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CONFERENCE ON FUNDAMENTAL SYMMETRIES AND FUNDAMENTAL CONSTANTS

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A STRING THEORIST TAMPERS WITH THE LAWS OF THERMODYNAMICS

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Thermal Duality Confronts Entropy:
A New Approach to String Thermodynamics?

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1. Introduction / review
2. The fundamental problem: Entropy vs. thermal duality
3. Solution #1: A thermal duality "bootstrap"
   → potentially yields exact, closed-form solutions for finite-temperature effective potentials in string theory
4. Solution #2: Towards a Duality-covariant String Thermodynamics
   → string modifications to the traditional laws of thermodynamics near the string scale!
5. Conclusions & Current Work

Some of the most intriguing features of string theory have been the existence of numerous DUALITIES which connect/relate physics in vastly dissimilar regimes

S duality : $g_s \longleftrightarrow 1/g_s$ strong / weak

T duality : $R \longleftrightarrow R_c^2/R$ large / small

⇒ Together, these have given tremendous insight into the full, non-perturbative structure of string theory.

M-theory! "Quantum geometry"...

Major progress since 1994/1995:
"SECOND SUPERSTRING REVOLUTION"

There is, however, an additional duality which has received far less scrutiny:

thermal duality : $T \longleftrightarrow T_c^2/T$ hot/cold

... follows directly from Lorentz invariance and T-duality

$\Longrightarrow$ ROOTS OF THERMAL DUALITY are just as deep as those dualities which occur at zero temperature!

Given the importance of string dualities in understanding the unique features of string theory,

WHAT NEW INSIGHTS MAY EMERGE FROM A STUDY OF THERMAL DUALITY?

As we shall see, our fundamental observation is that

THE RULES OF CLASSICAL THERMODYNAMICS ARE NOT INVARIANT WITH RESPECT TO THERMAL DUALITY!

Even if the free energies & internal energies are invariant, other quantities such as entropy & specific heat are not!

SOMETHING? needs to be modified or fixed!

Leads to a variety of new approaches towards thinking about the relations between string theory & traditional thermodynamics ...

Traditional Thermodynamics

vacuum amplitude $\quad V = \log Z$

free energy $\quad F = TV$

internal energy $\quad U = -T^2 \frac{d}{dT} V$

entropy $\quad S = -\frac{d}{dT} F$

specific heat $\quad c_V = \frac{d}{dT} U$

$$\begin{tikzcd}
 & V \arrow[dl, "T"'] \arrow[dr, "-T^2\frac{d}{dT}"] & \\
F \arrow[d, "-\frac{d}{dT}"] & & U \arrow[d, "\frac{d}{dT}"'] \\
S & & c_V
\end{tikzcd}$$

Starting from vacuum amplitude V, all relevant thermodynamic quantities F, U, S, c_V follow directly!

What does this imply for thermal duality?

String theory implies

$$V(T_c^2/T) = V(T) \qquad \text{"invariant"}$$

What about "descendent" thermodynamic quantities

$$F(T_c^2/T) = (T_c^2/T)^2 \; F(T) \qquad \checkmark$$

$$U(T_c^2/T) = -(T_c^2/T)^2 \; U(T) \qquad \checkmark$$

"covariant"

BUT...

$$S(T_c^2/T) = \frac{T^2}{T_c^2}\frac{d}{dT}\left[\frac{T_c^2}{T^2}F(T)\right] = -S(T) - \frac{2F(T)}{T}$$

$$c_V(T_c^2/T) = \frac{T^2}{T_c^2}\frac{d}{dT}\left[\frac{T_c^2}{T^2}U(T)\right] = c_V(T) - \frac{2U(T)}{T}$$

$\ast$ || Do not close back into themselves!
Improperly defined $\Longleftrightarrow$ not physical eigenquantities

"Solution" 0 :   So what ? Ignore this problem...

After all, thermodynamics is a successful, centuries-old branch of physics !

How firmly is thermal duality embedded in string theory ?

Maybe it is merely an "accident" of $V(T)$, but not fundamental ... ?

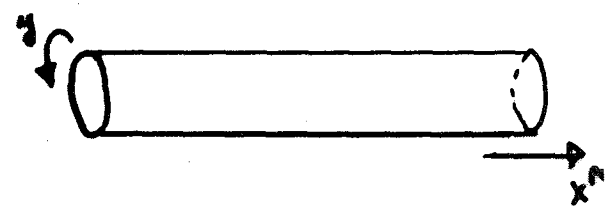
Where does thermal duality come from ?

First, recall

QFT at finite temp T $\approx$ QFT with (Euclidean) time dimension compactified on circle of radius R

$$T = \frac{1}{2\pi R}$$

thermal Matsubara modes $\approx$ Kaluza-Klein modes



fermions more complicated, will discuss later

$$\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi R)$$

$$\Rightarrow \quad \phi(x^\mu, y) = \sum_n \phi_n(x^\mu)\, e^{iny/R}$$

$$M_n^2 = m_0^2 + n^2/R^2$$

Thus, just do QFT on a circle!

Same holds in string theory!

Strings at temp T $\approx$ Strings with extra dimension compactified on circle $R = 1/2\pi T$.

But for closed strings, there is a new feature ...

NOT JUST

→ MOMENTUM (KK) MODES

$M_n \sim n/R$

BUT ALSO

→ WINDING MODES

$M_m \sim mR$

$m=3$

⇒ $\boxed{\text{T-DUALITY}}$

momentum ↔ winding

$n \leftrightarrow m$

$R \leftrightarrow 1/R$

big ↔ small

[really $R \to R_c^2/R$ where $R_c = \sqrt{\alpha'}$]

a fundamental symmetry of closed strings; holds to all orders in perturbation theory, even non-perturbatively!

However, since

strings at temp T $\sim$ strings with extra dim compactified on circle $R = 1/2\pi T$

T duality

$$R \leftrightarrow \frac{R_c^2}{R}$$

where $R_c = \sqrt{\alpha'}$

⇐

THERMAL DUALITY

$$T \leftrightarrow T_c^2/T$$

where $T_c = \dfrac{1}{2\pi\sqrt{\alpha'}}$

$$= \frac{M_s}{2\pi} = \mathcal{H}$$

- Also holds to all orders in string pert. th.
- Arises from symmetry between thermal Matsubara modes ↔ thermal "winding" modes
- Also fundamental symmetry of string theory!

Solution #1 :

Are there any special $V(T)$ such that S, c_V "accidentally" inherit the thermal duality symmetry anyway?

Can we impose this as a constraint on $V(T)$?

If so, we would be exploiting thermal duality in order to constrain $V(T)$ in a manner that transcends a direct, order-by-order, perturbative calculation!

$\Rightarrow$ A THERMAL DUALITY "BOOTSTRAP"!

First, some terminology...

$f(T)$ is covariant with $\begin{cases} \text{weight} & k \\ \text{sign} & \gamma \end{cases}$

if $$f(T_c^2/T) = \gamma \, (T_c/T)^k \, f(T)$$

Thus,

	k	γ
$V(T_c^2/T) = V(T)$	0	+1
$F(T_c^2/T) = (T_c/T)^2 F(T)$	2	+1
$U(T_c^2/T) = -(T_c/T)^2 U(T)$	2	−1
$S(T_c^2/T) = -S(T) - \frac{2F(T)}{T}$	X	X
$c_V(T_c^2/T) = c_V(T) - \frac{2U(T)}{T}$	X	X

This is the problem!

e.g.,

even though

$$F(T_c^2/T) = (T_c/T)^2 \, F(T) \quad \checkmark$$

and

$$S(T) \equiv -\frac{d}{dT} F(T)$$

$\Rightarrow$ does <u>NOT</u> imply that

$$S(T_c^2/T) = \gamma \, (T_c/T)^k \, S(T)$$

for <u>any</u> k, γ !

