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ASPECTS OF NON-COMMUTATIVE FIELD THEORIES

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Aspects of noncommutative field theories: exact results

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• Standard space-time = a manifold $\mathcal{M}$;

points $x \in \mathcal{M} \leftrightarrow$ finite number of real coordinates $x^\mu \in R^4$.

• Usual quantum mechanics:

$$
\begin{aligned}
\left[x_i, x_j\right] &= 0 \ , \ \left[p_i, p_j\right] = 0 \ , \\
\left[x_i, p_j\right] &= i\hbar\delta_{ij} \ .
\end{aligned}
$$

Wigner's contribution: change of canonical commutation relation of Heisenberg with special attention to its group theoretical properties.

• This picture of space-time is likely to break down at very short distances $\sim$ Planck length $\lambda_P \approx 1.6 \times 10^{-33}\, cm$.

• A possible approach to description of physics at short distances is QFT in a NC space-time.

The generalization of commutation relations for the canonical operators of the type

$$x^\mu \longrightarrow \hat{x}^\mu \ : \qquad [\hat{x}^\mu, \hat{x}^\nu] \neq 0 \ ,$$

was suggested long ago, in particular, by

Snyder (1947); Heisenberg (1954);

Golfand (1962)

According to a survey by J. Wess, Heisenberg conveyed to R. Peierls his idea that noncommutating space coordinates could resolve the problem of infinite self-energies. Apparently, Peierls also described it to Pauli, who pursued further this idea by explaining it to Oppenheimer who told it to Snyder, the latter having written the first paper on the subject.

- Practical motivation: the *hope* that QFTs in NC space-time have an improved UV-behaviour.

- Physical motivations:

  – black hole formation in the process of measurement at small distances ($\sim \lambda_P$) $\Rightarrow$ additional uncertainty relations for **coordinates**

  Doplicher, Fredenhagen, Roberts (1994)

  – open string + D-brane theory in the background with antisymmetric tensor

  Ardalan, Arfaei, Sheikh-Jabbari (1998);

  Seiberg, Witten (1999).

$*$ boundary conditions for open string in constant B-field background:

$$[g_{mn}(\partial-\bar{\partial})X^n+2\pi\alpha' B_{mn}(\partial+\bar{\partial})X^n|]_{z=\bar{z}}=0$$

$*$ corresponding propagator

$$\begin{aligned}
&< X^m(z,\bar{z})X^n(w,\bar{w}) >= \\
&- \alpha'(g^{mn}log|z-w| - g^{mn}log|z-\bar{w}| \\
&+ G^{mn}log|z-\bar{w}|^2 + \frac{1}{2\pi\alpha'}\theta^{mn}log(-\frac{z-\bar{w}}{\bar{z}-w})
\end{aligned}$$

$*$ in the limit when both z and w approach the real axis: $z=\bar{z}\rightarrow\tau_1$, $w=\bar{w}\rightarrow\tau_2$, the propagator becomes:

$$\begin{aligned}
< X^m(\tau_1)X^n(\tau_2) > &= -\alpha' G^{mn}log(\tau_1-\tau_2)^2 \\
&+ \frac{i}{2}\theta^{mn}sign(\tau_1-\tau_2)
\end{aligned}$$

implying the commutation relation:

$$[X^m,X^n]=i\theta^{mn}.$$

NC space-time and field theory; $\star$-product

Heisenberg-like commutation relations

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu} ,$$

$\theta^{\mu\nu}$ - *constant* antisymmetric matrix.

$$\mathbf{QFT} \;\rightarrow\; \mathbf{NC\text{-}QFT}: \quad \Phi(x) \;\longrightarrow\; \hat{\Phi}(\hat{x}) .$$

$$S^{(cl)}[\Phi] = \int d^4x [\frac{1}{2}(\partial^\mu\Phi)(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4] ,$$

$$\Downarrow$$

$$S^{(\theta)}[\hat{\Phi}] = \mathrm{Tr}\,[\frac{1}{2}(\hat{\partial}^\mu\hat{\Phi})(\hat{\partial}_\mu\hat{\Phi}) - \frac{1}{2}m^2\hat{\Phi}^2 - \frac{\lambda}{4!}\hat{\Phi}^4] .$$

The NC analogs $\hat{\partial}_\mu\hat{\Phi}$ of field derivatives $\partial_\mu\Phi$:

$$\hat{\partial}_\mu\hat{\Phi} \stackrel{\mathrm{def}}{\equiv} \epsilon_{\mu\nu}\frac{1}{\theta}[\hat{x}_\nu, \hat{\Phi}].$$

The perturbative field theory formulation can be based on operator (e.g. Weyl) symbols $\Phi(x) =$ functions on the **commutative** counterpart of the space-time:

$$\hat{\Phi}(\hat{X}) \longleftrightarrow \Phi(x) \ ;$$

$$\hat{\Phi}(\hat{X}) = \int e^{i\alpha\hat{X}}\phi(\alpha)d\alpha,$$

$$\Phi(x) = \int e^{i\alpha x}\phi(\alpha)d\alpha,$$

where α and x are real variables. Then, using the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) &= \int e^{i\alpha\hat{X}}\phi(\alpha)e^{i\beta\hat{X}}\psi(\beta)d\alpha d\beta \\ &= \int e^{i(\alpha+\beta)\hat{X}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{X}_\mu,\hat{X}_\nu]}\phi(\alpha)\psi(\beta)d\alpha d\beta,\end{aligned}$$

hence the **Moyal $\star$-product** is defined:

$$\hat{\Phi}(\hat{X})\hat{\Psi}(\hat{X}) \longleftrightarrow (\Phi \star \Psi)(x),$$

$$(\Phi \star \Psi)(x) \equiv \left[\Phi(x) e^{\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}}\Psi(y)\right]_{x=y}.$$

Thus, all the multiplications (e.g. in the Lagrangian) must be replaced by the $\star$-product

$$\Downarrow$$

$$S^\theta[\Phi] = \int d^4x [\frac{1}{2}(\partial^\mu\Phi)\star(\partial_\mu\Phi) - \frac{1}{2}m^2\Phi\star\Phi - \frac{\lambda}{4!}\Phi\star\Phi\star\Phi\star\Phi]$$

Moyal bracket will replace the commutators:

$$[\Phi(x), \Psi(x)]_{MB} \equiv \Phi(x)\star\Psi(x) - \Psi(x)\star\Phi(x).$$

Quantization of the theory: using path integrals

CPT and Spin-Statistics Theorems in Axiomatic Approach to NC QFT

CPT and spin-statistics theorems in NC QFT in Lagrangian formalism

M.C., K. Nishijima and A. Tureanu (2002)

Consider NC space-time (coming from string theory and other arguments) with the commutation relation:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \ , \tag{1}$$

where $\theta_{\mu\nu}$ is an antisymmetric constant matrix $\Rightarrow$ Lorentz invariance is violated.

