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Fuzzy spaces and gauge theory for non-associative algebras:

(Non-associative gauge theory and higher spin fields)

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These are preliminary lecture notes, *intended only* for *distribution to* participants.

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Non-associative gauge theory and higher spin fields

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Based on –

hep-th/0412027 (with Paul de Medeiros)
Non-associative gauge theory and higher spin interactions

JHEP0403 - hepth/0310153 ;
Towards Gauge theory for a class of commutative and non-associative fuzzy spaces

+ Earlier papers on higher dimensional fuzzy spheres

Hepth/0105006 – NPB610 – *Spherical harmonics for fuzzy spheres in diverse dimensions*

Hep-th/0111278 – NPB627 – with P.M. Ho – *Hidden dimensions in Matrix Brane constructons*

1 Introduction and Outline

- Non-commutative space is a deformation of ordinary space, which allows generalization of quantum field theory, gauge invariance ...

$$[Z_1, Z_2] = \theta$$

- More generally

$$[Z_\mu, Z_\nu] = \theta_{\mu\nu}$$

- Non-commutative space appears in D-brane wordvolume field theories when B-field is turned on.

- Field theory on these con-commutative spaces is equivalent to field theory on commutative spaces, with the product of fields replaced by a star product

$$F * G = e^{i\theta_{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} F(x)G(y)|_{x=y}$$

- A choice of Background field breaks the Euclidean symmetries of space.

- Commutative but non-associative deformations of the coordinate algebra can preserve Euclidean Invariance (or Lorentzian after a change of signature)

$$Z_\mu Z_\nu - Z_\nu Z_\mu = 0$$

$$(Z_\mu Z_\nu)Z_\lambda - Z_\mu(Z_\nu Z_\lambda) = \frac{\theta}{n^2}(-\delta_{\mu\nu}Z_\lambda + \delta_{\nu\lambda}Z_\mu)$$

- These algebras arise in the context of fuzzy spheres. Fuzzy sphere S^{D-1} constructions are based on constructions of Matrices Z_μ acting on a subspace of $V^{\otimes n}$, the n-fold tensor product of the spinor representation, such that

$$\sum_{\mu=1}^{D} Z_\mu Z_\mu = 1$$

$[Z_\mu, Z_\nu] =$ [illegible]) 0

$$(Z_\mu Z_\nu)Z_\lambda - Z_\mu(Z_\nu Z_\lambda) = \mathcal{O}(1/n)$$

The Z_μ and their symmetric products span an algebra which approaches the classical algebra of functions on a sphere in the large n limit, and admits action on $so(D)$ for any n. For $D > 2$ these deformed algebras are commutative but non-associative.

$$Z_{\mu_1\mu_2\cdots\mu_s} = \frac{1}{n^k} \sum_{r_1 \neq r_2 \cdots \neq r_s} \rho_{r_1}(\Gamma_{\mu_1})\rho_{r_2}(\Gamma_{\mu_2}) \cdots \rho_{r_s}(\Gamma_{\mu_s})$$

- These symmetric products of Z_μ form the space of fuzzy spherical harmonics. This space is only a subspace of the space of all Matrices acting on $\mathcal{R}_n$, which can be singled out by a projection P. The subspace does not close under Matrix multiplication. It does close under a new multiplication which is just the Matrix multiplication followed by the projection. This new multipication is non-associative but commutative.

- These fuzzy sphere Matrices describe the worldvolumes of D-branes in string theory. The D-branes carry gauge fields so there should be gauge theory for these algebras. We will explore gauge theory for these algebras.

- Fields which are usually functions of commutative and associative coordinates, will now be functions of these commutative, non-associative coordinates.

  If Φ is some charged scalar matter, and A_μ a gauge field, we will have

$$\Phi = \phi_0 + \phi_\mu Z_\mu + \phi_{\mu_1\mu_2} Z_{\mu_1\mu_2} + \cdots$$

$$A_\mu = a_\mu + a_{\mu\alpha} Z_\alpha + a_{\mu\alpha_1\alpha_2} Z_{\alpha_1\alpha_2} + \cdots$$

OUTLINE

Part I :

1A. Introduce the algebra $\mathcal{A}_n(\mathbb{R}^D)$

– A deformation of the algebra of polynomial functions.

1B. Covariant Derivative in the commutative, non-associative context.

– Leibniz Rule for derivatives.

– Associator

Part II :

– Deformed derivations on $\mathcal{A}_n(\mathbb{R}^D)$

– Derivations on $\mathcal{A}^*{}_n(\mathbb{R}^D)$

Part III:

– The Associator

– Examples

– Use of $m_2^* \leftrightarrow m_2^c$ to get the general expression

– general formula for the associator for any product which can be written in terms of m_2^c and expansions in derivatives.

Part IV : Generalized gauge fields for non-associative space

— The generalized gauge field $\hat{A}_\mu$.

— Generalized gauge transformations $\hat{\epsilon}$

— Covariant field strength

— Global form of transformations

— Generalized fields as functions on T^*M and higher spin fields.

Part V : The associative limit

— The associative limit of the higher spin theory :

— Construction of the action via relation to Wigner space formulation of quantum mechanics.

— Coordinate space form of action

– Embedding a theory on M in the bigger theory : projection and gauge fixing ; trace reduction ; Unfolding ;

Part VI :

– Physical Background – Fuzzy spheres in Matrix theory

Part VII :

– Open problems

Part I

IA : *The Algebras* –

- Recall the algebra of polynomial functions on $\mathbb{R}^D$. Let $\mu = 1, \cdots D$

$$1$$

$$Z_\mu$$

$$Z_{\mu_1} Z_{\mu_2} = Z_{(\mu_1 \mu_2)}$$

$$Z_{\mu_1} Z_{\mu_2} Z_{\mu_3} = Z_{(\mu_1 \mu_2 \mu_3)}$$

Elements of the algebra correspond to symmetric tensors

- There is a degree operator

$$\begin{aligned} D(1) &= 0 \\ D(Z_\mu) &= 1 \\ D(Z_{\mu_1} Z_{\mu_2}) &= 2 \\ &\vdots \end{aligned}$$

- We can also consider an ideal formed by elements of degree $D > n$ and form the quotient, to get an algebra of symmetric tensors of degree less than or equal to n.

