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Edge Current in Non-commutative Chern-Simons Theories on a Manifold with Boundary

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These are preliminary lecture notes, intended only for distribution to participants.

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EDGE CURRENT IN NON-COMMUTATIVE CHERN-SIMONS THEORIES ON A MANIFOLD WITH BOUNDARY.

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WORK DONE IN COLLABORATION WITH:

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2) SECKIN KÜRKÇÜOĞLU, DIAS, DUBLIN, IRELAND

PLAN OF THE TALK:

1) INTRODUCTION

2) EDGE CURRENTS IN COMMUTATIVE CHERN-SIMONS (CS) THEORY

3) NON-COMMUTATIVE (NC) CHERN-SIMONS THEORY ON THE INFINITE STRIP

4) LARGE M LIMIT

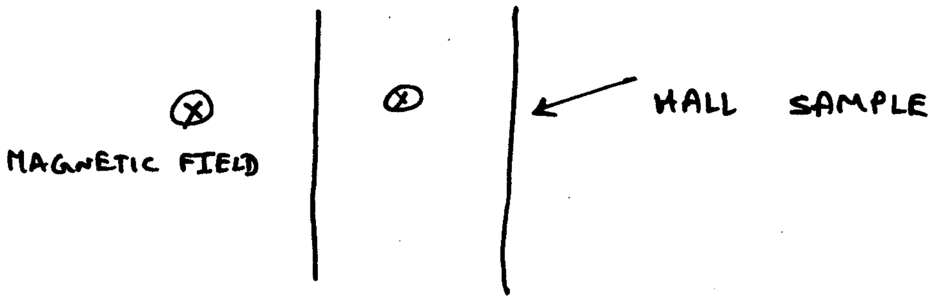
5) COMMUTATIVE LIMIT

6) CONCLUSION

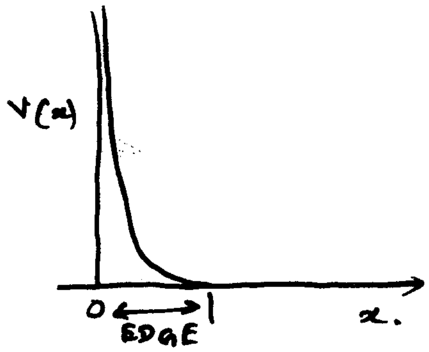
TWO IMPORTANT ASPECTS OF QHE:

1) QUANTIZATION OF HALL CONDUCTANCE

2) EXISTENCE OF EDGE CURRENTS



CONFINING POTENTIAL



GRADIENT OF $V(x) \neq 0$ ONLY AT "EDGE" AND ACTS AS EFFECTIVE ELECTRIC FIELD. THIS TOGETHER WITH THE EXTERNAL MAGNETIC FIELD PRODUCES EDGE CURRENT

ALTHOUGH THE SYSTEM IS 2+1 DIMENSIONAL, THE D.O.F. ARRANGE THEMSELVES IN A WAY SUCH THAT THE PHYSICS IS DESCRIBED IN TERMS OF A THEORY WHICH IS ESSENTIALLY 1+1 DIMENSIONAL

$\Rightarrow$ HOLOGRAPHY.

1) INTRODUCTION

→ NC CS IS THE GENERALIZATION OF THE COMMUTATIVE CS THEORY WHICH ARISES IN VARIOUS AREAS OF PHYSICS AND MATHEMATICS.

→ THE RELATION BETWEEN COMMUTATIVE CS AND QUANTUM HALL EFFECT (QHE) RECEIVED EXTENSIVE ATTENTION IN THE LITERATURE.

→ MORE RECENTLY, THE CONNECTION BETWEEN NC CS AND QHE HAS BEEN DISCUSSED BY VARIOUS AUTHORS

(SUSSKIND, hep-th/0101029; FRADKIN et al, COND-MAT/0205653; POLYCHRONAKOS, hep-th/0103013, 0106011; MORARIU & POLYCHRONAKOS, hep-th/0106011; HELLERMAN & RAAMSDONK, hep-th/0103179).

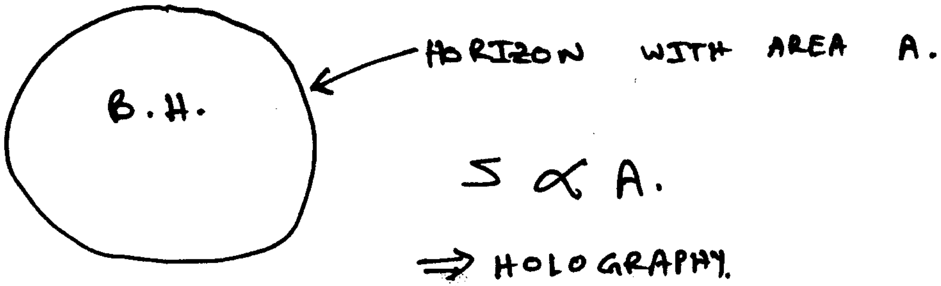
→ CS ON A DISC OR HALF PLANE IN PRESENCE OF SPATIAL NON-COMMUTATIVITY HAVE BEEN STUDIED, WHICH FACED OBSTACLES IN DEFINING THE STAR PRODUCT ON A MANIFOLD WITH BOUNDARY. (PINZUL & STERN, hep-th/0107179; LUGO, hep-th/0111064). CS ON NC PLANE WITH A HOLE HAS ALSO BEEN DISCUSSED (PINZUL & STERN, hep-th/0112220).
(ALSO CHEN & WU, hep-th/0111109)

→ IN OUR WORK WE STUDY THE NC CS ON A MANIFOLD WITH BOUNDARY AS A LIMIT OF A SUITABLE MATRIX MODEL. WE PAY SPECIAL ATTENTION TO THE PHYSICS AT THE "EDGE".
(A.P. BALACHANDRAN, KSG & S. KÜRKÇÜOĞLU, hep-th/0306255)

IN ADDITION TO QHE, CS THEORY HAS PLAYED A MAJOR ROLE IN 2+1 GRAVITY

THE CALCULATION OF THE ENTROPY OF 2+1 DIM. BTZ BLACK HOLE IS A CONSEQUENCE OF THE HOLOGRAPHIC NATURE OF CS THEORY.