DERIVATIVES OF COVARIANT FUNCTIONS ARE <u>NOT</u>, IN GENERAL, COVARIANT FUNCTIONS!

<u>Can see this explicitly</u>:

If $f(T_c^2/T) = \gamma \, (T_c/T)^k \, f(T)$, then

$$\left[\frac{df}{dT}\right](T_c^2/T) = \frac{d}{d(T_c^2/T)} f(T_c^2/T)$$

$$= -(T/T_c)^2 \frac{d}{dT}\left[\gamma \left(\frac{T_c}{T}\right)^k f(T)\right]$$

$$= \underbrace{-\gamma \left(\frac{T_c}{T}\right)^{k-2} \left(\frac{df}{dT}\right.}_{\text{covariant function } \checkmark} - \underbrace{\left.\frac{kf}{T}\right)}_{\text{problem } \times}$$

But can $[df/dT]$ be "accidentally" covariant for special functions $f(T)$?

Can see this explicitly:

If $f(T_c^2/T) = \gamma\,(T_c/T)^k\, f(T)$, then

$$\left[\frac{df}{dT}\right](T_c^2/T) = \frac{d}{d(T_c^2/T)}\, f(T_c^2/T)$$
$$= -(T/T_c)^2\, \frac{d}{dT}\left[\gamma\left(\frac{T_c}{T}\right)^k f(T)\right]$$
$$= \underbrace{-\gamma\left(\frac{T_c}{T}\right)^{k-2}}_{\text{covariant function ✓}} \boxed{\left(\frac{df}{dT} - \frac{kf}{T}\right)}$$

X problem!

But can $[df/dT]$ be "accidentally" covariant for special functions $f(T)$?

ONLY IF

$$= -\delta\left(\frac{T_c}{T}\right)^{\ell} \frac{df}{dT}$$

for some δ, ℓ

$\Rightarrow \frac{df}{dT}$ would then have $\begin{cases} \text{weight} & k+\ell-2 \\ \text{sign} & \gamma\delta \end{cases}$

So we must require

$$\frac{df}{dT} - \frac{kf}{T} = -\delta\left(\frac{T_c}{T}\right)^{\ell} \frac{df}{dT}$$

Solving this differential equation we thus obtain ($\ell \neq 0$):

$$f(T) \sim (T^\ell + \delta\, T_c^{\ell})^{k/\ell}$$

where $\delta^{k/\ell} = \gamma$

> Only this special form for $f(T)$ guarantees that
>
> $\left[\frac{df}{dT}\right]$ is covariant
>
> whenever $f(T)$ is covariant!

Strategy:

Use this result to "bootstrap" our way to closed-form solutions for $V(T)$ such that S, c_V are also thermal duality covariant!

Thermodynamics

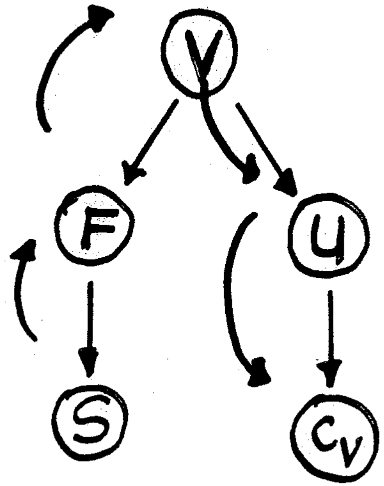
vacuum amplitude $\quad V = \log Z$

free energy $\quad F = T V$

internal energy $\quad U = -T^2 \quad V$

entropy $\quad S = - \quad F$

specific heat $\quad C_V = \quad U$



Bootstrap procedure

Start with entropy

Since we want S to be covariant, F must have form:

- $$F(T) \sim - \frac{(T^\ell + T_c^\ell)^{2/\ell}}{T_c}$$

k=2 for F

F<0 usually

inserted on dimensional grounds (choose $\delta = +1$)

This in turn implies that

- $$V(T) \sim - \frac{(T^\ell + T_c^\ell)^{2/\ell}}{T\,T_c}$$

which likewise implies the remaining solutions:

- $$U(T) \sim \frac{1}{T_c} (T^\ell + T_c^\ell)^{2/\ell - 1} (T^\ell - T_c^\ell)$$

- $$S(T) \sim 2 \frac{T^{\ell-1}}{T_c} (T^\ell + T_c^\ell)^{2/\ell - 1}$$

- $$C_V(T) \sim 2 \frac{T^{\ell-1}}{T_c} (T^\ell + T_c^\ell)^{2/\ell - 2} \times \times [T^\ell + (\ell - 1) T_c^\ell]$$

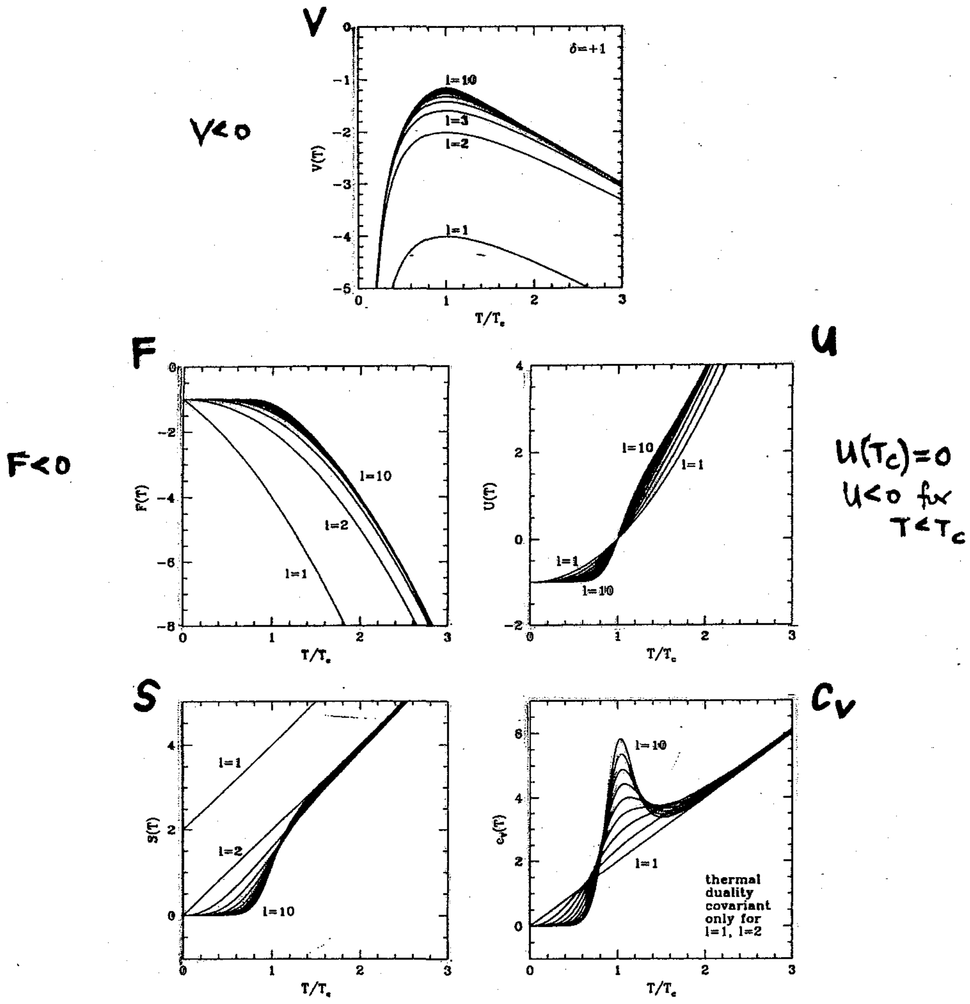


Figure 1: The thermodynamic quantities $\mathcal{V}$, F, U, S, and c_V in Eq. (3.11), plotted as functions of T for $1 \leq \ell \leq 10$ and $\delta = +1$, in units of $\mathcal{M} \equiv M_{\rm string}/2\pi = T_c$. All quantities except for c_V are thermal duality covariant for all ℓ, while c_V is covariant only for $\ell = 1, 2$. For these values of ℓ, the entropy and specific heat are exactly linear functions of T. Note that c_V develops a discontinuity as $\ell \to \infty$, suggesting the emergence of a second-order phase transition in this limit.

