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H4.SMR/1586-21

"7th Workshop on Three-Dimensional Modelling of Seismic Waves Generation and their Propagation"

25 October - 5 November 2004

Seismic Waves Propagation in Complex Media

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7th Workshop on Three-Dimensional Modelling of Seismic Waves Generation, Propagation and their Inversion Miramar, 2004

Seismic waves propagation in complex media

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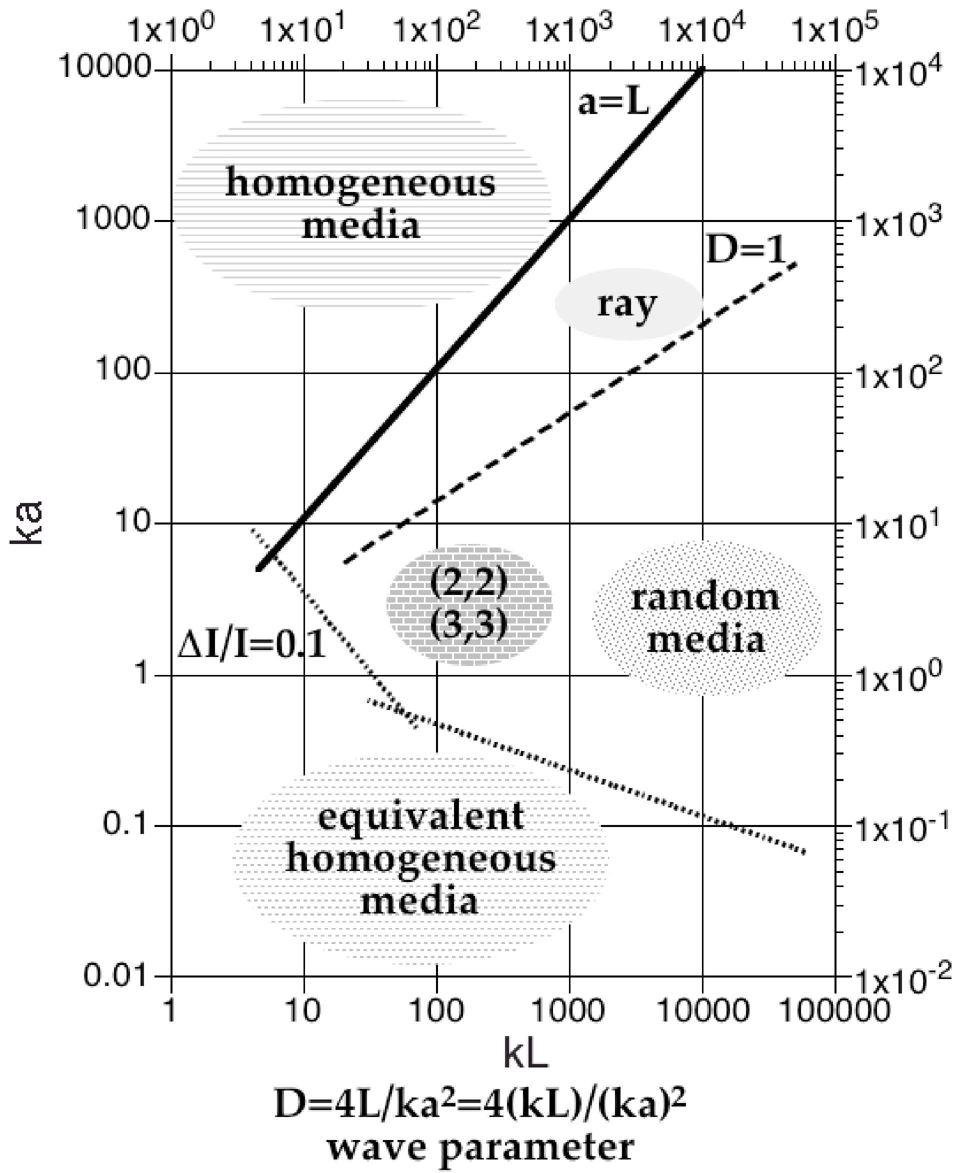


Seismic wave propagation problems can be classified using some parameters.

This classification is crucial for the choice of technique to calculate synthetic seismograms, but it needs a **deep comprehension of the physical meaning of the problem.**

(Adapted from Aki and Richards, 1980)

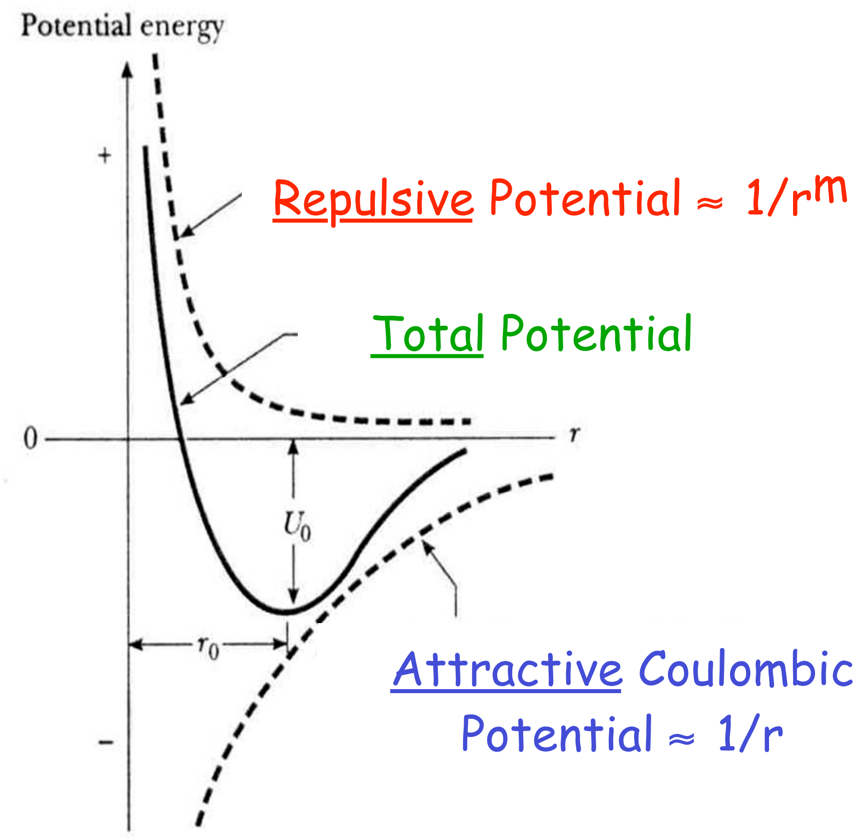
Seismic wave propagation in COMPLEX MEDIA

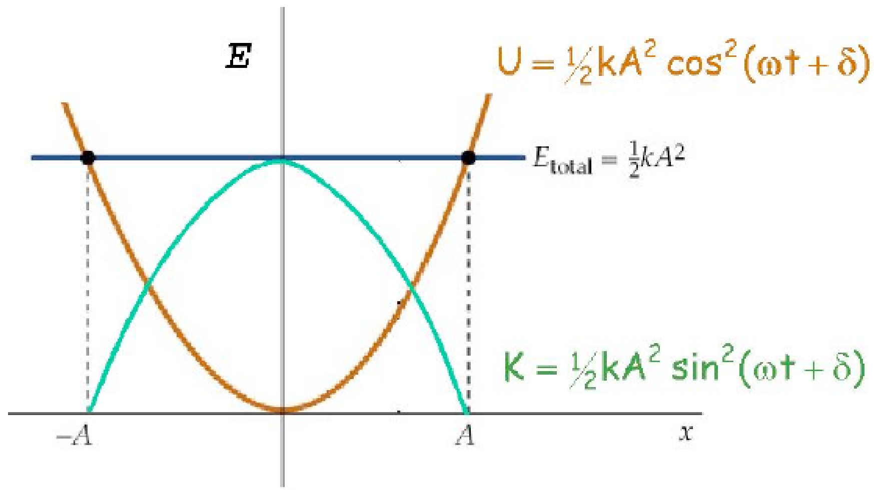
Part 1: Scattering classification

Outline

- **Basic physical concepts 1**
  - What is a wave?
  - Born of elastic wave equation
- Basic mathematical reference:
  - PDE: Poisson, diffusion and wave equation
- Basic physical concepts 2
  - EM scattering and diffusion
- Application to the seismic wavefield
  - Seismic scattering, diffusion
  - Methods for laterally heterogenous media

Small perturbations of a **stable** equilibrium point —— **Linear restoring force** —— **Harmonic Oscillation**