NC CS COULD BE IMPORTANT FOR ANALYZING THE QUANTUM ASPECTS OF GRAVITY IN A NC SETUP.

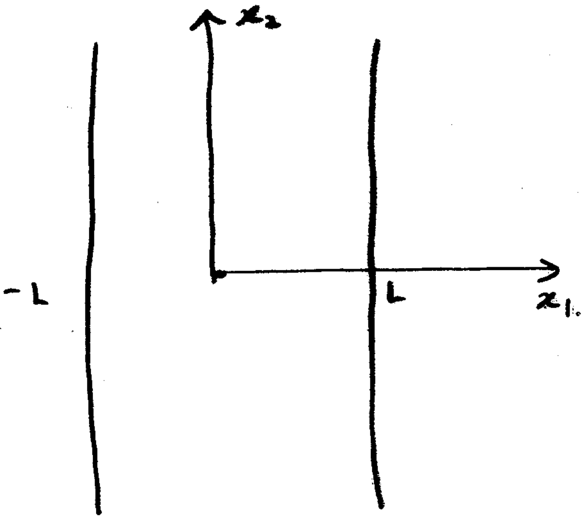


$$S \propto A.$$

$\Rightarrow$ HOLOGRAPHY.

2) EDGE CURRENTS IN COMMUTATIVE CS THEORY

OUR MANIFOLD IS AN INFINITE STRIP $\mathcal{T}$ IN $\mathbb{R}^2 \otimes \mathbb{R}^1 \leftarrow$ TIME.



$$\mathcal{T} = \{ (x_1, x_2) \in \mathbb{R}^2 \mid -L \leq x_1 \leq L \}.$$

THE ACTION FOR ABELIAN CS ON $\mathcal{T}$ IS

$$S = \frac{k}{4\pi} \int_{\mathcal{T}\times\mathbb{R}^1} A \wedge dA \,, \qquad A = A_\mu \, dx^\mu$$

THE EQUAL TIME P.B.'S ARE:

$$\{ A_i (x_0, x_1, x_2),\ A_j (x_0, x_1', x_2') \} = \frac{2\pi}{k} \varepsilon_{ij} \ \delta^2 (\vec{x} - \vec{x}') \,, \quad i,j = 1,2.$$

THE GAUSS LAW CONSTRAINT IS:

$$g(\Lambda^0) = \frac{k}{2\pi} \int_{\mathcal{T}} \Lambda^0 (x)\, dA(x) \approx 0. \qquad (\Lambda^0 : \text{TEST FN.})$$

DIFFERENTIABILITY OF $g(\Lambda^0) \Rightarrow \Lambda^0 |_{\partial\mathcal{T}} = 0$

$g(\Lambda^0)$ GENERATE THE GAUGE TRANSFORMATION $A \to A - d\Lambda^0$ AND ARE FIRST CLASS CONSTRAINTS:

$$\{ g(\Lambda_1^0),\ g(\Lambda_2^0) \} \approx 0$$

Q: WHAT ABOUT OBSERVABLES?

THE CHARGES OF THIS THEORY ARE:

$$q(\Lambda) = \frac{k}{2\pi} \int_{\tau} d\Lambda \wedge A \qquad (\Lambda : \text{TEST FN.}) \quad \Lambda|_{\partial\tau} \neq 0.$$

THEY ARE FIRST CLASS:

$$\{ q(\Lambda), g(\Lambda^0) \} \approx 0$$

AND ARE THE OBSERVABLES OF THE THEORY.

Q: WHAT ABOUT EDGE?

FOR ~~AB~~ $\Lambda_1 - \Lambda_2 = \Lambda^0$,

$$q(\Lambda_1) - q(\Lambda_2) = -g(\Lambda_1 - \Lambda_2) \approx 0$$

$\Rightarrow$ TEST FNS. Λ_1 AND Λ_2 WHICH ARE EQUAL ON $\partial\tau$ ($\because \Lambda^0|_{\partial\tau} = 0$) GENERATE GAUGE EQUIVALENT CHARGES.

$\Rightarrow$ $q(\Lambda)$ ARE INDEED EDGE OBSERVABLES.

ALGEBRA OF OBSERVABLES:

$$\{ q(\Lambda_1), q(\Lambda_2) \} = k/2\pi \int_{\tau} d\Lambda_1 \wedge d\Lambda_2 = k/2\pi \int_{\partial\tau} \Lambda_1 \, d\Lambda_2$$

CHOOSING A BASIS OF TEST FNS. ON $\partial\tau$ AS
$\Lambda_1|_{x_1 = L} = e^{ik_1 x_2}$ $\quad \Lambda_2|_{x_1 = L} = e^{ik_2 x_2}$ $\quad \Lambda_1|_{x_1 = -L} = 0 = \Lambda_2|_{x_1 = -L}$

WE GET:

$$\{ q(\Lambda_1), q(\Lambda_2) \} = i k k_2 \, \delta(k_1 + k_2)$$

$\uparrow$

U(1) KAC-MOODY ALGEBRA.

WE SHALL NOW CONSIDER AN EQUIVALENT REFORMULATION.