Thus, S is covariant for all $\underline{\ell}$.

What about c_V?

__Recall__:

$$c_V(T) \sim \frac{T^{\ell-1}}{T_c}\left(T^\ell+T_c^\ell\right)^{2/\ell-2}\underbrace{\left[T^\ell+(\ell-1)\,T_c^\ell\right]}_{\text{Problem?}}$$

__Not__ a problem if:

- $\ell=1$: final term vanishes, T^ℓ adds to weight
- $\ell=2$: combination $T^\ell+T_c^\ell$ is covariant

> Thus, special cases with
>
> $\ell=1$ and $\ell=2$
>
> ensure that both $\underline{S}$ and $\underline{c_V}$ are covariant simultaneously!

Thus far, we have simply solved a mathematics puzzle!

$\Longrightarrow$ We have obtained closed-form solutions for our thermodynamic quantities such that they are all simultaneously covariant with respect to thermal duality.

**BUT $V(T)$, $F(T)$, $U(T)$ ...
should come from actual calculations in string theory!**

$\Longrightarrow$ should emerge from one-loop (and higher-loop) modular integrals, etc.

Do our solutions for arbitrary ℓ

... agree ?

... disagree ?

... even come close ?

MUST COMPARE "TOP-DOWN" SYMMETRY APPROACH WITH "BOTTOM-UP" EXPLICIT CALCULATION!

Well, what is known about these thermodynamic quantities in string theory?

- HIGH- and LOW- TEMPERATURE LIMITS :

LOW T: $F(T) \sim \Lambda$ as $T \to 0$

HIGH T: $F(T) \sim \Lambda T^2$ as $T \to \infty$

where Λ is the one-loop cosmological constant.

Let's check our solutions...

Recall

$$F(T) \sim -\frac{(T^\ell + T_c^\ell)^{2/\ell}}{T_c^2}$$

LOW T: $F(T) \sim \text{constant} \to \Lambda$

HIGH T: $F(T) \sim (\text{constant})\, T^2 \to \Lambda\, T^2/T_c^2$

*** This determines the overall normalization for our solutions! Holds for all ℓ!**

Note: This behavior $\left\{ \begin{array}{ll} F(T) \sim \Lambda & \text{as } T \to 0 \\ F(T) \sim \Lambda T^2 & \text{as } T \to \infty \end{array} \right.$

is consistent with thermal duality, since

$$\underbrace{F(T)}_{\hookrightarrow\ \Lambda T^2/T_c^2} = (T/T_c)^2 \underbrace{F(T_c^2/T)}_{\hookrightarrow\ \Lambda}$$

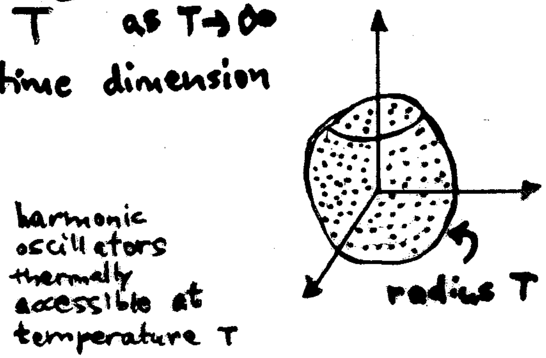
(high-temp limit) (low-temp. limit)

BUT IN QFT, this is very strange behavior!

Recall: in QFT,

$$F(T) \sim T^D \quad \text{as } T \to \infty$$

D = spacetime dimension



→ String theory behaves as though it has an effective dimensionality

$$D_{eff} = 2 \quad \text{as} \quad T \to \infty!$$

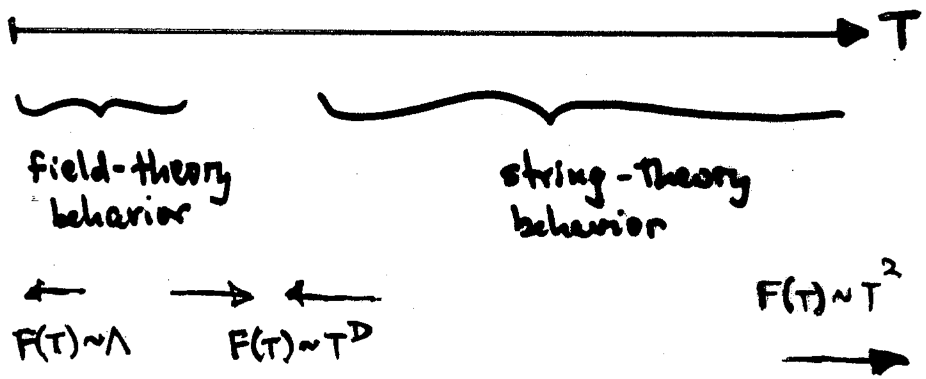
Conflict with field theory??

$\boxed{\text{No!}}$

Field theory should be the low-temperature limit of string theory!

Thus,

high-temperature limit of field theory
$\Leftrightarrow$ low-temperature limit of string theory!



In string theory,

$$F(T) \sim \Lambda + T^D$$

leading correction at small T!

Do our "top-down" bootstrap solutions have <u>this</u> property as well?

Recall:
$$F(T) \equiv \Lambda \, \frac{(T^{\ell} + T_c^{\ell})^{2/\ell}}{T_c^{2}}$$

Expand around $T \to 0$, keeping first subleading term in limit

$$\Rightarrow \quad F(T) \approx \Lambda + \frac{2\Lambda}{\ell}\left(T/T_c\right)^{\ell} + \dots$$

Thus, identify $\ell = D$!

Thus, our solutions are:

$$\boxed{F(T) = \Lambda \, \frac{(T^{D} + T_c^{D})^{2/D}}{T_c^{2}}}$$

- proper $T \to 0$ limit ✓
- proper $T \to \infty$ limit ✓
- proper field theory limit ✓

<u>But</u>:
What about a direct, <u>all</u>-temperature comparison with actual toy string models in various dimensions?

In string

$V(T) = \frac{-1}{2}$

$\mathcal{M} \equiv \frac{M_{strin}}{2\pi}$

- $Z_{model} \sim$
- $Z_{circ} \equiv$

S

N

For si

mod

by

First check simplest case : $\underline{\underline{D=2}}$.

Must compare

$$-\frac{1}{2}\mathcal{M}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\sqrt{\tau_2}\sum_{mn} \bar{q}^{(ma-n/a)^2/4}\, q^{(ma+n/a)^2/4}$$

V(T) for $\ell=2$.

$$\stackrel{?}{=} \underbrace{\frac{\pi}{6}}_{\text{cosmological constant } \Lambda \text{ for this "model"}} \frac{T^2+T_c^2}{T}$$

YES! Exact agreement for all T !

This is a remarkable (new?) mathematical identity, giving the complete temperature dependence emerging from this modular integral in closed polynomial form !

Provides important, compelling link between "top-down" symmetry-based "bootstrap" derivation and explicit one-loop "bottom up" modular integrals.

Suggests that this $D=2$ special case is "exact" already at one loop!

For $D>2$, agreement is not perfect

BUT our closed-form solutions agree to within a few percent ! $(\ell=D)$

[Difference would not be visible on previous diagrams ...]

Thus, we have highly accurately captured the leading-order temperature dependence of $\{F(T), V(T), u(T), s(T), c_V(T) \dots\}$ by a symmetry argument, leading to explicit closed-form results!