- Algebra : Vector space and a product
  $m_2^c : \mathcal{A}_n \otimes \mathcal{A}_n \to \mathcal{A}_n$
  The basis of the vector space is the space of symmetric tensors

$$1, Z_{\mu}, Z_{\mu_1\mu_2} \cdots Z_{\mu_1 \dots \mu_n}$$

  Product as given before.

- Now we will keep the same vector space and define a new product m_2

$$Z_{\mu_1}.Z_{\mu_2} = Z_{\mu_1\mu_2} + \frac{1}{n}\delta_{\mu_1\mu_2}$$

  Equivalently

$$m_2(Z_{\mu_1} \otimes Z_{\mu_2}) = Z_{\mu_1\mu_2} + \frac{1}{n}\delta_{\mu_1\mu_2}$$

  More generally

$$Z_{\mu_1}.Z_{\nu_1 \cdots \nu_s} = Z_{\mu_1\nu_1 \cdots \nu_s} + \sum_{i=1}^{s} \frac{n-s+1}{n^2}\delta_{\mu_1\nu_i} Z_{(\nu_1 \cdots \nu_s \setminus i)}$$

- $Z_{\mu_1\mu_2} \, . \, Z_{\nu_1 \cdots \nu_s}$ will have δ terms and $\delta\delta$ terms and some coefficients which are further suppressed in the $\frac{1}{n}$ expansion.

- n appears both in the size of the algebra and the structure constants of the algebra. We will be mostly interested in the large n limit where the $1/n$ factors in the structure constants of the algebra will be important but the cutoff on the Z's will not.

- *Remark:* **Non-associativity**

$$(Z_{\mu_1}.Z_{\mu_2}).Z_{\mu_3} - Z_{\mu_1}.(Z_{\mu_2}.Z_{\mu_3})$$
$$= \frac{-1}{n^2} \; [\; \delta_{\mu_1\mu_2} Z_{\mu_3} - \delta_{\mu_2\mu_3} Z_{\mu_1} \;]$$

This follows from the equations above.

- A general formula for the product :

$$Z_{\mu(S_1)}.Z_{\mu(S_2)}$$

$$= \sum_{T_1 \subset S_1} \sum_{T_2 \subset S_2; |T_2| = |T_1|} \frac{1}{n^{2|T_1|}} \frac{(n - |S_1| - |S_2| + 2|T_1|)!}{(n - |S_1| - |S_2| + |T_1|)!}$$

$$\tilde{\delta}(\; \mu(T_1)\;,\; \mu(T_2)\;) \quad Z_{\mu(S_1 \cup S_2 \setminus T_1 \cup T_2)}$$

- The product is *commutative* . Consider exchanging the μ-indices in the set S_1 with those in set S_2. These indices appear on the RHS as subscripts on the Z's or in the delta functions. Both are symmetric. Therefore the product is unchanged under the switch, hence commutative.

- It will also be useful to introduce $\mathcal{A}^*{}_n(\mathbb{R}^D)$, which has similar properties and a similar (somewhat simpler) form :

  - Same underlying vector space, different product m_2^*

  - Commutative and non-associative.

  - Non-associativity vanishes at large n.

  - Approaches the classical product at large n.

$$Z_{\mu(S_1)} * Z_{\mu(S_2)}$$

$$= \sum_{T_1 \subset S_1} \sum_{T_2 \subset S_2; |T_2| = |T_1|} \frac{1}{n^{2|T_1|}} \frac{(n)!}{(n - |T_1|)!}$$

$$\tilde{\delta}(\ \mu(T_1)\ ,\ \mu(T_2)\ )\quad Z_{\mu(S_1 \cup S_2 \setminus T_1 \cup T_2)}$$

IB – **Covariant derivatives**

- A scalar field takes values in the algebra of functions, more generally the deformed algebra.

$$\Phi \in \mathcal{A}_n(\mathbb{R}^D)$$

- We will consider $U(1)$ gauge theory. Φ is charged :

$$\delta_\epsilon \Phi = i\epsilon \cdot \Phi$$

The product is taken in the algebra $\mathcal{A}_n(\mathbb{R}^D)$

- Covariant derivative $D_\mu = \partial_\mu - iA_\mu$.

∂_μ are derivations : $\mathcal{A}_n \longrightarrow \mathcal{A}_n$

A_μ are in $\mathcal{A}_n$: they have an expansion in the $Z_{\mu_1 \cdots \mu_s}$.

We want D_μ to be covariant.

$$\delta_\epsilon(D_\mu \Phi) = i\epsilon \cdot (D_\mu \Phi)$$

- Expanding both sides :

$$\begin{aligned} LHS &= \delta_\epsilon((\partial_\mu - iA_\mu) \cdot \Phi) \\ &= i\partial_\mu(\epsilon \cdot \Phi) - i(\delta_\epsilon A_\mu) \cdot \Phi + A_\mu \cdot (\epsilon \cdot \Phi) \\ &= i(\partial_\mu \epsilon) \cdot \Phi + i\epsilon \cdot (\partial_\mu \Phi) - i(\delta_\epsilon A_\mu) \cdot \Phi + A_\mu \cdot (\epsilon \cdot \Phi) \end{aligned}$$

We assumed Leibniz rule in last line.