CONSIDER CS THEORY ON $\mathbb{R}^2 \otimes \mathbb{R}^1$, WITH

→ SPATIAL COMPONENTS OF A SUPPORTED IN $|x_1| \leq L$

→ A_0 SUPPORTED IN $|x_1| < L$

THE VARIATION OF S UNDER A_0 GIVES GAUSS LAW:

$$G(\Lambda^0) = \frac{k}{2\pi} \int_{\mathbb{R}^2} \Lambda^0 \, dA \approx 0$$

DIFFERENCE WITH GAUSS LAW BEFORE:

1) RANGE OF INTEGRATION NOW $\mathbb{R}^2$ (WAS T BEFORE)

2) Λ^0 NOW SUPPORTED IN $|x_1| < L$, i.e. $\Lambda^0 = 0$ FOR $|x_1| \geq L$
(THIS FOLLOWS FROM SUPPORT OF δA_0 WHICH IS SAME AS THAT OF A_0

NOTE THAT $G(\Lambda^0)$ STILL FIRST CLASS.

THE OBSERVABLES ARE NOW DEFINED AS:

$$Q(\Lambda) = \frac{k}{2\pi} \int_{\mathbb{R}^2} d\Lambda \wedge A$$

Λ CAN BE SUPPORTED ON ALL OF $\mathbb{R}^2$, BUT IF IT IS SUPPORTED ONLY ON $\mathbb{R}^2/\tau$ (i.e. $|x_1| > L$), THEN $Q(\Lambda) = 0$.

FOR $\Lambda_1 - \Lambda_2 = \Lambda^0$, $Q(\Lambda_1) - Q(\Lambda_2) = -G(\Lambda_1 - \Lambda_2) \approx 0$

$\Rightarrow Q(\Lambda)$ LOCALIZED AT $x_1 = \pm L$.

ALGEBRA OF OBSERVABLES

$$\{Q(\Lambda_1), Q(\Lambda_2)\} = \frac{k}{2\pi} \int_{\mathbb{R}^2} d\Lambda_1 \wedge d\Lambda_2$$

TO COMPUTE THIS ALGEBRA AT $x_1 = L$, CHOOSE

$$\Lambda_i = \theta(x_1 - L)\, e^{i k_i x_2} \qquad (i = 1, 2).$$

THEN: $\{Q(\Lambda_1), Q(\Lambda_2)\} = i k\, k_2\, (k_1 + k_2)$.

SIMILARLY WE CAN GET ALGEBRA AT $x_1 = -L$ WITH

$$\Lambda_i = (1 - \theta(x_1 + L))\, e^{i k_i x_2}$$

THIS REFORMULATION WOULD BE HELPFUL IN THE NC CASE.

3) NON-COMMUTATIVE CHERN-SIMONS THEORY ON AN INFINITE STRIP

A) PROBLEM WITH STRAIGHT FORWARD GENERALIZATION

CONSIDER CS ACTION ON A MOYAL PLANE WITH NO BOUNDARY

$$S_{NCCS} = -\frac{k}{4\pi}\int dx_0\, d^2x\, \varepsilon_{\mu\nu\lambda}\left(A_\mu * \partial_\nu A_\lambda + \frac{2}{3} A_\mu * A_\nu * A_\lambda\right)$$

$\varepsilon_{123} = 1$ AND $\mu, \nu, \lambda = 0, 1, 2$, x_0 IS TIME AND x_1, x_2 ARE COORDINATES ON THE MOYAL PLANE. THE $*$ PRODUCT IS GIVEN BY.

$$f * g\,(x_1, x_2) = f(x_1, x_2)\, e^{i\theta/2\left(\overleftarrow{\partial}_{x_1}\overrightarrow{\partial}_{x_2} - \overleftarrow{\partial}_{x_2}\overrightarrow{\partial}_{x_1}\right)}\, g(x_1, x_2)$$

THE SPATIAL NONCOMMUTATIVITY IS EXPRESSED AS

$$[x_1, x_2]_* = i\theta$$

WHERE θ IS THE NONCOMMUTATIVITY PARAMETER AND

$$[f, g]_* = f * g - g * f$$

WE MAY NAIVELY CONSIDER THE ABOVE S_{NCCS} ON THE INFINITE STRIP $\mathcal{T}$ AND DO THE CANONICAL ANALYSIS. BUT THERE IS A PROBLEM WITH THIS, NAMELY THE $*$-PRODUCT IS NOT WELL DEFINED ON $\mathcal{T}$. TO SEE THIS NOTE THAT THE FORMULA FOR $*$ PRODUCT CONTAINS THE EXPONENTIAL FOR THE DIFFERENTIAL OPERATOR $-i\partial_{x_1}$. WITH THE USUAL DEFN. OF ITS DOMAIN, $-i\partial_{x_1}$ GENERATES TRANSLATION, SO

$$e^{i(-ic\,\partial_{x_1})}\psi(x_1) = \psi(x_1 + c)$$

CONSEQUENTLY, IF ψ HAS SUPPORT $[-L, L]$, $e^{i(-ic\,\partial_{x_1})}\psi$ DOES NOT, AND $*$ PRODUCT IS NOT WELL DEFINED ON FUNCTIONS SUPPORTED IN $[-L, L]$.

B) MATRIX MODEL

IN VIEW OF THIS PROBLEM, WE CONSIDER A FINITE DIMENSIONAL MATRIX MODEL WHICH BECOMES THE CS THEORY ON A NC INFINITE STRIP IN THE LIMIT THE SIZE OF THE MATRICES $\rightarrow \infty$.