Take

$$\begin{aligned} \theta_{0i} &= 0 \ \ \text{(otherwise unitarity is violated)} \\ \theta_{12} &= -\theta_{21} = \theta \end{aligned} \tag{2}$$

$\Rightarrow$ theory is invariant under $SO(1,1) \times SO(2)$.

Translational invariance preserved.

Causality condition: corresponding to $SO(1,1)$

(**Álvarez-Gaumé et al., 2001)**

$$\begin{aligned} [\phi(x), \phi(y)]_\star \equiv \phi(x) \star \phi(y) - \phi(y) \star \phi(x) = 0, \\ \text{for } (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \end{aligned} \tag{3}$$

Spectral condition: physical momenta in forward light-wedge

$$\tilde{p}^2 = p_0^2 - p_3^2 > 0 \ \text{ and } \ p_0 > 0 \ . \tag{4}$$

Noncommutative Wightman functions

$$W_\star(x_1, x_2, ..., x_n) = \langle 0|\phi(x_1) \star \phi(x_2) \star ...\phi(x_n)|0\rangle \ , \tag{5}$$

where (5) is the Weyl form of the operator-valued Wightman functions $W(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$, with

$$\phi(x) \star \phi(y) = \phi(x) e^{\frac{i}{2}\theta^{\mu\nu} \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overrightarrow{\partial}}{\partial y^\nu}} \phi(y) \ , \tag{6}$$

the most natural generalization of the $\star$-product for noncoinciding points.

(**M.C., M. Mnatsakanova, K. Nishijima,**
A. Tureanu, Yu. Vernov, 2003)

- Another approach:

$$W(x_1, x_2, ..., x_n) = \langle 0|\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle \ , \tag{7}$$

but uses the $SO(1,1) \times SO(2)$ symmetry.

(**Álvarez-Gaumé et al., 2003)**

CPT theorem in commutative case: CPT invariance condition in terms of Wightman functions, e.g. in the case of a neutral scalar field,

$$W(x_1, x_2, ..., x_n) = W(-x_n, ..., -x_2, -x_1) \ , \tag{8}$$

for any values of x_1, x_2,...,x_n, is equivalent to the weak local commutativity (WLC) condition,

$$W(x_1, x_2, ..., x_n) = W(x_n, ..., x_2, x_1) \ , \tag{9}$$

where $x_1 - x_2$, ...,$x_{n-1} - x_n$ is a Jost point, i.e. it satisfies the condition that $\left(\sum_{j=1}^{n-1} \lambda_j (x_j - x_{j+1})\right)^2 < 0$, for all $\lambda_j \geq 0$ with $\sum \lambda_j > 0$.

CPT invariance condition in terms of **NC Wightman functions**

Use antiunitarity of CPT operation:

$$\langle \Theta\Phi | \Theta\Psi \rangle = \langle \Psi | \Phi \rangle \ . \tag{10}$$

Take the vector states as

$$\langle \Phi | = \langle 0 | \equiv \langle \Psi_0 |$$

and

$$|\Psi\rangle = \phi(x_n) \star ... \star \phi(x_2) \star \phi(x_1) |\Psi_0\rangle$$

and express both sides of (10) in terms of NC Wightman functions.

CPT invariance condition will read:

$$W_\star(x_1, x_2, ..., x_n) = W_\star(-x_n, ..., -x_2, -x_1) \ . \tag{11}$$

WLC condition in terms of **NC Wightman** functions (consequence of the locality condition (3)):

Remark that the $\star$-products contained in the definition of the Wightman functions do not influence in any way the coordinates involved in defining the light-wedge in (3), i.e. x_0 and x_3. Consequently, at space-like separated points in the sense of $SO(1,1)$ (denoted by $x_i \sim x_j$, $i, j = 1, 2, ..., n$) we can permute the field operators in (5) in accordance with (3), until we obtain the reversed order of operators compared to the order in (5):

$$\begin{aligned} W_\star(x_1, x_2, ..., x_n) \ &= \ W_\star(x_n, ..., x_2, x_1) \\ &\text{for} \ \ x_i \ \sim \ x_j \ , \ i, j = 1, 2, ..., n \ . \end{aligned} \tag{12}$$

Proof of the CPT theorem

• Show that the WLC condition (12) implies the CPT invariance condition (11).

• Rewrite (12) in terms of relative coordinates:

$$\mathcal{W}_\star(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_{n-1}) = \mathcal{W}_\star(-\tilde{\xi}_{n-1}, ..., -\tilde{\xi}_2, -\tilde{\xi}_1) \ . \tag{13}$$

The functions $\mathcal{W}_\star(\xi_1, ..., \xi_{n-1})$ and $\mathcal{W}_\star(-\xi_{n-1}, ..., -\xi_1)$ satisfy the spectral condition (4) and are invariant under $O(1,1)$ transformations. Thus, in accordance with the previous arguments, they are both analytical functions of the complex variables μ_i in the above-mentioned extended domain. Moreover, since they are equal at Jost points, they are also equal in the whole domain of analyticity.

• Using the invariance of $\mathcal{W}_\star(\mu_1, \mu_2, ..., \mu_{n-1})$ and $\mathcal{W}_\star(-\mu_{n-1}, ..., -\mu_2, -\mu_1)$ under the complex $SO(1,1)$ group, which includes the inversion $\mu_i^0 \to -\mu_i^0$ and $\mu_i^3 \to -\mu_i^3$ we arrive at the equality

$$\mathcal{W}_\star(\mu_1, ..., \mu_{n-1}) = \mathcal{W}_\star(-\mu'_{n-1}, ..., -\mu'_1) \ , \tag{14}$$

where $\mu'_i = (-\mu_i^0, \mu_i^1, \mu_i^2, -\mu_i^3) \equiv (-\mu_i^0, \tau_i^1, \tau_i^2, -\mu_i^3)$.

• Performing a $SO(2)$ rotation by π in the (τ^1, τ^2) plane and subsequently going to the real limit, we obtain that

$$\mathcal{W}_\star(\xi_1, \xi_2, ..., \xi_{n-1}) = \mathcal{W}_\star(\xi_{n-1}, ..., \xi_2, \xi_1) \ , \tag{15}$$

which is equivalent to the CPT invariance condition (11) in terms of x_1, x_2,...,x_n. Thus CPT invariance is the consequence of WLC. By similar considerations the converse can also be proven.