This is remarkable, since we are comparing

- a "bottom-up", one-loop, actual string calculation
- a result achieved through a non-perturbative, thermal-duality bootstrap argument

Would not have expected any relation!

Thus,

Symmetry argument

+

great agreement with explicit one-loop results

$\Rightarrow$ $\boxed{\text{CONJECTURE}}$:

Perhaps these simple forms are the correct, all-order results, and we miss having a precise agreement with the one-loop results for $D>2$ because of higher-order corrections (or even non-perturbative effects)!

If so, holds even when $Z_{model} \neq 1$

$\Rightarrow$ model-dependent effects are subleading and get "washed out" at higher loops

$\Rightarrow$ ALL STRING MODELS "flow" to a UNIVERSAL TEMPERATURE-DEPENDENCE!

These solutions also have another intriguing property...

$$\text{Recall} \quad F(T) \sim \begin{cases} \Lambda + T^D & \text{as } T\to 0 \\ T^2 & \text{as } T\to\infty \end{cases}$$

What happens for all T in between?

Define an effective scaling exponent D_{eff} at any temperature

$$F(T) \sim T^{D_{eff}} \quad \text{or} \quad S(T) \sim T^{D_{eff}-1}$$

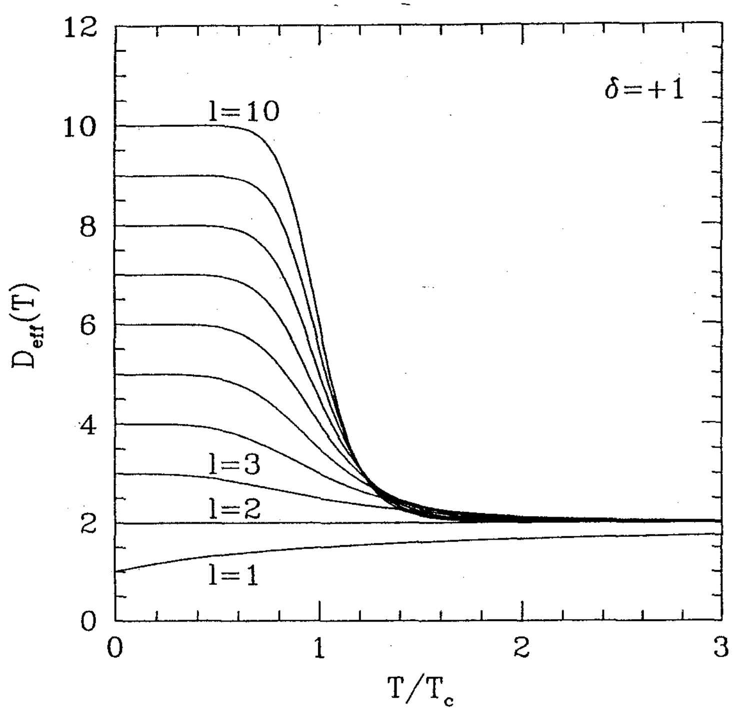
$$\Rightarrow \quad D_{eff}(T) = 1 + \frac{d\ln S}{d\ln T} = 1 + \frac{T}{S}\frac{dS}{dT} = 1 + \frac{C_V(T)}{S(T)}$$

For our explicit closed-form solutions, How does D_{eff} behave as a function of temperature?

All of our solutions interpolate between

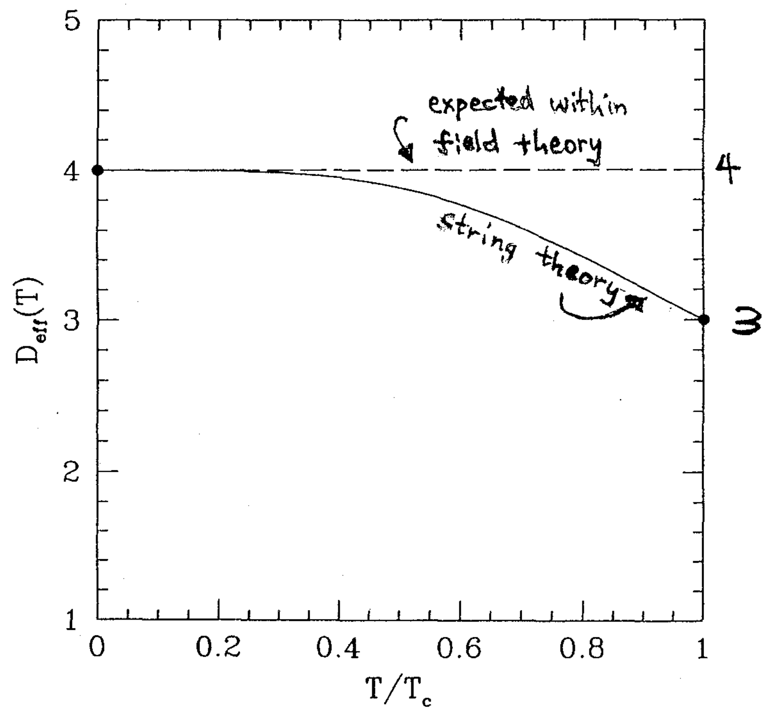
$\ell = D$ as $T \to 0$ (field-theory limit)

and $\ell = 2$ as $T \to \infty$ (string-Theory limit)



Moreover, focus on $\ell = D = 4$ solution:

What happens to D_{eff} as $T \to T_c$?



⇒ Entropy scales as though it loses exactly one dimension as $T \to T_c$!

Reminiscent of holography: $S_{BH} \sim$ Area not Volume!

Of course, this is not true holography:

HOLOGRAPHY IS NOT JUST STATE-COUNTING, BUT ALSO GEOMETRY!

For true holography, would need to formulate theory in a non-trivial background geometry with a clearly definable "bulk" and "boundary",

then formulate a map between the diminished entropy and a set of degrees of freedom on the boundary rather than the bulk.

Nevertheless, the possible relation between thermal duality and holography is worthy of further exploration...

SUMMARY OF SOLUTION #1.

- thermal duality symmetry restored to S, c_V!
- obtained explicit, closed-form results
  - → remarkable agreement with one-loop calculation
  - → conjectured to hold to all orders as exact results from string theory!

BUT: we are still unhappy from a theoretical standpoint!

- S covariant for all D, but c_V covariant only $D=2$
- More importantly, this "solution" requires a special ground state $V(T)$
  - → rules of thermodynamics still break the thermal duality symmetry which we are taking to be fundamental!

Note: this argument should hold even when thermal duality is spontaneously broken!

OK if $S(T)$ not covariant because $V(T)$ not covariant

but NOT ok if $S(T)$ not covariant because its definition in terms of $V(T)$ is faulty!

Back to the beginning :

recall $\begin{cases} F(T_c^2/T) = (T_c/T)^2 F(T) \\ S(T) \equiv dF(T)/dT \end{cases}$

$\not\Rightarrow$ $S(T)$ is covariant !

THE TEMPERATURE DERIVATIVE BREAKS THE DUALITY COVARIANCE !

Situation is completely analogous to LOCAL GAUGE INVARIANCE :

$$\phi(x) \to U(x)\, \phi(x)$$

but $\partial_\mu \phi(x) \not\to U(x)\, \partial_\mu \phi(x)$!

Do we restrict $\phi(x)$ to have a special form such that $\partial_\mu \phi(x)$ remains covariant ?

No!