IN ORDER TO SET UP THE MATRIX MODEL, CONSIDER THE MOYAL PLANE DESCRIBED BY OPERATORS $\hat{x}_1, \hat{x}_2$ WITH

$$[\hat{x}_1, \hat{x}_2] = i\theta$$

CONSIDER NOW A HARMONIC OSCILLATOR WITH HAMILTONIAN

$$\hat{H} = \frac{\hat{x}_2^2}{2m} + \frac{1}{2}\tilde{k}\hat{x}_1^2$$

AND WITH OSCILLATION FREQUENCY $\omega = \sqrt{\tilde{k}/m}$.
THE HILBERT SPACE $\mathcal{H}$ SPANNED BY THE EIGENSTATES OF THIS HAMILTONIAN WOULD ACT AS THE CARRIER SPACE OF THE OPERATORS IN OUR MODEL. THE NUMBER OF ENERGY EIGENSTATES OF $\hat{H}$ BELOW THE ENERGY $E = \frac{1}{2}\tilde{k}L^2$, L BEING THE CLASSICAL AMPLITUDE IS FINITE AND IS GIVEN B

$$M = \left[\frac{\tilde{k}L^2 + \theta\omega}{2\theta\omega}\right]$$

WHERE $[x]$ IS THE LARGEST INTEGER SMALLER THAN x.
THESE M STATES, LABELLED FROM 0 to $M-1$ CAN BE TAKEN AS AN ORTHONORMAL BASIS FOR A SUBSPACE $\mathcal{H}_M$ OF THE HARMONIC OSCILLATOR HILBERT SPACE $\mathcal{H}$.

M CAN BE INCREASED BY KEEPING L AND ω FIXED WHILE INCREASING $\tilde{k}$.

WE NOW SPLIT M AS $M = N + \eta$, $N, \eta \in \mathbb{Z}^+$, $N \neq 0$.
WE TAKE THE GAUGE FIELDS $\hat{A}_\mu$ ($\mu = 0, i$; $i = 1, 2$) AS ANTI-HERMITIAN OPERATORS WITH

$$\hat{A}_i \mathcal{H}_{N+\eta} \subseteq \mathcal{H}_{N+\eta} \qquad \hat{A}_i H^{\perp}_{N+\eta} = \{0\}$$

$$\hat{A}_0 \mathcal{H}_{N-1} \subseteq \mathcal{H}_{N-1} \qquad \hat{A}_0 \mathcal{H}^{\perp}_{N-1} = \{0\}.$$

IN TERMS OF THE OPERATOR $\hat{P}_{nm} = |n\rangle\langle m|$, WHERE $|n\rangle$ DENOTES THE n^{th} NORMALIZED EIGENSTATE, WE HAVE

$$\hat{A}_i = \sum_{n,m=0}^{N-1+\eta} i\,(\hat{A}_i)_{nm}\,\hat{P}_{nm}$$

$$(\hat{A}_i)_{nm} = 0 \quad \text{for} \quad n \text{ or } m > N-1+\eta.$$

AND $$\hat{A}_0 = \sum_{n,m=0}^{N-2} i\,(\hat{A}_0)_{nm}\,\hat{P}_{nm}$$

$$(\hat{A}_0)_{nm} = 0 \quad \text{for} \quad n \text{ or } m > N-2$$

IN TERMS OF THE PROJECTION OPERATOR

$$\hat{1}_K = \sum_{n,m=0}^{K-1} (\hat{1}_K)_{nm}\,\hat{P}_{nm} = \sum_{n,m=0}^{K-1} (\delta)_{nm}\,\hat{P}_{nm} = \sum_{n=0}^{K-1} \hat{P}_{nn}$$

$$(\hat{1}_K)_{nm} = (\delta)_{nm} = 0 \quad \text{for} \quad n \text{ or } m > K-1$$

WE HAVE

$$\hat{A}_i = \hat{1}_{N+\eta}\,\hat{A}_i\,\hat{1}_{N+\eta}$$

$$\hat{A}_0 = \hat{1}_{N-1}\,\hat{A}_0\,\hat{1}_{N-1}$$

THE LAGRANGIAN:

THE CS LAGRANGIAN FOR OUR MODEL IS

$$L_{NCCS} = -\frac{k\theta}{2}\varepsilon_{ij}\,\mathrm{Tr}\left(-\dot{\hat{A}}_i\hat{A}_j + 2\hat{A}_0\left(\partial_i\hat{A}_j + \hat{A}_i\hat{A}_j\right)\right)$$

WHERE $\dot{\hat{A}}_j = \partial_0 \hat{A}_j$ AND $\partial_i(\cdot) = i/\theta\ \varepsilon_{ij}\left[\hat{x}_j, (\cdot)\right]$

REMARK:

1) UNDER INFINITESIMAL GAUGE TRASFORMATION

$$\hat{A}_\mu \longrightarrow \hat{A}_\mu + \left(\partial_\mu\hat{\lambda} + i\left[\hat{A}_\mu, \hat{\lambda}\right]\right)$$

WHERE $\hat{\lambda}$ IS A MATRIX WITH INFINITESIMAL ELEMENTS, L_{NCCS} CHANGES BY A TOTAL DERIVATIVE

2) $\hat{\Pi}_0 \approx 0$ ($\hat{\Pi}_0$ CONJUGATE TO $\hat{A}_0$)
$\Longrightarrow$ $\hat{A}_0$ NOT OBSERVABLE AND CAN BE ELIMINATED FROM THE REST OF THE DISCUSSION

POISSON BRACKETS

$$\left\{(\hat{A}_i)_{nm}, (\hat{A}_j)_{rs}\right\} = \frac{1}{k\theta}\varepsilon_{ij}\left(\hat{1}_{N+\eta}\right)_{ns}\left(\hat{1}_{N+\eta}\right)_{mr} = \frac{1}{k\theta}\varepsilon_{ij}\,\delta_{ns}\,\delta_{mr}$$

$$n, m, r, s \in \left[0, N-1+\eta\right]$$

OR

$$\left\{\hat{A}_i, \hat{A}_j\right\} = -\frac{1}{k\theta}(N+\eta)\,\varepsilon_{ij}\sum_{n=0}^{N-1+\eta}\hat{P}_{nn} = -\frac{1}{k\theta}(N+\eta)\,\varepsilon_{ij}\,\hat{1}_{N+\eta}.$$

