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MARRIAGE OF 4-DIMENSIONAL GRAVITY WITH 3-DIMENSIONAL CHERN-SIMONS THEORY

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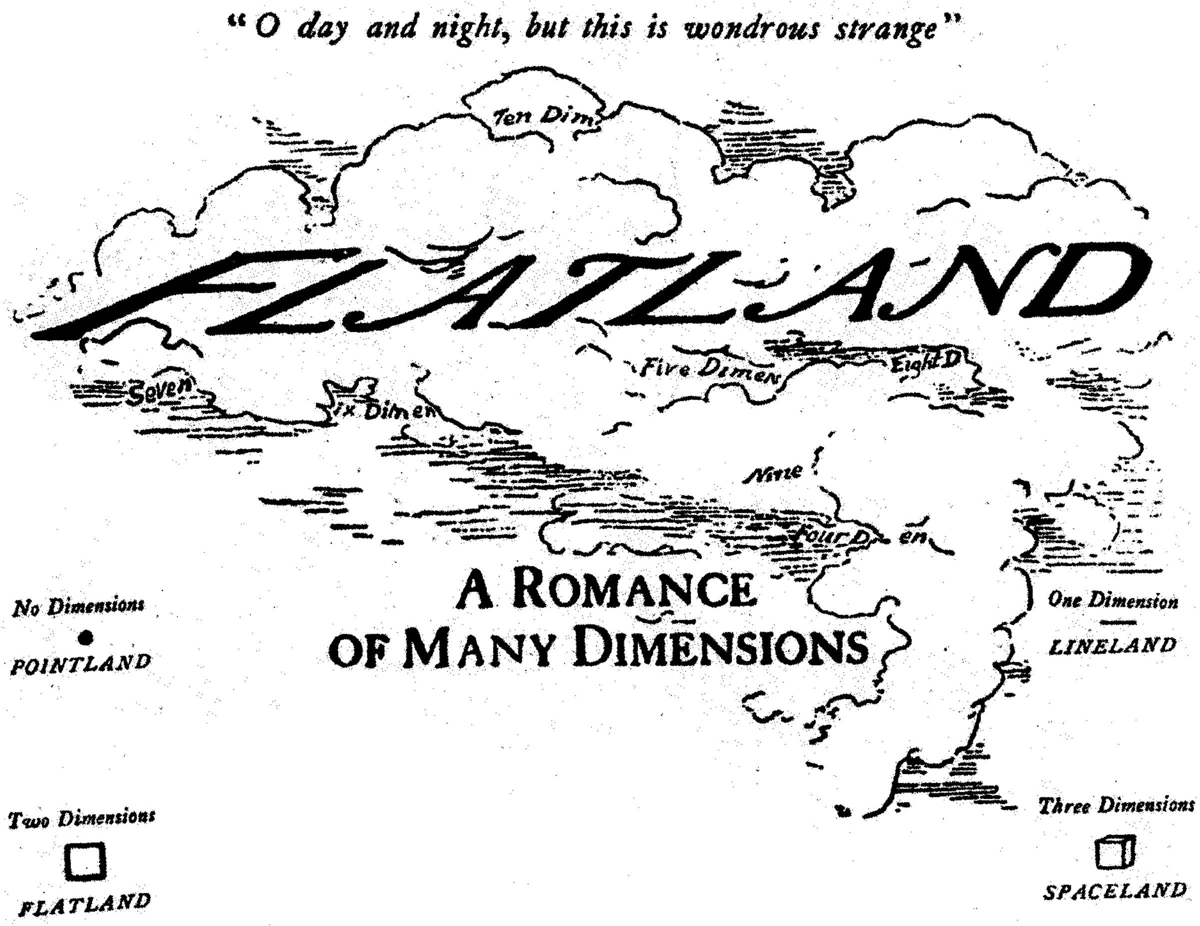

Edwin A. Abbott
(1884)

This Work is Dedicated
By a Humble Native of Flatland
In the Hope that
Even as he was Initiated into the Mysteries
Of THREE Dimensions
Having been previously conversant
With ONLY TWO
So the Citizens of that Celestial Region
May aspire yet higer and higher
To the Secrets of FOUR FIVE OR EVEN SIX DIMENSIONS
Thereby contributing
To the Enlargement of The IMAGINATION
And the possible Development
Of that most rare and excellent Gift of MODESTY

Exhortation by Abbott to the study of various dimensions. His observation that this will contribute “to the enlargement of the imagination” has been forcefully realized these days, even though his hope for a “development...of modesty” has not.

