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Quantum Physics with time-space Noncommutativity

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Quantum Physics with time-space Noncommutativity

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Plan of the talk

- Motivations - quantum gravity and space time geometry
- Noncommutative geometry and Quantum Physics
- Noncommutative space-time and unitary quantum physics
- Noncommutative space-time and quantised evolutions - Bal
- Conclusions

Acknowledgements: Collaborations with Paolo Teotonio Sobhrinho, Molina, Amilkar, Martins,.....

Motivations.....

- Quantum Gravity -at Planck length - must have - Noncommutative geometric structure - Limit of classical gravity - emerge - Commutative geometry of spacetime we know. Just like:

$$\lim_{\hbar \longrightarrow 0} Q.Physics = Cl.Physics$$

- Expectation:

$$\lim_{Planck\ length \longrightarrow 0} Non\ commutative\ geometry = Commutative\ Geometry$$

Folklore ..

- One should study - Semiclassical Gravity-Or-simple models - where - such information can be obtained
- Black hole Physics - near horizon geometry - specifically for small blackholes- may exhibit - such - behaviour ?
- With modern developements in understanding microstates of blackhole - one should - relook at the geometry of - horizon
- In most of the studies of noncommutative geometry in Physics - such structures are introduced from outside as additional parameter- whereas it should result by virtue of quantising gravity.

NCG and quantum physics.

- Recent times NC Geometry has become popular. Though it was introduced about 60 years back by Snyder (PR 1949). Just like introducing $\hbar$ descretises phase space and reduces the phase space degrees of freedom introducing a fundamental length will reduce the degrees of freedom further and hopefully reduce severity of divergences which plagued the QFT's then. The original idea was attributed to Heisenberg himself.
- Renormalisation program and its success with QED made this suggestion unpopular and quickly forgotten.

NCG and quantum physics..

- In string theory gets introduced through the background fields $B_{\mu\nu}$. This leads in addition to new physics, mixing of infrared and ultraviolet behaviour of field theories. Seiberg and Witten, Minwalla etal
- The NC geometry can also be studied in a discrete setting- examples of which are fuzzy spheres. These serve as alternatives to lattice regularisations with advantages such as avoidance of fermion doubling. Madore; Balachandran,trg and Ydri
- Nonlinear models such as CP^1 $model$ have interesting and novel topological behaviour in such spaces. bal and Immirzi,trg and Harikumar
- And many more....

Unitary NC spacetime..

- Having seen the history/philosophy of Noncommutative geometric spacetime we will first study a simpler version of spacetime with noncommutativity and its implications for quantum theory.
- We will start with $1+1$ dimensional theory. And look at the spacetime commutators of the form:

$$\{\hat{x}_\mu, \hat{x}_\nu\} = i\theta_{\mu\nu}\mathcal{I}$$

- Its naively remarked that this leads to non unitary quantum theory. This is due to incorrect appreciation of the role of "Time".

NC spacetime...

- But the correct statement is if a group of transformations cannot be implemented on the algebra $\mathcal{A}_\theta\left(\mathcal{R}^2\right)$ generated by $\hat{x}_\mu$ with our relation then it will not be a symmetry.
- We readily see that spacetime translations are automorphisms of $\mathcal{A}_\theta\left(\mathcal{R}^2\right)$: With $\mathcal{U}(\vec{a})\hat{x}_\mu = \hat{x}_\mu + a_\mu$ we see that,

$$[\mathcal{U}(\vec{a})\hat{x}_\mu, \mathcal{U}(\vec{a})\hat{x}_\nu] = i\theta\varepsilon_{\mu\nu} \ .$$

- The time-translation automorphism is:

$$U(\tau) := \mathcal{U}\left((\tau, 0)\right)$$

NC spacetime..

