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Noncommutative geometry and the quantum Hall plateau transition

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Noncommutative geometry and the quantum Hall plateau transition

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I. Introduction to the quantum Hall plateau transition: one of the last holes in our understanding of the 2D QHE

II. Percolation and the classical limit

III. Noncommutativity and the quantum case

IV. Hidden conservation laws & consequences

V. Possible extensions: higher dimensions & spin

Collaborations: A. Zee, J. Sinova
Acknowledgments: S. Girvin, N. Read

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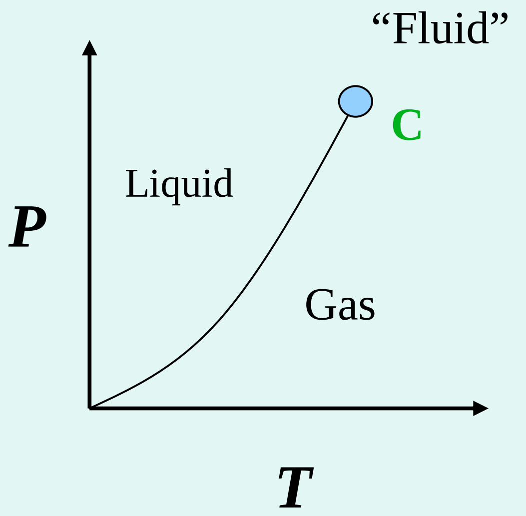
Why worry about disorder, i.e., randomness in the single-electron potential *V(x)*?

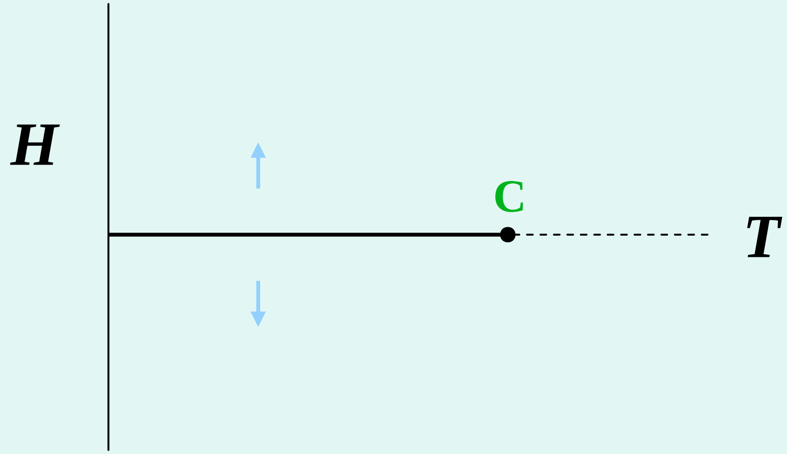
1. (Negative answer) The experimental observation of the quantum Hall effect requires breaking of Galilean invariance: otherwise the electric field vanishes in a properly chosen frame, and there are no Hall plateaus.

2. (Positive answers) The disorder-driven transition between Hall plateaus is (a) the best experimental demonstration of universal scaling at a quantum phase transition;

3. (b) the natural generalization to a noncommutative space of one of the best understood classical geometric critical points, 2D percolation.

Anomalous dimensions in condensed matter





Ising (uniaxial) ferromagnet

$$\rho_L - \rho_G \sim \left(\frac{T_C - T}{T_C}\right)^{\beta} \qquad M_{\uparrow} - M_{\downarrow} \sim \left(\frac{T_C - T}{T_C}\right)^{\beta}$$

$$\text{Experiment}: \quad \beta = 0.322 \pm 0.005$$

$$\text{Theory}: \quad \beta = 0.325 \pm 0.002$$

Classical and quantum phase transitions

I. Classical transitions are driven by competition between **energy** and **entropy**; at sufficiently low temperature, the system goes into an ordered state.

II. Near a critical point, there are strong **thermal** fluctuations on large length scales (critical opalescence). On these length scales, classical physics applies.