Marriage
of
4-dimensional gravity
with
3-dimensional Chern-Simons theory

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CS modification of E & M

CS: 3-d (Euclidean) embedded in 4-d space-time
(violates Lorentz boosts, CTP, ...)
[Carroll, Field, RJ, PRD **41**, 123 (90)]

Abelian gauge theory CS in 3-d

$$CS(A) = \tfrac{1}{4}\varepsilon^{ijk}F_{ij}A_k = \tfrac{1}{2}\mathbf{A}\cdot\mathbf{B}$$

$$\begin{aligned} I &= \int d^4x \left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{\mathrm{m}}{2}\mathbf{A}\cdot\mathbf{B}\right) \\ &= \int d^4x \Big(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{4}v_\mu \underbrace{\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}A_\nu}\Big) \quad \{v_\mu = (\mathrm{m}, \mathbf{0}) \end{aligned}$$

CS current in 4-d: $K^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}A_\nu = \varepsilon^{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma$

$$\partial_\mu K^\mu = \tfrac{1}{4}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F^{\mu\nu} = \tfrac{1}{2}\,{}^*F^{\mu\nu}F_{\mu\nu}$$

$${}^*F^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

$$\begin{aligned} I &= \int d^4x(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{4}\theta\,{}^*F^{\mu\nu}F_{\mu\nu}) \quad \begin{cases} v_\mu \equiv \partial_\mu\theta \\ \theta = \mathrm{m}t \end{cases} \\ &= \int d^4x(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\theta\,\partial_\mu K^\mu) \\ &= \int d^4x(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}v_\mu K^\mu) \end{aligned}$$

Physical Consequences of Modified Theory

Only Ampère's law is modified

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \mathbf{B} = \mathbf{J} + \mathrm{m}\,\mathbf{B}$$

$\Rightarrow$ gauge invariant, 2 polarizations, each travels with velocity

$\neq c$ (Lorentz boost invariance lost)

$\neq$ each other (parity lost)

$\Rightarrow$ Faraday rotation, light from distant galaxies shows no such effect in Nature.

CS Modification of Einstein Gravity

Gravity theory CS in 3-d

[Deser,Templeton, RJ, AP **140**, 372 (82)]

CS current in 4-d: $K^\mu \equiv 2\varepsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^\sigma_{\alpha\tau}\,\partial_\beta\Gamma^\tau_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma_{\alpha\tau}\,\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right]$

$$\partial_\mu K^\mu = \frac{1}{2}{}^*R^\sigma{}_\tau{}^{\mu\nu}\,R^\tau{}_{\sigma\mu\nu} \equiv \frac{1}{2}{}^*RR \qquad {}^*R^\sigma{}_\tau{}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R^\sigma{}_{\tau\alpha\beta}$$

Proposal: 4-d gravity with CS modification

[Pi, RJ, PRD **68**, 104012 (03)]

$$\begin{aligned} I &= \tfrac{1}{16\pi G}\int d^4x\left(\sqrt{-g}\,R + \tfrac{1}{4}\,\theta\,{}^*RR\right) \\ &= \tfrac{1}{16\pi G}\int d^4x\left(\sqrt{-g}\,R - \tfrac{1}{2}\,v_\mu K^\mu\right) \qquad \{v_\mu \equiv \partial_\mu\theta \end{aligned}$$

Equation of motion: $G^{\mu\nu} + C^{\mu\nu} = -8\pi\,G\,T^{\mu\nu}$

$$\begin{aligned} G^{\mu\nu} &\equiv R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\,R^\alpha_{\ \alpha} \\ \sqrt{-g}\,C^{\mu\nu} &\equiv \tfrac{\delta}{\delta g_{\mu\nu}}\tfrac{1}{4}\int d^4x\theta{}^*RR \end{aligned}$$

$$= -\frac{1}{2}(v_\sigma\varepsilon^{\sigma\mu\alpha\beta}D_\alpha R^\nu_\beta + v_{\sigma\tau}{}^*R^{\tau\mu\sigma\nu} + \mu \leftrightarrow \nu) \qquad \{v_{\sigma\tau} \equiv D_\sigma v_\tau$$

$G^{\mu\nu}$: "Einstein tensor" (symmetric, $D_\mu G^{\mu\nu} = 0$)

$C^{\mu\nu}$: "4-d Cotton tensor" (traceless, symmetric)

BUT $D_\mu C^{\mu\nu} = \frac{1}{8\sqrt{-g}}\,v^\nu\,{}^*RR$

Consistancy condition on new dynamics: ${}^*RR = 0$

NB: $\int d^4x\, \theta\, {}^*\!RR$ violates diffeomorphism
invariance when θ is external

But: Consistency condition $\Rightarrow {}^*\!RR = 0$
Symmetry breaking suppressed!