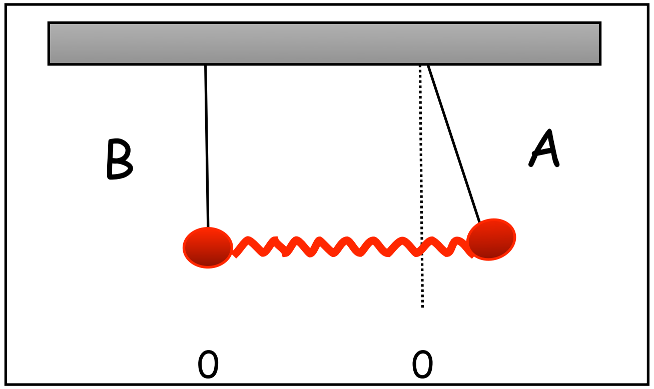




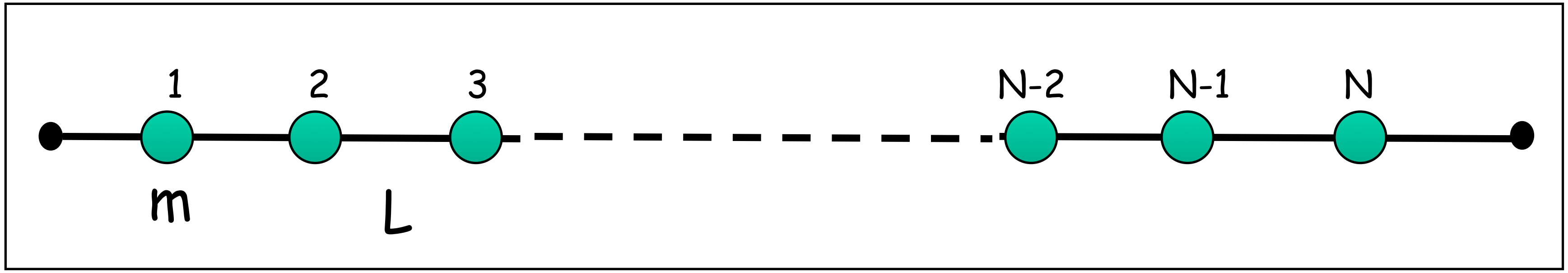
What is a **wave**? - 2

Small perturbations of a **stable** equilibrium point — **Linear restoring force** — **Harmonic Oscillation**

Coupling of harmonic oscillators — the disturbances can **propagate**, superpose and stand



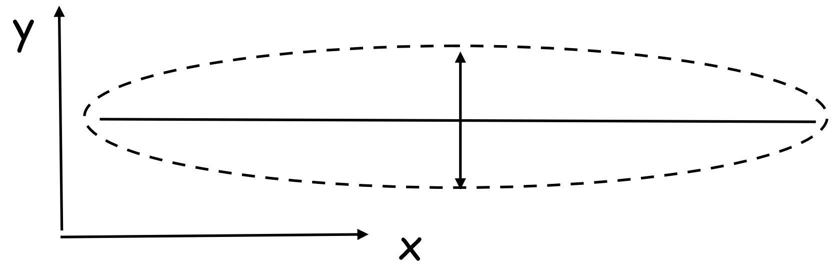
N coupled oscillators



Consider a flexible elastic string to which is attached N identical particles, each mass m, equally spaced a distance L apart.

The ends of the string are fixed a distance L from mass 1 and mass N. The initial tension in the string is T.

Consider small transverse displacements of the masses



$$F_p = -T\left[\frac{y_p - y_{p-1}}{L} + \frac{y_{p+1} - y_p}{L}\right]$$

but $F_p = m_p a_p$

$$\therefore \quad m\frac{d^2 y_p}{dt^2} = -T\left[\frac{y_p - y_{p-1}}{L} + \frac{y_{p+1} - y_p}{L}\right]$$

Substitute $T/mL = \omega_0^2$

$$\therefore \quad \frac{d^2 y_p}{dt^2} = -\omega_0^2 (y_p - y_{p-1}) + \omega_0^2 (y_{p+1} - y_p)$$

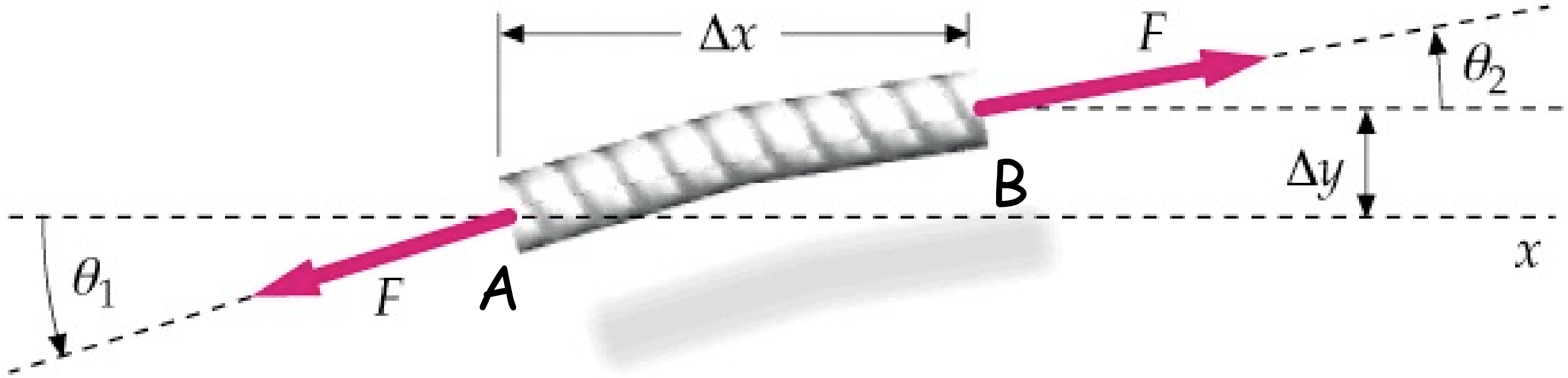
$$\text{or} \quad \frac{d^2 y_p}{dt^2} + 2\omega_o^2\, y_p\ \omega\omega_o^2\,(y_{p+1}\ \omega y_{p\omega 1}) = 0$$

We can write a similar expression for all N particles

Therefore we have a set of N differential equations one for each value of p from p=1 to p=N.

NB at fixed ends: $y_0 = 0$ and $y_{N+1} = 0$

Derivation of the wave equation



Consider a small segment of string of length ≈x and tension F on which a travelling wave is propagating.

The ends of the string make small angles ≈1 and ≈2 with the x-axis.

The vertical displacement ≈y is very small compared to the length of the string

Solution of the wave equation

Consider a wavefunction of the form $y(x,t) = A\sin(kx - \approx t)$

$$\frac{\partial^2 y}{\partial t^2} = \partial\partial^2 A\sin(kx\,\partial\,\partial\,t) \qquad \frac{\partial^2 y}{\partial x^2} = \partial k^2 A\sin(kx\,\partial\,\partial\,t)$$

If we substitute these into the linear wave equation

$$\frac{\mu}{F}(--^2 A\sin(kx--t)) = -k^2 A\sin(kx--t)$$

$$\frac{\mu}{F}\omega^2 = k^2$$

Using the relationship $v = \approx/k$, $v^2 = \approx^2/k^2 = F/\mu$ $\quad v = \sqrt{F/\mu}$

,

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

General form of LWE

What is a **wave**? - 3

Small perturbations of a **stable** equilibrium point — **Linear restoring force** — **Harmonic Oscillation**

Coupling of harmonic oscillators — the disturbances can **propagate**, superpose and stand

WAVE: organized propagating imbalance, satisfying differential equations of motion

Organization can be destroyed, when interference is destructive

non linearity

strong scattering

Turbulence

Diffusion

Exceptions

Solitons

Phonons

Seismic wave propagation in COMPLEX MEDIA

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Mathematic reference: Linear PDE

Classification of Partial Differential Equations (PDE)

Second-order PDEs of two variables are of the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2}+b\frac{\partial^2 f(x,y)}{\partial x \partial y}+c\frac{\partial^2 f(x,y)}{\partial y^2}+d\frac{\partial f(x,y)}{\partial x}+e\frac{\partial f(x,y)}{\partial y}=F(x,y)$$

$b^2-4ac<0$	elliptic	LAPLACE equation
$b^2-4ac=0$	parabolic	DIFFUSION equation
$b^2-4ac>0$	hyperbolic	WAVE equation

Elliptic equations produce **stationary and energy-minimizing** solutions

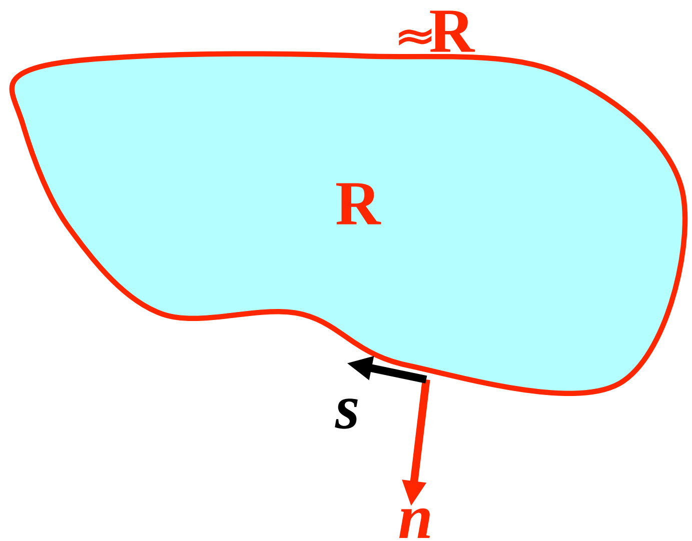
Parabolic equations a **smooth-spreading flow** of an initial disturbance