RHS of covariance condition :

$$i\epsilon.\partial_\mu \Phi + \epsilon.(A_\mu \Phi)$$

which implies

$$i(\delta_\epsilon A_\mu).\Phi = i(\partial_\mu \epsilon).\Phi + A_\mu.(\epsilon.\Phi) - \epsilon.(A_\mu.\Phi)$$

$$
\begin{aligned}
\delta_\epsilon A_\mu \cdot \Phi &= (\partial_\mu \epsilon) \cdot \Phi - i A_\mu \cdot (\epsilon \cdot \Phi) + i\epsilon \cdot (A_\mu \cdot \Phi) \\
&= (\partial_\mu \epsilon) \cdot \Phi - i(\epsilon \cdot \Phi) \cdot A_\mu + i\epsilon \cdot (\Phi \cdot A_\mu) \\
&= (\partial_\mu \epsilon) \cdot \Phi - i\mathcal{A}(\epsilon, \Phi, A_\mu)
\end{aligned}
$$

We may expect that

$$
\mathcal{A}(\epsilon, \Phi, A_\mu) = E(\epsilon, A_\mu)\Phi
$$

where E is some operator depending on ϵ and A_μ and acting on Φ. One finds

$$
E(\epsilon, A_\mu) = E_\alpha(\epsilon, A_\mu) \cdot \delta_\alpha \Phi + E_{\alpha_1\alpha_2}(\epsilon, A_\mu) \cdot \delta_{\alpha_1}\delta_{\alpha_2}\Phi + \cdots
$$

Successive terms are subleading in the $\frac{1}{n}$ expansion.

- *Remark* :

  When we do gauge theory for non-commutative but associative algebras, extra term is a commutator (which appears even for $U(1)$ gauge theory)

$$
\delta_\epsilon A_\mu = \partial_\mu \epsilon - i[A_\mu, \epsilon]
$$

The second operator is just a function of the Z's

- Here, in the non-associative case, we pick up associator instead of commutator, but the extra operators act not just by multiplication but involve derivatives.

$$\delta A_\mu = \partial_\mu \epsilon + E_\alpha(\epsilon, A_\mu)\delta_\alpha + E_{\alpha_1\alpha_2}(\epsilon, A_\mu)\delta_{\alpha_1}\delta_{\alpha_2} + \cdots$$

- A_μ starts off being just a function of the Z's but has to be generalized.

- We will return to the implications of this later.

Now we explore $\partial_\mu \to \delta_\mu$, Leibniz rule ;
and operators related to the associator.
in the context of $\mathcal{A}_n(\mathbb{R}^D)$ and $\mathcal{A}^*{}_n(\mathbb{R}^D)$.

Part II : Derivations and Deformed Derivations

- δ_α for $\alpha = 1 \cdots D$ will be defined as maps from $\mathcal{A}_n \to \mathcal{A}_n$. Define them on a basis :

$$\delta_\alpha Z_{\mu_1\mu_2\cdots\mu_s} = \sum_{i\in S} \delta_{\alpha\mu_i} Z_{\mu(S\setminus i)}$$

This is an obvious definition for finite n which yields the ordinary derivatives in the large n limit. But an easy check shows that they don't obey Leibniz rule in general.

$$\begin{aligned}
\delta_\alpha(Z_{\mu_1} \cdot Z_{\mu_2}) &= (\delta_\alpha Z_{\mu_1}) \cdot Z_{\mu_2} + Z_{\mu_1} \cdot (\delta_\alpha Z_{\mu_2}) \\
\delta_\alpha(Z_{\mu_1} \cdot Z_{\mu_2\mu_3}) &= \delta_\alpha Z_{\mu_1} \cdot Z_{\mu_2\mu_3} + Z_{\mu_1} \cdot (\delta_\alpha Z_{\mu_2\mu_3}) \\
&\quad - \frac{1}{n^2}(\delta_{\alpha\mu_2}\delta_{\mu_1\mu_3} + \delta_{\alpha\mu_3}\delta_{\mu_1\mu_2})
\end{aligned}$$

- The correction can be expressed in terms of

$$\frac{-1}{n^2}(\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha)(\delta_{\alpha_1} \otimes \delta_{\alpha_1})$$

- More precisely

$$\begin{aligned}
& -\tfrac{1}{n^2}(\delta_{\alpha\mu_2}\delta_{\mu_1\mu_3} + \delta_{\alpha\mu_3}\delta_{\mu_1\mu_2}) \\
= \;& -\tfrac{1}{n^2} m_2(\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha)(\delta_{\alpha_1} \otimes \delta_{\alpha_1})(Z_{\mu_1} \otimes Z_{\mu_2\mu_3})
\end{aligned}$$

- We can evaluate on a general pair of elements $Z_{\mu(S)}$, $Z_{\mu(T)}$ to obtain a general formula for the deformation of the Leibniz rule.

$$
\begin{aligned}
&\delta_\alpha(Z_{\mu(S)} \cdot Z_{\mu(T)}) \\
&= (\delta_\alpha Z_{\mu(S)}) \cdot Z_{\mu(T)} + Z_{\mu(S)} \cdot (\delta_\alpha Z_{\mu(T)}) \\
&= \sum_{l=1}^{n} \frac{(-1)^l}{n^{2l}} \delta_\alpha \delta_{\alpha_1} \cdots \delta_{\alpha_l} Z_{\mu(S)} \cdot \delta_{\alpha_1} \cdots \delta_{\alpha_l} Z_{\mu(T)} \\
&+ \sum_{l=1}^{n} \frac{(-1)^l}{n^{2l}} \delta_{\alpha_1} \cdots \delta_{\alpha_l} Z_{\mu(S)} \cdot \delta_\alpha \delta_{\alpha_1} \cdots \delta_{\alpha_l} Z_{\mu(T)}
\end{aligned}
$$

- In other words

$$
\delta_\alpha \cdot m_2 = m_2 \cdot (\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha). \sum_{l=0}^{n} \frac{(-1)^l}{n^{2l}} \delta_{\alpha_1} \cdots \delta_{\alpha_l} \otimes \delta_{\alpha_1} \cdots \delta_{\alpha_l}
$$