- Without the time-translation automorphism, we cannot formulate conventional quantum physics.
- The infinitesimal generators of $\mathcal{U}(\vec{a})$ can be obtained from

$$\mathcal{U}(\vec{a}) = e^{-ia_0 \hat{P}_0 + ia_1 \hat{P}_1} \ .$$

- Then we have

$$\hat{P}_0 = -\frac{1}{\theta} \text{ad}\, \hat{x}_1 \ ,$$

$$\text{ad}\, \hat{x}_\mu \hat{a} \equiv [\hat{x}_\mu, \hat{a}] \ , \ \hat{a} \in \mathcal{A}_\theta \left(\mathbb{R}^2\right) \ .$$

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NC spacetime..

- It is a special feature of two dimensions that the $(2+1)$ connected Lorentz group is an inner automorphism group
- Its generators are ad$\hat{J}_3$ and ad$\hat{K}_a$,
-

$$\hat{J}_3 = \frac{1}{4\theta}\left(\hat{x}_0^2 + \hat{x}_1^2\right) \ ,$$

$$,$$

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$$\hat{K}_2 = \frac{1}{4\theta}\left(\hat{x}_0^2 - \hat{x}_1^2\right) \; ,$$

NC spacetime..

- It is easy to see Parity is NOT an automorphism.

$$\mathcal{P} : [\hat{x}_0, \hat{x}_1] \rightarrow -[\hat{x}_0, \hat{x}_1] \ , \ i\theta\mathcal{I} \rightarrow i\theta\mathcal{I} \ .$$

- But time reversal IS: due to anti-linearity.

$$\mathcal{T} : i\theta\mathbb{I} \rightarrow -i\theta\mathbb{I} \ ,$$

- $\mathcal{CPT}$ will be broken! This will have implications for spin-statistics connection!. (neutrino puzzle?)
- The group of area preserving diffeomorphisms also generates automorphisms of $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$. This group includes the Lorentz group Susskind,Jackiw.

NC spacetime..

- Causality: It is impossible to localize (the representation of) "coordinate" time $\hat{x}_0$ in $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ sharply. This leads to failure of causality Chaichian et al.
- The following important point was emphasised to us by Doplicher. In quantum mechanics, if $\hat{p}$ is momentum, $\exp(i\xi\hat{p})$ is spatial translation by amount ξ. This ξ is not the eigenvalue of the position operator $\hat{x}$. In the same way, the amount τ of time translation is not "coordinate time", the eigenvalue of $\hat{x}_0$. It makes sense to talk about a state and its translate by $U(\tau)$
- Concepts like duration of an experiment for $\theta = 0$ are expressed using $U(\tau)$. They carry over to the noncommutative case too.

NC spacetime..

- With loss of causality, one loses local qft's as well. As the best proofs of the spin-statistics connection require locality.
- Precision experiments to test the spin-statistics connection are possible Capri meeting. If signals for this violation due to $\theta \neq 0$ can be derived, good phenomenological bounds on $|\theta|$ should be possible.

Representation theory..

- Observables, states and dynamics of quantum theory are to be based on the algebra $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$. Here we develop the formalism for their construction.
- To each $\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)$, we associate its left and right regular representations $\hat{\alpha}^L$ and $\hat{\alpha}^R$,

$$\hat{\alpha}^L \hat{\beta} = \hat{\alpha}\hat{\beta} \ , \ \hat{\alpha}^R \hat{\beta} = \hat{\beta}\hat{\alpha} \ , \ \hat{\beta} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right) \ ,$$

  with $\hat{\alpha}^L \hat{\beta}^L = \left(\hat{\alpha}\hat{\beta}\right)^L$ and $\hat{\alpha}^R \hat{\beta}^R = \left(\hat{\beta}\hat{\alpha}\right)^R$. The carrier space of this representation is $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ itself.
- An “inner”product on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ is needed for an eventual construction of a Hilbert space.

Representation theory..