Hyperbolic equations a **propagating disturbance**

Boundary and Initial conditions

Initial conditions: starting point for propagation problems

Boundary conditions: specified on domain boundaries to provide the interior solution in computational domain



$$\text{(i) Dirichlet condition}: u = f \quad \text{on} \quad \partial R$$

$$\text{(ii) Neumann condition}: \frac{\partial u}{\partial n} = f \ \text{ or } \ \frac{\partial u}{\partial s} = g \quad \text{on} \quad \partial R$$

$$\text{(iii) Robin (mixed) condition}: \frac{\partial u}{\partial n} + ku = f \quad \text{on} \quad \partial R$$

Elliptic PDEs

Steady-state two-dimensional heat conduction equation is prototypical elliptic PDE

Laplace equation - no heat generation

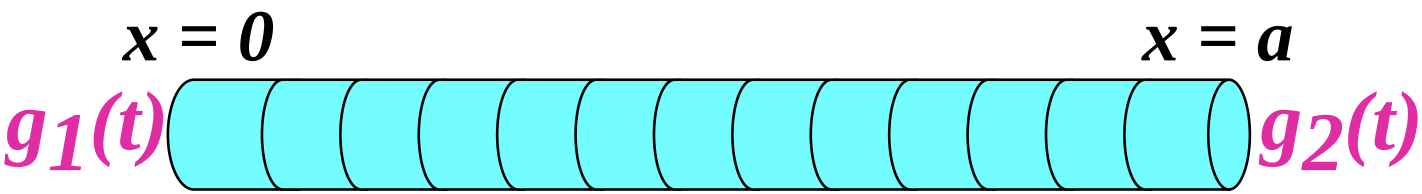
$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

Poisson equation - with heat source

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)}$$

Heat Equation: Parabolic PDE

Heat transfer in a one-dimensional rod



$$\boxed{\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2},} \quad 0 \le x \le a,\quad 0 \le t \le T$$

$$\text{I.C.s} \qquad u(x,0) = f(x) \qquad 0 \le x \le a$$

$$\text{B.C.s} \quad \begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \qquad 0 \le t \le T$$

Hyperbolic Equation

$b^2 - 4ac = 0 - 4(1)(-c^2) > 0$: Hyperbolic

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}, \quad 0 \le x \le a, \quad 0 \le t$$

$$\text{I.C.s} \quad \begin{cases} u(x,0) = f_1(x) \\ u_t(x,0) = f_2(x) \end{cases} \quad 0 \le x \le a$$

$$\text{B.C.s} \quad \begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \quad t > 0$$

Coupled PDE

Navier-Stokes Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Numerical Methods

- Complex geometry
- Complex equations (nonlinear, coupled)
- Complex initial / boundary conditions

- **No analytic solutions**
- **Numerical methods needed !!**

Objective: Speed, Accuracy at minimum cost

- **Numerical Accuracy** (error analysis)
- **Numerical Stability** (stability analysis)
- **Numerical Efficiency** (minimize cost)
- **Validation** (model/prototype data, field data, analytic solution, theory, asymptotic solution)
- **Reliability and Flexibility** (reduce preparation and debugging time)
- **Flow Visualization** (graphics and animations)

Computational **solution** procedures

Governing Equations ICS/BCS → Discretization → System of Algebraic Equations → Equation (Matrix) Solver → Approximate Solution

Continuous Solutions

Finite-Difference

Finite-Volume

Finite-Element

Spectral

Boundary Element

Discrete Nodal Values

$U_i\ (x,y,z,t)$

$p\ (x,y,z,t)$

$T\ (x,y,z,t)$

or

≈ (≈,≈,≈,≈)

Discretization

- **Time derivatives**
- almost exclusively by finite-difference methods
- **Spatial derivatives**

  - Finite-difference: Taylor-series expansion
  - Finite-element: low-order shape function and interpolation function, continuous within each element
  - Finite-volume: integral form of PDE in each control volume
  - There are also other methods, e.g. collocation, spectral method, spectral element, panel method, boundary element method

Finite Difference

- **Taylor series**

$$f(x) = f(x_o) + (x' \ x_o) f'(x_o) + \frac{(x' \ x_o)^2}{2!} f''(x_o) + \frac{(x' \ x_o)^3}{3!} f'''(x_o)$$

$$+ \cdots + \frac{(x' \ x_o)^n}{n!} f^{(n)}(x_o) + \cdots \qquad a' \ x' \ b, \quad a' \ x_o ' \ b$$

- **Truncation error**

$$T_E = \frac{(x \,\xi\, x_o)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \qquad a \,\xi\, \xi \,\xi\, b$$

- **How to reduce truncation errors?**
  - Reduce grid spacing, use smaller $\approx x = x - x_o$
  - Increase order of accuracy, use larger n

Finite Difference Scheme

- **Forward difference**
- **Backward difference**
- **Central difference**

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + O(\Delta x)$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + O(\Delta x)$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + O(\Delta x^2)$$

Seismic wave propagation in COMPLEX MEDIA

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Basic concepts of EM wavefield

Extinction and **emission** are two main types of the interactions between an electromagnetic radiation field and a medium (e.g., the atmosphere).

Extinction is due to **absorption** and **scattering**.

Absorption is a process that removes the radiant energy from an electromagnetic field and transfers it to other forms of energy.

Scattering is a process that does not remove energy from the radiation field, but **redirect** it. Scattering can be thought of as absorption of radiant energy followed by re-emission back to the electromagnetic field with negligible conversion of energy, i.e.can be a "source" of radiant energy for the light beams traveling in other directions.

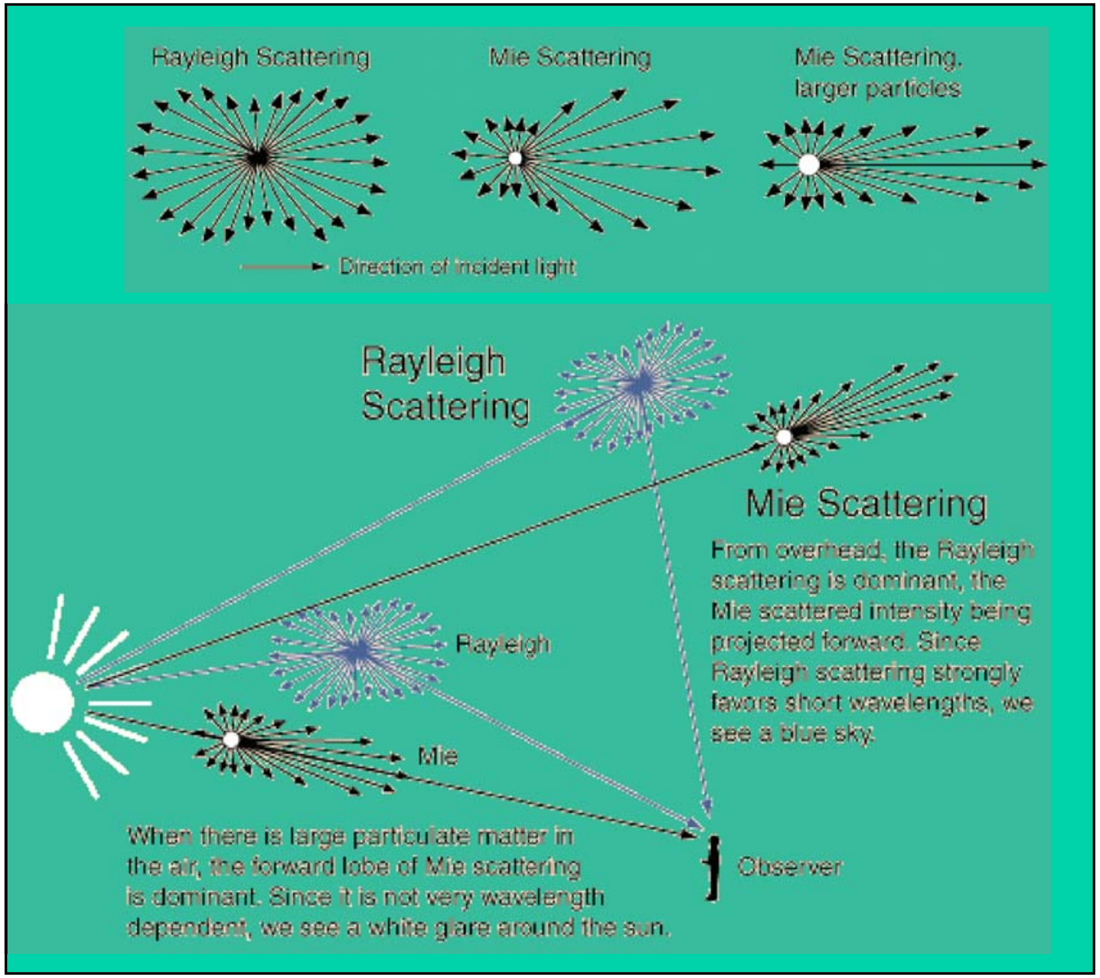
Scattering **occurs at all wavelengths** (spectrally not selective) in the electromagnetic spectrum, for any material whose refractive index is different from that of the surrounding medium (**optically inhomogeneous**).