- There is a deformed derivation rule. described by a co-product on the algebra $(\mathcal{U})$ of derivatives

$$
\Delta : (\mathcal{U}) \to (\mathcal{U}) \otimes (\mathcal{U})
$$

REMARKS

- 1. Proving the desired deformed derivation rule uses some interesting combinatoric identities.

$$
\sum_{k=0}^{A} (-1)^k \frac{(N+1+2A)!}{(A-k)!(N+1+A+k)!} = \frac{(N+2A)!}{A!(N+A)!}
$$

- 2. This modified Leibniz rule will have implications for the gauge transformations of A_μ because

$$\delta_\alpha(\epsilon \cdot \Phi) = (\delta_\alpha \epsilon) \cdot \Phi + \epsilon \cdot (\delta_\alpha \Phi) + \cdots$$

- 3. Another way we might think of characterizing the failure of δ_α from being a derivation

$$\delta_\alpha(1 + \frac{1}{n}\delta_\beta^2 + \cdots$$

does not work

- 4. Such co-products occur in the context of quantum groups. Lie algebra elements act on the algebra of functions on the group as derivations. These functions form representations of the Lie algebra. Products of these functions are related to tensor products of representations of the Lie algebra. The Leibniz rule for the action of derivatives on products of functions is related to the standard co-product which gives the action of Lie algebra on tensor products of representations.

Take $su(2)$ Lie algebra for example :

$$\Delta(J_+) = J_+ \otimes 1 + 1 \otimes J_+$$

In the case of the q-deformed $su(2)$ we have

$$\Delta(J_+) = J_+ \otimes e^{hJ_3/2} + e^{-hJ_3/2} \otimes J_+$$

- 5. Now we know from

a) quantum groups acting on non-commutative spaces

b) Seiberg-Witten map in non-commutative Yang Mills

that, often, the same physics can be described by two different kinds of products, which are related :

E.g in Seiberg-Witten map

$$\begin{aligned} a * b &= m_2^*(a \otimes b) \\ &= m_2^c . e^{i\theta_{\mu\nu}\partial_\mu \otimes \partial_\nu} \end{aligned}$$

$$m_2^* = m_2^c . e^{i\theta_{\mu\nu}\partial_\mu \otimes \partial_\nu}$$

Here SW show that both m_2^* and m_2 describe the same physics. i.e the DBI action written in terms of closed string parametrs (g, B, g_s)

$$\int \mathcal{L}_{DBI} = \int \sqrt{g + B + F}$$

and the Non-commutative DBI written in terms of the open string parameters (G, Θ, G_s).

$$\int \hat{\mathcal{L}}_{DBI} = \int \sqrt{G + \hat{F}}$$

- For an appropriate map $\hat{A}(A)$, $\hat{\lambda}(\lambda, A)$ such that

$$\hat{A}(A) + \delta_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A)$$

there is an equivalence of

$$\mathcal{L}_{DBI} = \hat{\mathcal{L}}_{DBI} + \text{ total der. } + \mathcal{O}(\partial F)$$

- In the context of quantum groups acting on non-commutative spaces, one can also consider different products related by a Drinfeld twist.

$$m_2' = m_2 . F$$

where F lives in $\mathcal{U}_q \otimes \mathcal{U}_q$.

- This suggests that we should look for a different product related to the original one by similar formulas, such that, δ_α are derivations with respect to the new product.

We try

$$m_2^* = \sum_l m_2 \, . \, c_l \, . \, \delta_{\alpha_1} \cdots \delta_{\alpha_l} \otimes \delta_{\alpha_1} \cdots \delta_{\alpha_l}$$

This does not work.

But a slight generalization works.

$$m_2^* = \sum_l \frac{h_l(D)}{n^{2l}} \, . \, m_2 \, . \, (\delta_{\alpha_1} \cdots \delta_{\alpha_l} \otimes \delta_{\alpha_1} \cdots \delta_{\alpha_l})$$

where D is the degree operator and $h_l(D) = \frac{D!}{l!(D-l)!}$. With this h, the derivation property holds

$$\delta_\alpha \, m_2^* = m_2^* \, (\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha)$$

Proof uses the combinatoric identity :

$$\sum_{l=0}^{p} h_l(s-1) \binom{n-s+1}{p-l} = \sum_{l=0}^{p} h_l(s) \binom{n-s}{p-l} = \binom{n}{p}$$

The star product is given by the formula :

$$Z_{\mu(S_1)} * Z_{\mu(S_2)}$$

$$= \sum_{T_1 \subset S_1} \sum_{T_2 \subset S_2; |T_2| = |T_1|} \frac{1}{n^{2|T_1|}} \frac{(n)!}{(n-|T_1|)!}$$

$$\tilde{\delta}(\ \mu(T_1)\ ,\ \mu(T_2)\ )\quad Z_{\mu(S_1 \cup S_2 \setminus T_1 \cup T_2)}$$

Following from work with G. Travaglini (unpublished)

- This has a nice expression for exponentials, which can be defined as

$$e^{ik_\mu Z_\mu} = \sum_{m=1}^{n} \frac{i^m}{m!} k_{\mu_1} \cdots k_{\mu_m} (Z_{\mu_1 \cdots \mu_m})$$

- There is a nice action of the derivations

$$\delta_\alpha e^{ik_\mu Z_\mu} = ik_\alpha e^{ik_\mu Z_\mu}$$

- The star product of exponentials takes the form :

$$\begin{aligned} e^{ik_{1\mu} Z_\mu} * e^{ik_{2\nu} Z_\nu} &= \sum_{m=0}^{n} (-)^m (\theta k_1.k_2)^m \frac{n!}{m!(n-m)!} e^{i(k_{1\mu}+k_{2\mu})Z_\mu} \\ &= \left(1 - \frac{\theta k_1.k_2}{n^2}\right)^n e^{i(k_{1\mu}+k_{2\mu})Z_\mu} \end{aligned}$$

- This is a product of the form

$$e^{ik_{1\mu} Z_\mu} * e^{ik_{2\nu} Z_\nu} = f(k_1.k_2) e^{i(k_{1\mu}+k_{2\mu})Z_\mu}$$