Spin-statistics theorem: The wrong statistics, for a neutral scalar field,

$$\{\phi(x), \phi(y)\}_\star = 0,\ (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0\ , \tag{16}$$

leads to $\phi(x) = 0$.

• Start by proving that, if $\psi(x)|0\rangle = 0$ and $\psi(x)$ is a local field operator, then $\psi(x) = 0$. To show this, take at the Jost points $\tilde{x}_1 - \tilde{x}_2$,...,$\tilde{x}_j - \tilde{x}$, $\tilde{x} - \tilde{x}_{j+1}$,...,$\tilde{x}_{n-1} - \tilde{x}_n$ the arbitrary NC Wightman function

$$\begin{aligned}&\langle 0|\phi(\tilde{x}_1) \star ... \star \phi(\tilde{x}_j) \star \psi(\tilde{x}) \star \phi(\tilde{x}_{j+1})... \star \phi(\tilde{x}_n)|0\rangle \\ &= \langle 0|\phi(\tilde{x}_1)...\phi(\tilde{x}_j) \star \phi(\tilde{x}_{j+1})...\phi(\tilde{x}_n) \star \psi(\tilde{x})|0\rangle = 0\ .\end{aligned} \tag{17}$$

By analytically continuing the first line of (17), one obtains $\langle 0|\phi(x_1) \star ... \star \phi(x_j) \star \psi(x) \star \phi(x_{j+1}) ... \star \phi(x_n)|0\rangle = 0$, i.e. all the matrix elements of the operator $\psi(x)$ between a complete set of states $\langle 0|\phi(x_1) \star ... \star \phi(x_j)$ and $\phi(x_{j+1})... \star \phi(x_n)|0\rangle$ are zero and thus $\psi(x) = 0$.

• Consider $W_\star(x, y) = \langle 0|\phi(x) \star \phi(y)|0\rangle$. According to (16) we have

$$W_\star(\tilde{x}, \tilde{y}) + W_\star(\tilde{y}, \tilde{x}) = 0\ . \tag{18}$$

Eq. (18) can be analytically continued, as in the previous section, into the extended domain. Performing a space-time inversion and taking the real limit for the coordinates, we obtain for the second term of (18) $W_\star(y, x) = W_\star(x, y)$. Thus, $W_\star(x, y) = 0$. At $y = x$ we get

$$\langle 0|\phi(x) \star \phi(x)|0\rangle = 0\ , \tag{19}$$

which is equivalent to $\langle\Psi|\Psi\rangle = 0$, with $|\Psi\rangle = \phi(x)|0\rangle$, if one adopts the definition for the norm of a state as in (19), or equivalently as $\langle\Psi|\Psi\rangle = \langle 0|\phi(\hat{x})\phi(\hat{x})|0\rangle$.

• Then $\phi(x)|0\rangle = 0$ and, due to the result first derived, $\phi(x) = 0$.

High-energy bounds on total cross-section in NC QFT

1. Analyticity of the scattering amplitude in $\cos\Theta$ (derived on the basis of the Jost-Lehmann-Dyson integral representation). Lehmann ellipse.

2. Extension of the analyticity domain to Martin ellipse, using unitarity constraint on the partial-wave amplitudes and dispersion relation.

3. Derivation of analog of Froissart-Martin bound on total cross-section.

I. Jost-Lehmann-Dyson Representation

(M.C. and A. Tureanu, 2004)

Scattering process $k + p \longrightarrow k' + p'$

The JLD representation (1957, 1958) is the integral representation for the Fourier transform of the matrix element of commutator of currents:

$$f(q) = \int d^4x e^{iqx} f(x) \ , \tag{20}$$

where

$$f(x) = \langle p' | [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] | p \rangle \ , \tag{21}$$

satisfying the causality and spectral conditions.

Consider now NC space-time (coming from string theory and other arguments) with the commutation relation:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \ , \tag{22}$$

where $\theta_{\mu\nu}$ is an antisymmetric constant matrix $\Rightarrow$ Lorentz invariance is violated.

Take

$$\begin{aligned} \theta_{0i} &= 0 \ \ \text{(otherwise unitarity is violated)} \\ \theta_{12} &= -\theta_{21} = \theta \end{aligned} \tag{23}$$

$\Rightarrow$ theory is invariant under $SO(1,1) \times SO(2)$.

Translational invariance preserved.

Causality condition: corresponding to $SO(1,1)$

(**Álvarez-Gaumé et al., 2001)**

$$f(x) = 0 \quad \text{for} \quad x_0^2 - x_3^2 = \tilde{x}^2 < 0. \tag{24}$$

Spectral condition: physical momenta in forward light-wedge

$$\tilde{p}^2 = p_0^2 - p_3^2 > 0 \quad \text{and} \quad p_0 > 0\ . \tag{25}$$

Due to translational invariance, region where

$$f(q) = 0$$

can be written analogously to usual case (in Breit frame, $\frac{1}{2}(p + p') = (p_0, 0, 0, 0)$), as the region outside the hyperbola:

$$p_0 - \sqrt{q_3^2 + \tilde{m}_2^2} < q_0 < -p_0 + \sqrt{q_3^2 + \tilde{m}_1^2}\ , \tag{26}$$

where $\tilde{m}_i^2 \leq m_i^2$, $i = 1, 2$. In case $\tilde{m}_i^2 = m_i^2$, $i = 1, 2$, the condition (26) is stronger than the usual one, which contains $|\vec{q}|^2$ instead of q_3. In general, $\tilde{m}_i^2 = f(m_i^2, p_x^2 + p_y^2)$; in Lorentz invariant case, $\tilde{m}_i^2 = m_i^2 - (p_x^2 + p_y^2)$.

• To derive the JLD representation, take the 6-dimensional space with the same metric $(+, -, -, -, -, -)$. Define vector $z = (x_0, x_1, x_2, x_3, y_1, y_2)$. Introduce also the 2-dimensional vector $\tilde{x} = (x_0, x_3)$. Define the function:

$$F(z) = f(x)\delta(\tilde{x}^2 - y^2) = f(x)\delta(\tilde{z}^2), \tag{27}$$

where

$$\tilde{z} = (z_0, z_3, z_4, z_5) = (x_0, x_3, y_1, y_2).$$

(Note however that $F(z)$ depends on all 6 coordinates!)