CANONICAL ANALYSIS

THE GAUSS LAW CONSTRAINT IS:

$$-k\theta\,\varepsilon_{ij}\,\mathrm{Tr}\left(\delta\hat{A}_0\left(\partial_i\hat{A}_j+\hat{A}_i\hat{A}_j\right)\right)\approx 0$$

NOTE: $\delta\hat{A}_0 \neq 0$ ONLY IN $\mathcal{H}_{N-1}$. HENCE

$$\boxed{g(\hat{\Lambda}^0) = k\theta\,\varepsilon_{ij}\,\mathrm{Tr}\left(\hat{\Lambda}^0\left(\partial_i\hat{A}_j+\hat{A}_i\hat{A}_j\right)\right)\approx 0}$$

WHERE $\hat{\Lambda}^0$ IS OF THE SAME FORM AS $\delta\hat{A}_0$

$$\hat{\Lambda}^0\mathcal{H}_{N-1}\subseteq\mathcal{H}_{N-1}\qquad \hat{\Lambda}^0\mathcal{H}_{N-1}^{\perp}=\{0\}$$

OR

$$\hat{\Lambda}^0=\hat{1}_{N-1}\,\hat{\Lambda}^0\,\hat{1}_{N-1}$$

OR

$$\hat{\Lambda}^0=\sum_{n,m=0}^{N-2} i\,(\hat{\Lambda}^0)\,\hat{P}_{nm}\,,\quad (\hat{\Lambda}^0)_{nm}=0 \text{ for } n,m>N-2$$

ALSO $\quad (\hat{\Lambda}^0)^*_{nm} = -(\hat{\Lambda}^0)_{mn}$ (ANTI-HERMITICITY)

NOW NOTE THAT WE CAN WRITE THE GAUSS LAW CONSTRAINT AS

$$g(\hat{\Lambda}_0)=k\theta\,\varepsilon_{ij}\,\mathrm{Tr}\left(-\partial_i\hat{\Lambda}^0\,\hat{A}_j+\hat{\Lambda}^0\,\hat{A}_i\,\hat{A}_j\right)\approx 0$$

WITH $\quad (\partial_i\hat{\Lambda}^0)\,\mathcal{H}_N\subseteq\mathcal{H}_N\qquad(\partial_i\hat{\Lambda}^0)\,\mathcal{H}_N^{\perp}=\{0\}$

OR $\quad (\partial_i\hat{\Lambda}^0)=1^N(\partial_i\hat{\Lambda}_0)\,1_N.$

CONSIDER NOW THE QUANTITY

$$q(\hat{\Sigma}) = k\theta\, \varepsilon_{ij}\, \mathrm{Tr}\, (-\partial_i \hat{\Sigma}\hat{A}_j + \hat{\Sigma}\hat{A}_i\hat{A}_j)$$

FOR AN ARBITRARY OPERATOR Σ. WE THEN GET

$$\{q(\hat{\Sigma}_1), q(\hat{\Sigma}_2)\} = -q([\hat{\Sigma}_1, \hat{\Sigma}_2]) - k\theta\, \varepsilon_{ij}\, \mathrm{Tr}\, \hat{1}_{N+n}(\partial_i \hat{\Sigma}_1)\hat{1}_{N+n}(\partial_j \hat{\Sigma}_2)$$

ALGEBRA OF GAUGE CONSTRAINTS

FOR $\hat{\Sigma}_i = \hat{\Lambda}_i^0$ $(i=1,2)$, WE HAVE $q(\hat{\Lambda}_i^0) = g(\hat{\Lambda}_i^0)$

WE THEREFORE GET

$$\{g(\hat{\Lambda}_1^0), g(\hat{\Lambda}_2^0)\} = -g([\hat{\Lambda}_1^0, \hat{\Lambda}_2^0]) \approx 0$$

$\Longrightarrow$ $g(\hat{\Lambda}^0)$ FIRST CLASS.

[THE CENTRAL TERM IN THIS CALCULATION VANISHES:

$$\begin{aligned} -k\theta\, \varepsilon_{ij}\, \mathrm{Tr}\, \hat{1}_{N+n}(\partial_i \hat{\Lambda}_1^0)\hat{1}_{N+n}(\partial_j \hat{\Lambda}_2^0) &= -k\theta\, \varepsilon_{ij}\, \mathrm{Tr}(\partial_i \hat{\Lambda}_1^0)(\partial_j \hat{\Lambda}_2^0) \\ &= -k\theta\, \mathrm{Tr}\, \partial_i(\hat{\Lambda}_1^0\, \partial_j \hat{\Lambda}_2^0) \\ &= 0 \end{aligned}$$

SINCE TRACE OF TOTAL DERIVATIVE VANISHES ON A FINITE DIM. HILBERT SPACE]

OBSERVABLES

TO CONSTRUCT THE OBSERVABLES (OR "CHARGES"), CHOOSE

$$\hat{\xi} = \hat{\Lambda} = \hat{\Lambda}' + \hat{\Lambda}^0$$

WITH $\hat{\Lambda}^0$ AS BEFORE AND

$$\hat{\Lambda}' \mathcal{H}_{N-1} = 0 \qquad \hat{\Lambda}' \mathcal{H}^{\perp}_{N-1} \subseteq \mathcal{H}^{\perp}_{N-1}$$
$$(\partial_i \hat{\Lambda}') \mathcal{H}_{N-2} = 0 \qquad (\partial_i \hat{\Lambda}') \mathcal{H}^{\perp}_{N-2} \subseteq \mathcal{H}^{\perp}_{N-2}$$

OR

$$0 = \hat{1}_{N-1} \hat{\Lambda}' \hat{1}_{N-1}$$

IT THEN FOLLOWS THAT

$$\{ q(\hat{\Lambda}), g(\hat{\Lambda}_0) \} \approx 0$$

$\Rightarrow$ $q(\hat{\Lambda})$ WITH $\hat{\Lambda} = \hat{\Lambda}' + \hat{\Lambda}^0$ ARE FIRST CLASS & OBSERVABLES

FURTHERMORE, FOR

$$\hat{\Lambda}_1 = \hat{\Lambda}' + \hat{\Lambda}_1^0 \qquad \hat{\Lambda}_2 = \hat{\Lambda}' + \hat{\Lambda}_2^0$$

$$q(\hat{\Lambda}_1) - q(\hat{\Lambda}_2) = g(\hat{\Lambda}_1 - \hat{\Lambda}_2) \approx 0.$$

$\Rightarrow$ $q(\hat{\Lambda}_1)$ & $q(\hat{\Lambda}_2)$ ARE GAUGE EQUIVALENT.

