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THE CPT THEOREM AND LORENTZ COVARIANCE

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The CPT Theorem and Lorentz Covariance

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What should you get from this talk?

- A new formulation in which locality follows from Lorentz covariance
- The general theorems about CPT and spin-statistics follow from Lorentz covariance and spectrum
- Why CPT is fundamental and how to calculate it directly on an arbitrary field

Outline

1. Choice of Lorentz covariance condition

   (a) On fields or vevs of products (Wightman)

   (b) On T-products

2. (b) CPT and spin-statistics theorems follow;
   CPT violation implies Lorentz violation

3. Why CPT is fundamental,
   but C, P, T, CP, PT, etc. are not

CPT is the only new symmetry that follows from Lorentz covariance of T-products.

Wightman axioms

O. Relativistic transformation of states

I. Domain and continuity of the fields

II. Transformation law of the fields

III. Local commutativity (microscopic causality)

Transformation law of the fields

$$U(a,A)\phi^{(p)}(x)U^{\dagger}(a,A) = S^{(p)}(A)^{-1}\phi^{(p)}(\Lambda x + a)$$

$$U(a,A)|0\rangle = |0\rangle$$

$$\langle 0|\phi^{(p_1)}(x_1)\cdots\phi^{(p_n)}(x_n)|0\rangle$$
$$= [\prod_1^n S^{(p_i)}(A)^{-1}]\langle 0|\phi^{(p_1)}(\Lambda x_1 + a)\cdots\phi^{(p_n)}(\Lambda x_n + a)|0\rangle$$

$$\langle 0|T(\phi^{(p_1)}(x_1)\cdots\phi^{(p_n)}(x_n))|0\rangle$$
$$= [\prod_1^n S^{(p_i)}(A)^{-1}]\langle 0|T(\phi^{(p_1)}(\Lambda x_1 + a)\cdots\phi^{(p_n)}(\Lambda x_n + a))|0\rangle$$

Proof that covariance of T-products implies locality of fields

$N_L + N_R + 2$ points

$$\tau(x_{-N_L}, \cdots, x_{-1}, x_{-0}, x_0, x_1, \cdots, x_{N_R}) =$$
$$\langle 0|T(\phi(x_{-N_L}) \cdots \phi(x_{-1})\phi(x_{-0})\phi(x_0)\phi(x_1) \cdots \phi(x_{N_R}))|0\rangle$$

$$\xi_{-N_L} = x_{-N_L} - x_{-N_L+1}, \cdots, \xi_{-1} = x_{-1} - x_{-0},$$
$$\xi = x_{-0} - x_0,\ \xi_1 = x_0 - x_1, \cdots,\ \xi_{N_R} = x_{N_R-1} - x_{N_R}$$

$$\sum_{l=-1}^{-N_L} \lambda_l \xi_l + \lambda\xi + \sum_{r=1}^{N_R} \lambda_r \xi_r \sim 0, \forall\lambda_l \geq 0, \forall\lambda_r \geq 0,\ \lambda \geq 0$$