- Consider a map $\chi : \mathcal{A}_\theta\left(\mathbb{R}^2\right) \to \mathbb{C}$ which is also positive,i.e.,

$$\chi\left(\hat{\alpha}^* \hat{\alpha}\right) \geq 0 \ .$$

- Then we can set:

$$\left\langle \hat{\alpha}, \hat{\beta} \right\rangle = \chi\left(\hat{\alpha}^* \hat{\beta}\right) .$$

- It will be a scalar product if $\chi\left(\hat{\alpha}^* \alpha\right) = 0$ implies $\hat{\alpha} = 0$. If that is not the case, it is necessary to eliminate null vectors.
- We illustrate these ideas briefly in the context of the commutative case, when $\theta = 0$

The Commutative case

- The algebra $\mathcal{C}$ in the commutative case is $\mathcal{A}_0\left(\mathbb{R}^2\right) = C^\infty\left(\mathbb{R}\times\mathbb{R}\right)$,
- There is no distinction now between $\hat{\alpha}^L$ and $\hat{\alpha}^R$: $\hat{\alpha}^L = \hat{\alpha}^R$.
- There is a family of positive maps χ_t of interest obtained by integrating i ψ in x_1 at “time” t:

$$\chi_t(\psi) = \int dx_1\, \psi(t, x_1)\ ,$$

- We get a family of spaces $\mathcal{C}_t$ with a positive-definite sesquilinear form $(.\,,\ .)_t$:

$$(\psi, \varphi)_t = \int dx_1\, \psi^*(t, x_1)\varphi(t, x_1)\ .$$

The Commutative case

- Every function $\hat{\alpha}$ which vanishes at time t is a two-sided ideal $\mathcal{I}_t^{\theta=0} = \mathcal{I}_t^0$ of $\mathcal{C}$. As elements of $\mathcal{C}_t$, they become null vectors.
- As in the GNS construction Haag, we can quotient by these vectors and work with $\mathcal{C}_t/\mathcal{N}_t^0$.
- The completion $\overline{\mathcal{C}_t/\mathcal{N}_t^0}$ of $\mathcal{C}_t/\mathcal{N}_t^0$ in this scalar product gives a Hilbert space $\widehat{\mathcal{H}}_t^0$
- For elements $\psi + \mathcal{N}_t^0$ and $\chi + \mathcal{N}_t^0$ in $\mathcal{C}_t/\mathcal{N}_t^0$, the scalar product is

$$(\psi + \mathcal{N}_t^0, \chi + \mathcal{N}_t^0)_t = (\psi, \chi)_t \quad .$$

The Commutative case

- The quantum mechanical Hilbert space however is not $\widehat{\mathcal{H}}_t^0$.
- It is constructed in a different way, starting from a subspace $\tilde{\mathcal{H}}_{0,t} \subset \mathcal{C}_t$ which contains only $\{0\}$ as the null vector: $\tilde{\mathcal{H}}_{0,t} \cap \mathcal{N}_t^0 = \{0\}$
- Then χ_t is a good scalar product on $\tilde{\mathcal{H}}_{0,t}$ and the quantum mechanical Hilbert space is given by $\mathcal{H}_t^0 = \overline{\tilde{\mathcal{H}}_{0,t}}$, the completion of $\tilde{\mathcal{H}}_{0,t}$.
- The subspace $\tilde{\mathcal{H}}_{0,t}$ depends on the Hamiltonian H and is chosen as follows.

The Commutative case

- Let H be a time-independent Hamiltonian on commutative spacetime, self-adjoint on the standard quantum mechanical Hilbert space $L^2(\mathbb{R})$.
- We now pick the subspace $\tilde{\mathcal{H}}_{0,t}$ of $\mathcal{C}_t$ by requiring that vectors in $\mathcal{C}_t$ obey the time-dependent Schrödinger equation:

$$\tilde{\mathcal{H}}_{0,t} = \{\psi \in \mathcal{C}_t : (i\partial_{x_0} - H)\,\psi(x_0, x_1) = 0\} \ .$$