When the causality condition in the sense of $SO(1,1)$ is fulfilled, $f(x)$ and $F(z)$ determine each other, since

$$\int dy_1 dy_2 F(z) = f(x)\theta(\tilde{x}^2) = \begin{cases} f(x) & \text{for} \quad \tilde{x}^2 > 0 \ , \\ 0 & \text{for} \quad \tilde{x}^2 < 0 \ . \end{cases} \tag{28}$$

- Take the Fourier transform of $F(z)$:

$$F(r) = \int d^6 z e^{izr} F(z) \tag{29}$$

and, using (27) and (28) obtain:

$$F(r) = \int dq D_1(r - \hat{q}) f(q), \tag{30}$$

where, denoting $\tilde{r} = (r_0, r_3, r_4, r_5)$

$$D_1(r) = \int d^6 z e^{izr} \delta(\tilde{z}^2) = \frac{\delta(r_1)\delta(r_2)}{\tilde{r}^2} = \delta(r_1)\delta(r_2) D_1(\tilde{r}). \tag{31}$$

Obviously, $D_1(\tilde{r}) = \frac{1}{\tilde{r}^2}$.

Defining $\hat{q} = (q_0, q_1, q_2, q_3, 0, 0)$ and in view of causality condition (24):

$$F(\hat{q}) = \int d^4 x f(x) \theta(\tilde{x}^2) e^{iqx} = f(q) \tag{32}$$

- $D_1(\tilde{r})$ satisfies the equation

$$\Box_4 D_1(\tilde{r}) = 0, \tag{33}$$

d'Alembertian defined with respect to coordinates r_0, r_3, r_4, r_5. Then, due to (30), it follows that

$$\Box_4 F(r) = 0. \tag{34}$$

Note that $F(r)$ depends on all 6 variables $r_0, ... r_5$:

$$F(r) = \int d^4 q f(q) D_1(\tilde{r} - \tilde{q}) \delta(r_1 - q_1) \delta(r_2 - q_2) = \int d\tilde{q} f(\tilde{q}, r_1, r_2) D_1(\tilde{r} - \tilde{q}),$$

where $\tilde{q} = (q_0, q_3, 0, 0)$.

• Write the solution of (34)

$$F(r') = \int d^3\Sigma_\alpha \int\int dr_1 dr_2 [F(r)\frac{\partial D(\tilde{r}-\tilde{r}')}{\partial \tilde{r}_\alpha} - D(\tilde{r}-\tilde{r}')\frac{\partial F(r)}{\partial \tilde{r}_\alpha}]\delta(r_1)\delta(r_2)\,,$$

where

$$\begin{aligned} D(r) = \int d^6 z e^{-izr}\epsilon(z_0)\delta(\tilde{z}^2) &= \epsilon(r_0)\delta(\tilde{r}^2)\delta(r_1)\delta(r_2) \\ &= D(\tilde{r})\delta(r_1)\delta(r_2). \end{aligned} \tag{35}$$

Note that the surface Σ is **3-dimensional** and not 5-dimensional as in commutative case!

Express $f(q)$ using (32):

$$\begin{aligned} &f(q) = F(\hat{q}) = \int dr_1 dr_2 \delta(r_1 - q_1)\delta(r_2 - q_2) \\ &\times \int d^3\Sigma_\alpha [F(r)\frac{\partial D(\tilde{r}-\tilde{q})}{\partial \tilde{r}_\alpha} - D(\tilde{r}-\tilde{q})\frac{\partial F(r)}{\partial \tilde{r}_\alpha}]. \end{aligned} \tag{36}$$

After integrating over r_4 and r_5, changing the notation of variables r_i to u_i and using the explicit form of $D(\tilde{r})$ from (35), we obtain:

$$\begin{aligned} &f(q) = \int du_1 du_2 \delta(u_1 - q_1)\delta(u_2 - q_2) \int d^1\Sigma_j d\kappa^2 \\ &\times\{F(u,\kappa^2)\frac{\partial}{\partial \tilde{u}_j}[\epsilon(u_0 - q_0)\delta((\tilde{u}-\tilde{q})^2 - \kappa^2)] \\ &-\epsilon(u_0 - q_0)\delta((\tilde{u}-\tilde{q})^2 - \kappa^2)\frac{\partial F(u,\kappa^2)}{\partial \tilde{u}_j}\}\ . \end{aligned} \tag{37}$$

After integration by parts in u_0, we get the JLD representation:

$$\begin{aligned} f(q) &= \int d^4 u d\kappa^2 \epsilon(q_0-u_0)\delta[(q_0-u_0)^2-(q_3-u_3)^2-\kappa^2] \\ &\times\ \delta(q_1-u_1)\delta(q_2-u_2)\phi(u,\kappa^2), \end{aligned} \tag{38}$$

where $\phi(u,\kappa^2) = -\frac{\partial F(u,\kappa^2)}{\partial \tilde{u}_0}$.

Analogously, with $\tilde{u} = (u_0, u_3)$, (38) can be written as:

$$f(q) = \int d\tilde{u} d\kappa^2 \epsilon(q_0-u_0)\delta[(\tilde{q}-\tilde{u})^2-\kappa^2]\phi(\tilde{u},q_1,q_2,\kappa^2)\ . \tag{39}$$

• The region in which $\phi(\tilde{u},q_1,q_2,\kappa^2)=0$ [**otherwise** $\phi(\tilde{u},q_1,q_2,\kappa^2)$ **is an arbitrary function**] is outside the region where the δ function in (39) vanishes,

$$(\tilde{q}-\tilde{u})^2-\kappa^2 = 0, \tag{40}$$

but with q in the region given by (26), where $f(q)=0$. Putting together (40) and (26), we obtain the region out of which $\phi(\tilde{u},q_1,q_2,\kappa^2)=0$:

$$a)\ \frac{1}{2}(\tilde{p}+\tilde{p}') \pm \tilde{u}\ \text{ are in the forward light-wedge;} \tag{41}$$

$$b)\ \kappa \geq max\left\{0, \tilde{m}_1 - \sqrt{\left(\frac{\tilde{p}+\tilde{p}'}{2}+\tilde{u}\right)^2}, \tilde{m}_2 - \sqrt{\left(\frac{\tilde{p}+\tilde{p}'}{2}-\tilde{u}\right)^2}\right\}\ .$$

For the **retarded commutator**

$$f_R(x) = \theta(x_0) f(x)$$

one obtains straightforwardly the **JLD representation for NC QFT**:

$$f_R(q) = \int du_0 du_3 d\kappa^2 \frac{\phi(\tilde{u},q_1,q_2,\kappa^2)}{(q_0-u_0)^2-(q_3-u_3)^2-\kappa^2}. \tag{42}$$

Compare with usual JLD representation:

$$f_R(q) = \int d^4 u d\kappa^2 \frac{\phi(u,\kappa^2)}{(q_0-u_0)^2-(\vec{q}-\vec{u})^2-\kappa^2}. \tag{43}$$

II. Analyticity of scattering amplitude in $\cos\Theta$.

Lehmann's ellipse

• Scattering process $k + p \to k' + p'$;

• Scattering amplitude in terms of JLD representation [recall $\tilde{u} = (u_0, u_3)$]:

$$M(E, \cos\Theta) = i \int d\tilde{u} d\kappa^2 \frac{\phi(\tilde{u}, \kappa^2, k+p, (k'-p')_{1,2})}{\left[\frac{1}{2}(\tilde{k}' - \tilde{p}') + \tilde{u}\right]^2 - \kappa^2} \,, \tag{44}$$

where $\phi(\tilde{u}, \kappa^2, ...)$ is a function of its $SO(1,1) \times SO(2)$-invariant variables: $u_0^2 - u_3^2$, $(k_0 + p_0)^2 - (k_3 - p_3)^2$, $(k_1 + p_1)^2 + (k_2 + p_2)^2$, $(k_1' - p_1')^2 + (k_2' - p_2')^2$,...

