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"Seventh Workshop on Non-Linear Dynamics and Earthquake Prediction"

29 September - 11 October 2003

Boolean Delay Equations: A New Type of Dynamical Systems and Its Applications to Climate and Earthquakes

Theory of BDEs

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7^{th} Workshop on NL Dynamics & EQ Prediction

ICTP, Trieste, 29 Sept.– 10 Oct., 2003

Boolean Delay Equations: A New Type of Dynamical Systems and Its Applications to Climate and Earthquakes

Michael Ghil
Ecole Normale Supérieure & UCLA

Motivation

1. Complexity of the phenomena and feedback networks in solid-earth geophysics and fluid-envelope problems.

2. Difficulty in formulating "classical" models (ODEs, PDEs, SDEs), ascertaining parameter values, and analyzing even qualitative behavior for such models.

3. Availability of new modeling tool – Boolean Delay Equations (BDEs): simpler, more flexible, easier to formulate and analyze.

Work with *D. Dee* (NASA Goddard), *V. Keilis-Borok* (IGPP, UCLA & MITP, Moscow), *A. P. Mullhaupt* (Wall Street), *P. Pestiaux* (TotalFina, France), *A. Saunders* (UCLA), *& I. Zaliapin* (IGPP, UCLA & MITP, Moscow).

http://www.atmos.ucla.edu/tcd/

F. Bretherton's "horrendogram" of Earth System Science

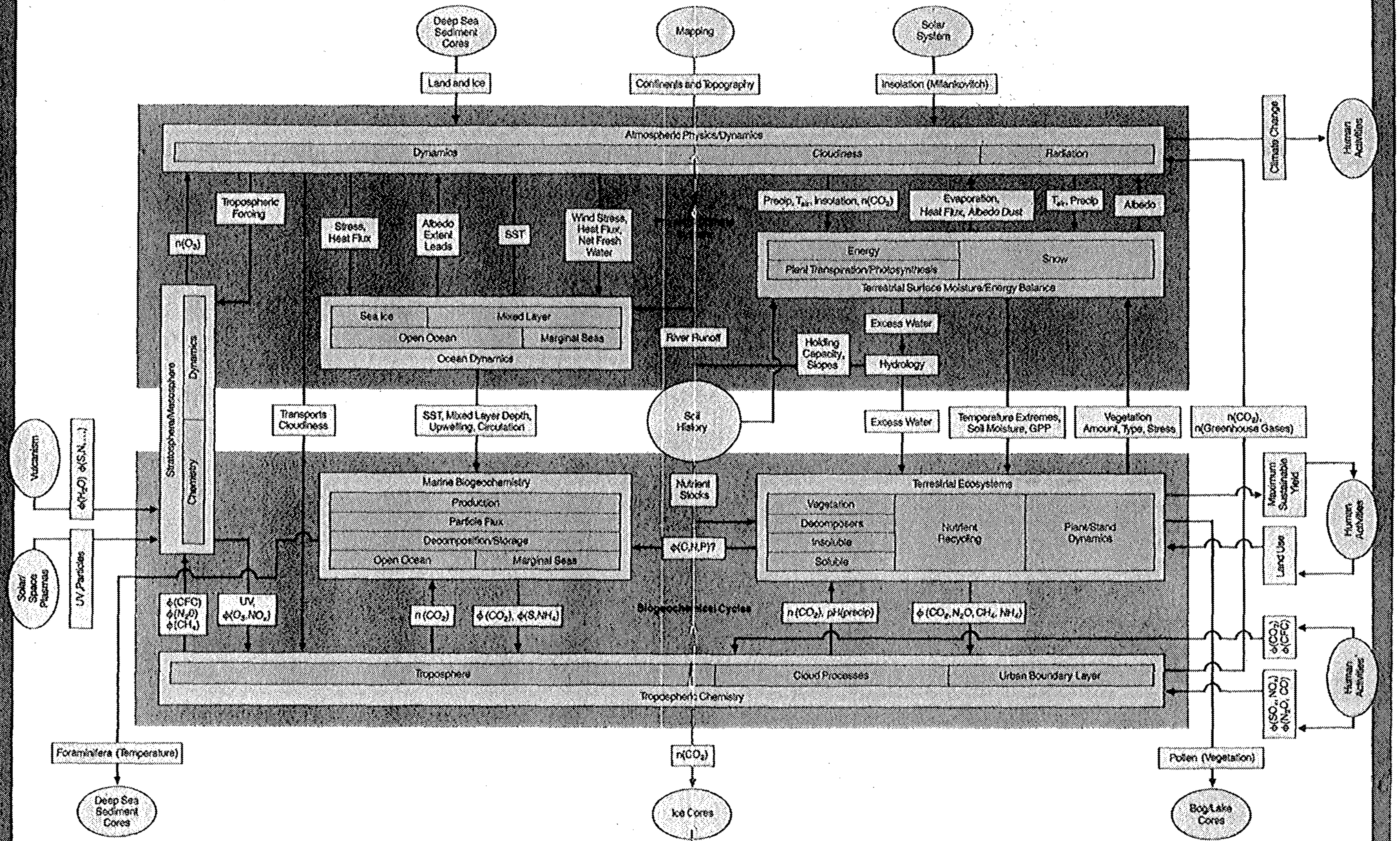


Figure 3. FLUID AND BIOLOGICAL EARTH PROCESSES: DETAILED INFORMATION FLOW
[φ(...) = flux, n(...) = concentration]