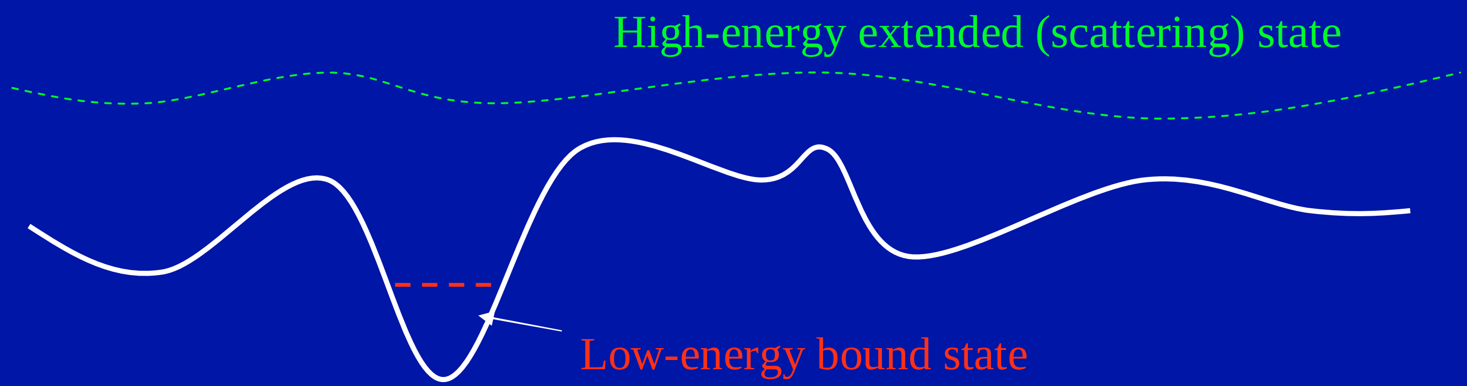
I. Quantum phase transitions are driven by a **nonthermal coupling** (doping, magnetic field, chemical potential, etc.) and exist at zero temperature.

II. Near a quantum critical point, there are strong **quantum** fluctuations on large length scales. Quantum mechanics remains essential.

Example: disorder-driven localization

There can be a phase transition even in a **noninteracting** system, driven by quantum interference. There are still electronic states at the Fermi level, but they are **localized** and carry no current at low temperature.

Consider the Schrodinger equation for one electron in a random potential.



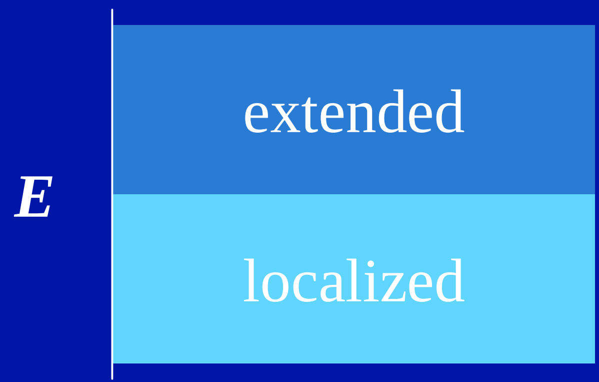
The mobility edge

The preceding picture of high-energy extended states and low-energy localized states is essentially correct in 3D. Furthermore, there is one energy (the "**mobility edge**") which separates the two classes.

Why?

Mott's argument:
Suppose there were both extended and localized states at the same energy.

This is unstable to any small perturbation: since the energy denominator is zero, the localized state and extended state will mix, leaving two extended states.

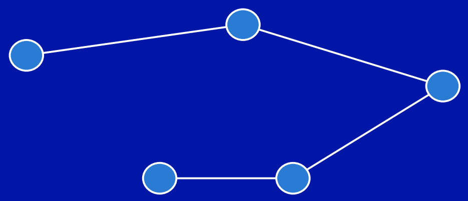


Importance of dimensionality for Anderson localization

As the Fermi level nears the mobility edge, there should be universal scaling of the localization length, conductivity, and other quantities.

The localization of free electrons by a random potential is called **Anderson localization**. In one and two dimensions, there is no mobility edge: **all states are localized** by even a weak random potential. How is this even possible?