ϕ is zero in a certain domain, determined by the causality and spectral conditions, but otherwise arbitrary.

• For the discussion of analyticity of $M(E, \cos\Theta)$ in $\cos\Theta$, it is of crucial importance that **all dependence on $\cos\Theta$ be contained in the denominator of (44)**. But, as the **arbitrary function** ϕ depends now on $(k' - p')_{1,2}$, **it also depends on $\cos\Theta$. This makes impossible the mere consideration of analyticity of the scattering amplitude in $\cos\Theta$!**

• However, all perturbative scattering calculations performed in NC QFT show that the scattering amplitude respects the Froissart-Martin bound!

$\Rightarrow$ the causality and spectrality hypotheses used for the present derivation of JLD representation are too weak, in the sense of their physical implications.

• Challenge the causality condition

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \quad \text{for} \;\; x_0^2 - x_3^2 = \tilde{x}^2 < 0 \; , \tag{45}$$

which takes into account **only** $SO(1,1)$ variables.

This causality condition would be suitable in the case when nonlocality in NC variables x_1 and x_2 is *infinite*, which is not the case:

$$[x_1, x_2] = i\theta \;\; \Rightarrow \;\; \Delta x_1 \Delta x_2 \geq \frac{\theta}{2} \;\; \Rightarrow \;\; (\Delta x_1)^2 + (\Delta x_2)^2 \geq \theta \; .$$

• Consequently, we propose as **locality condition**:

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \; , \quad \text{for} \quad x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0 \; ,$$

or, equivalently,

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \; , \quad \text{for} \quad x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2 \; . \tag{46}$$

• Admitting that the scale of nonlocality in x_1 and x_2 is $l \sim \sqrt{\theta}$, then the propagation of interaction in the noncommutative coordinates is instantaneous *only within this distance l*. It follows then that two events are correlated, i.e. $f(x) \neq 0$, when $x_1^2 + x_2^2 \leq l^2$ (where $x_1^2 + x_2^2$ is the distance in the NC plane with $SO(2)$ symmetry), provided also that $x_0^2 - x_3^2 \geq 0$ (the events are time-like separated in the sense of $SO(1,1)$). Adding the two conditions, we obtain that

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] \neq 0 \; , \; \text{for} \;\; x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) \geq 0 \; . \tag{47}$$

The negation of condition (47) leads to the conclusion that the locality condition should indeed be given by:

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0\,, \quad \text{for} \;\; \tilde{x}^2 - (x_1^2 + x_2^2 - l^2) \equiv x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0\,,$$

or, equivalently,

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0\;, \;\text{for}\; x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2\;, \tag{48}$$

where l^2 is a constant proportional to NC parameter θ. When $l^2 \rightarrow 0$, (48) becomes the usual locality condition.

• When $x_1^2 + x_2^2 > l^2$, for the propagation of a signal only the difference $x_1^2 + x_2^2 - l^2$ is time-consuming and thus in the locality condition it is the quantity $x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2)$ which will occur. Therefore, we shall have a again the locality condition in the form:

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0\;, \;\text{for}\;\; x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0\;,$$

which is equivalent to (48).

• Strong support for the new causality condition (48)

(**Seiberg, Susskind and Toumbas, 2000)**

There it was shown, through the study of a scattering process, that space-space NC ϕ^4 in 2+1 dimensions is causal at macroscopical level.

"Solid rod" argument: the scattered wave appears to originate from a position shifted by $\frac{1}{2}\theta p$, where p is the momentum of the incoming wave packet.

Physical interpretation: the incident particles should be viewed as extended rigid rods, of the size θp, perpendicular to their momentum. In other words, the noncommutativity introduces a scale θ of the spatial nonlocality. The effect is actually an amplification at macroscopic scale of the (micro)causality condition (48).

Note: if one admits the causality condition in the form (45) for 3+1-dimensional NC QFT, then for the 2+1-dimensional theory one simply could not write any (micro)causality condition, since all (two) spatial coordinates are noncommutative and the signal should propagate instantaneously in all directions.

• Correspondingly, the **spectral condition** will read:

$$p_0^2 - p_3^2 - (p_1^2 + p_2^2) \geq 0, \quad p_0 > 0 \ , \tag{49}$$

due to the twisted-Poincaré symmetry of noncommutative space-time. (For details see the paper hep-th/0408069, which is also reproduced in the end of these transparencies.)

• **Now in the literature there exists an old result, highly mathematical and extremely involved (more than the "edge of the wedge" theorem).**

This theorem states that the causality condition

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 < -l^2$$

implies

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0 \ .$$

This **question** had been **first posed by Wightman (1960)** and **rigorously proven later** by

V.S. Vladimirov (1960), D.Ya. Petrina (1961)
A.S. Wightman (1962)

The proof utilizes translational invariance, spectral condition and mathematically the properties of functions of several complex variables.

The **proof goes through also in our case**, i.e.

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2 \ ,$$

$$\Downarrow$$

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad x_0^2 - x_3^2 - (x_1^2 + x_2^2) < 0 \ . \qquad (50)$$

• With the locality condition (50) and spectral condition (49) we find the analog of JLD representation, which we use to derive analyticity domain in $\cos\Theta \Rightarrow$ **Lehmann ellipse for NC case**, which behaves at **high energies E the same way as in the commutative case**, i.e. with the semi-major axis as

$$y_L = 1 + \frac{const}{E^4} \ .$$

• With the new causality conditions, **dispersion relations**, both forward and nonforward, **can be proven**. Note that with the causality condition

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad \tilde{x}^2 = x_0^2 - x_3^2 < 0 \ .$$

forward dispersion relations cannot be derived.