ESS Overview NASA Adv Council 1986

The climate system
on long time scales

1) *"Ambitious" diagram*

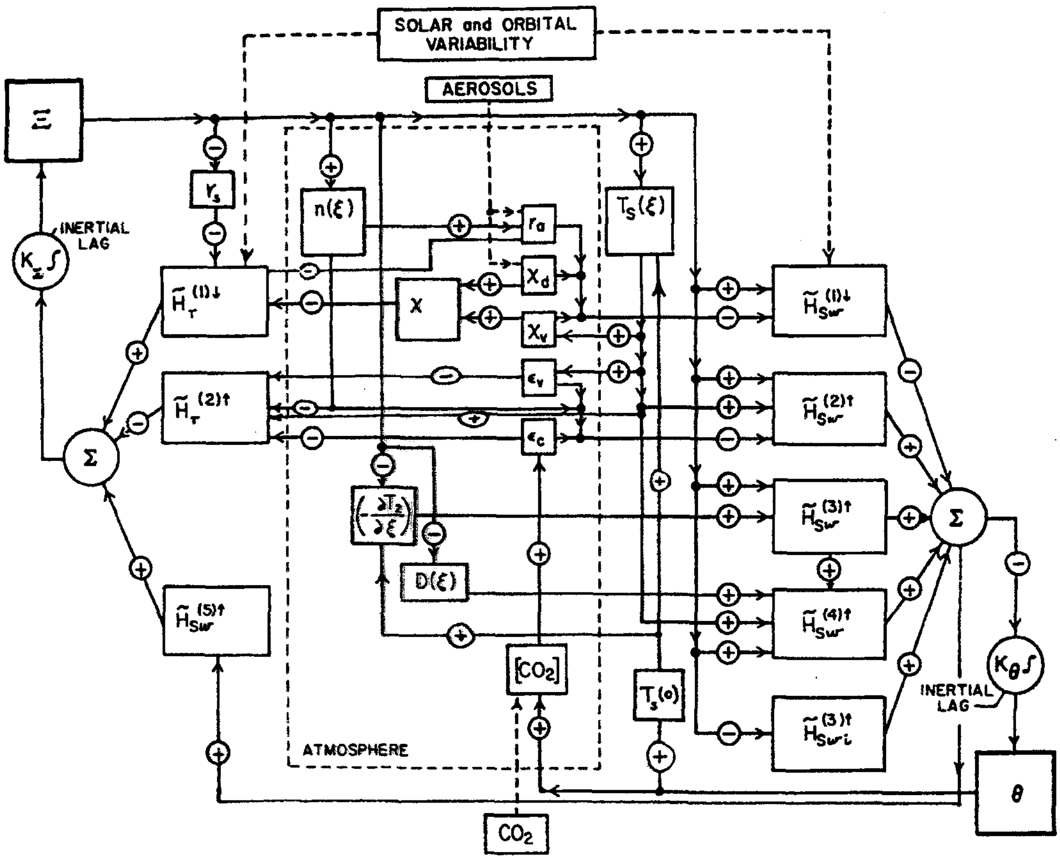


Fig. 19. Flow diagram showing feedback loops contained in the proposed dynamical system for ice-mass (Ξ) and ocean temperature (θ) variations.

m *T*

Constants for ODEs/PDEs poorly known.
Mechanisms & effective delays easier to ascertain

B. Saltzman, Climatic systems analysis, *Adv. Geophys.*, **25**, 1983.

2) "*Modest*" *model*

(Ghil & associates, 1979–1981)

(i) $\dot{T} \cong -m$: ice-albedo feedback

$\dot{m} \cong T$: precipitation-temperature feedback

(ii) $\dot{m} \cong p$: ice-load feedback

$\dot{p} = -m$: snow-accumulation feedback

Källén, Crafoord & Ghil (1979), *JAS*

Ghil & Le Treut (1981), *JGR*

Outline

1. *What for BDEs?*
   - life is sometimes too complex for ODEs & PDEs

2. *What are BDEs?*
   - formal models of complex feedback webs
   - classification & major results

3. *Applications to climate modeling*
   - paleoclimate — Quaternary glaciations
   - interdecadal climate variability in the Arctic
   - ENSO — interannual variability in the Tropics

4. *Applications to earthquake modeling*
   - colliding-cascades model of seismic activity
   - advances in prediction

5. *Concluding remarks*

http://www.atmos.ucla.edu/tcd/

Short BDE Bibliography

Theory: Dee & Ghil (l984, *SIAM J. Appl. Math.*); Ghil & Mullhaupt (l985, *J. Stat. Phys.*)

Applications to climate: Ghil *et al.* (1987, *Climate Dyn.*); Mysak *et al.* (1990, *Climate Dyn.*); Wright *et al.* (1990, *Climate Dyn.*); Darby & Mysak (1993, *Climate Dyn.*); Saunders & Ghil (2001, *Physica D*)

Application to solid-earth problems: Zaliapin, Keilis-Borok & Ghil (2002a, b, *J. Stat. Phys.*)

1. Introduction

Binary systems

Examples: Yes/No, True/False (ancient Greeks)

- *Classical logic* (Tertium non datur)
  Boolean algebra (19th cent.)
  Propositional calculus (20th cent.)
  (syllogisms as trivial examples)

- Genes: on/off
  Descriptive – Jacob & Monod (1961)
  Mathematical *genetics* – L. Glass, S. Kauffman,
  M. Sugita (1960s)
- Symbolic dynamics of differentiable dynamical systems (DDS): S. Smale (1967)

- Switches: on/off, 1/0
  Modern *computation* (EE & CS)
  – *cellular automata* (CAs)
  J. von Neumann (1940s, 1966), S. Ulam, Conway (the game of life), S. Wolfram (1970s, '80s)
  – spatial increase in complexity – infinite number of channels
  – *conservative logic*
  Fredkin & Toffoli (1982)

– *kinetic logic*: importance of distinct delays
R. Thomas (1973, 1979, ...) to achieve
temporal increase in complexity(*)

M. G.'s immediate motivation:

Climate dynamics – complex interactions
(reduce to binary)
C. Nicolis (1982)

Joint work on developing and applying
BDEs to climate dynamics
with D. Dee, A. Mullhaupt & P. Pestiaux (1980s)
& with A. Saunders (late 1990s)
Work of L. Mysak and associates (early 1990s)

Recent applications to solid-earth geophysics
(earthquake modeling & prediction) with V. Keilis-Borok & I. Zaliapin

(*)Synchronization, operating systems & parallel computation

What are BDEs?

Short Answer :

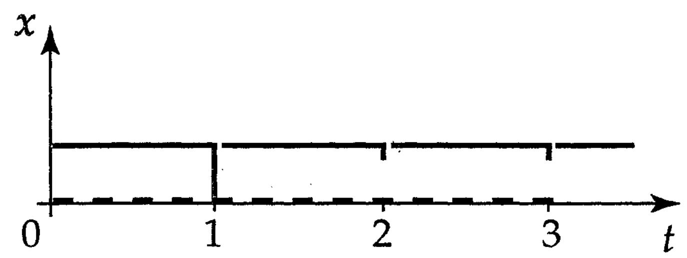
Maximum simplification of nonlinear dynamics
(nondifferentiable time-continuous dynamical system)

Longer Answer :