- The operator $i\partial_{x_0}$ is not hermitian on all of $\mathcal{C}_t$:

$$(\psi, i\partial_{x_0}\chi)_t \neq (i\partial_{x_0}\psi, \chi)_t \ \text{ for generic } \psi, \chi \in \mathcal{C}_t \ ,$$

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$$(\psi, i\partial_{x_0}\chi)_t = (i\partial_{x_0}\psi, \chi)_t \text{ for generic } \psi, \chi \in \tilde{\mathcal{H}}_{0,t} \ ,$$

- but on $\tilde{\mathcal{H}}_{0,t}$, it fulfills this property:

The Commutative case

- We notice since,

$$\psi(x_0+\tau, x_1) = \left(e^{-i\tau(i\partial_{x_0})}\psi\right)(x_0, x_1)$$

$$= \left(e^{-i\tau H}\psi\right)(x_0, x_1) \ ,$$

  time evolution preserves the norm of $\psi \in \tilde{\mathcal{H}}_{0,t}$. Therefore if it vanishes at $x_0 = t$, it vanishes identically and is the zero element of $\tilde{\mathcal{H}}_{0,t}$: the only null vector in $\tilde{\mathcal{H}}_{0,t}$ is 0:
- The completion of $\tilde{\mathcal{H}}_{0,t}$ is the quantum Hilbert space $\mathcal{H}_t^0$. There is no convenient inclusion of $\mathcal{H}_t^0$ in $\widehat{\mathcal{H}}_t^0$.

The Commutative case

- Under time evolution by amount τ, ψ becomes

$$e^{-i\tau H}\psi = e^{-i(\hat{x}_0+\tau)H}\psi_0 \in \tilde{\mathcal{H}}_{0,t} \ .$$

where ψ_0 is a constant function of x_0 so that $i\partial_{x_0}\psi_0 = 0$. This conceptual difference between coordinate time $\hat{x}_0$ and time translation τ is crucial for NC spacetime.

- An observable $\hat{K}$ has to respect the Schrödinger constraint and leave $\tilde{\mathcal{H}}_{0,t}$ (and hence $\mathcal{H}_t^0$) invariant. This means that

$$\left[i\partial_{x_0} - H, \hat{K}\right] = 0 \ .$$

The commutative case

- Under time translation, $\hat{x}_0$ in $\hat{K}$ shifts to $\hat{x}_0 + \tau$ as it should:

$$\hat{K}(\tau) = e^{-i\tau H}\hat{K}e^{+i\tau H} = e^{-i(\hat{x}_0+\tau)H}\hat{L}e^{+i(\hat{x}_0+\tau)H} \; .$$

  where $\hat{L}$ is defined by:

$$\hat{K}(0) = e^{-i\hat{x}_0 H}\hat{L}e^{+i\hat{x}_0 H}$$

- What we have described above leads to conventional physics. As expected $\hat{x}_0$ is not an observable as it does not commute with $i\partial_{x_0} - H$:

$$[\hat{x}_0, i\partial_{x_0} - H] = -i\mathbb{I} \; .$$

The commutative case

- In conventional quantum physics, the Hilbert space has no time-dependence, whereas $\mathcal{H}_t^0$ has a label t. This is puzzling. But the puzzle is easy to resolve: $\mathcal{H}_t^0$ is independent of t.
- There is thus only one Hilbert space which we call $\mathcal{H}_0$
- Further the observables have no explicit t-dependence and act on $\mathcal{H}_0$ as in standard quantum theory.

The Noncommutative Case

- The above discussion shows that for quantum theory, what we need are: (1) a suitable inner product on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$; (2) a Schrödinger constraint on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$; and (3) a Hamiltonian $\hat{H}$ and observables which act on the constrained subspace of $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$.
- We also require that (1) is compatible with the self-adjointness of $\hat{H}$ and classically real observables.
- We now consider these one by one.