- In the large n limit, it looks a natural generalization of the star product that appears in non-commutative gauge theory, except that there is scalar rather than a tensor deformstion parameter.

$$\begin{aligned} f &= e^{-\gamma k_1.k_2 - \frac{\gamma^2 (k_1.k_2)^2}{2n} - \frac{\gamma^3 (k_1.k_2)^3}{3n^2} - \cdots} \\ &= e^{-\gamma k_1.k_2} \left(1 - \frac{\gamma^2 (k_1.k_2)^2}{2n} + \cdots\right) \end{aligned} \tag{1}$$

Part IV : The Associator and Related Operators

- For an arbitrary triple of elements Φ_1, Φ_2, Φ_3, we want to write

$$\begin{aligned}&\mathcal{A}[\Phi_1, \Phi_2, \Phi_3]\\ &= (\Phi_1\Phi_2)\Phi_3 - \Phi_1(\Phi_2\Phi_3)\\ &= E(\Phi_1, \Phi_3)\Phi_2\\ &= F(\Phi_1, \Phi_2)\Phi_3\end{aligned}$$

- To obtain E and F operators we observe that m_2^* (and m_2) are related to a simple commutative, associative product m_2^c – the concatenation product.

$$m_2^c(Z_{\mu_1\cdots\mu_s} \otimes Z_{\nu_1\cdots\nu_t}) = Z_{\mu_1\cdots\mu_s\nu_1\cdots\nu_t}$$

This product m_2^c is commutative and associative.

$$m_2^* = \sum_k m_2^c \frac{n!}{k!(n-k)!} \sum_{\alpha_1\cdots\alpha_k} \delta_{\alpha_1}\cdots\delta_{\alpha_k} \otimes \delta_{\alpha_1}\cdots\delta_{\alpha_k}$$

This is of the form

$$m_2^* = m_2^c f(\delta_\alpha \otimes \delta_\alpha)$$

where $f(x) = (1 + \frac{x}{n^2})^n$.

- We can invert this relation and write an expression for m_2^c in terms of m_2^*.

$$
\begin{aligned}
m_2^c &= m_2^* f^{-1}(\delta_\alpha \otimes \delta_\alpha)\Big) \\
&= \sum_{l=0} \frac{(-1)^l}{l! n^{2l-1}} \frac{(n+l-1)!}{n!} \\
&\quad m_2^* \sum_{\alpha_1 \cdots \alpha_l} \delta_{\alpha_1} \cdots \delta_{\alpha_l} \otimes \delta_{\alpha_1} \cdots \delta_{\alpha_l}
\end{aligned}
$$

- We also need to know how the commute strings of partial derivatives through the multiplication. For a single derivative, we have Leibniz rule

$$\delta_\alpha m_2^* = m_2^*(\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha)$$

Define a map Δ from the space of multiple derivatives of $\mathcal{A}_n^*(\mathbb{R}^D)$ to the tensor product of two copies of this space. That is

$$\Delta(\partial_{\mu(S)}) := \sum_{U \cup V = S} \partial_{\mu(U)} \otimes \partial_{\mu(V)}$$

Consequently, we can write $\partial_{\mu(S)} m_2^* = m_2^* \Delta(\partial_{\mu(S)})$ when acting on $\mathcal{A}_n^*(\mathbb{R}^D) \otimes \mathcal{A}_n^*(\mathbb{R}^D)$. The same equation also holds when we replace m_2^* with m_2^c, since ∂_μ also obeys the Leibnitz rule with respect to m_2^c.

- Consider the product $A * (B * C)$ for any three functions on $\mathcal{A}_n^*(\mathbb{R}^D)$,

$$
\begin{aligned}
& A * (B * C) = m_2^*(1 \otimes m_2^*)(A \otimes B \otimes C) \\
= \;& m_2^c\, f(\partial_\mu \otimes \partial^\mu)(1 \otimes m_2^c) f(1 \otimes \partial_\nu \otimes \partial^\nu)(A \otimes B \otimes C) \\
= \;& m_2^c\,(1 \otimes m_2^c)((1 \otimes \Delta) f(\partial_\mu \otimes \partial^\mu)) f(1 \otimes \partial_\nu \otimes \partial^\nu)(A \otimes B \otimes C)\,,
\end{aligned}
$$

which can be rearranged using associativity of m_2^c to give

$$
\begin{aligned}
& m_2^* f^{-1}(\partial_\mu \otimes \partial^\mu)(m_2^* \otimes 1) f^{-1}(\partial_\nu \otimes \partial^\nu) \\
& \quad ((1 \otimes \Delta) f(\partial_\rho \otimes \partial^\rho)) f(1 \otimes \partial_\sigma \otimes \partial^\sigma)(A \otimes B \otimes C) \\
= \; & m_2^* (m_2^* \otimes 1)((\Delta \otimes 1) f^{-1}(\partial_\mu \otimes \partial^\mu)) f^{-1}(\partial_\nu \otimes \partial^\nu) \\
& \quad ((1 \otimes \Delta) f(\partial_\rho \otimes \partial^\rho)) \times f(1 \otimes \partial_\sigma \otimes \partial^\sigma)(A \otimes B \otimes C) .
\end{aligned}
$$

This manipulation has expressed the product $A * (B * C)$ in terms of a sum of products involving derivatives of the functions A, B and C (where the $*$-multiplication of the first two entries is done first and so is similar in structure to $(A * B) * C$). Thus the F operator can be read off from

$$
\begin{aligned}
\mathsf{F}(A, B)C \; = \; & m_2^* (m_2^* \otimes 1)[1 - ((\Delta \otimes 1) f^{-1}(\partial_\mu \otimes \partial^\mu)) f^{-1}(\partial_\nu \otimes \partial^\nu) \\
& \times ((1 \otimes \Delta) f(\partial_\rho \otimes \partial^\rho)) f(1 \otimes \partial_\sigma \otimes \partial^\sigma)](A \otimes B \otimes C) ;
\end{aligned}
$$

in terms of derivatives acting on C. The analogous derivative expansion of the E operator immediately follows, since $\mathsf{E}(A, B) = \mathsf{F}(A, B) - \mathsf{F}(B, A)$.