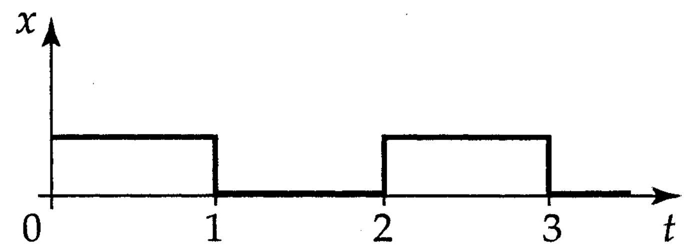
$x \in \mathbb{B} = \{0,1\}$

1) $x(t) = x(t\text{-}1)$

(simplest EBM: $x = T$)



2) $x(t) = \bar{x}(t\text{-}1)$



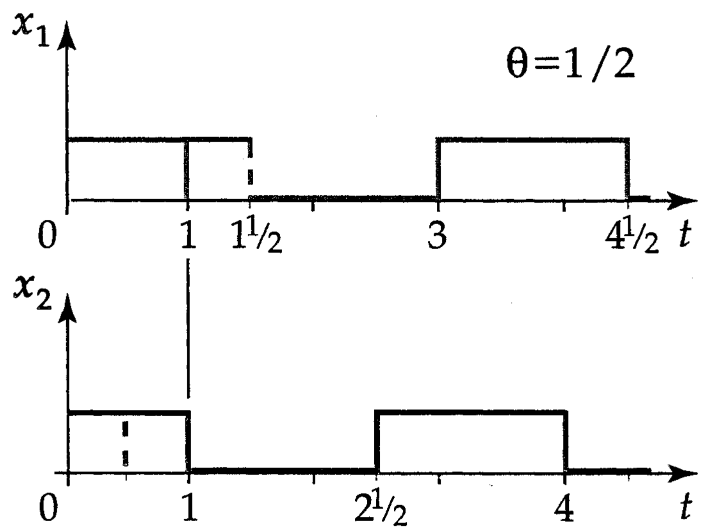
3) $x_1, x_2 \in \mathbb{B}$, $0 < \theta \leq 1$

$x_1(t) = x_2(t\text{-}\theta)$

$x_2(t) = \bar{x}_1(t\text{-}1)$

Eventually periodic
with a period = $2(1+\theta)$

(simplest OCM: $x_2 = T$, $x_1 = m$)



$x_1(t) = x_2(t-\theta)$

$x_2(t) = x_2(t-1) \nabla x_1(t-\theta)$, θ irrational

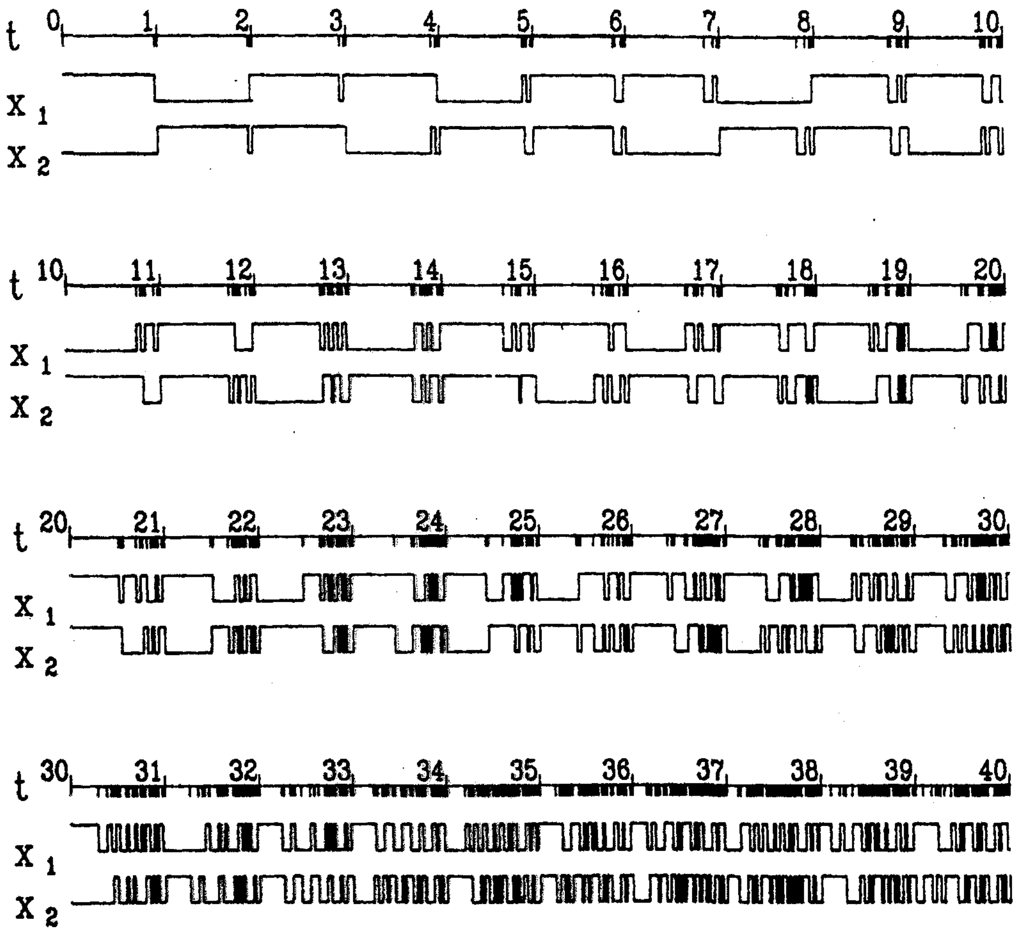


Fig. 1

Increase in complexity!
Evolution: biological, cosmogonic, historical.

But how much?

Dee & Ghil, *SIAM J. Appl. Math.* (1984), **44**, 111-126.