ALGEBRA OF OBSERVABLES

$$\{ q(\hat{\Lambda}_1), q(\hat{\Lambda}_2) \} = - q([\hat{\Lambda}_1, \hat{\Lambda}_2]) - k\theta\, \varepsilon_{ij}\, \mathrm{Tr}\, \hat{\mathbb{1}}_{N+\eta} (\partial_i \hat{\Lambda}_1) \hat{\mathbb{1}}_{N+\eta} (\partial_j \hat{\Lambda}_2)$$

WHICH IS A FINITE DIMENSIONAL ALGEBRA ANALOGOUS TO THE NON-ABELIAN KAC-MOODY ALGEBRA.

NOTE:

1) IT CAN BE SHOWN THAT

$$\{ (\hat{\Lambda}_\ell)_{mn}, q(\Lambda^0) \} \approx 0$$

FOR n OR $m \geq N$ AND FOR $n = m = N-1$, THUS $(\hat{\Lambda}_\ell)_{mm}$ FOR n OR $m \geq N$ AND FOR $m = n = N-1$ ARE OBSERVABLES OF OUR THEORY

2) INDEPENDENT OF VALUE OF η, $q(\hat{\Lambda}) \neq 0$ FOR NONZERO ENTRIES $\hat{\Lambda}_{N-1\,N-1}$, $\hat{\Lambda}_{N-1\,N}$ AND $\hat{\Lambda}_{N\,N-1}$ IN A GIVEN $\hat{\Lambda}$. WE THUS HAVE 3 UNIQUE NON-ABELIAN OBSERVABLES.

4) THE LARGE M LIMIT.

WE FOCUS ON THE OPERATOR $\hat{P}_{M-1\,M-1} = |M-1\rangle\langle M-1|$

IN TERMS OF THE COHERENT STATE $|z\rangle$

$$|z\rangle = e^{-\frac{1}{2\theta}|z|^2} \sum_{r=0}^{\infty} \frac{z^r}{\sqrt{\theta^r\, \underline{|r}}}\, |r\rangle$$

THE DIAGONAL COHERENT STATE ELEMENT OF $\hat{P}_{M-1\,M-1}$ IS

$$\tilde{P}_{M-1}(z,\bar{z}) = \frac{1}{\pi\theta}\langle z|M-1\rangle\langle M-1|z\rangle = \frac{e^{-\frac{|z|^2}{\theta}}\,|z|^{2(M-1)}}{\pi\,\theta^M\,\underline{|(M-1)}}$$

$\tilde{P}_{M-1}$ HAS MAXIMA AT $z = z_0$ WITH $(z_0)^2 = \theta(M-1)$.

IN TERMS OF THE VARIABLES

$$x_1 = \langle z|\hat{x}_1|z\rangle = \frac{L}{\sqrt{(M-1/2)\theta}}\, z_1$$

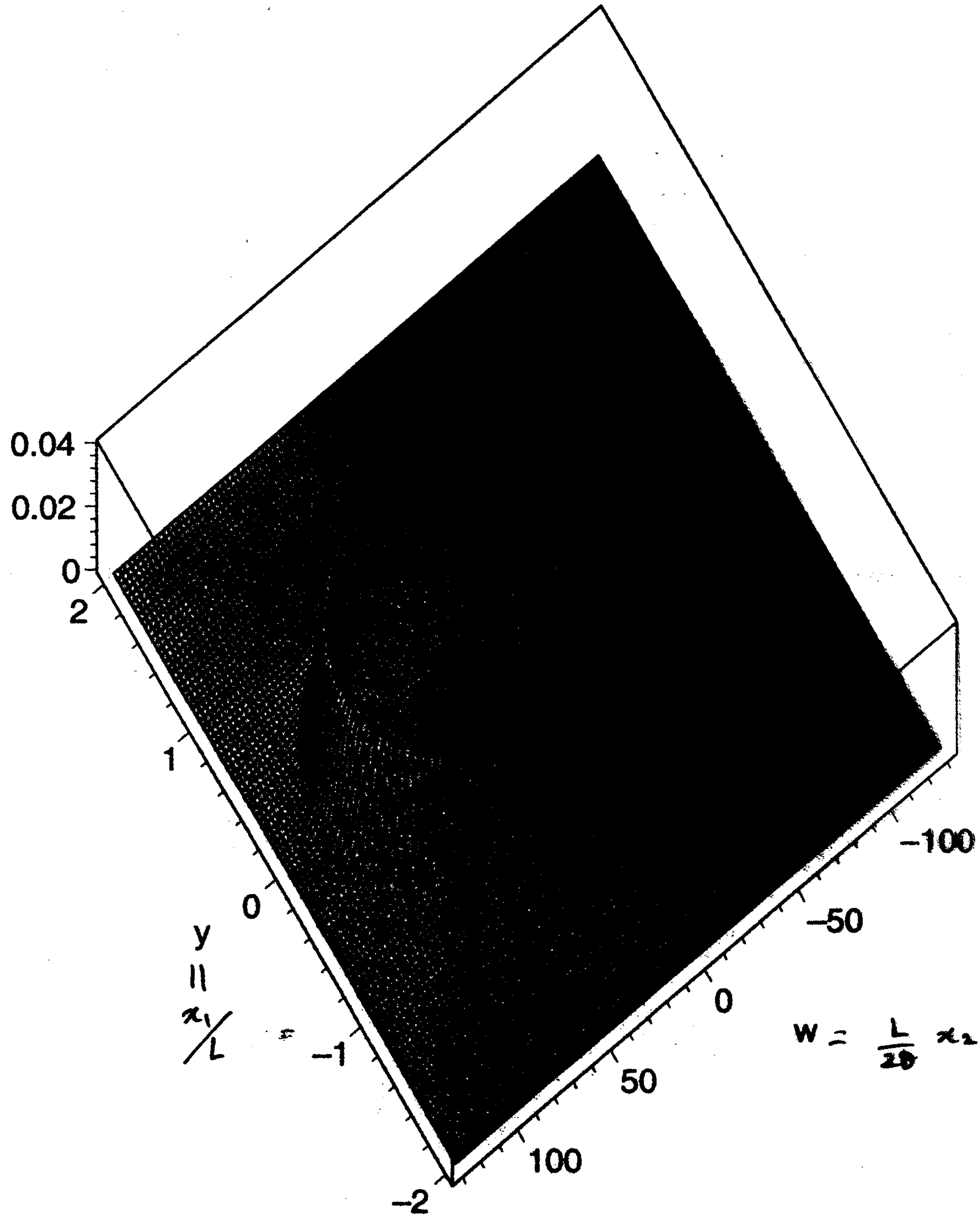
$$x_2 = \langle z|\hat{x}_2|z\rangle = \frac{2\sqrt{(M-1/2)\theta}}{L}\, z_2$$