- Using these ingredients, we can express the associator explicitly in terms of the operators E and F as desired.

$$
E(\Phi_1, \Phi_3)\Phi_2 = E_\alpha(\Phi_1, \Phi_3) * \delta_\alpha \Phi_2 + E_{\alpha_1 \alpha_2}(\Phi_1, \Phi_3) * \delta_{\alpha_1} \delta_{\alpha_2} \Phi_2 + \cdots
$$

- The E, F operators can be constructed generally for any product m_2 written in terms of the associative m_2^c using an expansion in $\delta_\alpha \otimes \delta_\alpha$. The components $E_{\alpha_1 \alpha_2 \cdots}$ are written in terms of star products of derivatives acting on Φ_1, Φ_3.

Part IV : Generalized gauge fields, gauge transformations and Field Strengths

- Since the coordinate dependent gauge fields pick up derivative corrections under gauge transformations

$$\delta A_\mu = \delta_\mu \epsilon + E_\alpha(\epsilon, A_\mu)\delta_\alpha + E_{\alpha_1\alpha_2}(\epsilon, A_\mu)\delta_{\alpha_1}\delta_{\alpha_2} + \cdots$$

- We should generalize the notion of gauge fields

$$\hat{A}_\mu = A_\mu + A_{\mu\alpha}\delta_\alpha + A_{\mu\alpha_1\alpha_2}\delta_{\alpha_1}\delta_{\alpha_2} + \cdots$$

- This suggests that gauge parameters should also be generalized

$$\hat{\epsilon} = \epsilon + \epsilon_\alpha\delta_\alpha + \epsilon_{\alpha_1\alpha_2}\delta_{\alpha_1}\delta_{\alpha_2} + \cdots$$

- Analogs of the operators E, F defined before can be extended to these more general objects having an expansion in derivatives.

$$\hat{E}(\hat{A}, \hat{B})C = \hat{B}(\hat{A}C) - \hat{A}(\hat{B}C)$$
$$\hat{F}(\hat{A}, \hat{B})C = (\hat{A}\hat{B})C - \hat{A}(\hat{B}C)$$

These can be computed in terms of the E, F defined before, using in addition the commutators of δ_α with the coefficients $A_{\alpha_1 \cdots}$.

$$E(A,B) = F(A,B) - F(B,A)$$

$$E(A,B)C = -A(BC) + B(AC)$$
$$= -A(CB) + (AC)B$$

$$F(A,B)C = -A(BC) + (AB)C$$

$$F(B,A)C = -B(AC) + (BA)C$$

$$= -B(AC) + (AB)C$$

$$F(A,B)C - F(B,A)C$$

$$= B(AC) - A(BC)$$

$$= E(A,B)C$$

$$\hat{B}(\hat{A}C) = \sum_{s,t} \frac{1}{s!t!} B^{\alpha_1..\alpha_s}\partial_{\alpha_1}...\partial_{\alpha_s}(A^{\beta_1...\beta_t}\partial_{\beta_1}...\partial_{\beta_t}C)$$

$$= \sum_{s,t}\frac{1}{s!t!}\sum_k \binom{s}{k} B^{\alpha_1...\alpha_s}(\partial_{\alpha_1}..\partial_{\alpha_k}A^{\beta_1...\beta_t}\partial_{\alpha_{k+1}}....\partial_{\alpha_s}\partial_{\beta_1}...\partial_{\beta_t}C)$$

$$= \sum_{s,t}\frac{1}{s!t!}\sum_k \binom{s}{k}\Big(\ (B^{\alpha_1...\alpha_s}\partial_{\alpha_1}..\partial_{\alpha_k}A^{\beta_1...\beta_t})\partial_{\alpha_{k+1}}...\partial_{\alpha_s}\partial_{\beta_1}....\partial_{\beta_t}C$$

$$+F(B^{\alpha_1..\alpha_s}, \partial_{\alpha_1}..\partial_{\alpha_k}A^{\beta_1...\beta_t})\partial_{\alpha_{k+1}}..\partial_{\alpha_s}\partial_{\beta_1}...\partial_{\beta_t}C \ \Big)$$

- This gives $\hat{B}(\hat{A}C)$ as a sum of operators on C. Same thing can be done for $\hat{B}(\hat{A}C)$. Hence we can calculate $\hat{E}(\hat{A},\hat{B})C = \hat{B}(\hat{A}C) - \hat{A}(\hat{B}C)$ to get something of the form

$$\hat{\mathsf{E}}(\hat{A},\hat{B}) \ = \ \sum_{s=0}^{\infty}\frac{1}{s!}(\hat{E}^{\mu_1...\mu_s}(\hat{A},\hat{B}))(x) \ * \ \partial_{\mu_1}...\partial_{\mu_s} \ ,$$

where the components $\hat{E}^{\mu_1...\mu_s}$ are written in terms of multiplication by products of derivatives of the components of $\hat{A},\hat{B}$ and the F-operators with arguments which are derivatives of these components.