An electron in an extended state does a random walk:

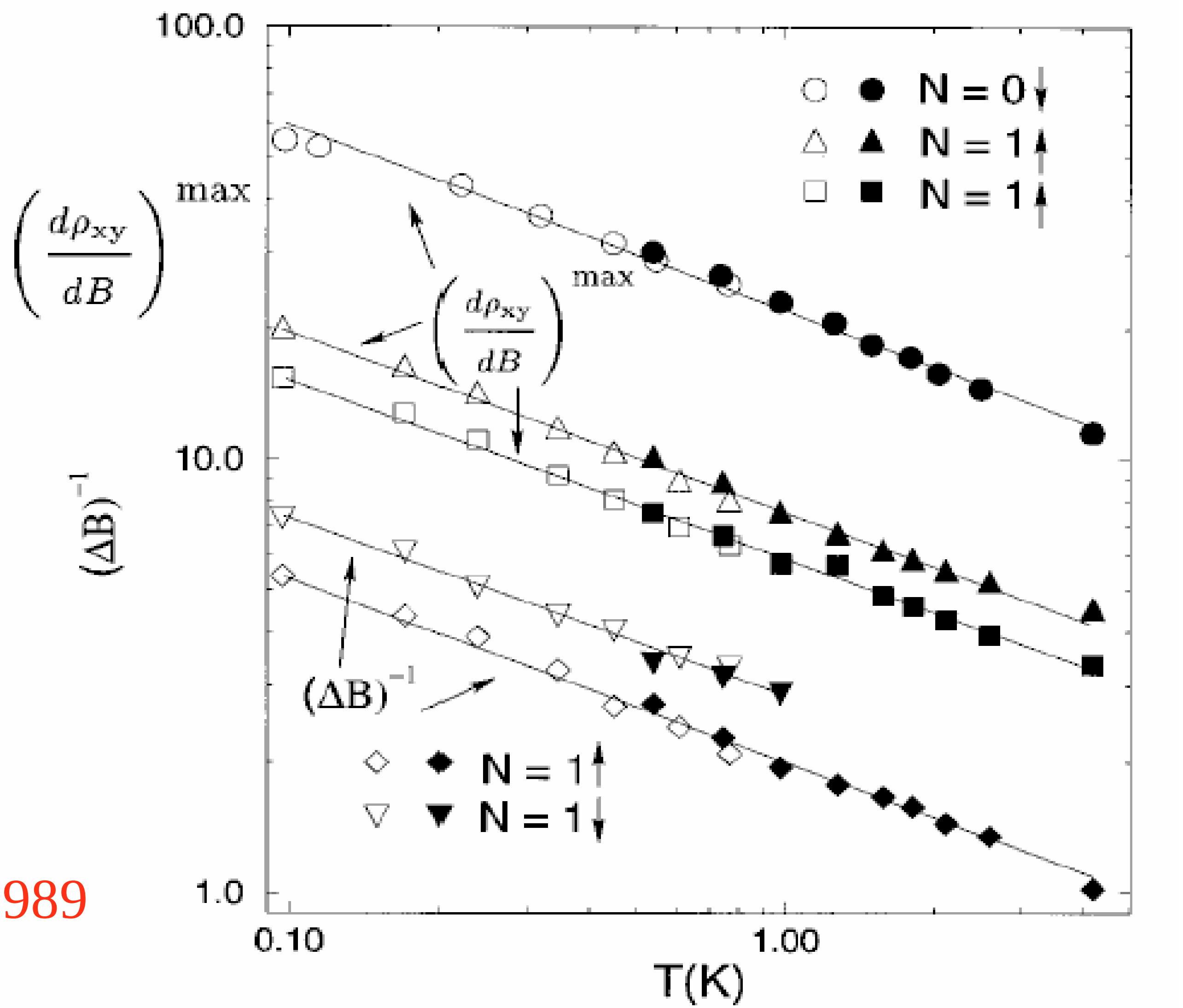


A random walk in 3D of length **N** returns to an impurity “almost never”.
A random walk in 2D of length **N** returns to an impurity **log N** times.
A random walk in 1D of length **N** returns to an impurity **sqrt(N)** times.

All these returns effectively amplify even a weak impurity potential, in 1D and 2D, because of constructive interference.

It turns out that an accessible delocalization transition occurs for 2D electrons in a magnetic field, which will be discussed later. The magnetic field spoils the interference mentioned above.

Temperature scaling of transition width



H. P. Wei et al., PRL 1989

Random level surfaces

Consider the following 2D **classical** model of the quantum Hall phase transition:

An electron moves in a random 2D electric potential and strong magnetic field.

The electron makes fast circular orbits in the magnetic field; its "drift velocity" is along ***E*** x ***B*** and hence along a level surface of the random potential.

What are the statistical properties of "random level surfaces", contours of constant energy in a random potential? (Trugman)

Some intuition: consider a contour map of the Earth's crust.
Contours at high elevation (8000m) are small.
Contours at sea level can be large (continental coastlines).