NB: $\int d^4x\, \theta\, {}^*\!RR$ preserves diffeomorphism
invariance when θ is dynamical

vary $g_{\mu\nu} \Rightarrow$ equation as before
vary $\theta \Rightarrow {}^*\!RR = 0$

$\Rightarrow$ Equations of Motion the same!

Physical Conseqences of Modified Theory

with $\theta = t/\mathrm{m}, v_\sigma = (\frac{1}{\mathrm{m}}, \mathbf{0}), v_{\sigma\tau} = -\frac{1}{\mathrm{m}}\Gamma^0_{\sigma\tau}$

1) Schwarzschild solution presists

   because:

   $$C^{\mu\nu}\Big|_{\substack{stationary\ radially \\ symmetric\ metric}} = 0$$

   Classical tests of GR satisfied!

   $$\text{NB:} {}^*RR\Big|_{Schwarzschild} = 0$$

   $$\text{NB:} {}^*RR\Big|_{Kerr} \neq 0$$

   Kerr solution must be deformed to satisfy modified equations.

2) Linear analysis $\qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

   $$G^{linear}_{\mu\nu} + C^{linear}_{\mu\nu} = -8\pi G T_{\mu\nu}$$

   $$\partial_\mu (\mathrm{LHS})^{\mu\nu} = 0$$

   gauge symmetry of LHS: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu$

   Decompose $h_{\mu\nu}$ and extract h^{ij}_{TT}

   In Einstein theory, only h^{ij}_{TT} propagates with d 'Alembertian; other components non-propagating or gauge excitations.

   Same is true when $C^{linear}_{\mu\nu}$ is included.

(a) Wave Motion

Equation of motion for h^{ij}_{TT}

$$\left(\delta^{im}\,\delta^{jn} + \frac{1}{2\mathrm{m}}\,\varepsilon^{ipn}\,\delta^{mj}\,\partial_p + \frac{1}{2\mathrm{m}}\,\varepsilon^{jpn}\,\delta^{mi}\,\partial_p\right)\Box\, h^{mn}_{TT}$$

$$= \mathcal{O}^{ij|mn}_{CS}\,\Box\, h^{mn}_{TT} = -16\pi G\, T^{ij}_{TT}$$

$$\Box h^{mn}_{TT} = -16\pi G\, \mathcal{P}^{mn|ij}_{CS}\, T^{ij}_{TT} = -16\pi G\, \tilde{T}^{ij}_{TT}$$

$$\mathcal{P} \equiv \mathcal{O}^{-1}$$

- Plane waves move with velocity c for both polarizations; NO Faraday rotation

- intensity of definite helicity waves differs

$$\frac{-}{+} \sim \left(1 + \frac{4k}{\mathrm{m}}\right) \qquad \omega = kc$$

2 helicities in conventional gravity: usually a consequence of diffeomorphism invariance.

In our deformed theory we again find just 2 helicities!

(b) Energy momentum (pseudo-) tensor in Einstein theory

$$
\begin{aligned}
G_{\mu\nu} &= -8\pi\, G\, T_{\mu\nu} \\
G_{\mu\nu}^{linear} &= -8\pi\, G \left(T_{\mu\nu} + \tfrac{1}{8\pi G}\, G_{\mu\nu}^{non-linear} \right) \\
D^{\mu} G_{\mu\nu} &= 0 \Rightarrow \partial^{\mu}\, G_{\mu\nu}^{linear} = 0 \\
\tau_{\mu\nu} &= T_{\mu\nu} + \tfrac{1}{8\pi G}\, G_{\mu\nu}^{non-linear} \\
\partial^{\mu} \tau_{\mu\nu} &= 0
\end{aligned}
$$

$\tau_{\mu\nu}$: symmetric, ordinarily conserved
energy-momentum (pseudo-) tensor.

Same is true when $C_{\mu\nu}$ is included
(NB: $D^{\mu} C_{\mu\nu} = quadratic$, $\partial^{\mu}\, C_{\mu\nu}^{linear} = 0$)

$$
\begin{aligned}
\tau_{\mu\nu} &= T_{\mu\nu} + \frac{1}{8\pi\, G} \left(G_{\mu\nu}^{non-linear} + C_{\mu\nu}^{non-linear} \right) \\
\partial^{\mu} \tau_{\mu\nu} &= 0
\end{aligned}
$$

$\tau_{\mu\nu}$: symmetric, ordinarily conserved
energy-momentum (pseudo-) tensor
in spite of apparent symmetry breaking!