and

$$\sum_{l=1}^{N_L-1} \lambda_l + \lambda + \sum_{r=1}^{N_R-1} \lambda_r > 0$$

Proof that covariance of T-products implies locality of fields

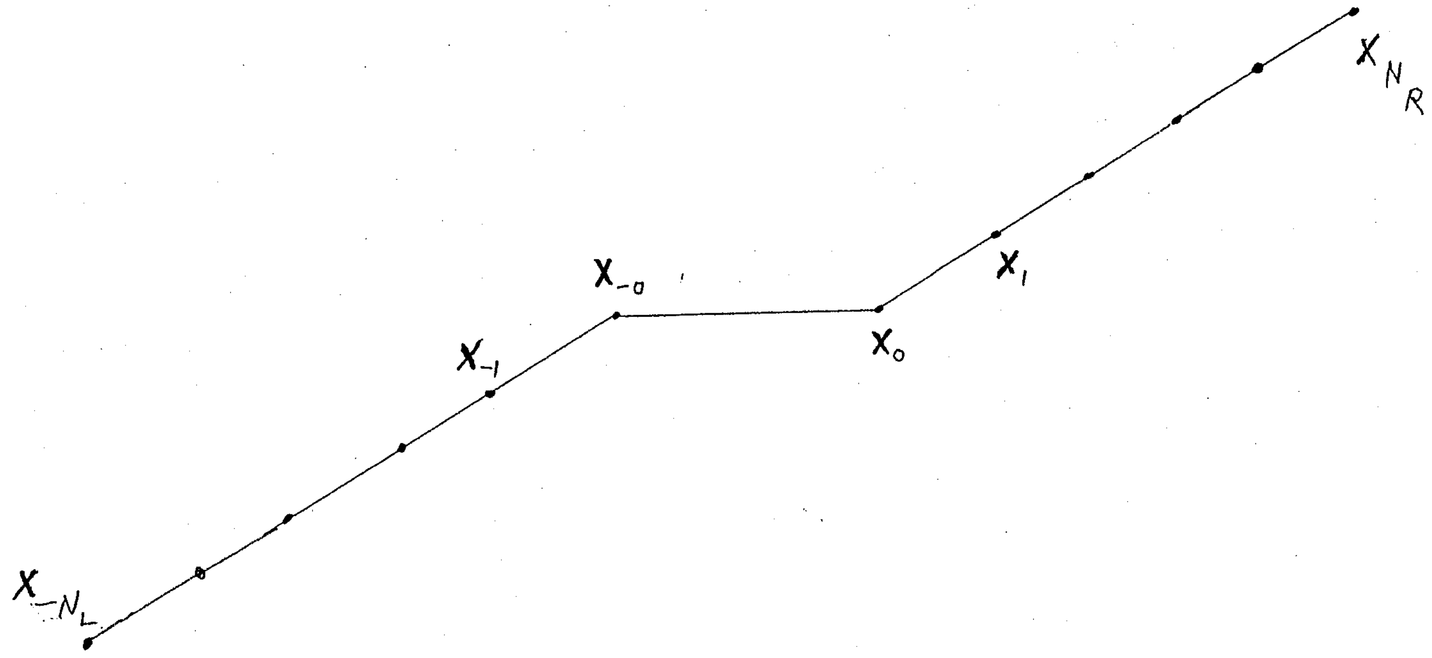
(continued)

We can choose the points so that a small boost will reverse the time order of x_{-0} and x_0

$$\mathcal{W}(x_{-N_L}, \cdots, x_{-1}, x_{-0}, x_0, x_1, \cdots, x_{N_R})$$
$$= \langle 0|\phi(x_{-N_L}) \cdots \phi(x_{-1})\phi(x_{-0})\phi(x_0)\phi(x_1) \cdots \phi(x_{N_R})|0\rangle$$

$$\zeta_{-N_L} = \xi_{-N_L} + i\eta_{-N_L}, \cdots, \zeta_{-1} = \xi_{-1} + i\eta_{-1};$$
$$\zeta_1 = \xi_1 + i\eta_1, \cdots, \zeta_{N_R} = \xi_{N_R} + i\eta_{N_R}.$$

$$\langle 0|\phi(x_{-N_L}) \cdots \phi(x_{-1})[\phi(x_{-0}), \phi(x_0)]_{\mp}\phi(x_1)n \cdots \phi(x_{N_R})|0\rangle = 0$$



X_{-N_L}
X_{-1}
X_{-0}
X_0
X_1
X_{N_R}

Can take boundary values to get

$$\langle 0|T(\phi(x_{-N_L})\cdots\phi(x_{-1})\phi(x_{-0})\phi(x_0)\phi(x_1)\cdots\phi(x_{N_R}))|0\rangle =$$

$$\langle 0|T(\phi(x_{-N_L})\cdots\phi(x_{-1})\phi(x_0)\phi(x_{-0})\phi(x_1)\cdots\phi(x_{N_R}))|0\rangle;$$