Instead, we covariantize the derivative !

$$\partial_\mu \to D_\mu$$

$\Longrightarrow$ $\boxed{\text{SOLUTION \#2}}$: Change the rules of thermodynamics to preserve thermal duality directly !

$$d/dT \to D_T = ?$$

Let $D_T \equiv \dfrac{d}{dT} + \dfrac{g(T)}{T}$

extra term needed for thermal duality covariance

Solve for $g(T)$:

Let $f(T_c^2/T) = \gamma\, (T_c/T)^k\, f(T)$

Then $[D_T f](T) = \dfrac{df(T)}{dT} + \dfrac{f(T)}{T}\, g(T)$

and
$$[D_T f](T_c^2/T) = -\frac{T^2}{T_c^2}\frac{d}{dT} f(T_c^2/T) + \frac{T^2}{T_c^2}\frac{f(T_c^2/T)}{T}\, g(T_c^2/T)$$
$$= -\gamma \left(\frac{T_c}{T}\right)^{k-2}\left[\frac{df(T)}{dT} + \frac{f(T)}{T}\left(-k - g(T_c^2/T)\right)\right]$$

Thus, $D_T f$ will be thermal duality covariant with $\begin{cases} \text{weight } k-2 \\ \text{sign } -\gamma \end{cases}$

iff $\boxed{g(T) + g(T_c^2/T) = -k}$

Also wish to demand

$$\lim_{T\to 0} D_T = \frac{d}{dT} \quad \Longrightarrow \quad \boxed{\lim_{T\to 0} \frac{g(T)}{T} = 0}$$

in order to recover traditional thermodynamics for $T \ll T_c$.

In principle, lots of solutions for $g(T)$ exist! Therefore, let us take thermal duality as our guide and demand

$$\boxed{g(T_c^2/T) = \gamma_g \left(\frac{T_c}{T}\right)^{\alpha} g(T)}$$

Combining these constraints, we then obtain a <u>unique</u> result:

$$g(T) = -k\,\frac{T^{\alpha}}{T^{\alpha} + \gamma_g T_c^{\alpha}} \qquad \alpha > 1$$

Take $\gamma_g = +1$ for simplicity

$$\Longrightarrow \quad \boxed{D_T = \frac{d}{dT} - \frac{k}{T}\,\frac{T^{\alpha}}{T^{\alpha}+T_c^{\alpha}}}$$

THERMAL DUALITY COVARIANT DERIVATIVE when operating on a function of weight k.

Thus, we have [illegible]

$$D_T = \frac{d}{dT} - \underbrace{k\,\frac{T^{\alpha-1}}{T^{\alpha}+T_c^{\alpha}}}_{} \qquad (\alpha > 1)$$

functions as a "string correction":
- vanishes as $T/T_c \to 0$
- grows significant as $T \to T_c$

k : analogue of gauge charge ("duality charge") of function on which D_T acts

$\dfrac{T^{\alpha-1}}{T^{\alpha}+T_c^{\alpha}}$: analogue of gauge field (connection); α is a free parameter > 1

PRESUMABLY D_T can be calculated directly in string theory by taking into account gravitational backgrounds & backreactions, dilaton tadpoles, etc

[analogous to calculation for $\frac{d}{d\tau} \to D_\tau$ in threshold correction calculations]

$\Longrightarrow$ WOULD fix precise form of D_T and value of α!

In any case, we know

$$\begin{cases} g(T) \to 0 & \text{as } T \to 0 \\ g(T) \to -k/2 & \text{as } T \to T_c \end{cases}$$

Thus, specific form of $g(T)$ merely describes how to interpolate between these fixed limits!

String-Corrected, Duality-Covariant Thermodynamics

vacuum amplitude $\quad V = \log Z$

free energy $\quad F = T V$

internal energy $\quad U = -T^2 D_T V$

entropy $\quad \tilde{S} = -D_T F$

specific heat $\quad \tilde{C}_V = D_T U$

$$\begin{tikzcd} & V \arrow[dl, "T"'] \arrow[dr, "-T^2 D_T"] & \\ F \arrow[d, "-D_T"'] & & U \arrow[d, "D_T"] \\ S & & C_V \end{tikzcd}$$

$\Rightarrow$ Leads to new definitions for "string-corrected entropy" $\tilde{S}$ and string-corrected specific heat $\tilde{C}_V$!

We thus have

$$V(T_c^2/T) = V(T)$$
$$F(T_c^2/T) = (T_c/T)^2 F(T)$$
$$U(T_c^2/T) = -(T_c/T)^2 U(T)$$
$$\tilde{S}(T_c^2/T) = -\tilde{S}(T)$$
$$\tilde{C}_V(T_c^2/T) = \tilde{C}_V(T)$$

(weight) k	(sign) γ
0	+1
2	+1
2	−1
0	−1
0	+1

D_T $\quad$ D_T

<u>Note</u>: this implies $\tilde{S}(T)=0$ at $T=T_c$, just like $U(T)=0$ at $T=T_c$

$\tilde{S}$ is <u>odd</u> under thermal duality;
$\tilde{C}_V$ is <u>**even**</u> under thermal duality.

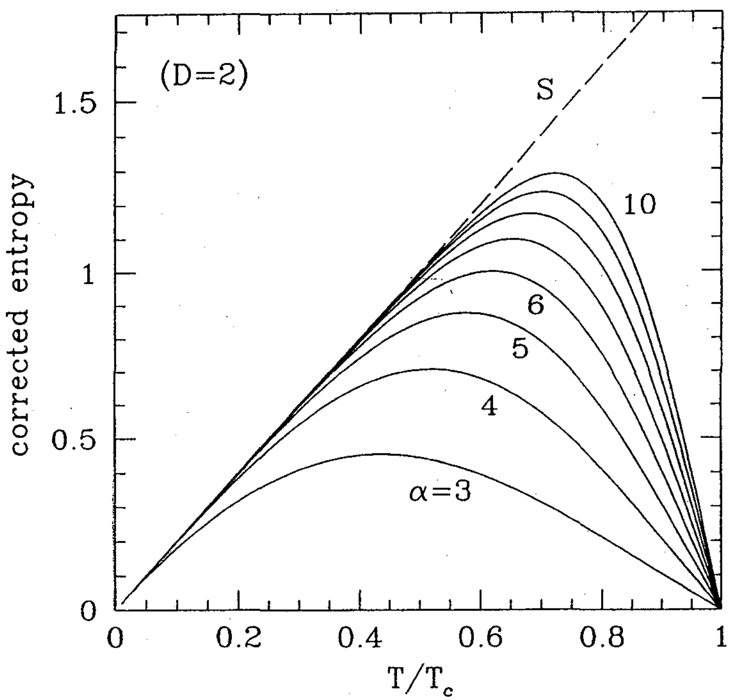
What do $\tilde{S}$ and $\tilde{C}_V$ look like as functions of temperature?

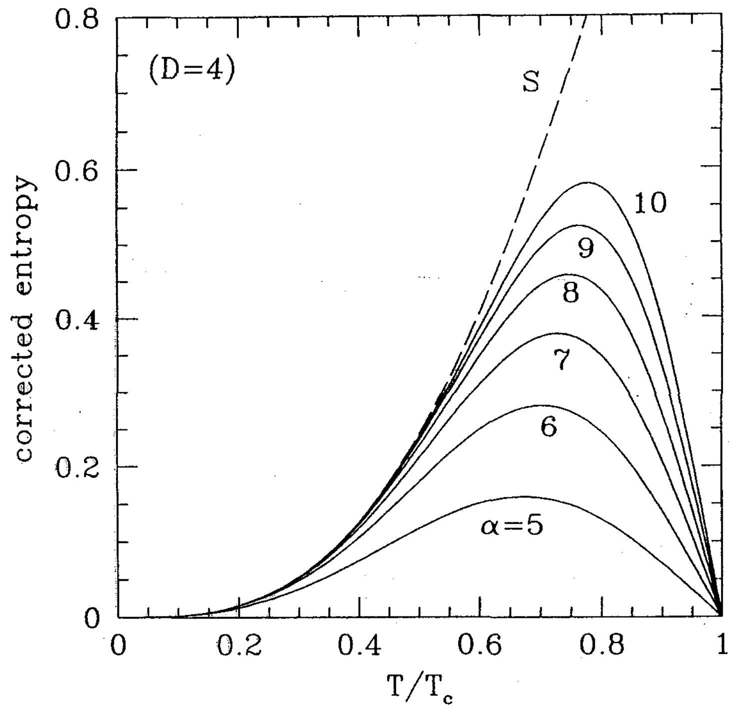
$$\tilde{S}(T) \equiv S(T) + \frac{2T^\alpha}{T^\alpha + T_c^\alpha}\frac{F(T)}{T}$$

(proper field theory limit if $\alpha > D$)

Similar for $D=4$...