Analogy with Stückelberg method for mass term in Abelian gauge theory

$$I_m = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu \right)$$

$$\delta_{gauge}\, I_m = \int d^4x\; m^2 A^\mu\, \partial_\mu \lambda = -\int d^4x\; m^2 \partial_\mu A^\mu \lambda$$

Not invariant: gauge invariance broken by $\partial_\mu A^\mu$

BUT: $\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu$

$$m^2 \partial_\nu A^\nu = 0$$

restores gauge invariance!

$$\partial_\mu F^{\mu\nu} + m^2 \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\Box} \right) A_\mu = J^\nu$$

Difference: $m^2 \to$ dynamical field $m^2(x)$

$$\frac{\delta I_m}{\delta m^2(x)} = \frac{1}{2} A^\mu(x) A_\mu(x) = 0$$

NOT an equation of original theory!

(Higgs mechanism in unitary gauge provides cure: kinetic + potential terms for "$m^2(x)$" field.)

Physical Conseqences of Modified Theory
with $\theta = t/m, v_\sigma = (\frac{1}{m}, \mathbf{0}), v_{\sigma\tau} = -\frac{1}{m}\Gamma^0_{\sigma\tau}$

1) Scharwzschild solution presists
proof:

(a) evaluate $C^{\mu\nu}$ on stationary metric tensor

$$g_{\mu\nu} = \begin{pmatrix} N & 0 \\ 0 & g_{ij} \end{pmatrix}$$

N, g_{ij} time-independent $\Rightarrow$

$C^{0\mu}$vanishes
$\sqrt{-g}\, C^{mn} = \sqrt{g}\, {}^3C^{mn}$

(b) evaluate $\sqrt{-g}\, C^{mn} = \sqrt{g}\, {}^3C^{mn}$
on radially symmetric metric $\Rightarrow$
$C^{mn} = 0$

2) Linear analysis $\qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$G^{linear}_{\mu\nu} + C^{linear}_{\mu\nu} = -8\pi\, G\, T_{\mu\nu}$$

$$\partial_\mu(\mathrm{LHS})^{\mu\nu} = 0$$

gauge symmetry of LHS: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu$
Decompose $h_{\mu\nu}$ and extract h^{ij}_{TT}. In Einstein theory, only h^{ij}_{TT} propagates with d 'Alembertian; other components non-propagating or gauge excitations Same is true when $C^{linear}_{\mu\nu}$ is included.

Understanding consistency condition:
infinitesimal diffeomorphism

$$\begin{aligned} \delta x^\mu &= -f^\mu(x) \\ \delta g_{\mu\nu} &= D_\mu f_\nu + D_\nu f_\mu \\ \delta\,({}^*RR) &= \partial_\mu(f^\mu\ {}^*RR) \end{aligned}$$

$$\begin{aligned} \delta \mathrm{I}_{CS} &= \tfrac{1}{4}\int d^4x\ \theta\ \delta({}^*RR) = -\tfrac{1}{4}\int d^4x\ f^\mu v_\mu {}^*RR \\ \delta \mathrm{I}_{CS} &= \int d^4x\, \frac{\delta I_{CS}}{\delta g_{\mu\nu}}\ \delta g_{\mu\nu} \quad = \ 2\int d^4x\, \sqrt{-g} C^{\mu\nu}\, D_\mu f_\nu \\ &= -2\int d^4x\sqrt{-g}\ \ (D_\mu C^{\mu\nu}) f_\nu \\ &\Rightarrow\ D_\mu C^{\mu\nu} = \frac{1}{8\sqrt{-g}}\ v^\nu\, {}^*RR \end{aligned}$$

$\Rightarrow$ θ is external, does NOT transform, symmetry is broken by non-vanishing $v^{\nu *}RR$

But consistency of equations requires ${}^*RR = 0$
symmetry restored?

$\Rightarrow$ if θ is dynamical, transforms by $\delta\theta = f^\mu \partial_\mu \theta$
symmetry is preserved: $\delta I_{CS} = 0$
equations of motion: vary $g_{\mu\nu} \Rightarrow$ as before
vary $\theta \Rightarrow {}^*RR = 0$
already present

symmetric theory has the SAME equations of motion as broken theory.