- Gauge transformations can be defined in terms of these.

$$\begin{aligned}
\delta\Phi \ &= \ \hat{\epsilon}\Phi \\
\delta\hat{A}_\mu \ &= \ [\delta_\mu,\hat{\epsilon}] - \hat{E}(\hat{A}_\mu,\hat{\epsilon}) \\
\delta\hat{D}_\mu \ &= \ \hat{E}(\hat{D}_\mu,\hat{\epsilon}) \\
\hat{F}_{\mu\nu} \ &= \ [\delta_\mu,\hat{A}_\nu] + [\delta_\nu,\hat{A}_\mu] + \hat{E}(\hat{A}_\nu,\hat{A}_\mu) \\
\delta\hat{F}_{\mu\nu} \ &= \ \hat{E}(\hat{F}_{\mu\nu},\hat{\epsilon})
\end{aligned}$$

- These formulas look very much like those for a theory on non-commutative theory or for non-abelian theory on commutative space, with $\hat{E}(A,B)$ playing the role of $[A,B]$. In a sense $\hat{E}(\hat{A},\hat{B})C = (\hat{A}(\hat{B}C)) - (\hat{B}(\hat{A}C))$ is indeed the "commutator" in the space of linear operators, if the action of operator are understood with a specified choice of brackets written from the right and outwards. The fact that the transformation of $F_{\mu\nu}$ takes the usual form because the "Jacobi identity" holds for $\hat{E}$, and follows from the usual manipulations leading to Jacobi identity, but with brackets inserted from the right ourward.

- We can also exponentiate the action of the gauge transformation parameter, and define the action of the exponential with a choice of brackets ("right outwards").

$$g\Phi \equiv e^{\hat{\epsilon}}\Phi = \Phi + (\hat{\epsilon}\Phi) + \frac{1}{2}(\hat{\epsilon}(\hat{\epsilon}\Phi)) + \frac{1}{6}(\hat{\epsilon}(\hat{\epsilon}(\hat{\epsilon}\Phi))) + \cdots$$

- The formulae for global transformations look similar to non-abelian gauge transformations, withcare taken to correctly order the brackets :

$$\begin{aligned} \hat{D}_\mu\Phi &\rightarrow \hat{g}\,(\hat{D}_\mu(\,\hat{g}^{-1}\,\Phi'))\,, \\ \hat{F}_{\mu\nu}\Phi &\rightarrow \hat{g}\,(\hat{F}_{\mu\nu}(\,\hat{g}^{-1}\,\Phi'))\,, \end{aligned}$$

- This theory can be viewed as naturally defined on a deformation of $T^*\mathbb{R}^D$ (the co-tangent bundle of the non-associative deformation of $\mathbb{R}^D$) and is a theory of higher spin fields on (the deformation of) $\mathbb{R}^D$. We can take a limit where the non-associativity goes to zero and we have a theory which is non-trivial due to the non-vanishing commutators $[\delta_\alpha, A(Z)]$

Part V : **The Associative limit**

- In the associative limit, we have the standard algbera of functions on $\mathbb{R}^D$ but the fields include higher spin gauge fields $A_\mu^{\alpha_1 \ldots \alpha_s}$ and higher spin gauge parameters $\epsilon^{\alpha_1 \ldots \alpha_s}$. The generalized gauge fields $\hat{A}_\mu$ are functions of Z^μ and ∂_μ. The Z^μ are coordinates on $\mathbb{R}^D$ and can be viewed as position variables in quantum mechanics. The derivatives are momentum operators since $P_\mu = -i\delta_\mu$.

- It is useful to use the map between operators $\hat{A}(Z^\mu, P_\mu)$ and functions $\mathcal{A}(z^\mu, p_\mu)$ which appears in the (Wigner-Weyl-Moyal) phase space formulation of quantum mechanics. This allows us to write explicit actions which are invariant under the gauge transformations. The map is :

$$A(Z^\mu, P_\mu) = \int dydqdzdp\mathcal{A}(z,p)exp(iq_\mu(Z^\mu - z^\mu)exp(iy^\mu(P_\mu - p_\mu)$$

It is a variation on the more familiar operator-function map which uses the Weyl-ordering (symmetrized operators).

- There is an inverse relation taking functions to normal-ordered operators $\hat{A}$ of the type considered before.

$$\begin{aligned} \tilde{A}(z,p) &= \frac{1}{(2\pi)^{2D}} \int dy\, dq\, \exp(i\,(q_\mu z^\mu + y^\mu p_\mu)) \qquad (2) \\ & \mathrm{Tr}\left(\exp(-i\, q_\mu Z^\mu)\, \hat{A}\, \exp(-i\, y^\mu P_\mu)\right) . \end{aligned}$$

- Traces of operators are related to integrals of the functions.

$$Tr(\hat{A}(Z,P)) = \int dzdp\mathcal{A}(z,p) \qquad (3)$$

- As in non-commutative gauge theory, gauge invariant actions are traces of operators which can be mapped to integrals over z, p

$$Tr(\hat{F}_{\mu\nu}\hat{F}_{\mu\nu}) = \int dz\, dp\, \mathcal{F}_{\mu\nu}{}'(z,p)\mathcal{F}_{\mu\nu}{}'(z,p) \tag{4}$$

where $\tilde{F}'_{\mu\nu} := \exp\left(\frac{i}{2}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial p_{\mu}}\right)\tilde{F}_{\mu\nu}$. where $\tilde{F}_{\mu\nu}$ is the function obatined from the operator $\hat{F}$ by the equation given before.

- After integrating out the momentum variables, we are left with a coordinate space action involving the higher spin fields integrated over coordinate space.

- This associative theory based on the phase space is in fact obtained from ordinary non-commutative theory for $T^*\mathbb{R}^D$ by a projection. Note that the theory described here contains fields similar to this non-commutative theory but only half the number of covariant derivatives. In the non-commutative theory, we can consider "large" field solutions to the equations of motion such that half the covariant derivatives vanish. Constraining fluctuations around these solutions to respect the vanishing of these derivatives is a gauge invariant condition.

$$F_{\mu\nu} + F_{\mu\nu}^{\alpha} p_{\alpha} \cdots$$

For Non-commutative Theory on

$T^* \mathbb{R}^D$ (2D-dimensional space)

$x^\nu \in \mathbb{R}^D$

$$D_A = \partial_A + A_A (x^\nu, p_\nu)$$

$$A = 1 \ldots 2D$$

$\partial_{D+\nu}$ can be realized as adjoint action of x^ν

because

$$x^{D+\nu} = p_\nu \qquad x^\nu$$

$$[x^\nu, p_\nu] = +i \Rightarrow [x^\nu, x^{D+\nu}] = i$$

$$\Rightarrow \frac{\partial}{\partial x^{D+\nu}} = [x^\nu,$$

∴ Can consider "large gauge fields which cancel this to give $D_{D+\nu} = 0$

$$D \rightarrow u D u^{-1}$$

∴ $D = 0$ gauge invariant.

