \mathrm{r.ann}(a_2) \cap \mathrm{r.ann}(a_3) \cap E).$$

Since R is right Goldie, this process must stop. So for some k, $(\mathrm{r.ann}(a_1) \cap \cdots \cap \mathrm{r.ann}(a_k) \cap E) = 0$. If we assume E is essential, then $\mathrm{r.ann}(a_1) \cap \cdots \cap \mathrm{r.ann}(a_k) = 0$.

Let $c_1 = a_1 + \cdots + a_k$. Since $\sum a_iR$ is direct, $\mathrm{r.ann}(c_1) = \mathrm{r.ann}(a_1) \cap \cdots \cap \mathrm{r.ann}(a_k) = 0$.

Now let $c = c_1^n$ where $c_1^nR \oplus \mathrm{r.ann}(c_1^n) \leq_e R_R$ by Lemma 7.6. Now $\mathrm{r.ann}(c_1^n) = \mathrm{r.ann}(c_1) = 0$. If $zc = 0$, then $\mathrm{r.ann}(z) \supset cR$ and cR is essential. So $\mathrm{r.ann}(z)$ is essential, and so is its superset $\mathrm{r.ann}(z^m)$.

Again, there is m such that $z^mR \cap \mathrm{r.ann}(z^m) = 0$. So $z^mR = 0$ since $\mathrm{r.ann}(z^m)$ is esential. So by Lemma 7.5, $z = 0$. Thus $zc = 0$ implies $z = 0$ and since $\mathrm{r.ann}(c) = 0$, we have that $c \in E$ is regular. □

We leave it as an exercise to prove

Proposition 7.8. *Let R be semiprime right Goldie. Then the quotient ring Q of R is semisimple.*

Proof. Hint: Show that Q is also semiprime right Goldie. Then use Claim 7.1 along with Proposition 6.2 to show that Q is semisimple. □

References

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