4) $x_1, x_2 \in \mathbb{B}$, $\theta = 0.977$

$$x_1(t) = x_2(t-\theta)$$

$$x_2(t) = x_1(t-\theta) \triangledown x_2(t-1)$$

$\triangledown$:= 'exclusive or' ($x = y \triangledown z$: $x = 1$ iff $y \neq z$)

This can also be written as a single, 2nd-order BDE, rather than 1st-order system of 2 BDEs

$$x(t) = x(t-\tau) \triangledown x(t-1)$$

by changing time scale s.t.

$$\max\{2\theta, 1\} := 1$$

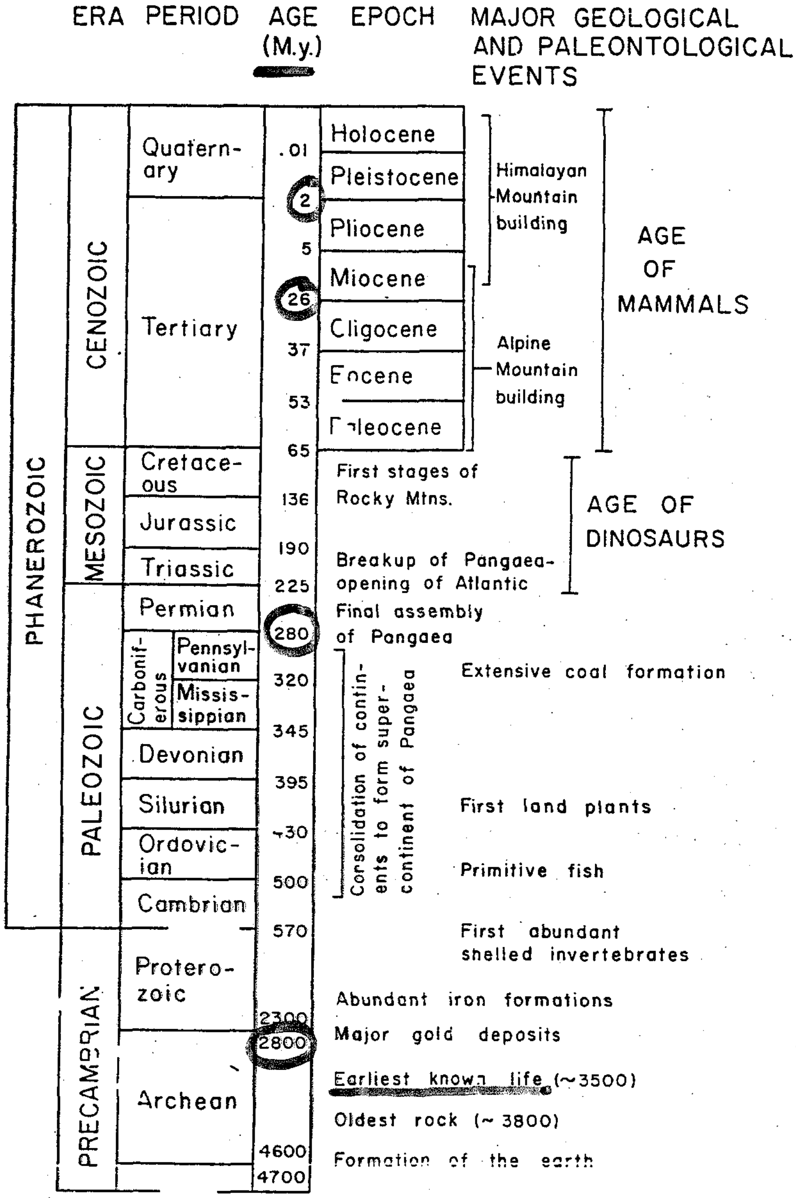


Fig. 2. The geological time scale.

Density of events ≈ log t

grained constituents of limestone are of biologic origin. For example, on the present south Florida carbonate shelf a single genus of calcified green algae produces the vast majority of the carbonate mud in the surrounding environments [*Stockman et al.*, 1967]. Precambrian carbonate formation may also have been controlled by organisms, for the carbonates commonly occur in the form of stromatolites (finely laminated rock). The origin of this rock type seems dependent on the presence of bacteria or blue-green algae [*Schopf*, 1980].

Early Precambrian CO_2 partial pressure estimates of 0.01–0.10 atm exceed present values by 30–300 times [*Garrels and Perry*, 1974; *Pollack and Yung*, 1980]. The high Mg concentrations in Precambrian carbonates [e.g., *Tucker*, 1982] are consistent with such an increased partial pressure of CO_2 [*Holland*, 1976]. *Owen et al.* [1979] calculate that an enhanced CO_2-H_2O greenhouse effect could have produced a mean temperature of 310°K at 4.2 b.y. (the present mean is 287°K). Some empirical support for high temperatures is provided by preliminary isotopic analyses of cherts, which indicate groundwater temperatures of 340°K as late as 2.8 b.y. [*Knauth and Epstein*, 1976].

In summary, climatic cooling induced by a faint sun may have been offset by an atmospheric greenhouse effect caused by high CO_2-H_2O concentrations. The first evidence for glaciation is at about 2.3 b.y. This date coincides with a rapid expansion of stromatolites [e.g., *Frakes*, 1979], an event that may have signaled an increased withdrawal of CO_2 from the atmosphere by photosynthetic organisms.

3. LATE PRECAMBRIAN (2.5–0.57 B. Y.)

The late Precambrian (Proterozoic) contains evidence for two phases of continental glaciation, at 2.3 and 0.9–0.6 b.y. [*Frakes*, 1979]. Before discussing these events it is useful to

The Historic Time Scale

Beginnings of *Homo expectorans* – 5 Myr ago (Ma),

First modern humans in Ambrica – 1 Ma

First tools, Palo Tinto civilization – 200 ka

Palo Tinto replaced by Long Ears – 20 ka

First Short Ear state in New Colorado – 5 ka ago

Period of rival Ear and Nose states - 2000–1500 B. C.

Amhurstex, legendary founder of

Lower Kingdom - 500 B.C.

Historic Plectorectic Dynasty - 200 B.C.-100 A.D.

....