Scattering of EM wavefield (1)

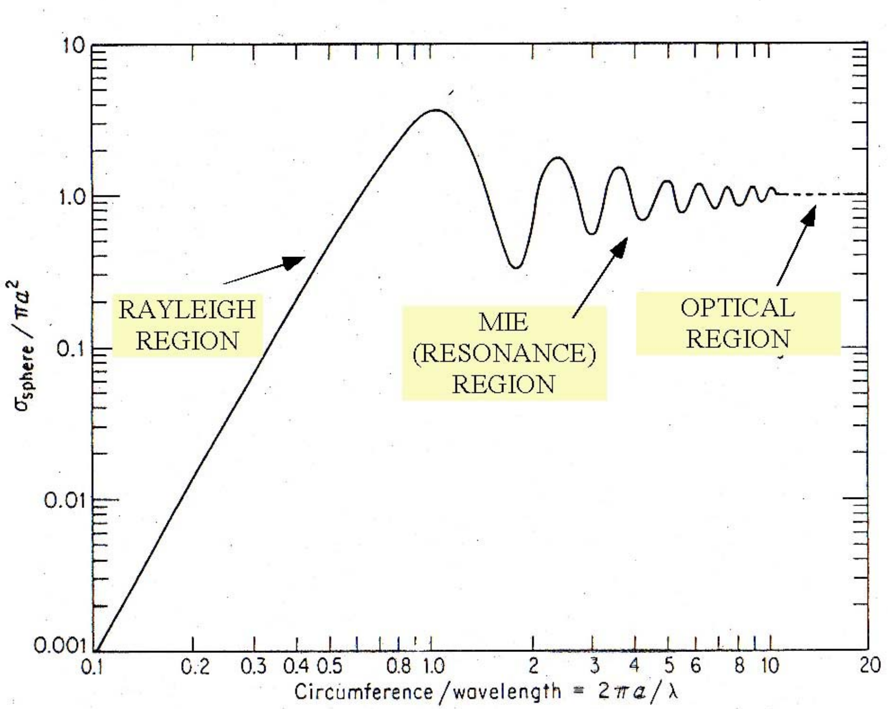
The amount of scattered energy depends strongly on the ratio of: particle size (a) to wavelength (≈) of the incident wave

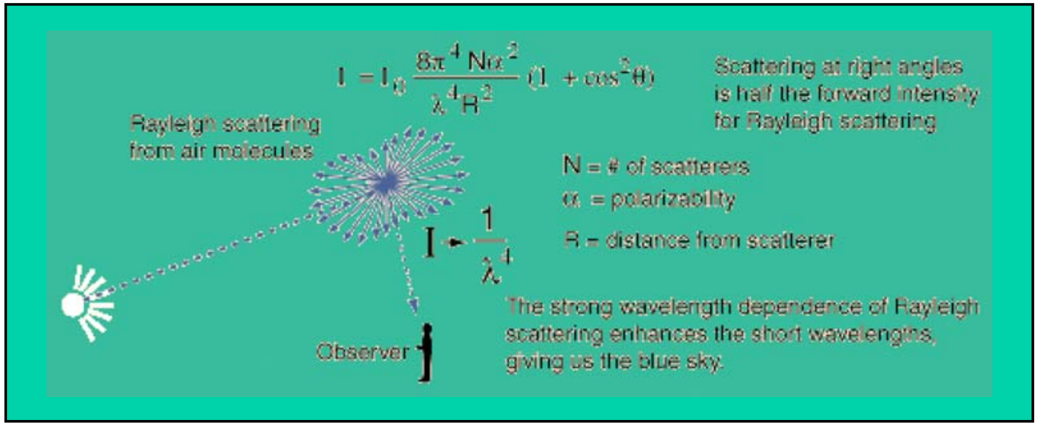
When **(a < ≈/10)**, the scattered intensity on both forward and backward directions are equal. This type of scattering is called <u>Rayleigh scattering</u>.

For **(a > ≈)**, the angular distribution of scattered intensity becomes more complex with more energy scattered in the forward direction. This type of scattering is called <u>Mie scattering</u>



Scattering of EM wavefield (2)





Single Scattering

≈=2≈**a**/≈

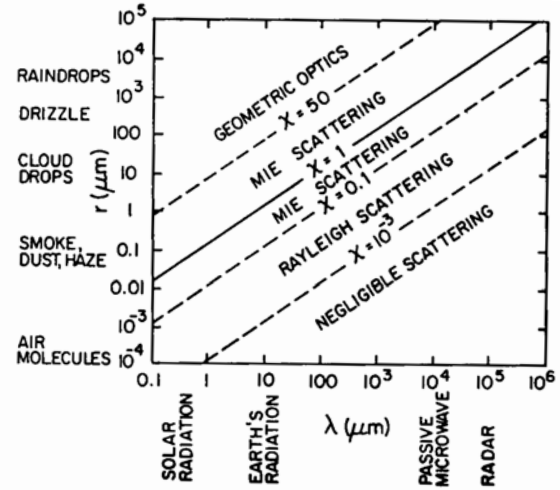
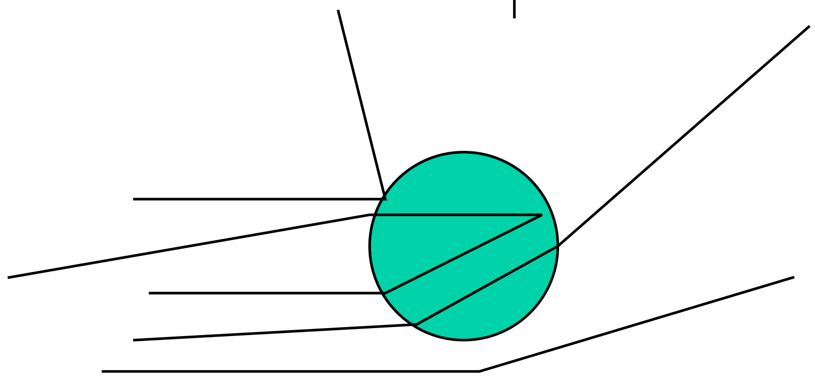


FIGURE 3.18. Scattering regimes. [Adapted from Wallace and Hobbs (1977). Reprinted by permission of Academic Press.]



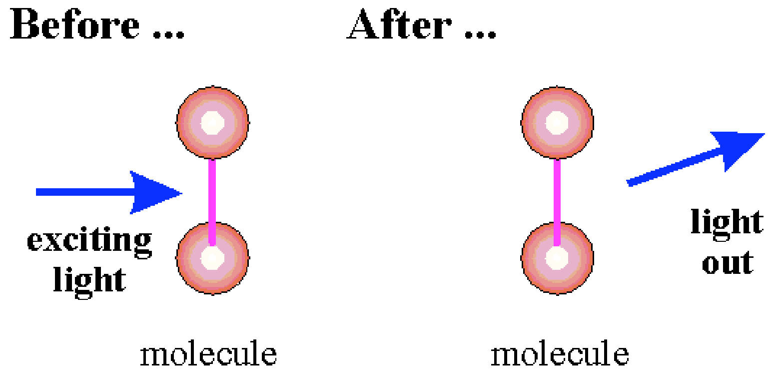
For (**a >> ≈**), the Scattering characteristics are determined from explicit Reflection, Refraction and Diffraction: **Geometric "Ray" Optics**

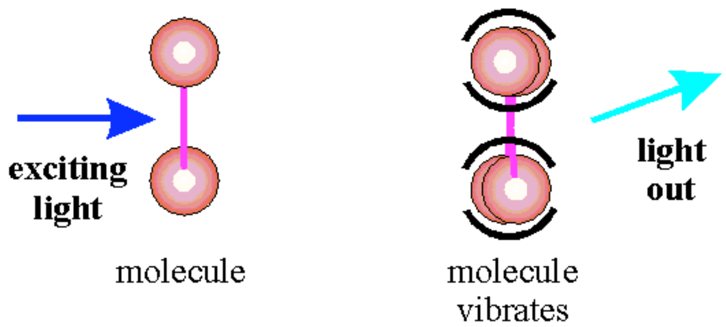
Scattering of EM wavefield (3)

Composition of the scatterer (n) is important!

The interaction (and its redirection) of electromagnetic radiation with matter May or may not occur with **transfer of energy**, i.e., the scattered radiation has a slightly different or the same wavelength.

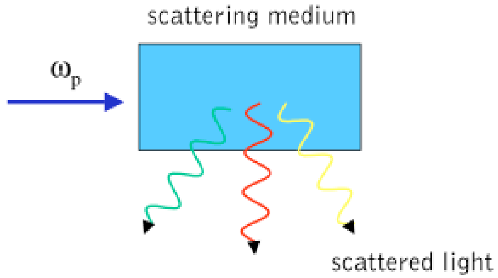
Rayleigh scattering - Light out has same frequency as light in, with scattering at many different angles.