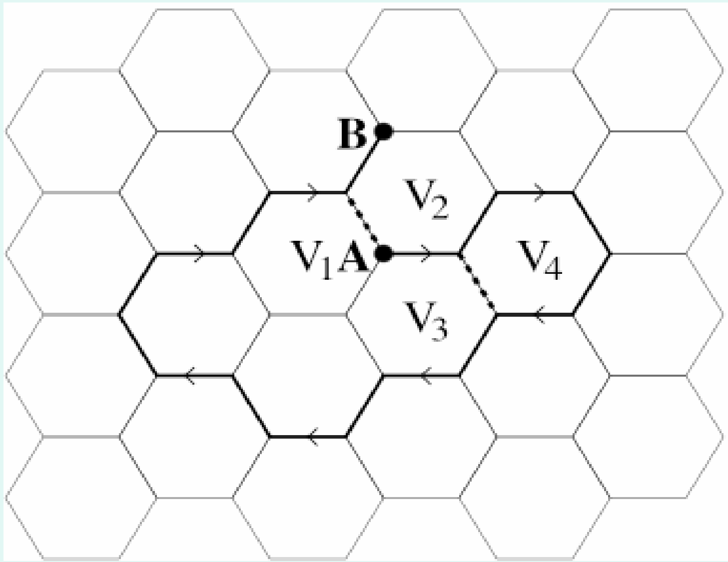
Contour maps and criticality

Suppose the random potential is uniformly distributed over [-1,1].

Then the typical contour size as a function of energy, $\xi(E)$, is small for E near 1 (peaks) or -1 (valleys), but large as E nears 0.

To understand this divergence, make a lattice model:

The electron of energy E moves on edges of the lattice subject to the constraint that faces to the left (right) have potential higher (lower) than E.



Percolation hulls and the classical limit

Using the lattice model, one can show that random level surfaces at criticality are equivalent to percolation hulls.

Idea of percolation, a **geometrical critical phenomenon**: Randomly color a fraction *p* of faces red. At some *p*, there appears an infinitely large connected cluster of red faces (the red faces “percolate”). This cluster’s boundary is a percolation hull.

This cluster is **fractal**: most of its fractal dimensions and other properties are known via properties of minimal models in conformal field theory (e.g., q->1 q-state Potts model).

The critical value of *p* corresponds to E = 0, and the mean cluster radius diverges as

$$\xi(E) \sim \frac{1}{(E-E_c)^{4/3}}$$

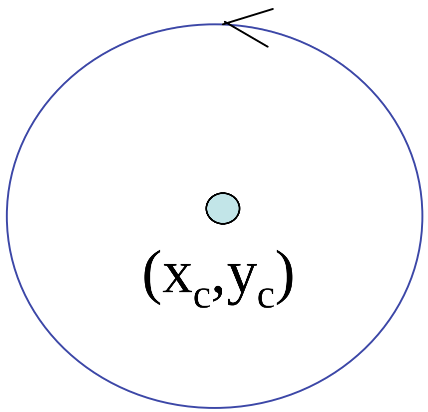
Origin of noncommutative geometry

The classical electron motion in a magnetic field conserves the **x** and **y** coordinates of the guiding-center.

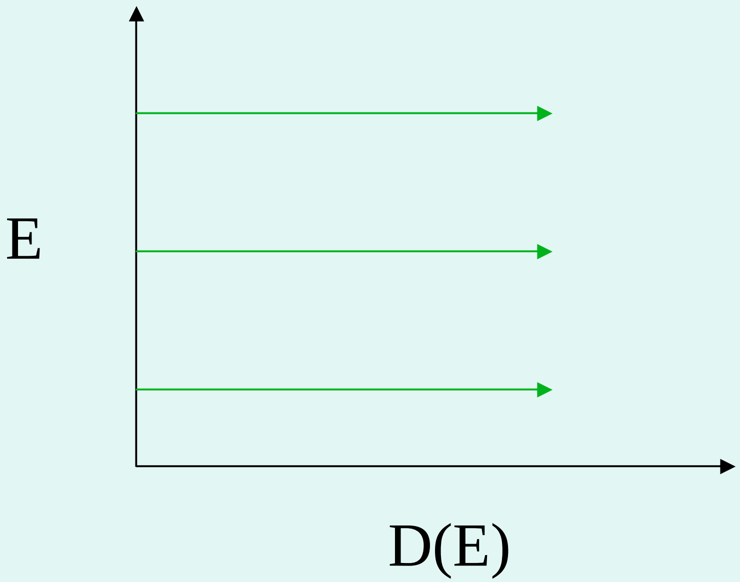
However, these two conserved quantities do not commute (the Poisson bracket or commutator is nonzero).