Ambrican revolution – 1792

1st war of independence – 1815

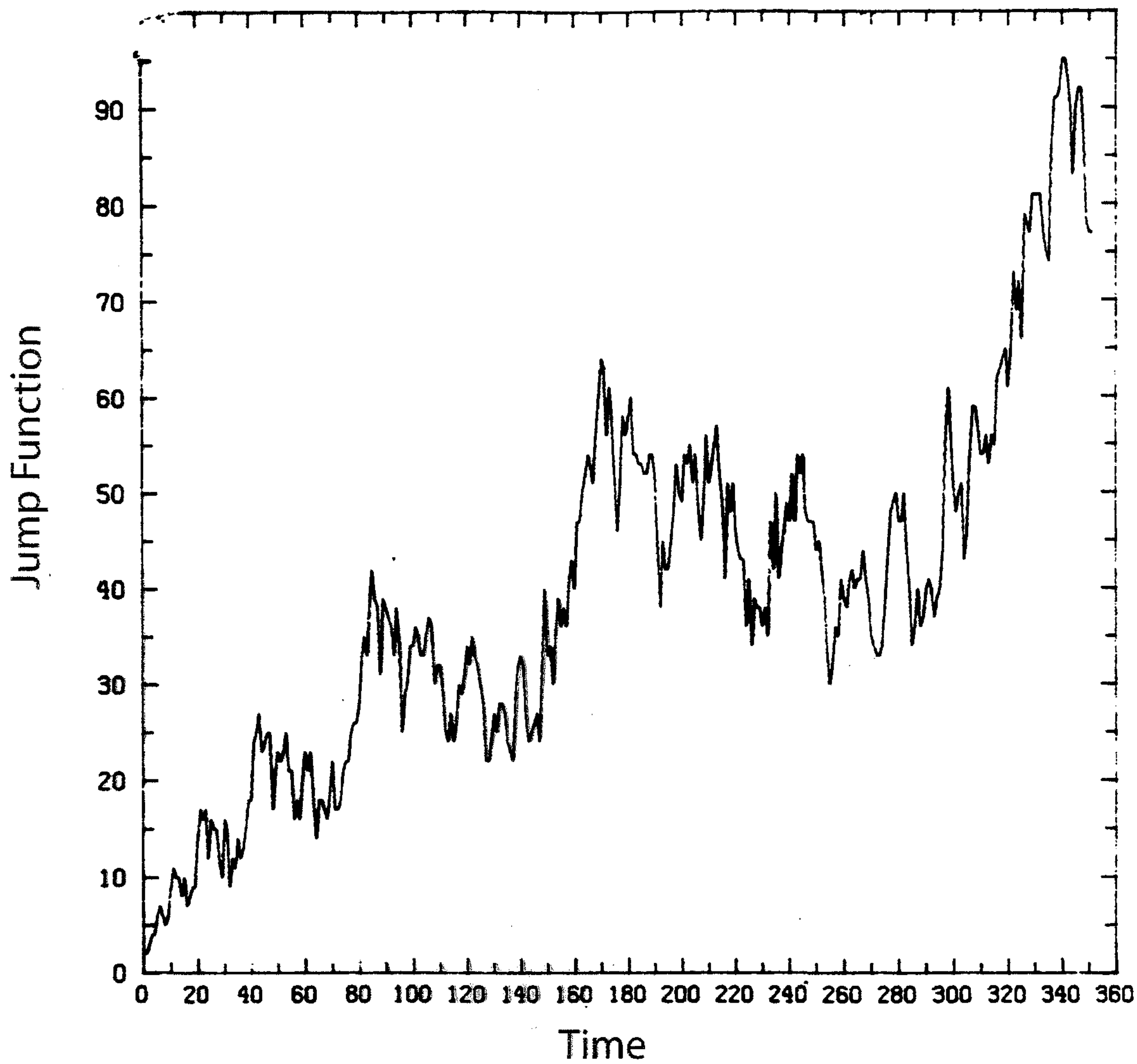
....

35 governments since WWII (1945)

Ghil2/Documents & CVs/Transparencies/CSEOL

Aperiodic solutions
with increasing complexity

$x(t) = x(t\text{-}1)\ \nabla\ x(t\text{-}\theta), \quad \theta = (\sqrt{5}\ -1)/2 =$ "golden ratio"

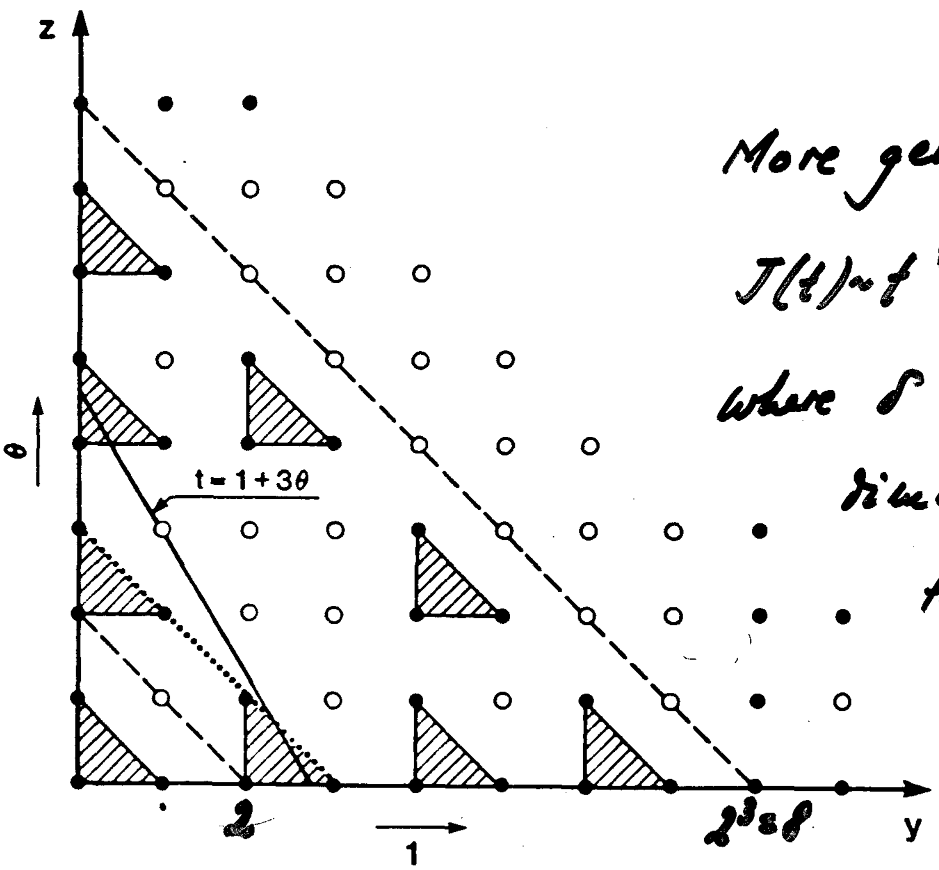


Theorem. Conservative BDEs with irrational delays have aperiodic solutions with a power-law increase in complexity.

N.B. Log-periodic behavior!

Sketch of pf. for 2 indep. delays

(11



More generally $J(t) \sim t^{\log_2(\delta+1)-1}$ where δ is the dimension of the delay lattice (here δ=2)

Figure 1. The delay lattice for Eq. (3.5). • - jump occurs, o - jump does not occur. See text for details.