(Y. Liao and K. Sibold, 2002)

III. Enlargement of the domain of analyticity in $\cos\Theta$ and use of unitarity - Martin's ellipse

In NC case the number of variables in the $2 \to 2$ scattering amplitude is bigger than in the commutative case: 5 variables in the most general NC case versus 2 variables in the commutative case.

Partial-wave expansion:

$$A(k+p \to k'+p') = \sum_{l,l',m,m'} a_{lm,l'm'}(E) Y_{lm}(\Theta_{12}, \phi_{12}) Y_{l'm'}(\Theta_{34}, \phi_{34}) \ .$$

M. C., C. Montonen, A. Tureanu (2003)

Here we take the specific situation with $\theta_{0i} = 0$ (compatible with unitarity) and the incoming particle orthogonal to the NC plane, $\vec{p}_1 || \vec{\beta} || Oz$ ($\beta_i \equiv \epsilon_{ijk}\theta_k$), when the NC scattering amplitude depends again only on 2 variables, E and $\cos\Theta$.

- In this case, the unitarity constraint on the partial-wave expansion is the same as in commutative case,

$$Im\ a_l(E) \geq |a_l(E)|^2 \ . \tag{51}$$

Enlarging the analyticity domain of scattering amplitude to Martin's ellipse with the semi-major axis at high energies as

$$y_M = 1 + \frac{const}{E^2} \tag{52}$$

and using unitarity constraint on partial-wave amplitudes, together with the assumption of polynomial boundedness, we obtain the NC analog of the Froissart-Martin bound on the total cross-section (among other bounds):

$$\sigma_{tot}(E) \leq c \quad \ln^2 \frac{E}{E_0} \ .$$

- For the incoming particle momentum not orthogonal on the NC plane, a simple unitarity constraint on partial-wave amplitudes can not be found. However this does not exclude the possibility of obtaining high-energy bounds for this case.

Conclusions

• In space-space NC QFT with $\theta_{12} = -\theta_{21} = \theta$, on the basis of the causality condition

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \quad \text{for} \quad x_0^2 - x_3^2 = \tilde{x}^2 < 0. \tag{53}$$

analyticity domain in $\cos\Theta$ for scattering amplitude can not be obtained.

• Physical arguments compel us to change this causality condition to

$$[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0 \ , \quad \text{for} \quad x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2 \ , \tag{54}$$

leading to the same domain analyticity in $\cos\Theta$ as in commutative case (Lehmann ellipse).

• For incoming particle momentum orthogonal to the NC plane, unitarity constraint on partial waves is the same as in commutative case and enlargement of analyticity domain to Martin ellipse is possible, leading finally to the analog of Froissart-Martin bound.

• For incoming particle momentum not orthogonal to NC plane, simple unitarity constraint on partial waves can not be derived. The possibility of enlarging analyticity domain and obtaining any high-energy bound is not yet clear.

On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT

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Abstract By invoking the concept of twisted Poincaré symmetry of the algebra of functions on a Minkowski space-time, we demonstrate that the noncommutative space-time with the commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$, where $\theta_{\mu\nu}$ is a *constant* real antisymmetric matrix, can be interpreted in a Lorentz-invariant way. The implications of the twisted Poincaré symmetry on QFT on such a space-time is briefly discussed. The presence of the twisted symmetry gives justification to all the previous treatments within NC QFT using Lorentz invariant quantities and the representations of the usual Poincaré symmetry.

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1 Introduction

Quantum field theories on noncommutative space-time have been lately thoroughly investigated, especially after it has been shown [1] that they can be obtained as low-energy limits of open string theory in an antisymmetric constant background field (for reviews, see [2], [3]). However, the issue of the lack of Lorentz symmetry has remained a challenge to this moment, since the field theories defined on a space-time with the commutation relation of the coordinate operators

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \ , \tag{1.1}$$

where $\theta_{\mu\nu}$ is a *constant* antisymmetric matrix, are obviously not Lorentz-invariant.

In spite of this well-recognized problem, all fundamental issues, like the unitarity [4], causality [5], UV/IR divergences [6], have been discussed in a formally Lorentz invariant approach, using the representations of the usual Poincaré algebra. These results have been achieved using the Weyl-Moyal correspondence, which assigns to every field operator $\phi(\hat{x})$ its Weyl symbol $\phi(x)$ defined on the commutative counterpart of the noncommutative space-time. At the same time, this correspondence requires that products of operators are replaced by Moyal $\star$-products of their Weyl symbols:

$$\phi(\hat{x})\psi(\hat{x}) \to \phi(x) \star \psi(x), \tag{1.2}$$

where the Moyal $\star$-product is defined as

$$\phi(x) \star \psi(x) = \phi(x) e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} \psi(y)|_{x=y} \ . \tag{1.3}$$

Consequently, the commutators of operators are replaced by Moyal brackets and the equivalent of (1.1) is

$$[x_\mu, x_\nu]_\star \equiv x_\mu \star x_\nu - x_\nu \star x_\mu = i\theta_{\mu\nu} \ , \tag{1.4}$$

In fact, admitting that noncommutativity should be relevant only at very short distances, the noncommutativity has been often treated as a perturbation and only the corrections to first order in θ were computed. As a result, the NC QFT was practically considered Lorentz invariant in zeroth order in $\theta_{\mu\nu}$, with the first order corrections coming only from the $\star$-product.

Later the fact that QFT on 4-dimensional NC space-time is invariant under the $SO(1,1)\times SO(2)$ subgroup of the Lorentz group was used [7] (for several applications, see [8], [9], [10], [11]). However, a serious problem arises from the fact that the representation content of the $SO(1,1) \times SO(2)$ subgroup is very different from the representation content of the Lorentz group: both $SO(1,1)$ and $SO(2)$ being abelian groups, they have only one-dimensional unitary irreducible representations and thus no spinor, vector etc. representations. In this respect, one encounters a contradiction with previous calculations, in which the representation content for the NC QFT was assumed to be the one of the Poincaré group.

In this letter we shall show that indeed the transformation properties of the NC space-time coordinates x_μ can still be regarded as the transformations under the usual Poincaré algebra, with their representation content identical to the one of the commutative case. At the same time, the commutation relation (1.4) appears as the consequence of the noncommutativity of the coproduct (called noncocommutativity) of the twist-deformed (Hopf) Poincaré algebra when acting on the products of the space-time coordinates $x_\mu x_\nu$. As a consequence, the QFT constructed with $\star$-product on such a NC space-time, though it explicitly violates the

Lorentz invariance, possesses the symmetry under the proper twist-Poincaré algebra.