Choose the $N_L + N_R + 2$ points as follows,

$$x_{-l} = (-la, 0, -3la, 0),\ 1 \leq l \leq N_L$$

$$x_r = (ra, 0, 3ra, 0),\ 1 \leq r \leq N_R$$

$$x_{-0} = (0, 0, 0, -a),\ x_0 = (0, 0, 0, a)$$

Lorentz covariance of T-products implies the CPT theorem

With the Wightman axioms, weak local commutativity of fields was a necessary additional assumption needed for the CPT theorem. With Lorentz covariance of T-products, weak local commutativity follows without further assumptions.

Corollary: Covariant-looking nonlocal actions need not give covariant theories

$$A_I = \int d^4x d^4y d^4z f(x-y, y-z)\phi_1(x)\phi_2(y)\phi_3(z)$$

The fields must be local in order for the T-products to be covariant.

The old literature shows that such actions violate local comm. of fields,

Physical conditions for CPT

- Lorentz covariance

- Spectrum

Technical tools for CPT

- Vacuum matrix elements for fields
- Extension of covariance of vacuum matrix elements under the real Lorentz group to covariance under the complex Lorentz group.
- Analytic continuation of vacuum matrix elements of fields

Four disconnected components of the real Lorentz group

$$\Lambda_{+}^{\uparrow}, \det\Lambda = 1, \Lambda_{0}^{0} \geq 1; \Lambda = (1,1,1,1)$$

$$\Lambda_{+}^{\downarrow}, \det\Lambda = 1, \Lambda_{0}^{0} \leq -1; \Lambda = (-1,-1,-1,-1)$$

$$\Lambda_{-}^{\uparrow}, \det\Lambda = -1, \Lambda_{0}^{0} \geq 1; \Lambda = (1,-1,-1,-1)$$

$$\Lambda_{-}^{\downarrow}, \det\Lambda = -1, \Lambda_{0}^{0} \leq -1; \Lambda = (-1,1,1,1)$$

The complex Lorentz group

L (C) the group of complex 4x4 matrices that preserve the metric

$$g = \Lambda^T g \Lambda$$

Complex Lorentz group, 2

Still have $\det \Lambda = 1$

however now the sign of $\Lambda^0{}_0$ can be changed continuously, so $L_+^\downarrow$ is connected to the identity.

Covering groups

SL(2,C) is the covering group of SO(1,3)

SL(2,C) x SL(2,C) is the covering group of $L_+(C)$.

Representations of SL(2,C)

- Fundamental representations

$$u'_\alpha = A_{\alpha\beta}\, u_\beta\,; \quad v'_{\dot\alpha} = A^*_{\dot\alpha\dot\beta}\, v_{\dot\beta}\,, \qquad A \in SL(2,C)$$

- General representation

$$t'^{(k,l)}_{\alpha_1\cdots\alpha_k;\dot\beta_1\cdots\dot\beta_l} = A_{\alpha_1\alpha_1'} \cdots A_{\alpha_k\alpha_k'}\, A^*_{\dot\beta_1\dot\beta_1'} \cdots A^*_{\dot\beta_l\dot\beta_l'}\, t^{(k,l)}_{\alpha_1'\cdots\alpha_k';\dot\beta_1'\cdots\dot\beta_l'}$$

- Irreducible if S is symmetric in α's and in $\dot\beta$'s.

$\epsilon_{\alpha\beta}$ and $\epsilon_{\dot\alpha\dot\beta}$ are invariant.

Alternative description

Each index corresponds to spin ½, so these spinors have spin k/2 and l/2 under SU(2) x SU(2) formed from J + iK, where J are the rotation generators and K are the boost generators.