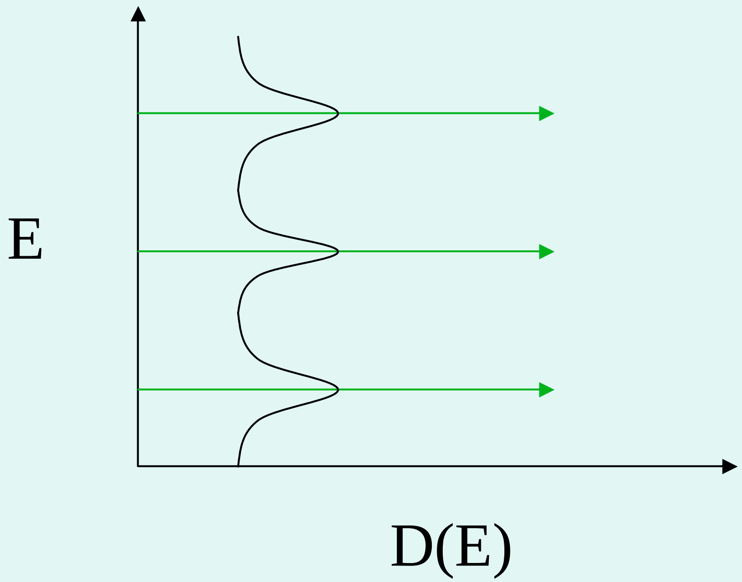
Recall that in our derivation of classical percolation, we essentially followed the guiding-center motion along level surfaces.

Claim: For a correct quantum-mechanical description, it is sufficient to redefine the space in which the classical percolation theory lives.



2D electrons in a magnetic field and random potential





Clean spectrum: degenerate Landau levels spaced by cyclotron energy.

Disorder pushes some weight into localized states at other energies.

$$\xi(E) \sim \frac{1}{(E - E_c)^{\nu}}$$

The quantum phase transition between Hall plateaus: experimental summary

Continuous quantum phase transitions, like their classical analogues, are believed to contain universal scaling laws near criticality.

The transitions between neighboring quantum Hall plateaus and between the lowest Hall plateau and the insulator are some of the best studied examples of such QPT's.

The power-law scaling of the transition width at very low samples measures a combination of critical exponents ν and z:

$$\Delta B \sim T^{z\nu}$$

$$\xi(E) \sim \frac{1}{(E - E_c)^{\nu}}$$

$$\xi_\tau \sim \xi^z$$

For more background see Sondhi, Girvin, Carini, and Shahar, RMP 69, 315 (1997).

Another measurement can be used to extract a different combination of ν and z.

In a few samples, there is scaling over two decades in temperature consistent with ν approximately 2.3 and z approximately 1. This value for ν is consistent with numerical results on **noninteracting** models of the transition, and considerable analytic effort has been expended on such models.

In many other samples, however, scaling of the plateau width does not hold to the lowest temperatures or is not consistent with these values…we'll see a possible reason for this later.

For now, focus on the plateau transition without interactions as the most experimentally relevant case of a **quantum interference** phase transition **without interactions**.

Localization and the Liouvillian, part I

Test for presence of extended states in a disordered electronic system:
(Anderson, 1958)
1. Start with a lattice tight-binding model.

2. Add an electron at one site and follow the evolution of the electron density over time.

3. If there are extended states in the system, then "typically" the electron density will diffuse:

$$\langle R^2 \rangle = \int |\psi(\mathbf{x})|^2 (\mathbf{x} - \mathbf{x}_0)^2 \, d\mathbf{x}$$

$$\overline{\langle R^2(t) \rangle} \sim Dt$$

4. With no extended states (all eigenstates fall off exponentially beyond some region), the mean squared radius approaches a constant.