The two dashed lines in the figure are drawn through the pairs of lattice points $(2^k,0)$, $(0,2^k\theta)$ for k = 1 and k = 3, respectively. The number of jumps occurring before t = 1 + 3θ can be estimated from above by the number of jumps in the lattice triangle with the dashed line for k = 3 as its base; it can be estimated from below by the jumps in the triangle corresponding to k = 1. These two numbers can be computed explicitly in the case at hand.

The computation proceeds by noticing the self-similarity in the pattern of jumps. The lowest-level pattern is given by the small

$J(t) \sim t^{\log_2 3-1} \cong t^{0.6}$, $\alpha = \log_2 3-1 \cong 0.6$

4. Classification of BDEs

Theorem. BDEs with rational delays only have periodic solutions only.

Proof. Main idea: solution space is reducible to a finite number of points (cf. cellular automata, kinetic logic).

Details: Let q be l.c.d. of delays p_{ij}/q_{ij}, & subdivide $[0,1]$ into q subintervals. The phase space $X = B^n(I)$ is finitely generated & "words" of length q have to repeat. If they repeat once, they do forever.

Q: What about increased complexity in Fig. 1?

A: Approximation theorem for delays.

Definition. A BDE is *conservative* if its solutions are immediately periodic, i.e. no transients; otherwise it is *dissipative*.

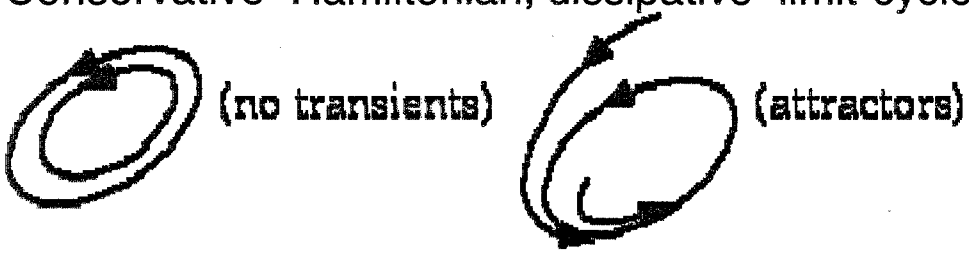
Remark. Rational vs. irrational delays.

Ex: 1) conservative $x(t) = \bar{x}(t-1)$

2) dissipative $x(t) = x(t-1) \wedge x(t-\theta)$

Analogy with ODEs

Conservative–Hamiltonian; dissipative–limit cycle,



M. Ghil & A. Mullhaupt, *J. Stat. Phys.*, 1985

Examples. Convenient shorthand for scalar 2nd-order BDEs

$$x = y \circ z \Leftrightarrow x(t) = x(t-1) \circ x(t-\theta)$$

1. Conservative

$$x = y \triangledown z = y \oplus z = y + z \text{ (mod 2)}$$

$$x = y \vartriangle z = 1 \oplus y \oplus z$$

Remarks: i) Conservative $\cong$ linear (mod 2)

ii) $\exists$ few conservative connections (~ODEs)

2. Dissipative

$$x = y \wedge z \tilde{\Rightarrow} x \to 0$$

$$x = y \vee z \tilde{\Rightarrow} x \to 1$$

Theorem 4.2. Conservative $\Leftrightarrow$ reversible

$\Leftrightarrow$ invertible

A. P. Mullhaupt, *Ph.D. thesis*, May 1984, CIMS/NYU.
M. Ghil & A. P. Mullhaupt, *J. Stat. Phys.*, 1985, 41, 125-173.

3) Dissipative

1) totally dissipative

2) partially conservative, i.e.,

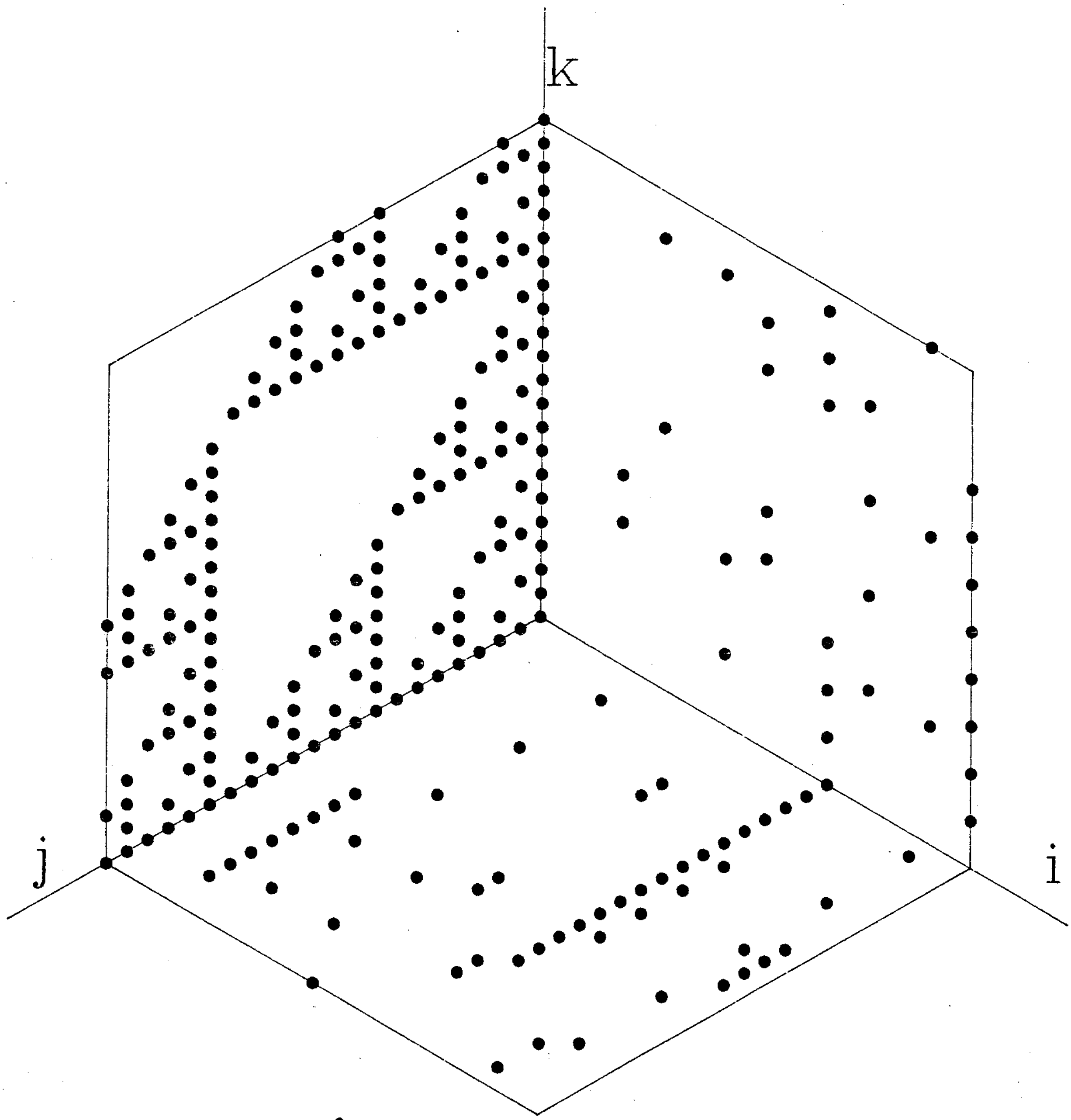
having some —"— connectives (1-1)