The symbol calculus

- The first inner product is based on symbol calculus. If $\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)$, we write it as

$$\hat{\alpha} = \int d^2k\, \tilde{\alpha}(k) e^{ik_1\hat{x}_1} e^{ik_0\hat{x}_0} \ ,$$

and associate the symbol α_S with $\hat{\alpha}$ where

$$\alpha_S(x_0, x_1) = \int d^2k\, \tilde{\alpha}(k) e^{ik_1 x_1} e^{ik_0 x_0} \ .$$

- The symbol is a function on $\mathbb{R}^2$. It is NOT the MOYAL symbol. Using this symbol, we can define a positive map S_t by

$$S_t\left(\hat{\alpha}\right) = \int dx_1\, \alpha_S(t, x_1) \ .$$

The Voros map

- The second inner product comes from the VOROS map, based on the coherent states. Let

$$a = \frac{\hat{x}_0 + i\hat{x}_1}{\sqrt{2\theta}} \ , \ a^\dagger = \frac{\hat{x}_0 - i\hat{x}_1}{\sqrt{2\theta}} \ , \ \left[a, a^\dagger\right] = \mathbb{I} \ ,$$

and introduce the coherent states

$$|z = x_0 + ix_1\rangle = e^{\frac{1}{\sqrt{2\theta}}\left(za^\dagger - \bar{z}a\right)} |0\rangle \ \ .$$

- The Voros symbol of an operator $\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)$ is the function α_V on $\mathbb{R}^2$ where

$$\alpha_V\left(x_0, x_1\right) = \langle z| \, \hat{\alpha} \, |z\rangle \ \ .$$

The Voros map

- The positive map V_t is then defined by

$$V_t\left(\hat{\alpha}\right) = \int dx_1\, \alpha_V\left(t, x_1\right) \;\; .$$

- The scalar product is:

$$\left(\hat{\alpha}, \hat{\beta}\right)_{V_t} = V_t\left(\hat{\alpha}^* \hat{\beta}\right)$$

  Hilbert space is obtained only after constraining the vector states by the noncommutative Schrödinger equation.

- It can be argued that both these maps lead to equivalent results Bal, trg, Molina, Paulo.

The Schrödinger constraint

- The noncommutative analogue "$i\frac{\partial}{\partial x_0}$" is

$$i\frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta}\mathrm{ad}\,\hat{x}_1 \ ,$$

- If the Hamiltonian $\hat{H}$ is time-independent,

$$[i\partial_{x_0}, \hat{H}] = 0$$

- We can write Hamiltonian as $\hat{H} = \hat{H}\left(\hat{x}_1^L, \hat{P}_1\right)$.
- If $\hat{H}$ has time-dependence then • is not correct, it will have $\hat{x}_0^L$, $\hat{x}_0^R$. But $\hat{x}_0^L = \theta\hat{P}_1 + \hat{x}_1^R$, so in the time-dependent case we write $\hat{H} = \hat{H}(\hat{x}_0^R, \hat{x}_1^L, \hat{P}_1)$

The Schrödinger constraint

- The states constrained by the Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta = \left\{ \hat{\psi} \in \mathcal{A}_\theta \left(\mathbb{R}^2 \right) : \left(i\partial_{x_0} - \hat{H} \right) \hat{\psi} = 0 \right\} \ ,$$

- The solutions are easy to construct:

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \Longrightarrow \hat{\psi} = e^{-i\left(\hat{x}_0^R - \tau_I\right)\hat{H}\left(\hat{P}_1, \hat{x}_1^L\right)} \hat{\chi}\left(\hat{x}_1\right)$$

- If $\hat{H}$ depends on $\hat{x}_0^R$, we can easily generalise the formula

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \Longrightarrow \hat{\psi} = U\left(\hat{x}_0^R, \tau_I\right) \hat{\chi}\left(\hat{x}_1\right)$$