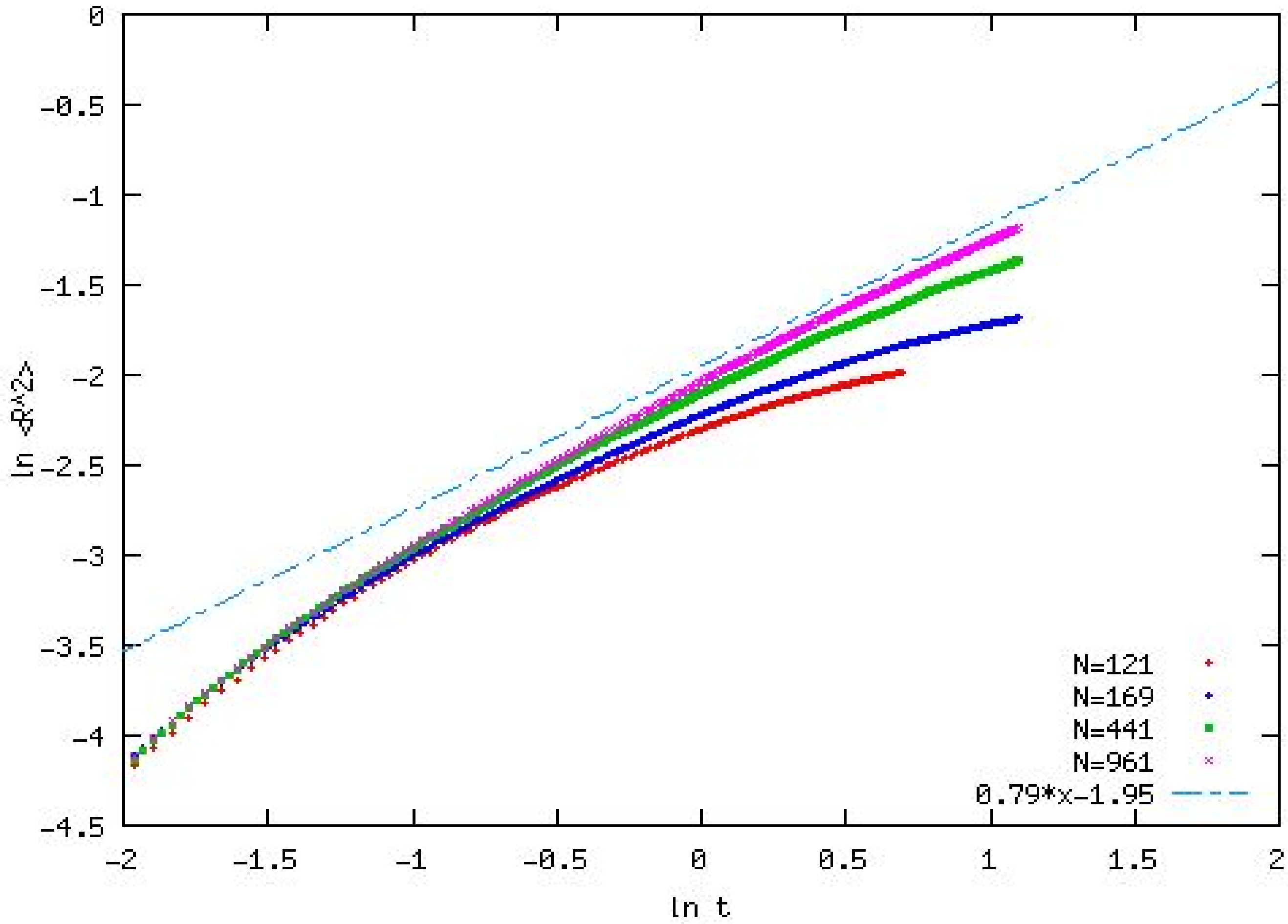
There is another possibility, however, which occurs in two dimensions with a magnetic field (the quantum Hall universality class).

Assume the magnetic field is strong and concentrate on the lowest Landau level.

Start a maximally localized wave packet at t=0. The mean squared displacement increases with a power law slower than diffusion:

$$\overline{\langle R^2(t) \rangle} \sim t^{1-\frac{1}{2\nu}}$$

Numerics for mean squared displacement in LLL



Localization and the Liouvillian, part II

Previously we claimed that the mean squared displacement of a wave packet in the LLL satisfied

$$\overline{\left\langle R^2(t) \right\rangle} \sim t^{1-\frac{1}{2\nu}}$$

The quantity ν appearing in the exponent is the same as appears in the conventional localization length scaling law

$$\xi(E) \sim \frac{1}{(E-E_c)^{\nu}}$$

(The equality of exponents can be understood by assuming that the starting wave packet projects equally onto states of all energies in the LLL, and that propagation at each energy is diffusive until the localization length is reached.)