Raman scattering - Light is scattered due to vibrations in molecules or optical phonons in solids. Light is shifted by as much as 4000 wavenumbers and exchanges energy with a molecular vibration.

Scattering of EM wavefield (4)

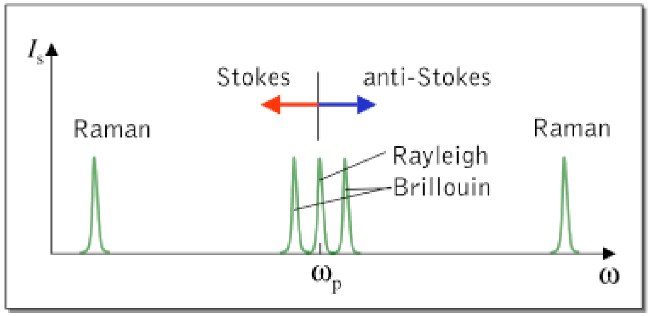


Rayleigh scattering
: scattering from *nonpropagating* density fluctuations (elastic)

Brillouin scattering
: scattering from *propagating* pressure waves (sound waves, acoustic phonons)

Raman scattering
: interaction of light with vibrational modes of molecules or lattice vibrations of crystals (scattering from optical phonons)

spectrally resolved detection



Raman scattering

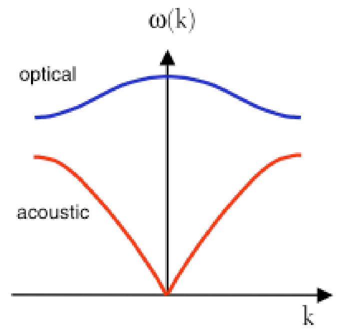
✓ optical phonon

Brillouin scattering

✓ acoustic phonon

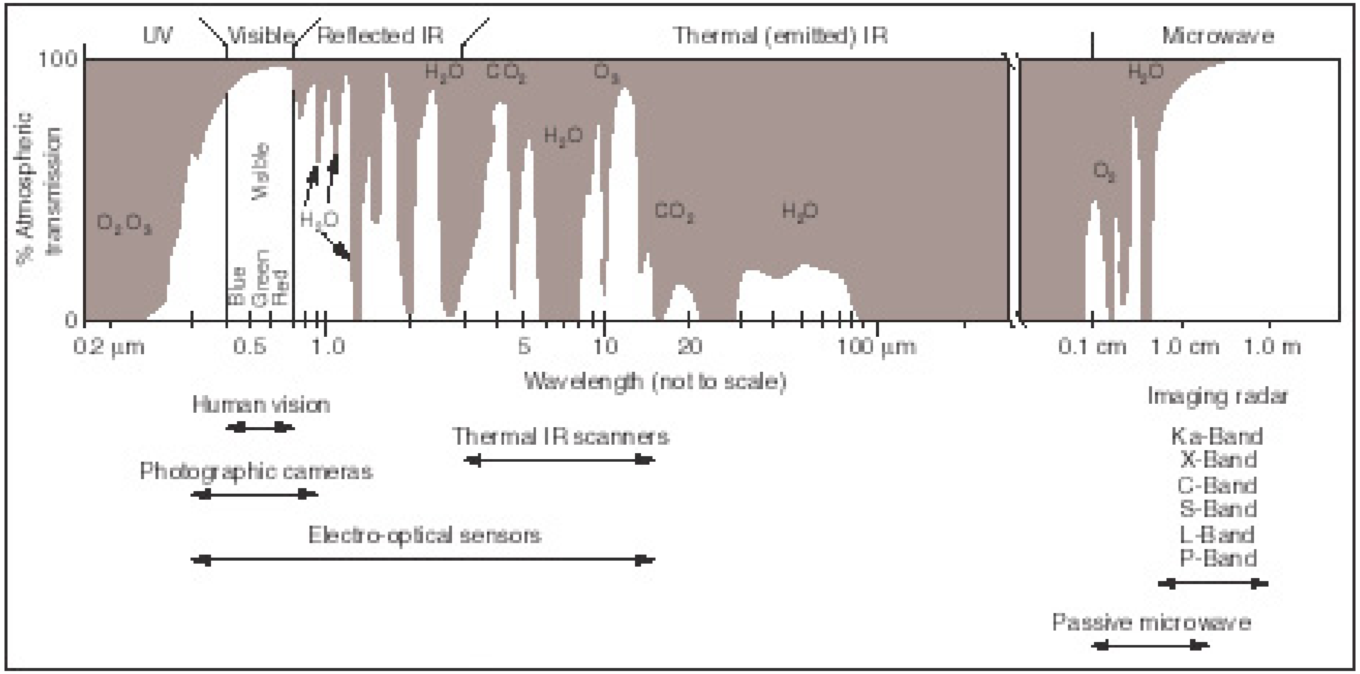
Phonons
quanta of the ionic displacement field in a solid

phonon dispersion curve



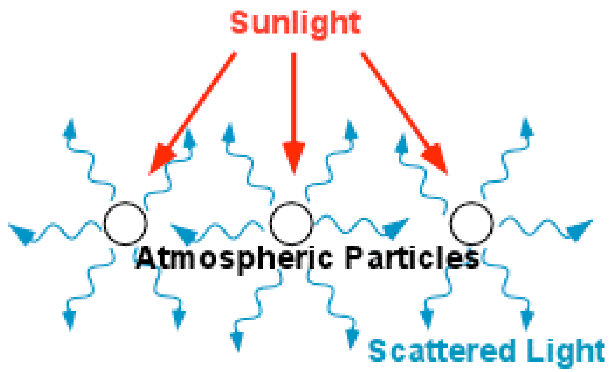
When the photon is absorbed and re-emitted at a different wavelength, this is absorption.

Transmissivity of the Earth's atmosphere



Scattering and **Diffusion**

In single scattering, the properties of the scatterer are important , but multiple scattering erases these effects - eventually **all** wavelengths are scattered in **all** directions.



Works for turbid media: clouds, beer foam, milk, etc...

Example: when a solid has a very low temperature, phonons behave like waves (long mean free paths) and heat propagate following ballistic term.
At higher temperatures, the phonons are in a diffusive regime and heat propagate following Maxwell law.

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Basic parameters for **seismic wavefield**

The governing parameters for the seismic scattering are:

wavelength of the wavefield (or wavenumber k)

≈ (10^0-10^5 m)

correlation length or dimension, of the heterogeneity

a ($10^?$-10^3 m)

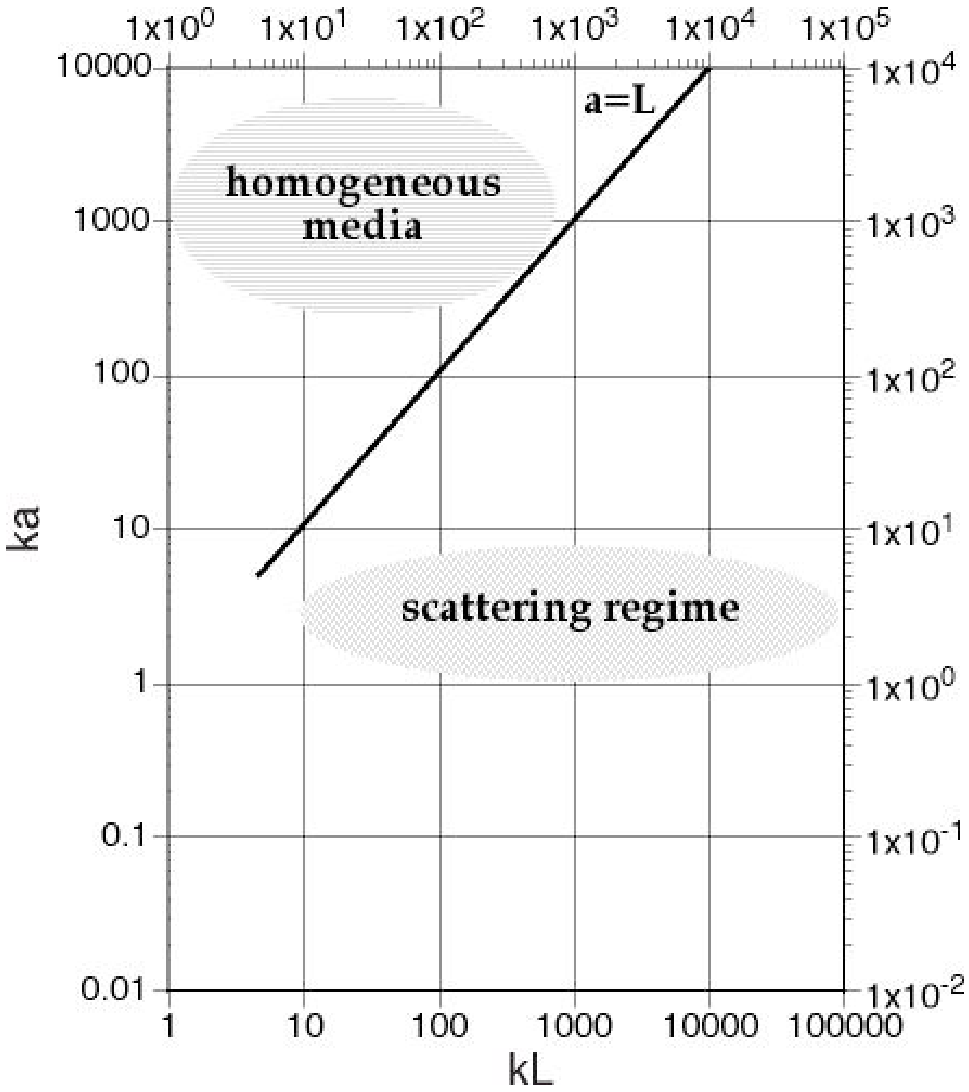
distance travelled in the heterogeneity

L (10^0-10^5 m)

With special cases:

- a = L homogeneous region
- a >> ≈ ray theory is valid
- a ≈ ≈ strong scattering effects

Seismic Scattering (1)



Wave propagation problems can be classified using the parameters just introduced.