- In this associative limit, there are two ways to embed an ordinary abelian theory on $\mathbb{R}^D$ in the higher spin theory.

  One is simply to set all the higher spin gauge fields to zero and gauge parameters to zero. This is consistent with the gauge transformations of the higher spin theory.

  The other is to restrict the higher spin components to equal derivatives of the leading components

$$\begin{aligned} A_{\mu;\alpha_1\cdots\alpha_s} &= \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_s}A_\mu \\ \epsilon_{\alpha_1\cdots\alpha_s} &= \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_s}\epsilon \end{aligned}$$

- For this choice of $\hat{A}_\mu$ we check that the field strength takes the abelian form

$$\hat{F}_{\mu\nu} = \delta_\mu \hat{A}_\nu - \delta_\nu \hat{A}_\mu$$

  and gauge transformation in the higher spin theory for the above gauge parameters is the same as performing the gauge transformation in the abelian theory and then mapping by the above rule.

- It remains to extend these embeddings to the non-associative case. For the trivial embedding : We attempt to use the extra gauge parameters to restrict the gauge fields to be just coordinate-dependent, i.e we want to set $A_{\mu\alpha}, A_{\mu\alpha_1\alpha_2}, \cdots$ to zero.

- This leads to a gauge transformation rule for A_μ which depends on $\epsilon, \epsilon_\alpha \epsilon_{\alpha_1\alpha_2} \cdots$

$$\delta_{\hat{\epsilon}} A_\mu = \delta_\mu \epsilon - i\epsilon_{\alpha_1} * \delta_{\alpha_1} A_\mu + \cdots$$

and some constraints that have to be satisfied by the $(\epsilon, \epsilon_\alpha, \epsilon_{\alpha_1\alpha_2}, \cdots)$

$$-iE_{\beta_1}(\epsilon, A_\mu) - iF_{\beta_1}(\epsilon_{\alpha_1}, \delta_{\alpha_1} A_\mu) + \cdots = 0$$

from Vanishing of first derivative corrections

$$-iE_{\beta_1\beta_2}(\epsilon, A_\mu) - iF_{\beta_1\beta_2}(\epsilon_{\alpha_1}, \delta_{\alpha_1} A_\mu) + \cdots = 0$$

from Vanishing of second derivative corrections.

If a solution exists to this system, which approaches the solution $\epsilon \neq 0; \epsilon_\alpha = 0; \epsilon_{\alpha_1\alpha_2} = 0 \cdots)$ at $n = \infty$

it would give a non-trivial embedding of the $U(1)$ inside the large gauge group generated by $(\epsilon, \epsilon_\alpha, \cdots)$ as we turn on the non-associativity parameter $1/n$.

Part VI : The Physical Background

- Given these commutative non-associative deformations of the algebra of functions on $\mathbb{R}^D$ we can also define deformations of the algebra of functions on S^{D-1}.
- Impose constraints such as

  $Z_{\mu\mu} = c$
- The Yang-Mills action for the sphere can be written in terms of these Cartesian coordinates of the embedding space. Derivatives δ_μ have to be projected on the sphere. Such projections can also be defined for these deformed algebras, and allows us to write deformations of the Yang-Mills action on the sphere.

- The existence of such Yang Mills theories on commutative/non-asssociative spheres is expected from "Matrix Theory"

In Matrix Theory one starts with a non-abelian $U(N)$ action for zero-branes

$$TR \int dt \ \ (D_o \Phi^a)^2 + \sum_{a,b} \ \ [\Phi^a, \Phi^b]^2 + \cdots$$

The equations of motion admit solutions where the Φ_i are set equal to matrices obeying an $SU(2)$ relations

$$\Phi_i \sim X_i$$

$$[X_i, X_j] = i\epsilon_{ijk}X_k$$
$$X_i^2 = \frac{(N^2-1)}{4}$$

- These solutions can be interpreted in terms of "fuzzy 2-spheres" – The collection of zero branes form spherical 2-branes (Myers effect)
- An important ingredient in this interpretation is that the algebra of functions on S^2 can be truncated in an $SO(3)$ covariant manner to give a Matrix algebra of size N.
- This is not possible with higher spheres S^4, S^6 etc. While S^2 is a co-adjoint orbit the other spheres are not.
- However S^{2k}, for higher k are bases of a fibration with total space $SO(2k+1)/U(k)$ – which are co-adjoint orbits – and do admit Matrix approximations.
- There is a projection of these Matrix approximations of $SO(2k+1)/U(k)$ to give "fuzzy higher spheres". The algebra structure on these fuzzy higher spheres is commutative and non-associative of the sort we have discussed.
- These higher fuzzy spheres play a role in the construction of higher spherical 4, 6- etc. branes from 0-branes. All these branes carry gauge fields.
- So we expect gauge theory on these commutative, non-associative algebras to exist.

Open problems and Directions

- Extending embeddings of abelian theory on M to theory on TM in the non-associative case.
- Lesson : Theories with infinitely many higher spin fields. Is the same true for Vassiliev theories in ADS space ? Application to ADS/CFT –deformations away from symmetric point ?
- The "Matrix Theory" applications suggest that there should be a duality between non-abelian theories on the comm/nass S^{2k} and abelian theories on non-commutative $SO(2k+1)/U(k)$. Explicit construction of the duality is an important direction.
- The Matrix algebras with $SO(5)$ symmetry underlying the fuzzy spheres have applications in condensed matter physics. Is there a role for the (gauge) field theories on non-associative algebras in this context ? Higher spins ?