Hence ν can be obtained

- using only the density, not the wavefunction;
- without studying states at specific energy, only integrated over energy.

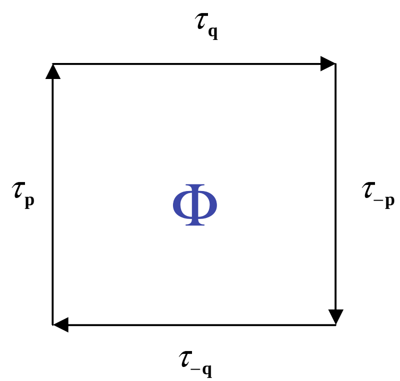
The Liouvillian formalism (Sinova, Meden, Girvin 1999) is a means to calculate disorder-averaged density correlations and hence obtain ν and possibly other critical properties.

Dynamics in the lowest Landau level

- Nonlocality: The most localized state in the LLL has spread of order the magnetic length

$$l = \sqrt{eB/hc}$$

- Moving an electron around a closed path generates an Aharonov-Bohm phase equal to the magnetic flux through the path.



- The Fourier components of the LLL-projected electron density are proportional to the “magnetic translation operators” (Girvin and Jach, 1984):

$$\bar{\rho}_{\mathbf{q}} = \overline{e^{i\mathbf{q}\cdot\mathbf{r}}} = e^{-q^2 l^2/4}\tau_{\mathbf{q}}$$

$$[\tau_{\mathbf{p}}, \tau_{\mathbf{q}}] = 2i\sin(l^2(p_x q_y - q_x p_y)/2)$$

- Main conclusion: the equations of motion for the electron density are **closed**: knowing the electron density ρ(x) at one time determines it for all times, for the noninteracting Hamiltonian

$$\frac{d}{dt}\bar{\rho}_{\mathbf{q}} = L_{\mathbf{q}\mathbf{q}'}\bar{\rho}_{\mathbf{q}'}$$

$$H = \sum_{\mathbf{q}} V_{-\mathbf{q}}\bar{\rho}_{\mathbf{q}}$$

$$\mathcal{L}_{\mathbf{q}\mathbf{q}'} = \frac{2i}{\hbar} v(\mathbf{q}-\mathbf{q}')\sin\left(\frac{\ell^2 \mathbf{q}\wedge\mathbf{q}'}{2}\right) e^{-\frac{\ell^2}{4}|\mathbf{q}-\mathbf{q}'|^2}.$$

Effective theory in noncommutative space

After some algebra, one can obtain a simple field theory which if solved would contain the exact critical exponents:
in terms of Fourier components, the propagator that contains the anomalous dimension we would like to calculate is

$$\hat{\Pi}(q,\omega) = -i \int D\bar{\phi}\, D\phi \int D\bar{\psi}\, D\psi\, \bar{\phi}_q \phi_q e^{-F(\omega)}$$

$$F(\omega) = -i\omega \int d\mathbf{q}\, (\bar{\phi}_{\mathbf{q}}\phi_{\mathbf{q}} + \bar{\psi}_{\mathbf{q}}\psi_{\mathbf{q}}) + \int_{1,2,3,4} f(1,2,3,4) \left[\bar{\phi}_{\mathbf{q}_1}\bar{\phi}_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\phi_{\mathbf{q}_4} + 2\bar{\psi}_{\mathbf{q}_1}\bar{\phi}_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\psi_{\mathbf{q}_4} + \bar{\psi}_{\mathbf{q}_1}\bar{\psi}_{\mathbf{q}_2}\psi_{\mathbf{q}_3}\psi_{\mathbf{q}_4}\right].$$

$$f(1,2,3,4) = \frac{1}{\pi} e^{-\frac{1}{2}|\mathbf{q}_1 - \mathbf{q}_4|^2} \delta(\mathbf{q_1} + \mathbf{q}_2 - \mathbf{q}_3 - \mathbf{q}_4) \times \sin\left(\frac{1}{2}\mathbf{q}_1 \wedge \mathbf{q}_4\right) \sin\left(\frac{1}{2}\mathbf{q}_2 \wedge \mathbf{q}_3\right).$$

In **real space** this sine (the “star product” in a noncommutative space) corresponds to higher and higher derivatives, so the theory is nonlocal.
Linearizing the sine gives a field-theory representation of **classical percolation**.
Standard practice is to integrate out the V fields to give an interacting clean problem with normal and Grassmann fields (the ``supersymmetry trick’’).