A in SL(2,C) induces Λ in $L_+^\uparrow$

$X_\alpha{}^{\dot\beta} = x^\mu (\sigma_\mu)_\alpha{}^{\dot\beta}$ $\quad \sigma_0 = \mathbb{1}$, $\sigma_{1,2,3}$ =Pauli matrices

X is 2x2 hermitian matrix, det $X = x^\mu x_\mu = x^2$

$X' = AXA^\dagger \Longleftrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

A and –A $\Longleftrightarrow$ same Λ

SL(2,C) covers $L_+^\uparrow$ twice

Going continuously from the identity to x → -x.

$$X = AXB.$$

Choose A=1, B=-1 or A=-1, B=1 to get x→-x.

For the first case, take A=1,

$$B = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \qquad \phi = 2\pi \Longrightarrow B = -1$$

For ℓ dotted indices, get $(-1)^{\ell}$.

Explicit complex Lorentz transformations to get x→ -x.

$$\begin{pmatrix} x'^0 \\ x'^3 \\ x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2} & i\ \sin\frac{\phi}{2} & 0 & 0 \\ i\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ 0 & 0 & \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^3 \\ x^1 \\ x^2 \end{pmatrix}$$

Condition on vacuum matrix elements

$$\mathcal{W}^{(n;p_1\cdots p_n)}(x_1,\cdots,x_n) \equiv \langle 0|\phi^{(p_1)}(x_1)\cdots\phi^{(p_n)}(x_n)|0\rangle$$

$$= \left[\prod_i^n S^{(p_i)}(A)^{-1}\right] \mathcal{W}^{(n;p_1\cdots p_n)}(\Lambda x_1+a,\cdots \Lambda x_n+a)$$

Difference vectors $\xi_j = x_j - x_{j+1}$

$$W^{(n;p_1\cdots p_n)}(x_1-x_2,\cdots,x_{n-1}-x_n) \equiv \langle 0|\phi^{(p_1)}(x_1)\cdots\phi^{(p_n)}(x_n)|0\rangle$$

$$= \left[\prod_1^n S^{(p_i)}(A)^{-1}\right] W^{(n;p_1\cdots p_n)}(\xi_1,\cdots,\xi_{n-1})$$

Support in momentum space

scalar case

$$W^{(n)}(x_1-x_2, \dots, x_{n-1}-x_n) = \int d^4q_1 \dots d^4q_{n-1} \exp\left(-i \sum_1^{n-1} q_i \cdot (x_i - x_{i+1})\right) \tilde{W}^{(n)}(q_1, \dots q_{n-1})$$

q_i are physical momenta

$q_i \in \overline{V}_+$

Analytic functions in a tube

From Fourier transforms to Laplace transforms

$e^{-iq_j \cdot \xi_j} \longrightarrow e^{-iq_j \cdot (\xi_j + i\eta_j)}$, decreasing if $\eta_j \in V_-$

$\zeta_j = \xi_j + i\eta_j$

Get an analytic function of $4(n-1)$ complex variables,

$W^{(n)}(\zeta_1, \cdots, \zeta_{n-1})$

Analytic in tube $\mathcal{T}_{n-1}$, $\operatorname{Im} \zeta_j \in V_-$.

The BHW theorem

V. Bargmann, D.W. Hall, A.S. Wightman

If $W^{(n;\rho)}(\Lambda\zeta_1, \cdots \Lambda\zeta_{n-1}) = \left[\prod_1^n S^{(\rho_i)}(A)\right] W^{(n;\rho)}(\zeta_1, \cdots, \zeta_{n-1})$

for $A \in SL(2,C)$, then $W^{(n;\rho)}$ can be analytically continued to $\mathcal{T}'_{n-1}$, the "extended tube."

$$\mathcal{T}'_{n-1} = \cup \{\Lambda \mathcal{T}_{n-1}\}, \quad \Lambda \in L_+(C).$$

Two crucial results from the BHW theorem

- $\mathcal{T}'_{n-1}$ has real points of analyticity.
- $\mathcal{T}'_{n-1}$ is invariant under L (C)

  which includes -1, i.e., spacetime inversion.

$$W^{(n;p)}(\zeta_1, \dots \zeta_{n-1}) = (-1)^L \, W^{(n;p)}(-\zeta_1, \dots -\zeta_{n-1}), \quad L = \sum l_i$$

in $\mathcal{T}'_{n-1}$.

$$S^{(k,l)}(1, -1) = (-1)^l \, \mathbb{1}$$

- This $S^{(k,l)}$ is unitary, not antiunitary.