2 Twist deformation of the Poincaré algebra

The usual Poincaré algebra $\mathcal{P}$ with the generators $M_{\mu\nu}$ and P_α has abelian subalgebra of infinitesimal translations. Using this subalgebra it is easy to construct a twist element of the quantum group theory [12] (for detailed explanations, see the monographs [13], [14]), which permits to deform the universal enveloping of the Poincaré algebra $\mathcal{U}(\mathcal{P})^*$.

This twist element $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ does not touch the multiplication in $\mathcal{U}(\mathcal{P})$, i.e. preserves the corresponding commutation relations among $M_{\mu\nu}$ and P_α,

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha}) \ , \\ [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu) \ , \end{aligned} \tag{2.1}$$

with the essential physical implication that the representations of the algebra $\mathcal{U}(\mathcal{P})$ are the same. However, the action of $\mathcal{U}(\mathcal{P})$ in the tensor product of representations is defined by the coproduct given, *in the standard case*, by the symmetric map (primitive coproduct)

$$\begin{gathered} \Delta_0 : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P}) \\ \Delta_0(Y) = Y \otimes 1 + 1 \otimes Y \ , \end{gathered} \tag{2.2}$$

for all generators $Y \in \mathcal{P}$. The twist element $\mathcal{F}$ changes the coproduct of $\mathcal{U}(\mathcal{P})$ [12]

$$\Delta_0(Y) \mapsto \Delta_t(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1} \ . \tag{2.3}$$

*For a deformed Poincaré group with twisted classical algebra, see [15].

This similarity transformation is consistent with all the properties of $\mathcal{U}(\mathcal{P})$ as a Hopf algebra if $\mathcal{F}$ satisfies the following twist equation[†]:

$$\mathcal{F}(\Delta_0 \otimes id)\mathcal{F} = \mathcal{F}(id \otimes \Delta_0)\mathcal{F} \ . \tag{2.4}$$

Taking the twist element in the form of an abelian twist [16],

$$\mathcal{F} = exp(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu) \ , \tag{2.5}$$

one can check that the twist equation (2.4) is valid.

Since the generators of translations P_α are commutative, their coproduct is not deformed ($\Delta_t = \Delta_0$ is primitive)

$$\Delta_t(P_\alpha) = \Delta_0(P_\alpha) = P_\alpha \otimes 1 + 1 \otimes P_\alpha \ . \tag{2.6}$$

However, the coproduct of the Lorentz algebra generators is changed:

$$\Delta_t(M_{\mu\nu}) = Ad\ e^{\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) = e^{\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) e^{-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \ . \tag{2.7}$$

Using the operator formula $Ad\ e^B C = e^B\ C\ e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[B,[B,...[}_{n} B, C]] = \sum_{n=0}^{\infty} \frac{(AdB)^n}{n!}\ C$ and the commutation relation between $M_{\mu\nu}$ and P_α (last line of (2.1)), we obtain the explicit form of the coproduct[‡] $\Delta_t(M_{\mu\nu})$:

$$\begin{aligned}\Delta_t(M_{\mu\nu}) &= Ad\ e^{\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) \\ &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu} P_\nu - \eta_{\alpha\nu} P_\mu) \otimes P_\beta\end{aligned}$$

[†]See more detailed explanations in monographs on quantum groups (e.g. [13], [14]).

[‡]After the submission of the present work to the hep-th Archive, we were informed that the result (2.8) appears also in [17], which is an extended version of the talk given by Julius Wess in the "Balkan Workshop 2003".

$$+ \quad P_\alpha \otimes (\eta_{\beta\mu} P_\nu - \eta_{\beta\nu} P_\mu)] \ . \tag{2.8}$$

It is known (cf. [13], [18]) that having a representation of a Hopf algebra $\mathcal{H}$ in an associative algebra $\mathcal{A}$ consistent with the coproduct Δ of $\mathcal{H}$ (a Leibniz rule)

$$h(a \cdot b) = h_1(a) \cdot h_2(b) \ , \quad \Delta(h) = h_1 \otimes h_2 \ , \tag{2.9}$$

the multiplication in $\mathcal{A}$ has to be changed after twisting $\mathcal{H}$. The new product of $\mathcal{A}$ consistent with the twisted coproduct Δ_t is defined as follows: let $\mathcal{F} = \sum f_1 \otimes f_2$, then

$$a \star b = \sum (\bar{f}_1(a)) \cdot (\bar{f}_2(b)) \ , \tag{2.10}$$

where $\bar{\mathcal{F}} = \sum \bar{f}_1 \otimes \bar{f}_2$ denotes the representation of $\mathcal{F}^{-1}$ in $\mathcal{A} \otimes \mathcal{A}$, and the action of elements $\bar{f} \in \mathcal{H}$ on elements $a, b \in \mathcal{A}$ is the same as without twisting.

Let us now consider the commutative algebra $\mathcal{A}$ of functions, $f(x)$, $g(x)$,..., depending on coordinates x_μ, $\mu = 0, 1, 2, 3$, in the Minkowski space M. In $\mathcal{A}$ we have the representation of $\mathcal{U}(\mathcal{P})$ generated by the standard representation of the Poincaré algebra:

$$P_\mu f(x) = i\partial_\mu f(x) \ , \ M_{\mu\nu} f(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f(x) \ , \tag{2.11}$$

acting on coordinates as follows:

$$P_\mu x_\rho = i\eta_{\mu\rho} \ , \ M_{\mu\nu} x_\rho = i(x_\mu \eta_{\nu\rho} - x_\nu \eta_{\mu\rho}) \ . \tag{2.12}$$

The Poincaré algebra acts on the Minkowski space x_μ, $\mu = 0, 1, 2, 3$ with commutative multiplication:

$$m(f(x) \otimes g(x)) := f(x) g(x) \ . \tag{2.13}$$

When twisting $\mathcal{U}(\mathcal{P})$, one has to redefine the multiplication according to (2.10), while retaining the action of the generators of the Poincaré algebra on the coordinates as in (2.12):

$$\begin{aligned} m_t(f(x) \otimes g(x)) &=: f(x) \star g(x) = m \circ e^{-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta}(f(x) \otimes g(x)) \\ &= m \circ e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \otimes \partial_\beta}(f(x) \otimes g(x)) \ . \end{aligned} \tag{2.14}$$

Specifically, one can now easily compute the commutator of coordinates:

$$\begin{aligned} m_t(x_\mu \otimes x_\nu) &= x_\mu \star x_\nu = m \circ e^{-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta}(x_\mu \otimes x_\nu) \\ &= m \circ [x_\mu \otimes x_\nu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\mu} \otimes \eta_{\beta\nu}] \\ &= x_\mu x_\nu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\mu}\eta_{\beta\nu} \ , \\ m_t(x_\nu \otimes x_\mu) &= x_\nu \star x_\mu = x_\nu x_\mu + \frac{i}{2}\theta^{\alpha\beta}\eta_{\alpha\nu}\eta_{\beta\mu} \ . \end{aligned} \tag{2.15}$$

Hence,

$$[x_\mu, x_\nu]_\star = \frac{i}{2}\theta^{\alpha\beta}(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}) = i\theta_{\mu\nu} \ , \tag{2.16}$$

which is indeed the Moyal bracket (1.4).