This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

Scattering in a **perturbed** model

Let us consider a **perturbed** model:
reference+perturbation (in elastic parameters)

$$\rho = \rho_0 + \delta\rho \qquad \lambda = \lambda_0 + \delta\lambda \qquad \mu = \mu_0 + \delta\mu$$

resulting in a velocity perturbation

$$c = c_0 + \delta c$$

solution: **Primary** field + **Scattered** field

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1(\delta\rho, \delta\lambda, \delta\mu)$$

satisfying equations of motion:

$$\rho_0 \ddot{u}_i^0 - (\lambda_0 + \mu_0)(\nabla \cdot \mathbf{u}^0)_{,i} - \mu_0 \nabla^2 \mathbf{u}_i^0 = 0$$

$$\delta\rho_0 \ddot{u}_i - (\delta\lambda \nabla \cdot \mathbf{u})_{,i} - \left[\delta\mu(u_{i,j} + u_{j,i})\right]_{,j} = 0$$

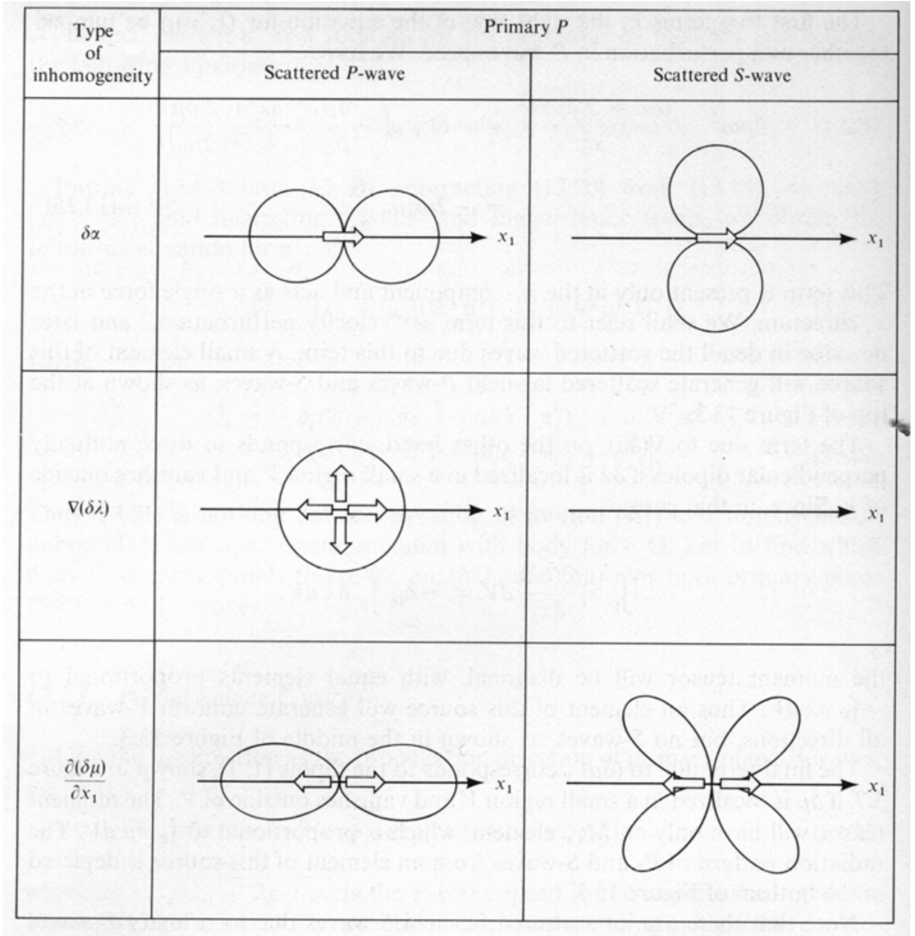
$$\rho_0 \ddot{u}_i^1 - (\lambda_0 + \mu_0)(\nabla \cdot \mathbf{u}^1)_{,i} - \mu_0 \nabla^2 \mathbf{u}_i^1 = Q_i$$

Point Scatterers

How does a point-like perturbation of the elastic parameters affect the wavefield?

Perturbation of the different elastic parameters produce characteristic radiation patterns. These effects are used in diffraction tomography to recover the perturbations from the recorded wavefield.

(Figure from Aki and Richards, 1980)



Correlation distance

When velocity varies in all directions with a finite scale length, it is more convenient to consider spatial fluctuations

Autocorrelation function (a is the **correlation distance)**:

$$N(\mathbf{r}_1) = \frac{\left\langle \frac{\delta c(\mathbf{r})}{c_0(\mathbf{r})} \frac{\delta c(\mathbf{r}+\mathbf{r}_1)}{c_0(\mathbf{r}+\mathbf{r}_1)} \right\rangle}{\left\langle \left(\frac{\delta c(\mathbf{r})}{c_0(\mathbf{r})} \right)^2 \right\rangle} = \begin{cases} e^{-|r_1|/a} \\ e^{-(|r_1|/a)^2} \end{cases}$$

Power Spectra of scattered waves

$$\left\langle |\mathbf{u}_1|^2 \right\rangle \propto \begin{cases} k^4 \left(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2} \right)^{-2} \\ k^4 \exp\left(-k^2 a^2 \sin^2 \frac{\theta}{2} \right) \end{cases}$$

$\propto k^4$ if $ka << 1$ (Rayleigh scattering)
if ka is large (forward scattering)

Energy loss through a cube of size L (Born approximation)

$$\frac{\infty I}{I} \infty \begin{matrix} \infty \\ \infty \\ \infty \\ \infty \\ \infty \end{matrix} \begin{matrix} k^4 a^3 L\left(1+4k^2a^2\right)^{\infty 1} \\ \\ k^2 a L\left(1 \infty e^{k^2a^2}\right)^{\infty 1} \end{matrix}$$

but violates the energy conservation law and it is valid if **(<0.1)**

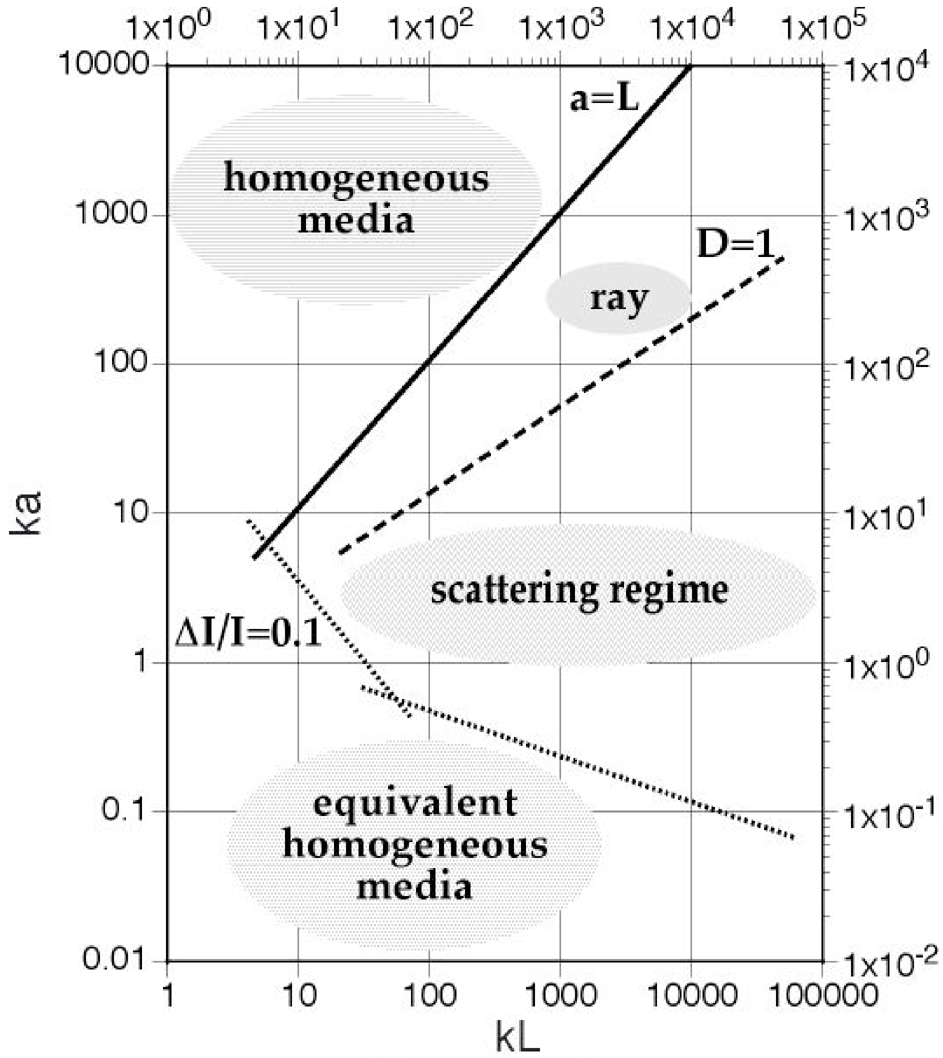
the **perturbations** (P &A) are function of the **wave parameter**:

$$D = \frac{4L}{ka^2}$$

$$D = \begin{matrix} \infty \\ \infty \\ \infty \end{matrix} \begin{matrix} \infty 0 & \text{phase perturbation} \\ & \\ \infty\infty & \text{phase = amplitude} \end{matrix}$$

when D<1, geometric ray theory is valid

Seismic Scattering (2)



D=4L/ka²=4(kL)/(ka)²
wave parameter

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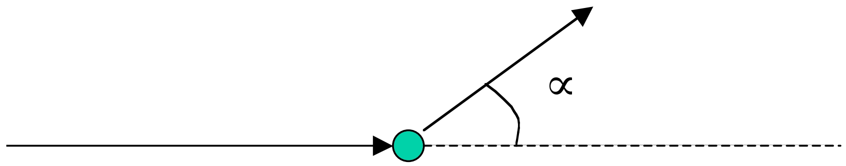
This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

Multiple scattering process leads to **attenuation** (spatial loss non a true dissipative one) and **energy mean free path**



$\infty(\infty)$ is the **differential scattering cross-section** and after a wave has travelled a distance x, the energy is reduced by an amount of

$$e^{\infty\infty x} \qquad \infty = \infty_{\infty 1}^{+1} \infty(\cos\infty)d\cos\infty$$

and the **average path length** between scattering events is

$$l = \infty_{0}^{\infty} e^{\infty\infty x} dx = \frac{1}{\infty}$$

forward scattering tendency

$$| \,' = \int_{|1}^{+1} (\cos |)\,|\,(\cos |)\,d\cos | \quad \left| \begin{array}{l} >0 \text{ forward} \\ | \; 0 \text{ isotropic} \\ <0 \text{ backward} \end{array} \right.$$

Multiple scattering randomizes the phases of the waves adding a diffuse (incoherent) component to the average wavefield.
Statistical approaches can be used to derive **elastic radiative transfer equations**

Diffusion constants
use the definition of a diffusion (transport) mean free path

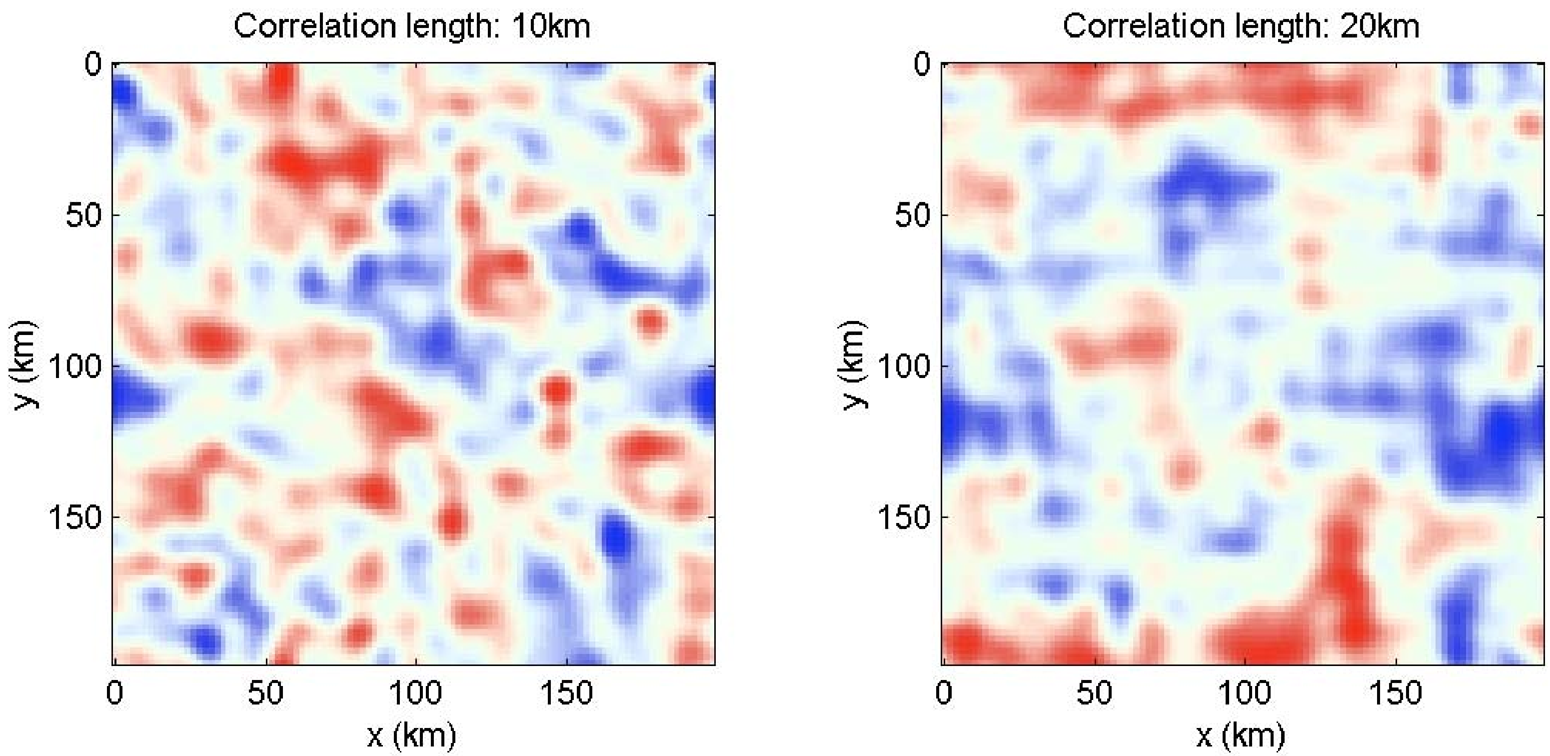
$$d = \frac{cl^*}{3} \quad l^* = \frac{l}{| - |\,'} \quad \text{(acoustic)}$$

$$d = \frac{1}{1+2K^3} \left| \frac{c_p l_p^*}{3} + 2K^2 \frac{c_s l_s^*}{3} \right| \quad \text{(elastic)}$$

for non-preferential scattering l* coincides with energy mean free path, l
for enhanced forward scattering l*>l

Experiments for ultrasound in materials can be applied to seismological problems…

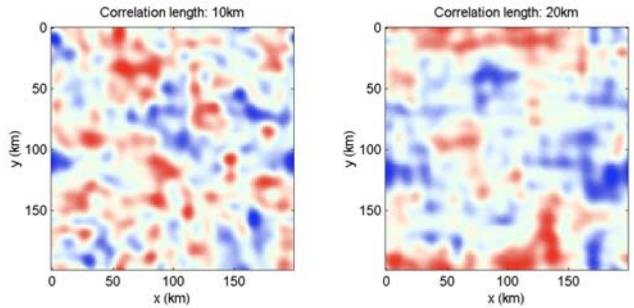
Scattering in random media

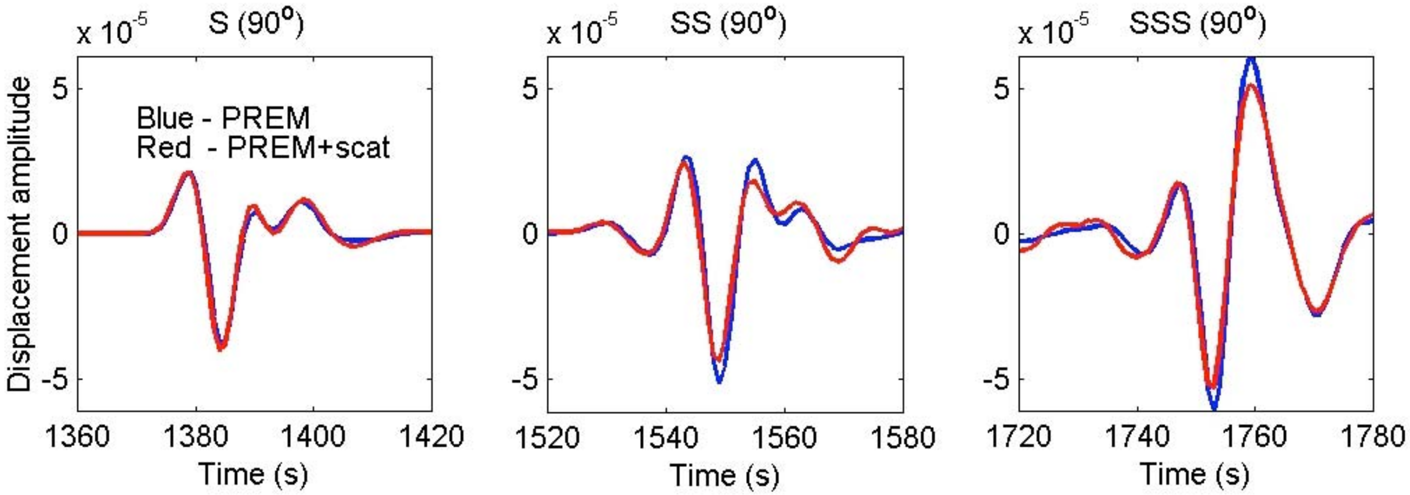


How is a propagating wavefield affected by random heterogeneities?