Real points in the extended tube

Jost's result : real points ρ_j in $\mathcal{T}_{n-1}'$:

$$\sum_1^{n-1} \lambda_i \rho_i \sim 0, \quad \lambda_i \geq 0, \quad \sum_1^{n-1} \lambda_i > 0.$$

(So all $\rho_i \sim 0$.)

Jost's result for $W^{(2)}$

scalar case

$W^{(2)}(\Lambda \zeta) = W^{(2)}(\zeta)$

Find ζ^2 that come from $\Lambda\zeta$, $\zeta = \xi + i\eta$,

$\eta \in V_-$. $\zeta^2 = \xi^2 - \eta^2 + 2i\,\xi\cdot\eta$. Real points have $\xi\cdot\eta = 0$

$\eta \in V_-$. So $\xi \sim 0$, which agree with Jost.

Spacetime inversion at Jost points

$$\langle 0| \phi^{(k_1,l_1)}(x_1)\cdots \phi^{(k_n,l_n)}(x_n)|0\rangle$$

$$= (-1)^L \langle 0| \phi^{(k_1,l_1)}(-x_1)\cdots \phi^{(k_n,l_n)}(-x_n)|0\rangle$$

$$x_j - x_{j+1} = \xi_j\,, \qquad \zeta_j = \xi_j + i\eta_j$$

$$-x_j - (-x_{j+1}) = -\xi_j\,, \qquad -\zeta_j = -\xi_j - i\eta_j$$

Incompatibility of analytic continuations

If $\mathrm{Im}\, \tilde{\zeta}_j \in V_-$, then $\mathrm{Im}\, -\tilde{\zeta}_j \in V_+$.

Invert the order of the fields

Consider $\langle 0| \phi^{(p_n)}(-x_n) \cdots \phi^{(p_1)}(-x_1) |0\rangle$

$\simeq W^{(n;\,i\beta)}(\xi_{n-1}, \cdots, \xi_1).$

This $W^{(n;\,i\beta)}$ is also analytic in $\mathcal{T}'_{n-1}$.

Now we can analytically continue the relation.

Count the transpositions of the Fermi fields

- To invert the order of F Fermi fields, need $I=(F-1)+(F-2)+\cdots+1=1/2\,F(F-1)$

  transpositions.

  Get phase $(-1)^{\frac{F(F-1)}{2}} = \left((-1)^{(F-1)}\right)^{\frac{F}{2}} = (-1)^{\frac{F}{2}} = i^{F}$

  since F must be even.

The relations valid at all spacetime points

$$\langle 0 | \phi^{(k_1,l_1)}(x_1) \cdots \phi^{(k_n,l_n)}(x_n) | 0 \rangle =$$

$$i^F (-1)^L \langle 0 | \phi^{(k_n,l_n)}(-x_n) \cdots \phi^{(k_1,l_1)}(-x_1) | 0 \rangle$$

Restoring the order of the fields

Hermiticity $\quad (\Psi, \Xi) = (\Xi, \Psi)^*$

$$\langle 0| \phi^{(k_1, l_1)}(x_1) \cdots \phi^{(k_n, l_n)}(x_n) |0\rangle =$$

$$i^F (-1)^L \langle 0| \phi^{(k_1, l_1)\dagger}(-x_1) \cdots \phi^{(k_n, l_n)\dagger}(-x_n) |0\rangle^*.$$