$D=2$ ...





<u>Note</u>:

- S and $\tilde{S}$ agree at low temperatures.
  $\Rightarrow$ $\tilde{S}$ also obeys third law of thermodynamics
- Usually $F(T) < 0$ $\Rightarrow$ thus, $\tilde{S} < S$!
- $\tilde{S} = 0$ at $T = T_c$ because $\tilde{S}$ is <u>odd</u>.

As $T \to T_c$, we head towards a (Hagedorn-like) <u>phase transition</u>:

$\Rightarrow$ system begins to convert into new d.o.f.'s

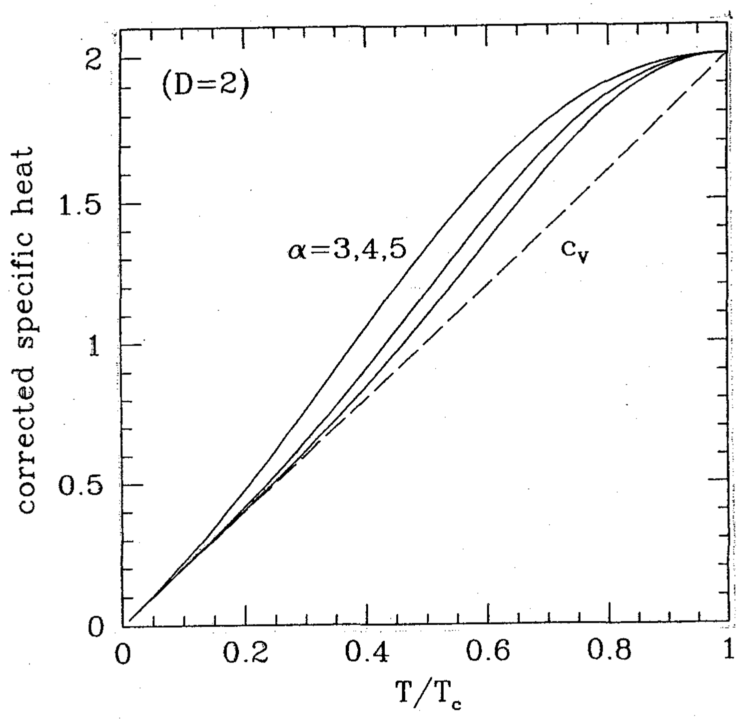
$\Rightarrow$ entropy associated with original string degrees of freedom begins to decrease and ultimately vanish!

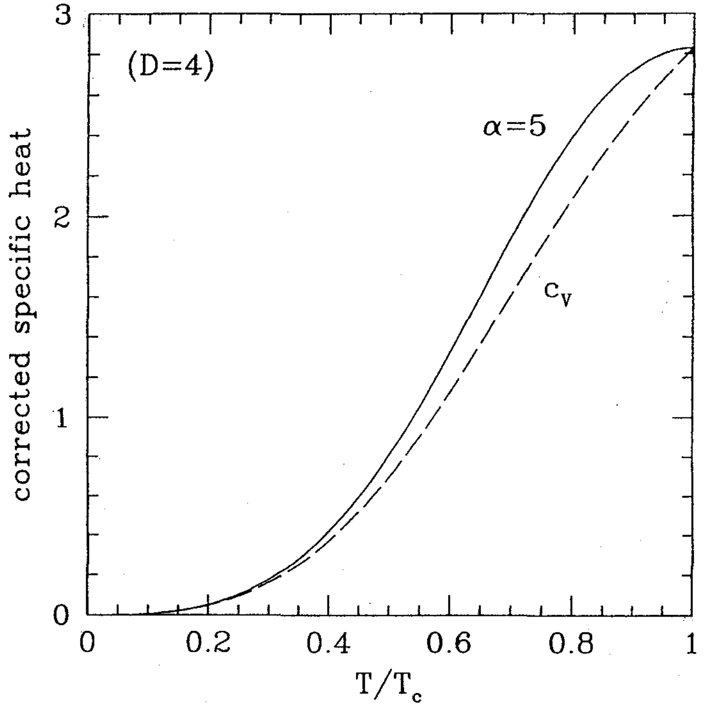
$$\tilde{c}_V(T) = c_V(T) - \frac{2T^\alpha}{T^\alpha + T_c^\alpha}\,\frac{u(T)}{T}$$

(proper field theory limit if $\alpha > D$)

Similar for $D=4$ ...

$D=2$ ...





Note:
- $u(T) < 0$ for $T < T_c$ $\Rightarrow$ $\tilde{c}_V \geq c_V$ for $T \leq T_c$
- $u(T) = 0$ at $\begin{pmatrix} T=0 \\ T=T_c \end{pmatrix}$ $\Rightarrow$ $\tilde{c}_V = c_V$ at $\begin{pmatrix} T=0 \\ T=T_c \end{pmatrix}$
- $\frac{d\tilde{c}_V(T)}{dT} = 0$ at $T = T_c$ because $\tilde{c}_V$ is even!

Note: $\tilde{c}_V \geq 0$ always

$\Rightarrow$ NO THERMAL INSTABILITIES!

Now we can understand why the traditional "uncorrected" S, c_V fail to be covariant:

$$S(T) = \tilde{S}(T) - \frac{2T^\alpha}{T^\alpha + T_c^\alpha}\,\frac{F(T)}{T}$$

$$c_V(T) = \underbrace{\frac{2T^\alpha}{T^\alpha + T_c^\alpha}\,\frac{u(T)}{T}}_{\substack{\text{odd} \\ \gamma=-1}} + \underbrace{\tilde{c}_V(T)}_{\substack{\text{even} \\ \gamma=+1}}$$

$\Rightarrow$ Both S and c_V are improper admixtures of odd and even functions

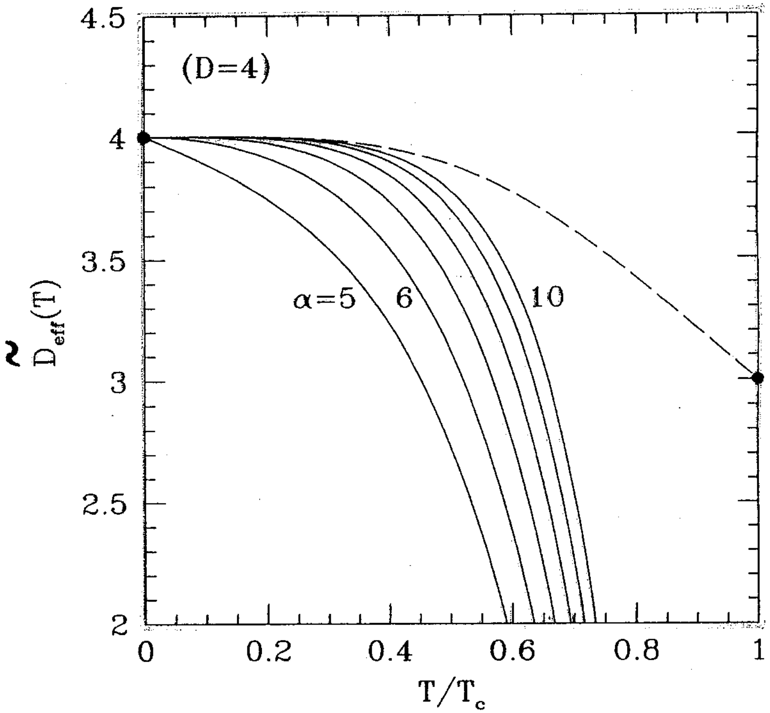
The covariant derivative D_T disentangles the even/odd behavior naturally, producing $\tilde{S}$ and $\tilde{c}_V$ which are now naturally "grouped together" just like (F, u):

	$\gamma=+1$	$\gamma=-1$	
$k=2$	$(F$,	$u)$	$\leftarrow$ $k=2$ "multiplet"
$k=0$	$(\tilde{c}_V$,	$\tilde{S})$	$\leftarrow$ $k=0$ "multiplet"

What is scaling behavior of $\tilde{S} \sim T^{\tilde{D}_{\rm eff}-1}$?