Hence the quantum Hall universality class is the generalization of classical critical percolation to a 2D noncommutative space.

Outline of remainder

So far everything looks quite promising: we have obtained a reasonably compact noncommutative field theory with normal and Grassmann variables that describes the plateau transition. The validity of this description can be verified by numerics.

Already this is something of an accomplishment: the intuitive notion (Trugman et al) that quantum tunneling modifies classical percolation can be put on solid theoretical footing.

It seems that the final goal is in sight: a controlled expansion of the anomalous dimension of the plateau transition should just be found by standard expansion techniques (e.g., 1/N) applied to this theory.

However, this theory for the density operators has an unusual class of symmetries that seem to frustrate standard approaches.

Extra conservation laws in the LLL

The ordinary Schrodinger equation is unitary and hence conserves the norm of the wavefunction (the total probability to find an electron somewhere in the system).

After projection to the LLL, there is also a **closed unitary** evolution of the density operators:

$$\frac{d}{dt}\tau_{\mathbf{q}} = \tilde{L}_{\mathbf{qq}'}\tau_{\mathbf{q}'}, \qquad \tilde{L}_{\mathbf{qq}'} = -(\tilde{L}_{\mathbf{q}'\mathbf{q}})^*$$

$$\Rightarrow \frac{d}{dt}\sum_{\mathbf{q}}\left\langle \tau_{\mathbf{q}}\right\rangle\left\langle \tau_{-\mathbf{q}}\right\rangle = \frac{dC}{dt} = 0$$

$$C \equiv \sum_{\mathbf{q}}\left\langle \tau_{\mathbf{q}}\right\rangle\left\langle \tau_{-\mathbf{q}}\right\rangle = \sum_{\mathbf{q}} e^{q^2 l^2/2}\left\langle \bar{\rho}_{\mathbf{q}}\right\rangle\left\langle \bar{\rho}_{-\mathbf{q}}\right\rangle$$

This is only the first of a series of many conservation laws. Think of the terms in this sum as paths from 0 to q and back again. With a proper phase and numerical factor, one can construct ternary, quaternary, etc. conservation laws (involving 6, 8, … electron operators).

Hence, in addition to particle number and energy, there is an **additional conservation law** in the LLL, independent of the realization of disorder.

The conserved quantity measures the variation in the density, and is minimized by the uniform density state.

Summary of plateau transition results

1. The picture of the quantum Hall plateau transition as a noncommutative generalization of classical percolation can be made precise in a "Liouvillian" theory based on the density operators.

2. The analysis of the resulting theory is made more difficult by the existence of an infinite (but incomplete) number of conservation laws. These are consistent with the notion that for N states in the LLL, there are N^2 magnetic translation operators.

3. In simpler models, the existence of an infinite but incomplete symmetry is known to have dramatic consequences: an example is the reduction of dimensionality in certain lattice Ising-like models from d to d-n, when there is an n-dimensional infinite set of conservation laws.

References: J. Sinova, V. Meden, and S. M. Girvin, PRB 2000; JEM, J. Sinova, and A. Zee, PRL 2001; JEM, Nucl. Phys. B 2004

What about higher dimensions? (a future direction)

1. The 4D QHE realizes a **different universality class** of mesoscopic physics (symplectic rather than unitary), which should have physical consequences. Also the ``natural'' objects for a localization theory may be membranes.

2. Physics **on a conventional quantum Hall plateau** is rather robust to disorder, because the chiral nature of edge states (boundary fermions) makes them resistant to backscattering. The **spin Hall effect**, which is a 3D cousin of the 4D QHE, is protected from disorder in a more subtle way;

3. Experiments can distinguish between **extrinsic** and **intrinsic** contributions to the spin QHE (tomorrow morning). In the ordinary QH case, the extrinsic contribution is 0; it can be nonzero in the SQHE (Dya'konov-Perel'). However, at least one recent experiment is interpreted as dominated by the **intrinsic** part.

Importance of dimensionality in QPT's

Jack Sowards

"He is intelligent, but not experienced. His pattern indicates two-dimensional thinking."

ICTP, 2 March 2005