CPT for an arbitrary field

$$\Theta\, \phi^{(k,l)}(x)\, \Theta^\dagger = (-1)^l\, i^f\, \phi^{(k,l)\dagger}(-x).$$

CPT for simple cases

Dirac field: $\Theta \phi^{(1,0)}(x) \Theta^{\dagger} = i \phi^{(1,0)\dagger}(-x)$

$\Theta \phi^{(0,1)}(x) \Theta^{\dagger} = -i \phi^{(0,1)\dagger}(-x)$

$$\psi_{Dirac} = \phi^{(1,0)} + \phi^{(0,1)}$$

Scalar, pseudoscalar fields $\phi^{(0,0)}$

antisymmetric $T^{\mu\nu} = \phi^{(2,0)} + \phi^{(0,2)}$

traceless symmetric tensor $S^{\mu\nu} = \phi^{(2,2)}$

all get phase 1.

Properties of (CPT)^2

$$\Theta\, \phi^{(k,l)}(x)\, \Theta^{\dagger} = (-1)^{l}\, i^{f}\, \phi^{(k,l)\dagger}(-x)$$

$$\Theta\, \phi^{(k,l)\dagger}(x)\, \Theta^{\dagger} = (-1)^{l}\, (-i)^{f}\, \phi^{(k,l)}(-x)$$

$$\Theta^{2}\, \phi^{(k,l)}(x)\, \Theta^{\dagger 2} = \Theta\, (-1)^{l}\, i^{f}\, \phi^{(k,l)\dagger}(-x)\, \Theta^{\dagger}$$

$$= (-1)^{l}\, (-i)^{f}\, \Theta\, \phi^{(k,l)}(-x)\, \Theta^{\dagger}$$

$$= (-1)^{f}\, \phi^{(k,l)}(x),$$

so $[\Theta^{2}, B]_{-} = 0$

$[\Theta^{2}, F]_{+} = 0.$

CPT transform of the S-matrix

$$S_{\alpha,\beta} \equiv {}_{out}\langle\alpha|\beta\rangle_{in} = {}_{out}\langle\hat{\beta}|\hat{\alpha}\rangle_{in} = S_{\hat{\beta},\hat{\alpha}}$$

$$|\hat{\alpha}\rangle = \Theta|\alpha\rangle, \quad |\hat{\beta}\rangle = \Theta|\beta\rangle$$

$|\hat{\alpha}\rangle$ has particle $\rightleftarrows$ antiparticle,

$\sigma \rightarrow -\sigma$

helicity $\rightarrow$ $-$helicity

$p^\mu \rightarrow p^\mu$

$$\Theta\, S\, \Theta^\dagger = S^{-1}, \quad \text{or} \quad \Theta\, S = S^{-1}\, \Theta$$

Summary, slide 1

- Formulating Lorentz covariance on T-products instead of unordered products leads to local commutativity.
- With this formulation, the CPT and spin-statistics theorems follow from covariance and spectrum.

Summary, slide 2

In the complex Lorentz group,

is connected to the identity.

The BHW theorem allows analytic continuation to the complex Lorentz group.

The larger domain of analyticity includes real points of analyticity, Jost points.

Summary, slide 3

The analytic continuations of the original matrix element and the spacetime inverted matrix element have different domains of analyticity, so we can't take limits to equate the boundary values.

We can equate the boundary values if we invert the order of the fields. We need the spin-statistics connection to do this.

Summary, slide 4

Hermiticity of the matrix elements allows us to restore the original order of the fields and requires complex conjugation of the matrix elements, which makes CPT antiunitary, as we expect.

The final result is

Relation of CPT and Lorentz Covariance

- For Lorentz covariance, need n! relations
- For CPT, need only one relation. P = inversion

$$\langle 0|\, \phi(x_{P1}) \cdots \phi(x_{Pn})\, |0\rangle = \langle 0|\, \phi(x_1) \cdots \phi(x_n)\, |0\rangle$$

at Jost points.

- Lorentz covariance implies CPT, but not conversely.