Define, as before,

$$\tilde{D}_{\rm eff} \equiv 1 + \frac{d\ln\tilde{S}}{d\ln T} = 1 + \frac{T}{\tilde{S}}\frac{d\tilde{S}}{dT}$$



Net effect of string corrections is to ENHANCE the "holographic" behavior

$\Rightarrow$ Lowers the scale at which holography becomes significant!

FERMIONS ?

In QFT, **bosons:** periodic under temperature "circle"
modes $n \in \mathbb{Z}$
"untwisted"

fermions: antiperiodic around circle
modes $r \in \mathbb{Z} + 1/2$
"twisted"

Same in string theory!

Consequences? Thermal duality spontaneously broken!
(just like Scherk-Schwarz)

Must compactify on a "twisted" circle:

- Model M_1 at $R = \infty$ $(T = 0)$
- Model M_2 at $R = 0$ $(T = \infty)$

Thermal effects interpolate between different models!

finite-temperature string model $\longrightarrow T$

Model M_1 Model M_2

$F(T) \sim \Lambda_1$ as $T \to 0$

$F(T) \sim \Lambda_2 T^2/T_c^2$ as $T \to \infty$

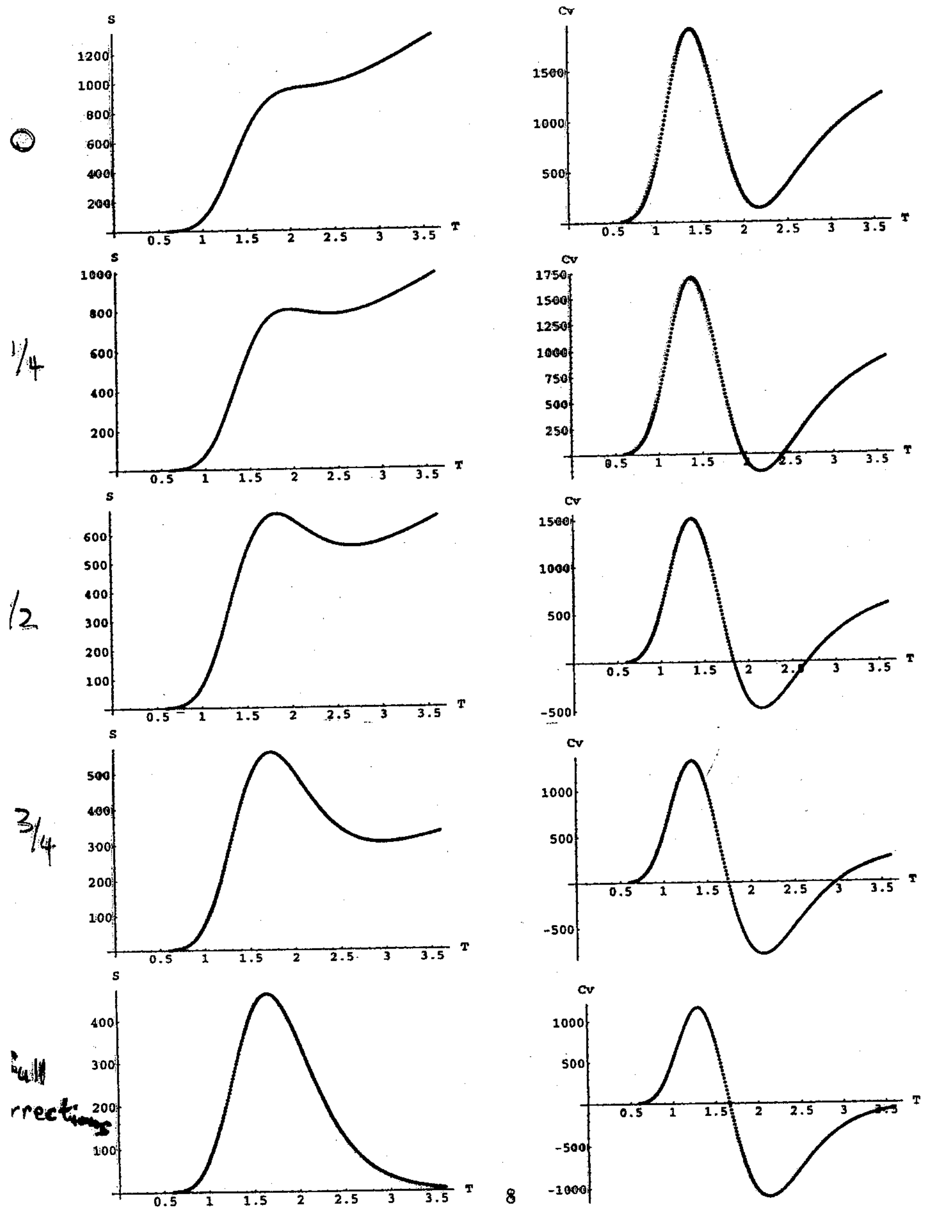
Even though thermal duality is spontaneously broken in such theories,

STILL NEED RULES OF THERMODYNAMICS TO RESPECT THE UNDERLYING SYMMETRY!

After all, OK if $S(T)$ is non-covariant because $V(T)$ is itself non-covariant
[because of spontaneously broken ground state]

BUT NOT OK if $S(T)$ is non-covariant because of its definition in terms of $V(T)$!

THUS, APPLY DUALITY-COVARIANT THERMODYNAMICS TO THIS CASE AS WELL, AND FIND...



EVEN! $\leftarrow (T^* \simeq 1.6) \rightarrow$ **ODD!**

- $T^* \neq T_c$
- $\tilde{S}$ is <u>even</u>, $\tilde{C}_V$ is <u>odd</u>

Nevertheless, our string-corrected thermodynamics exposes this additional, hidden duality!

— ie, this is the surviving remnant of thermal duality after spontaneous breaking by fermion twists!

USE THIS TO DEVELOP A "BOOTSTRAP" as before

$\boxed{\text{Yes!}}$

find $\quad T_*/T_c = \sqrt{\Lambda_1/\Lambda_2} \qquad (\Lambda_1, \Lambda_2 \neq 0)$

and the new closed-form solutions

$$F(T) \sim (T^D + T_c^D)(T^D T_c^D + T_*^{2D})^{1/D}$$

or

$$F(T) = \Lambda_1 \left[1 + \left(\frac{T}{T_c}\right)^D\right]^{1/D} \left[1 + \left(\frac{\Lambda_2}{\Lambda_1}\right)^D \left(\frac{T}{T_c}\right)^D\right]^{1/D}$$

Verify $\quad F(T) \sim \Lambda_1$ as $T \to 0$

$\sim \Lambda_2 (T/T_c)^2$ as $T \to \infty$ ✓

Conclusions & Outlook

- Developed a "bootstrap" approach to obtain closed-form solutions which capture leading-order temperature dependence of various thermodynamic quantities
  - → restore duality covariance for S, c_V
  - → conjectured that these solutions hold as <u>exact</u> results
- Developed a self-consistent, manifestly duality-covariant generalization of thermodynamics
  - → new definitions of both $\tilde{S}$ and $\tilde{c}_V$
  - → agrees with standard thermodynamics at low temperatures, but introduces "string corrections" which grow as $T \to T_C$
- Intriguing connections to
  - → holography near the string scale
  - → new versions of thermal duality
  - → Hagedorn phase transitions and other Planck-scale phenomena...