WE CAN REWRITE $\tilde{P}_{M-1}(z,\bar{z}) \equiv Q_{M-1}(x_1, x_2)$.

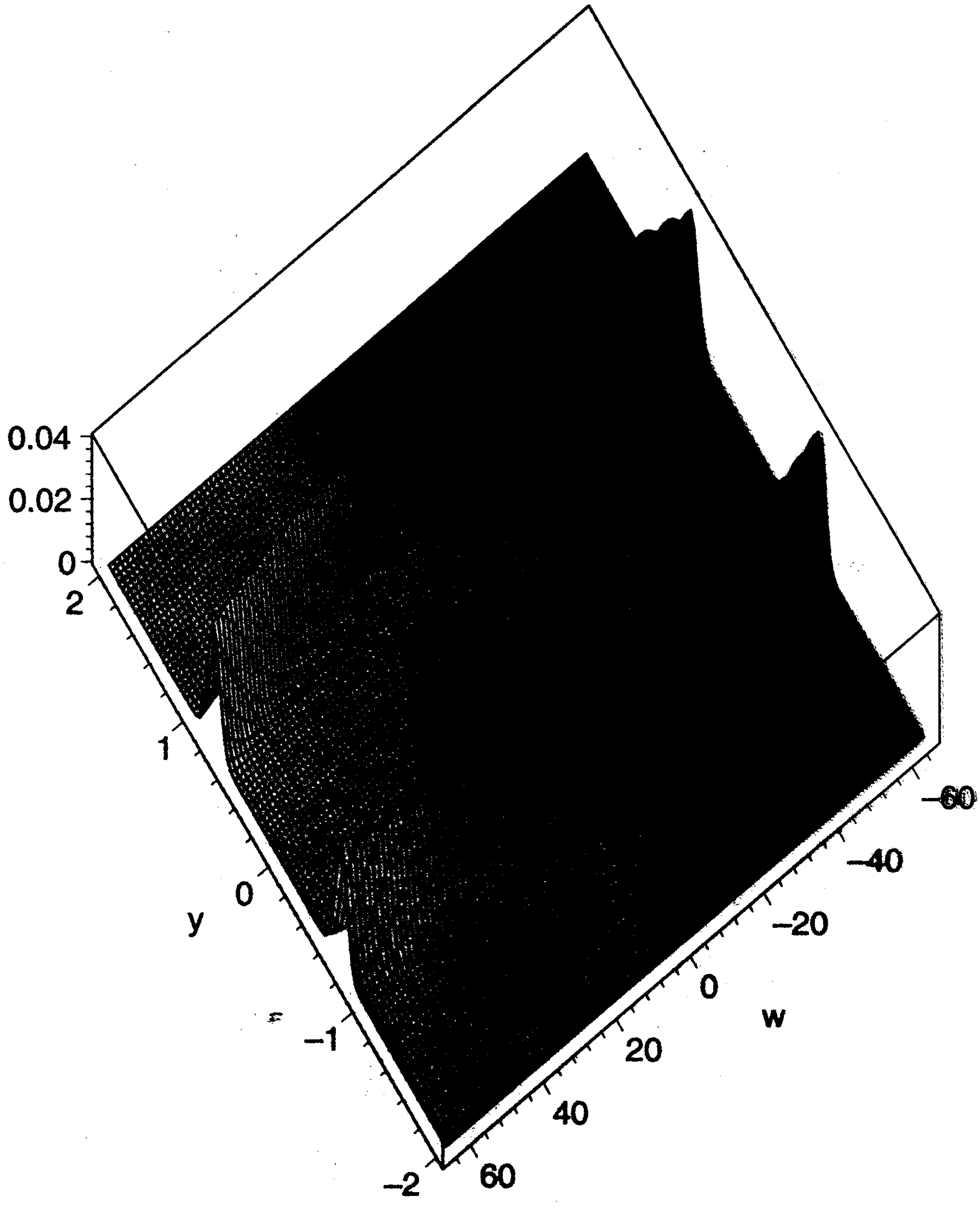
THE FUNCTION $Q_{M-1}(x_1,x_2)$ HAS MAXIMA ON THE ELLIPSE

$$\frac{x_1^2}{\left(\frac{M-1}{M-1/2}\right)L^2} + \frac{x_2^2}{\frac{4\theta^2}{L^2}(M-1)(M-1/2)} = 1.$$

WITH AXES GIVEN BY $\sqrt{\frac{M-1}{M-1/2}}\,L$ & $\frac{2\theta}{L}\sqrt{(M-1)(M-1/2)}$



PLOT OF $Q_{M-1}(x_1, x_2)$ FOR M=100.