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$$U\left(\hat{x}_0^R, \tau_I\right) = T\exp\left[-i \left(\int_{\tau_I}^{x_0} d\tau\, \hat{H}\left(\tau, \hat{x}_1^L, \hat{P}_1\right)\right)\right]\Bigg|_{x_0 = \hat{x}_0^R}$$

Some observations

- An alternative useful form for $\hat{\psi}$ is

$$\hat{\psi} = V\left(\hat{x}_0^R, -\infty\right)\hat{\chi}\left(\hat{x}_1\right)$$

$$V\left(\hat{x}_0^R, -\infty\right) = T\exp\left[-i\int_{-\infty}^{0} d\tau\, \hat{H}\left(\hat{x}_0^R + \tau, \hat{x}_1^L, \hat{P}_1\right)\right]$$

where the integral can be defined at the lower limit using the usual adiabatic cut-off.

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  where the integral can be defined at the lower limit using the usual adiabatic cut-off.

- The Hilbert spaces $\mathcal{H}_\theta^S$ and $\mathcal{H}_\theta^V$ based on scalar products $(.,.)_S$ and $(.,.)_V$ are obtained from $\tilde{\mathcal{H}}_\theta$ by completion. Our basic assumption is that $\hat{H}$ is self-adjoint in the chosen scalar product.

Some observations

- In the passage from H to $\hat{H}$, there is an apparent ambiguity. We replaced x_0 by $\hat{x}_0^L$, but we may be tempted to replace x_0 by $\hat{x}_0^R$. But it is incorrect to replace x_0 by $\hat{x}_0^R$ and at the same time x_1 by $\hat{x}_1^L$. Time and space should NOT commute when θ becomes nonzero whereas $\hat{x}_0^R$ and $\hat{x}_1^L$ commute.
- Note that $\hat{x}_0^L = -\theta \hat{P}_1 + \hat{x}_0^R$ and that $\hat{x}_0^R$ behaves much like the $\theta = 0$ time x_0. *Thus if H has time-dependence, its effect on $\hat{H}$ is to induce new momentum-dependent terms* leading to nonlocal (“acausal”) interactions.
- We can construct observables as before and no complications are encountered.

Examples:

- Plane Waves: Let us consider $\hat{H}_0 = \frac{\hat{P}_1^2}{2m}$ Its eigenstates and eigenvalues are:

$$\hat{\psi}_k = e^{ik\hat{x}_1} e^{-i\omega(k)\hat{x}_0} \ , \ \omega(k) = \frac{k^2}{2m} \ , \ k \in \mathbb{R} \ .$$

- The spectrum of $\hat{H}_0$ is completely conventional while the noncommutative plane waves too resemble the ordinary plane waves. But phenomena like beats and interference show new features Bal.
- The coincidence of spectra of the free Hamiltonians in commutative and noncommutative cases is an illustration of a more general result which we now point out.

A Spectral Map:

- For $\theta = 0$ let the Hamiltonian be: $H = -\frac{1}{2m}\frac{\partial^2}{\partial x_1^2} + V(\hat{x}_1)$ with eigenstates ψ_E fulfilling the Schrödinger constraint:

$$\psi_E\,(\hat{x}_0, \hat{x}_1) = \varphi_E(\hat{x}_1)e^{-iE\hat{x}_0}\;, H\varphi_E = E\varphi_E\;.$$

- The Hamiltonian $\hat{H}$ associated to H for $\theta \neq 0$ is

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1)\;.$$

- Then $\hat{H}$ has exactly the same spectrum as H and its eigenstates $\hat{\psi}_E$ are obtained from ψ_E.

$$\hat{\psi}_E = \varphi_E(\hat{x}_1)e^{-iE\hat{x}_0}\;, \hat{H}\varphi_E(\hat{x}_1) = E\varphi_E(\hat{x}_1)\;.$$

Conserved Current:

- The existence of a current j_λ which fulfills the continuity equation has a particular importance when $\theta = 0$. It is this current which after second quantization couples to electromagnetism.
- There is such a conserved current also for $\theta \neq 0$. It follows in the usual way from

$$\left(\hat{P}_0\hat{\psi}\right)^* - \hat{\psi}^*\hat{H} = -\hat{P}_0\hat{\psi}^* - \hat{\psi}^*\hat{H} = 0 \ .$$

- The noncommutative charge and current density:

$$\hat{\rho} = \hat{\psi}^*\hat{\psi} \ , \ \hat{j} = \frac{1}{2m}\left[\hat{\psi}^*\left(\hat{P}_1\hat{\psi}\right) - \left(\hat{P}_1\hat{\psi}^*\right)\hat{\psi}\right]$$

satisfies noncommutative continuity equation.

QFT.....:

- We can also see how to do perturbative qft's, our approach can be inferred from the work of Doplicher et al. We require of $\hat{\Phi}$ that it is a solution of the massive Klein-Gordon equation: $\left(\text{ad}\hat{P}_0^2 - \text{ad}\hat{P}_1^2 + \mu^2\right)\hat{\Phi} = 0$.
- The plane wave solutions are $\hat{\phi}_k = e^{ik\hat{x}_1}e^{-i\omega(k)\hat{x}_0}\ ,\ \omega(k)^2 - k^2 = \mu^2$.
- So for $\hat{\Phi}$, we write:

$$\hat{\Phi} = \int \frac{dk}{2\omega(k)}\left[a_k\hat{\phi}_k + a_k^\dagger\hat{\phi}_k^\dagger\right]\ ,$$

where a_k and $a_k^\dagger$ commute with $\hat{x}_\mu$ and define harmonic oscillators: $\left[a_k, a_k^\dagger\right] = 2\omega(k)\delta(k - k')$.

QFT.....:

- The "free" field $\hat{\Phi}$ "coinciding with the Heisenberg field initially" after time translation by amount τ using the free Schrödinger Hamiltonian $\hat{H}_0 = \int \frac{dk}{2\omega(k)} a_k^\dagger a_k$, becomes

$$U_0(\tau)\left(\hat{\Phi}\right) = e^{i\tau\hat{H}_0}\hat{\Phi}e^{-i\tau\hat{H}_0} \quad ,$$

- The interaction Hamiltonian is accordingly

$$\hat{H}_I\,(x_0) = \lambda : S_{x_0}\left(U_0(\tau)\left(\hat{\Phi}\right)^4\right) : = \lambda : S_{x_0+\tau}\left(\hat{\Phi}^4\right) : , \lambda > 0 \ ,$$

where $:\ \ :$ denotes the normal ordering of a_k and $a_k^\dagger$.

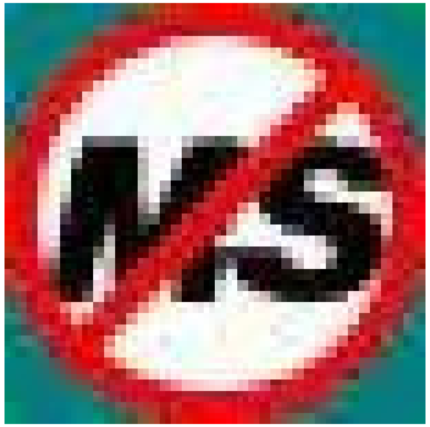
- The S-matrix S can be worked out as usual.

To conclude......

- The study involved space-time noncommutativity and it is obvious lot more has to be done.
- Some of these structures like quantised evolutions have made their appearance already while studying $2+1\ \ D$ gravity.
  I will conclude with a quotation:
- • ... a major revolution in our physical theory must be waiting in the wings... whatever the nature of this revolution might be the final theory ...... must have a fundamentally non-local character.
  - Roger Penrose, The Geometric Universe, 1998

FINAL REMARK....

- As we all Know - "Power corrupts".
  BUT "PowerPoint corrupts absolutely."



- The material presented here are prepared using publicly available software Prosper.