Ex: 1) $x(t) = x(t-\theta) \wedge x(t-1)$

2) $x = (p \vee q) \wedge r$

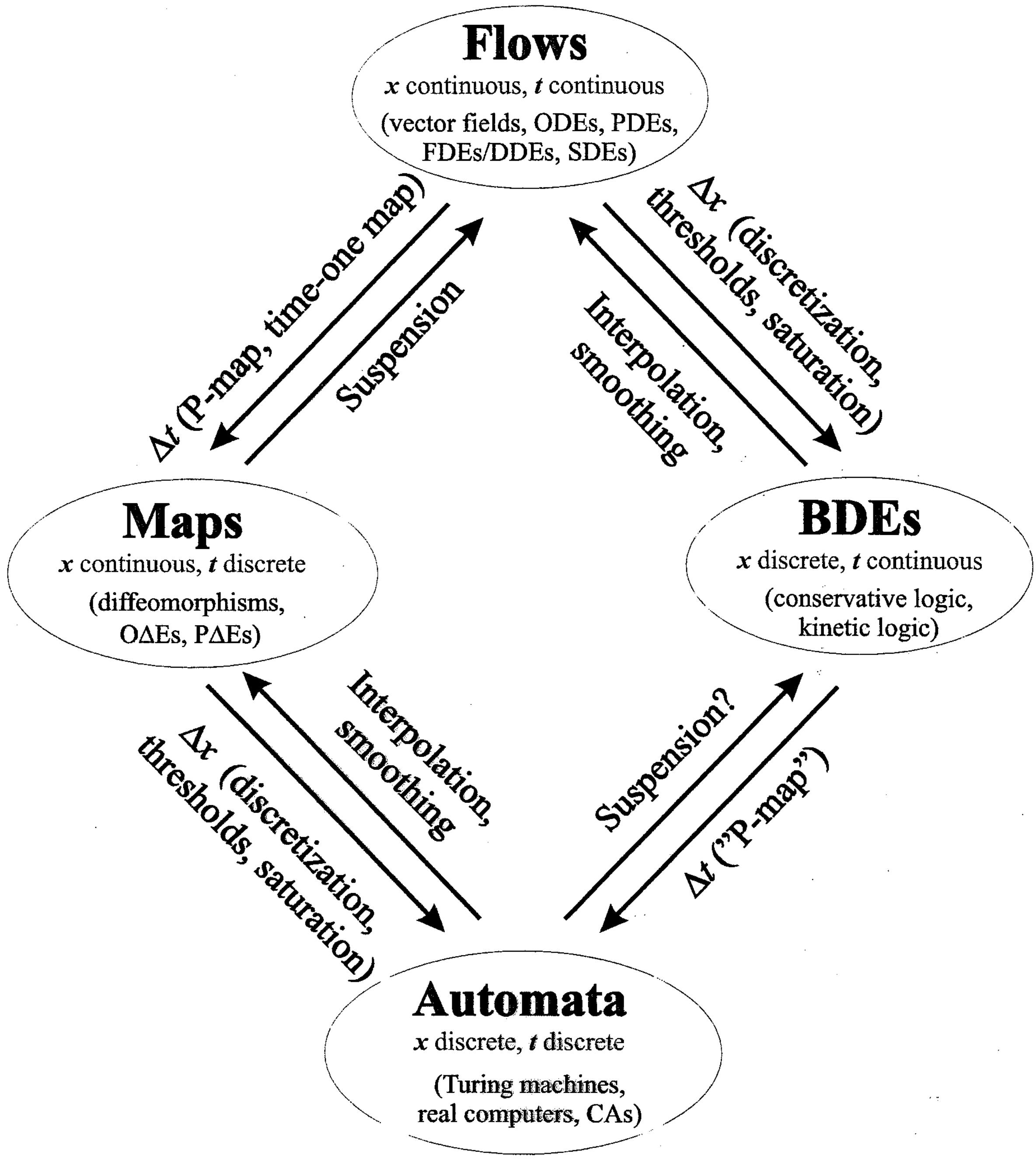
56. Partially linear BOEs

$$x(t) = x(t-1) \nabla x(t-\theta) \wedge \bar{x}(t-\tau) \quad (2)$$



$\theta, \tau, \theta/\tau \notin \mathbb{Q}$, $\delta = 3$

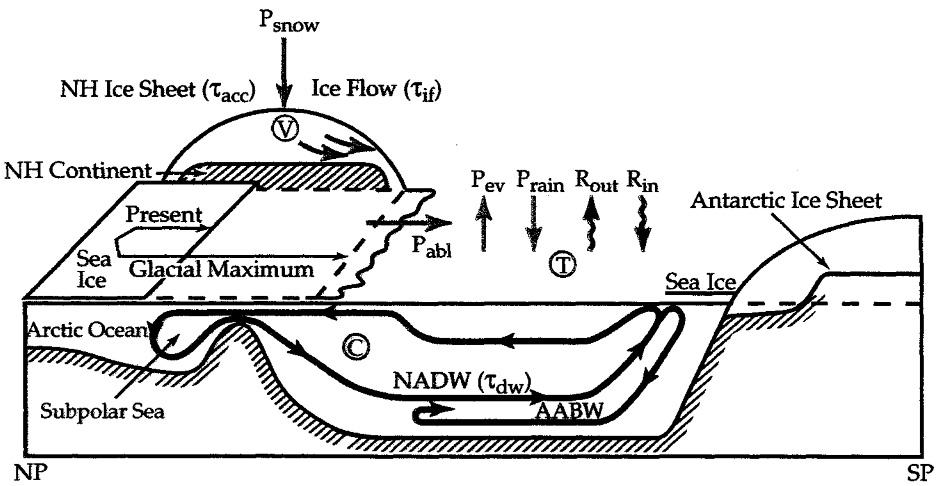
One can define the char. polyn. of the linear part of (2), & the index ν of reversibility.