3 QFT on space-time with twisted Poincaré symmetry

Comparing (1.3) and (2.14) (or equivalently (1.4) and (2.16)), it is obvious that building up the noncommutative quantum field theory through Weyl-Moyal correspondence is equivalent to the procedure of redefining the multiplication of functions, so that it is consistent with the twisted coproduct of the Poincaré generators (2.6), (2.8). The QFT so obtained is invariant under the twisted Poincaré algebra. The benefit of reconsidering NC QFT in the latter approach is that it makes transparent the *invariance* under the twist-deformed Poincaré algebra, while the first approach highlights the *violation* of the Lorentz group.

To show this invariance, let us take, as an instructive example, the product $f_{\rho\sigma}(x) = x_\rho x_\sigma$. In the standard non-twisted case, the action of the Lorentz generators on this product reads as:

$$M_{\mu\nu} f_{\rho\sigma} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f_{\rho\sigma} = i(f_{\mu\sigma}\eta_{\nu\rho} - f_{\nu\sigma}\eta_{\mu\rho} + f_{\rho\nu}\eta_{\mu\sigma} - f_{\rho\mu}\eta_{\nu\sigma}) \ , \tag{3.17}$$

expressing the fact that $f_{\rho\sigma}$ is a rank-two Lorentz tensor. In the twisted case, $f_{\rho\sigma}$ should be replaced, according to (2.14), by the symmetrized expression[§] $f^t_{\rho\sigma} = x_{\{\rho} \star x_{\sigma\}} = \frac{1}{2}(x_\rho \star x_\sigma + x_\sigma \star x_\rho)$, and correspondingly the action of the Lorentz generator should be applied through the twisted coproduct:

$$M^t_{\mu\nu} f^t_{\rho\sigma} = m_t \circ (\Delta_t(M_{\mu\nu})(x_\rho \otimes x_\sigma)) \ . \tag{3.18}$$

In the above equation, $M^t_{\mu\nu}$ denotes the usual Lorentz generator, but with the action of a twisted coproduct. A straightforward calculation gives:

$$M^t_{\mu\nu} f^t_{\rho\sigma} = i(f^t_{\mu\sigma}\eta_{\nu\rho} - f^t_{\nu\sigma}\eta_{\mu\rho} + f^t_{\rho\nu}\eta_{\mu\sigma} - f^t_{\rho\mu}\eta_{\nu\sigma}) \ , \tag{3.19}$$

which is analogous to (3.17), confirming the (expected) covariance under the twisted Poincaré algebra. This argument extends to any symmetrized tensor formed from the $\star$-products of x's. For example, the invariance of Minkowski length $s^2_t = x_\mu \star x^\mu = x_\mu x^\mu$ is obvious: multiplying (3.19) by $\eta^{\rho\sigma}$, one obtains $M^t_{\mu\nu} s^2_t = 0$.

[§]We use the symmetrization because, due to the commutation relation $[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}$ (where $\theta_{\mu\nu}$ is twisted-Poincaré invariant, as shown also in the consistency check performed below), every tensorial object of the form $x_\mu \star x_\nu \star \cdots \star x_\sigma$ can be written as a sum of symmetric tensors of lower or equal ranks, so that the basis of the representation algebra $\mathcal{A}_t$ is symmetric. This statement is valid in general in the case of the universal enveloping algebras of Lie algebras.

As a consistency check, we shall calculate the action of $M^t_{\mu\nu}$ on the antisymmetric combination $2x_{[\rho} \star x_{\sigma]} = [x_\rho, x_\sigma]_\star$:

$$
\begin{aligned}
M^t_{\mu\nu}([x_\rho, x_\sigma]_\star) &= ([x_\mu, x_\sigma]_\star - i\theta_{\mu\sigma})\eta_{\nu\rho} - ([x_\nu, x_\sigma]_\star - i\theta_{\nu\sigma})\eta_{\mu\rho} \\
&- ([x_\mu, x_\rho]_\star - i\theta_{\mu\rho})\eta_{\nu\sigma} + ([x_\nu, x_\rho]_\star - i\theta_{\nu\rho})\eta_{\mu\sigma} = 0 \ . \qquad (3.20)
\end{aligned}
$$

Thus, we have $M^t_{\mu\nu}\theta_{\rho\sigma} = 0$, since $\theta_{\rho\sigma} = -i[x_\rho, x_\sigma]_\star$, i.e. the antisymmetric tensor $\theta_{\rho\sigma}$ is twisted-Poincaré invariant.

Therefore, the Lagrangian obtained by replacing all the usual products of fields in the corresponding commutative theory with $\star$-products, though it breaks the Lorentz invariance in the usual sense, it is, however, invariant under the twist-deformed Poincaré algebra.

Another important feature of the QFT with twist-deformed Poincaré symmetry deserves a special highlighting: the representation content of the NC QFT is exactly the same as for its commutative correspondent. It is easy to see that the action of the Pauli-Ljubanski operator, $W_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}M^{\beta\gamma}P^\delta$ is not changed by the twist (due to the commutativity of the translation generators) and P^2 and W^2 retain their role of Casimir operators. Consequently, the representations of the twisted Poincaré algebra will be, just as in the commutative case, classified according to the eigenvalues of these invariant operators, m^2 and $m^2 s(s+1)$, respectively. Besides justifying the validity of the results obtained so far in NC QFT using the representations of the Poincaré algebra, this aspect will cast a new light on other closely-related fundamental issues, such as the CPT and the spin-statistics theorems in NC QFT [9, 10, 19].

4 Conclusions

In this letter we have shown that the quantum field theory on NC space-time possesses symmetry under a twist-deformed Poincaré algebra. The twisted Poincaré symmetry exists provided that: (i) we consider $\star$-products among functions instead of the usual one and (ii) we take the proper action of generators specified by the twisted coproduct. As a byproduct with major physical implications, the representation content of NC QFT, invariant under the twist-deformed Poincaré algebra, is identical to the one of the corresponding commutative theory with usual Poincaré symmetry. Some of the applications of the present treatment of the symmetry properties of NC QFT will be considered in a forthcoming communication [20].

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