BLOWUP OF $Q_{M-1}(x_1, x_2)$ FOR $M=100$

WE CAN GRAPHICALLY SEE THAT $Q_{M-1}(x_1, x_2)$ HAS THE GEOMETRY OF AN ELLIPSE EXTENDED ALONG x_2 DIRECTION. ALSO

$$\frac{\text{SEMI MAJOR AXIS}}{\text{SEMI MINOR AXIS}} = \frac{2/L \, \theta M}{L}$$

AS $M \to \infty$, THE ELLIPTICAL GEOMETRY CONVERGES TO AN INFINITE STRIP WITH PEAKS AT $x_1 = \pm L$.

ALSO, AS $M \to \infty$, THE "SHARPNESS" OF THE EDGE IS GIVEN BY

$$\Delta x_1 \Big|_{M \to \infty} = \sqrt{\frac{2}{M}} \, L + \mathcal{O}\left(\frac{1}{M}\right)$$

ORDER.

$\Rightarrow$ IN OUR MODEL THE EDGE IS FUZZY WITH WIDTH Δx_1

AS $M \to \infty$, FOR FIXED L, $\Delta x_1 \to 0$ AND WE GET SHARP EDGES.

IT IS IN THIS SENSE THAT THE LIMIT $M \to \infty$ DEFINES A NON-COMMUTATIVE CHERN-SIMONS THEORY ON THE INFINITE STRIP.

5) COMMUTATIVE LIMIT

HERE WE CONSIDER THE LIMIT $M \to \infty$ AND $\theta \to 0$.

USING STANDARD RELATIONS FOR THE $*$ PRODUCT, WE CAN EXPRESS THE ALGEBRA OF THE OBSERVABLES IN THE DIAGONAL COHERENT STATE REPRESENTATION AS

$$\{ q(\Lambda_1), q(\Lambda_2) \} = - q([\Lambda_1, \Lambda_2]_*) - \frac{k}{2\pi} \varepsilon_{ij} \int d^2 z \, \mathbb{1}_M * (\partial_i \Lambda_1) * \mathbb{1}_M * (\partial_j \Lambda_2)$$

AS $\theta \to 0$, THE FIRST TERM ON RHS $\to 0$.

ALSO AS $M \to \infty$ $\mathbb{1}_M \to \mathbb{1}$. HENCE WE RECOVER THE STANDARD $U(1)$ KAC-MOODY ALGEBRA OF THE EDGE OBSERVABLES OF THE COMMUTATIVE CS THEORY

$$\{ q(\Lambda_1), q(\Lambda_2) \} = - \frac{k}{4\pi} \varepsilon_{ij} \int d^2 x \, (\widetilde{\partial_i \Lambda_1})(\widetilde{\partial_j \Lambda_2})$$

WHERE WE HAVE USED $\int d^2 z \, F(z, \bar{z}) = \frac{1}{2} \int d^2 x \, \tilde{F}(x_1, x_2)$ AND WHERE $\tilde{F}(x_1, x_2)$ DENOTES THE MOYAL REPRESENTATION OF THE OPERATOR $\hat{F}$.

<u>OTHER LIMITS</u> AS $M \to \infty$, MAXIMA OF Q ON $\frac{x_1^2}{L^2} + \frac{x_2^2}{4\theta^2/L^2 \, M^2} = 1$.

FOR ALL LIMITS OF $\theta \to 0$ SUCH THAT $\theta M \to \infty$ AS $M \to \infty$ WE GET STRIP GEOMETRY.

ON THE OTHER HAND IF $\theta \to 0$ SUCH THAT θM IS FIXED AS $M \to \infty$, WE GET THE GEOMETRY OF A DISC. THIS RESULT WAS ALSO INDEPENDENTLY OBTAINED BY LIZZI et al (hep-th/0306247)

6) CONCLUSION

→ WE HAVE GENERALIZED THE WELL KNOWN USUAL RESULTS OF THE COMMUTATIVE CS THEORY TO NON-COMMUTATIVE MANIFOLDS WITH BOUNDARY.

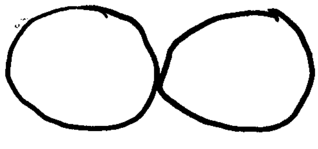
→ WE HAVE DEVELOPED A NEW MATRIX MODEL WHICH IN THE LIMIT OF LARGE MATRIX SIZE DEFINES THE NC CS ON AN ∞ STRIP

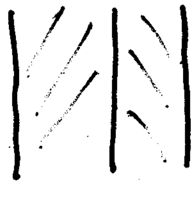
→ THE COMMUTATIVE LIMIT OF THIS THEORY AGREES WITH THE RESULTS OF COMMUTATIVE CS ON A MANIFOLD WITH BOUNDARY

→ THE CLASSICAL TOPOLOGY OBTAINED DEPENDS ON HOW LIMITS ARE TAKEN.

→ THERE ARE MANY OPEN PROBLEMS:

- TREATMENT OF DIFFOMORPHISMS IN NC SET UP
- SUGAWARA CONSTRUCTION
- ANALYSIS ON OTHER TYPE OF GEOMETRY, e.g.





- TOPOLOGY CHANGE IN THIS PICTURE.

→ GRAVITY ?