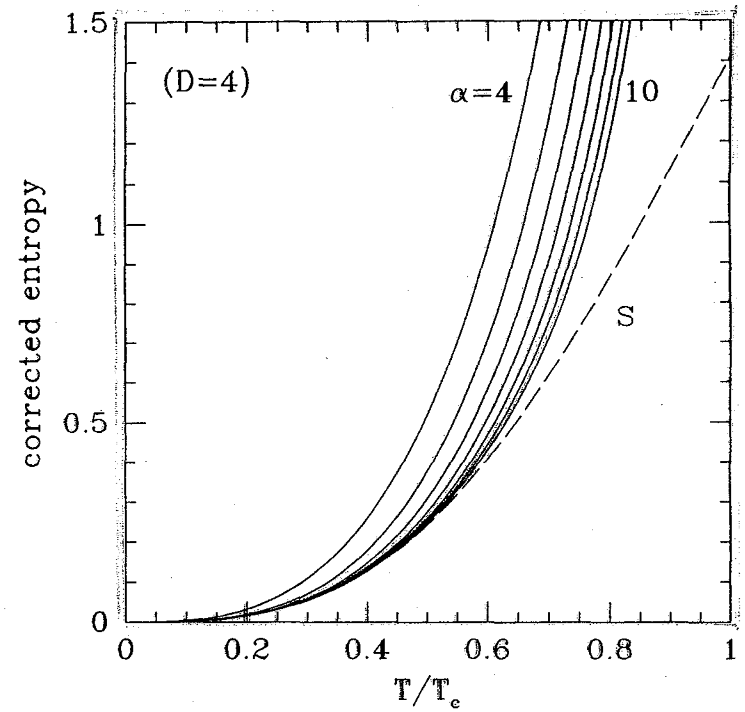
Current & future work & open questions

- Spontaneous breaking of thermal duality [in progress]
- extend results to open strings & branes [in progress]
- explore alternative generalizations of thermodynamics using different thermodynamic structures [done]
- examine holography in detail
  - → implications for curved spacetimes?
  - → AdS/CFT?
- relation to the Hagedorn transition?
- apply to early-universe cosmology
  - → entropy generation?
  - → phase transitions?
- full string calculation of $g(T)$ from first principles
- thermal effects <u>creating</u> SUSY rather than <u>breaking</u> SUSY? [in progress]
- a new approach towards <u>moduli stabilization</u>? [in progress]

Clearly, lots of exciting ideas are ahead...

Work is just beginning!

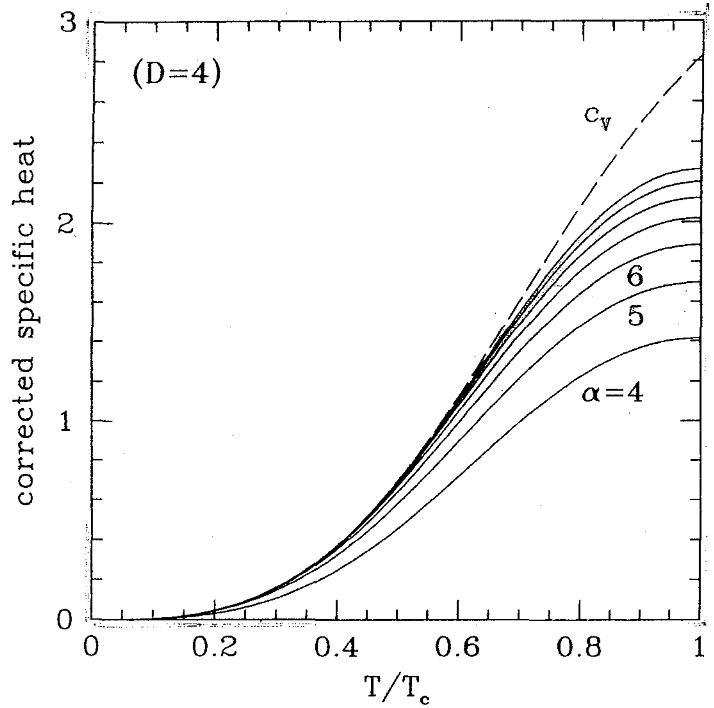
$$\tilde{S} \equiv S - \frac{2T^\alpha}{T_c^\alpha + T^\alpha}\,\frac{F(T)}{T}$$



Note:
- Now have $\tilde{S} \gtrsim S$
- $\tilde{S}$ diverges as $T \to T_c$!

(Hagedorn-related phenomenon?)

$$\tilde{C}_V \equiv C_V + \frac{2T^\alpha}{T_c^\alpha - T^\alpha}\,\frac{U}{T}$$



Note:

- As $T \to T_c$, denominator diverges but $U \to 0$!
- effects cancel $\Rightarrow$ $\tilde{C}_V$ remains finite!

$$\tilde{C}_V \to 2^{2/D-1}\, D\,(1 - 2/\alpha) \quad \text{as } T \to T_c$$

- NO THERMAL INSTABILITIES!

Allternative thermodynamic structures ...

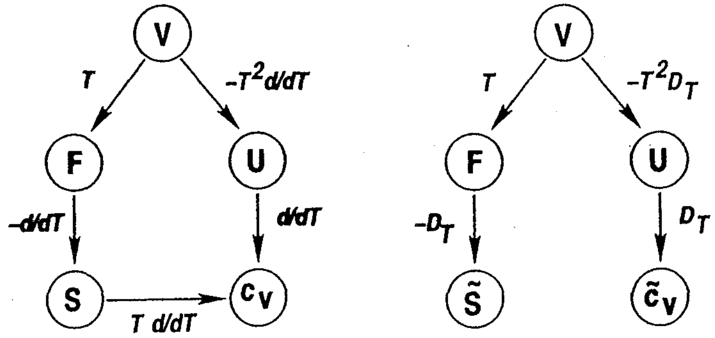


Figure 5: Relations between thermodynamic quantities. (a) Traditional thermodynamics: All thermodynamic quantities are related to each other through temperature multiplications and differentiations. (b) Our string-corrected thermodynamics: we replace the usual temperature derivatives by duality covariant derivatives, maintaining the definitions of $\tilde{S}$ and $\tilde{c}_V$ in terms of their respective thermodynamic potentials F and U. However, $\tilde{c}_V$ is no longer related to $\tilde{S}$ through either type of temperature derivative.

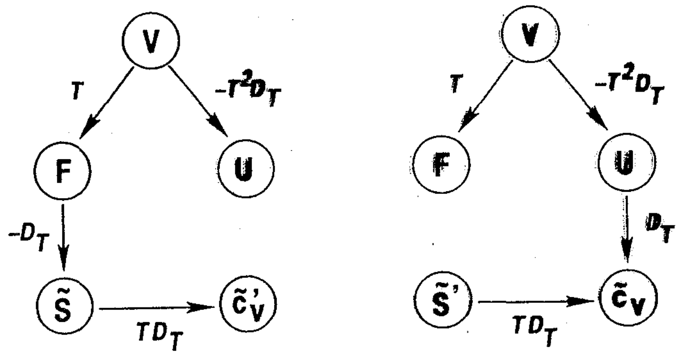


Figure 6: Relations between thermodynamic quantities in alternative formulations of duality covariant thermodynamics. (a) In this version based on Eq. (8.6), the corrected entropy $\tilde{S}$ is defined through the free energy, but the corrected specific heat $\tilde{c}_V$ is defined through the corrected entropy. (b) In this version based on Eq. (8.7), the corrected specific heat is defined through the internal energy, and the corrected entropy is defined implicitly through the corrected specific heat.