Synthetic seismograms

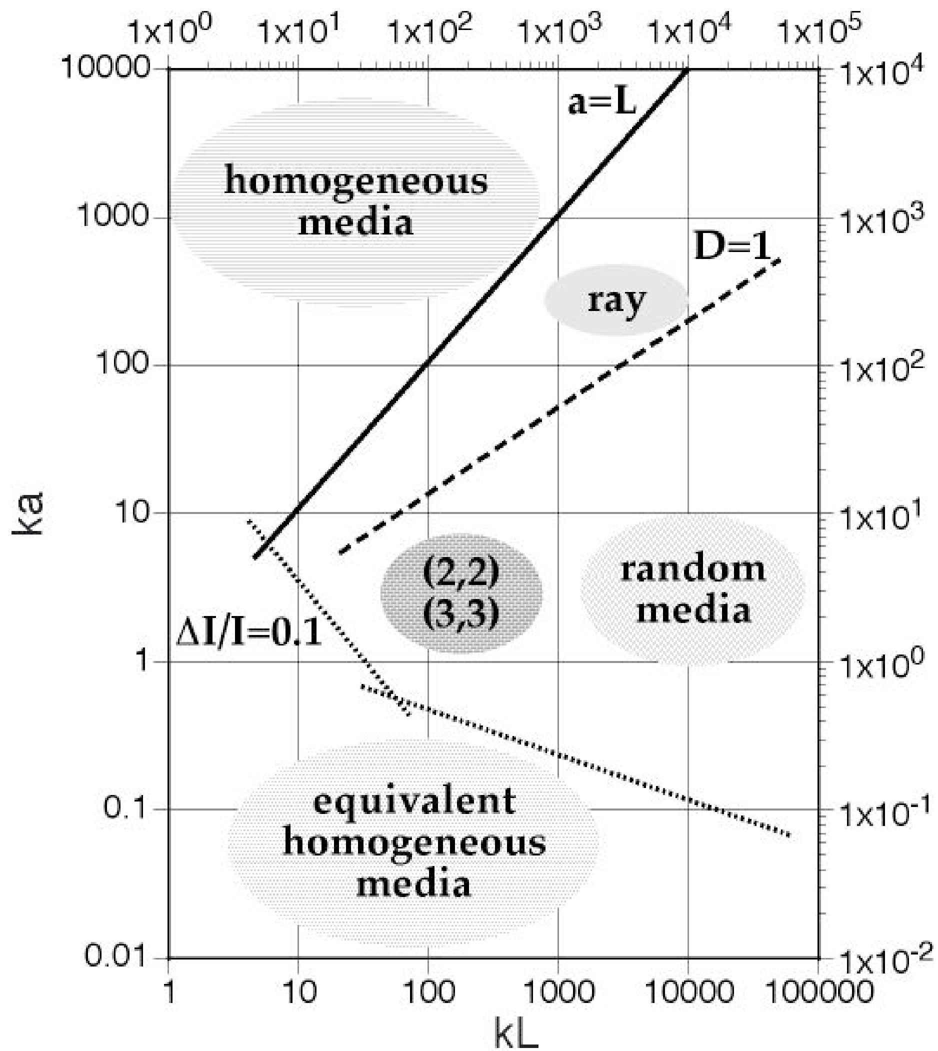
Synthetic seismograms for a global model with random velocity perturbations.





When the wavelength is long compared to the correlation length, scattering effects are difficult to distinguish from intrinsic attenuation.

Seismic Scattering Classification



$D=4L/ka^2=4(kL)/(ka)^2$

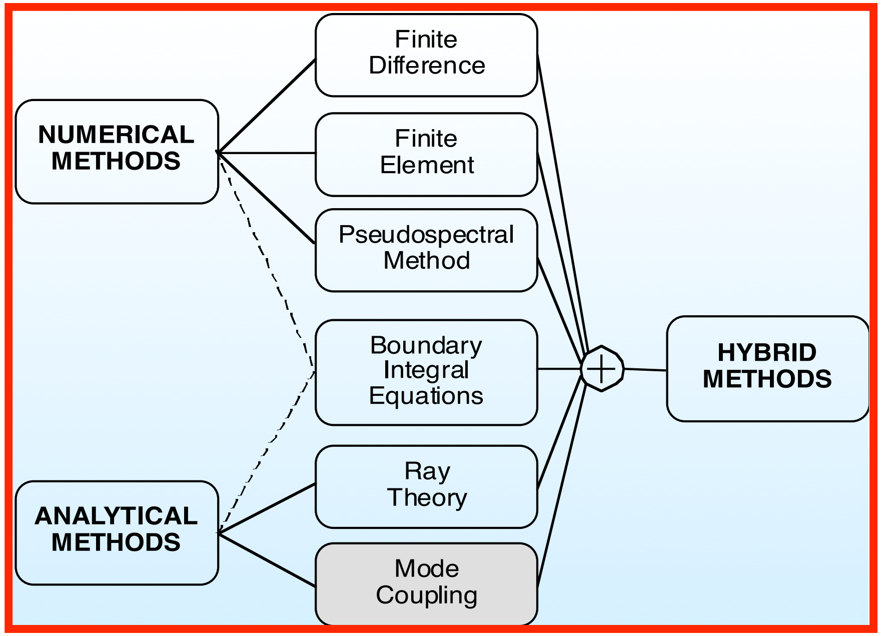
wave parameter

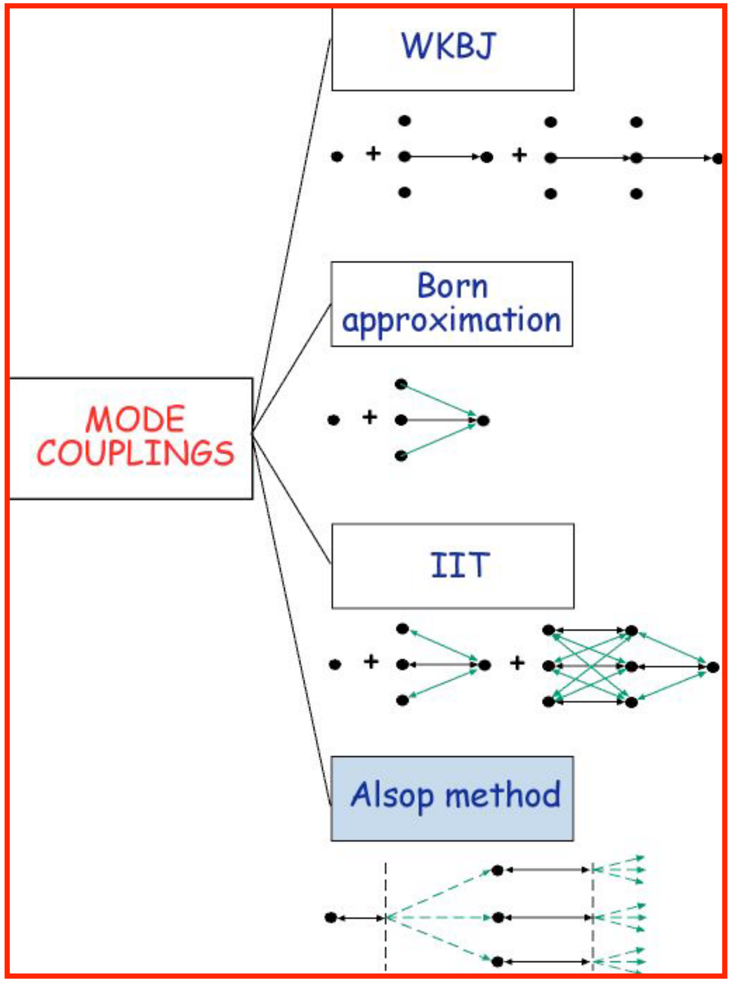
Wave propagation problems can be classified using the parameters just introduced.

This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

Techniques for synthetic seismograms





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