After Mullhaupt (1984), Courtesy of I. Zaliapin (2001)

Paleoclimate application:

Thermohaline circulation and glaciations



Logical variables

T - global surface temperature;
V_N - NH ice volume, $V_N = V$;
V_S - SH ice volume, $V_S \equiv 1$;
C - deep-water circulation index

Ghil, Mullhaupt & Pestiaux, *Climate Dyn.*, **2** (1987), 1-10.

Physical mechanisms at work

i) precipitation–temperature effect

$$V_N(t) = T(t - \tau_{acc}) \tag{1}$$

ii) ice–albedo effect + ocean cooling effect

$$T(t) = \overline{V}_N(t - \tau_{ifl}) \wedge \overline{C}(t - \tau_{dw}) \tag{2}$$

iii) deep-water formation effects

– melt water does not sink

– freezing increases salinity & sinking

$$\begin{aligned} \overline{C}(t) &= \{V_N(t) \triangledown V_N(t - \tau_{dw})\} \wedge \overline{V}_N(t) \\ &= V(t - \tau_{dw}) \wedge \overline{V}(t) \end{aligned} \tag{3}$$

Delay values

$\tau_{acc} \cong 10\text{kyr}$ – time for growth of ice sheet

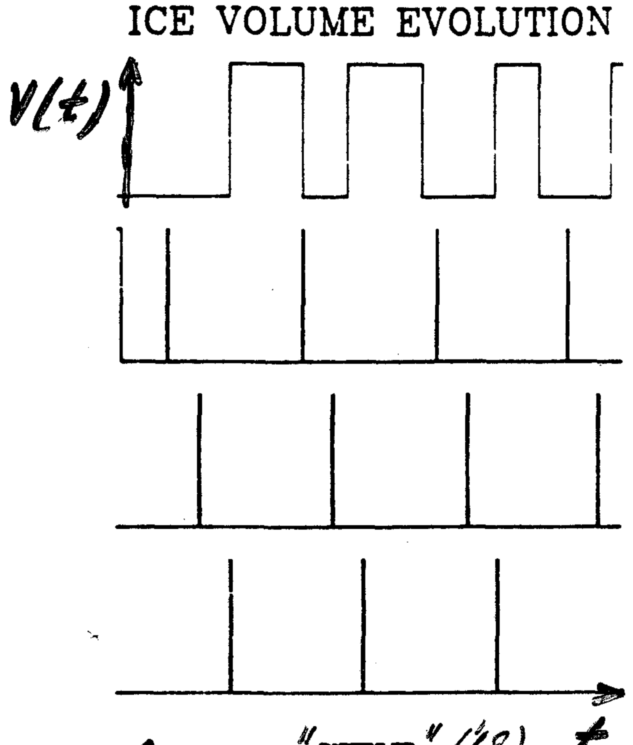
$\tau_{ifl} \cong 3\text{kyr}$ – characteristic ice flow time, with basal sliding

$\tau_{dw} \cong 1\text{kyr}$ – recirculation time of deep water

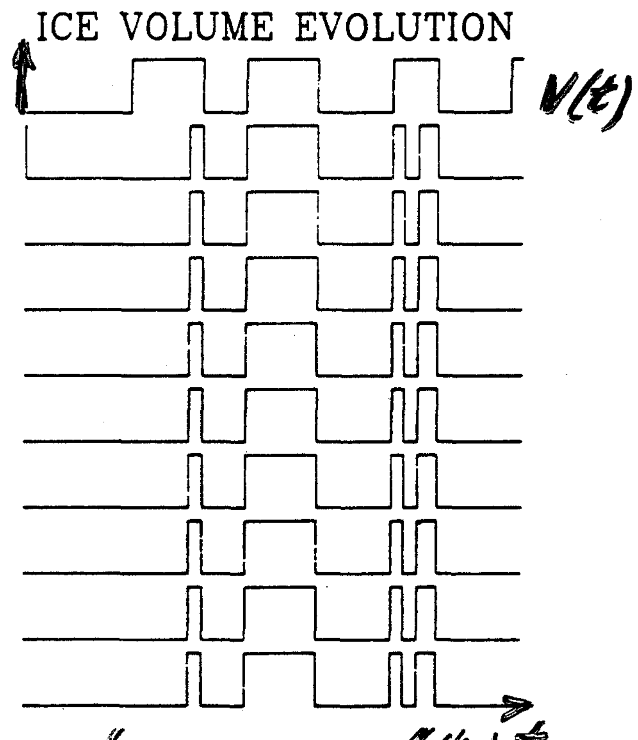
Substituting $\bar{C}(t)$ from Eq. (3) into Eq. (2) & the resulting $T(t)$ from (2') so obtained into (1) yields

$$V(t) = \bar{V}(t-\theta_1) \wedge V(t-\theta_2) \wedge \bar{V}(t-\theta_3). \qquad (4)$$

where $\theta_1 = \tau_{acc} + \tau_{if}$, $\theta_2 = \tau_{acc} + 2\tau_{dw}$, & $\theta_3 = \tau_{acc} + \tau_{dw}$.



$\tau_{if} > 2\tau_{dw}$: "WEYL" ('68)
(weakening of THC only)



$\tau_{if} < 2\tau_{dw}$: "WORTHINGTON" ('68)
(weakening & slowing down)

Asymptotic simplification of Eq. (4) yields

$$V(t) = V(t-\theta_2). \qquad (5)$$

So sol'ns are periodic eventually, w/per. $\theta_2 \cong 12$-18ka

"Weyl" - short glacial episodes only } "Weyl"
"Worthington" - warm episodes slightly longer } "Worthington"

6. Structural stability & bifurcations

Theorem. BDEs with periodic solutions only are structurally stable, & conversely.

Remark. They are dissipative.

Meta-theorems, by example.

The asymptotic behavior of

$$x(t) = x(t - \theta) \wedge \bar{x}(t - \tau) \tag{3}$$

is given by

$$x(t) = x(t - \theta).$$

Hence, if $\tau < \theta = 1$,

then solutions are asymptotically periodic;

if $\theta < \tau = 1$, however,

then solutions tend asymptotically to 0.

Therefore, as θ passes through τ, one has "Hopf bifurcation."