The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR 1646 - 14

Conference on
Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and
Non-Commutative Geometry in Condensed Matter Physics and Field Theory
1 - 4 March 2005

Duality in N=2 supersymetric Yang-Mills and the quantum Hall effect

Brian DOLAN
Department of Mathematical Physics,
National University of Ireland, Maynooth, Ireland

These are preliminary lecture notes, intended only for distribution to participants.

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Duality in $N = 2$ SUSY Yang-Mills and the Quantum Hall Effect

Brian Dolan

Department of Mathematical Physics

National University of Ireland

Maynooth

e-mail:bdolan@thphys.may.ie

ICTP, 3rd March 2005

- SUSY Yang-Mills and the running of θ
  - Duality and Modular Symmetry of $N = 2$ SUSY Yang-Mills
  - Callan-Symanzik β-functions and modular forms
- The Quantum Hall Effect (QHE)
  - Law of Corresponding States and Modular Symmetry in the QHE
  - Scaling and Crossover
  - Selection Rule, Semi-circle Law
- Hierarchies in 2-d bosonic systems

$N = 2$ SUSY QCD: $SU(2)$

- Field content (adjoint rep.): $A_\mu,\ \psi_1,\ \psi_2$ (Weyl), ϕ (complex)
- Action:

$$\mathbf{S} = \int \mathbf{dx^4} \left\{ \left(-\frac{1}{4g^2}\mathbf{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) + \frac{\theta}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathbf{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}) \right) + \frac{1}{g^2}\mathbf{tr}\left((\mathbf{D}_\mu\phi)^\dagger \mathbf{D}^\mu\phi - \frac{1}{2}[\phi^\dagger,\phi]^2 \right) + \cdots \right\}$$

- Only two independent couplings: g and θ.
- Degenerate vacua parameterised by $< \phi >$, (or $u = \frac{1}{2}tr < \phi^2 >$). $SU(2)$ broken to $U(1)$, $< \phi >$ gives gauge fields a mass.

Duality and the Modular Group

- **Duality:** $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$ is a symmetry of the vacuum Maxwell equations.
- This is not a symmetry when charges are included unless magnetic monopoles are introduced, $g \rightarrow g_D = 4\pi/g$, (Dirac).
- For QCD, when the vacuum parameter θ is included, this generalises to $\tau \rightarrow -1/\tau$ $\quad \tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ (Im$\tau > 0$).
- The **Modular Group**, $\Gamma(1)$, is the infinite discrete group of transformations, $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$, with a, b, c and d integral and $ad - bc = 1$. This generalises $\tau \rightarrow -1/\tau$, (Witten).

- Elements of $\Gamma(1)$ preserve Im$\tau > 0$ and can be represented by 2×2 matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c and d integral and det$(\gamma) = 1$.
  The modular group $\Gamma(1)$ is the double cover of $Sl(2, \mathbf{Z})$.
- $\Gamma(1)$ is generated by

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\tau \to \tau + 1} \qquad \boxed{\tau \to -1/\tau}.$$

- The dual theory has $\boxed{\tau_D = -1/\tau}$.

Modular Symmetry of $N = 2$ SUSY Yang-Mills

Low energy SUSY effective action **is** symmetric under $\tau \to \frac{a\tau+b}{c\tau+d}$ with a, b, c and d integers, $ad - bc = 1$ and both b and c even.

(Seiberg+Witten).

This is a sub-group $\Gamma(2) \subset \Gamma(1)$ generated by

$$\gamma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \boxed{\tau \to \tau + 2} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \boxed{\tau \to \frac{\tau}{2\tau+1}}.$$

- Weak coupling (ultra-violet, $u \to \infty$, $\tau \to i\infty$) gluons and squarks are relevant degrees of freedom.
- Strong coupling (infra-red, $u \to 1$, $\tau \to 0$, $\tau_D \to i\infty$) gluons and monopoles are relevant degrees of freedom.

Callan-Symanzik β-functions

- $u = (1/2)tr < \phi^2 >$ is a mass^2. Given $\tau(u)$ define $\beta = -(u-1)\frac{d\tau}{d(u-1)} \approx -u\frac{d\tau}{du}$ for large u.

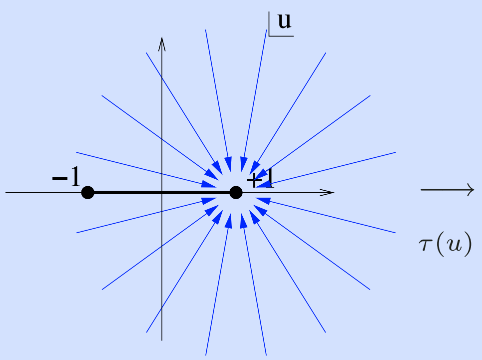
$$\beta(\tau) = -\frac{\mathbf{i}}{\pi}\frac{\mathbf{1}}{\vartheta_{\mathbf{3}}(\tau)^{\mathbf{4}}}$$

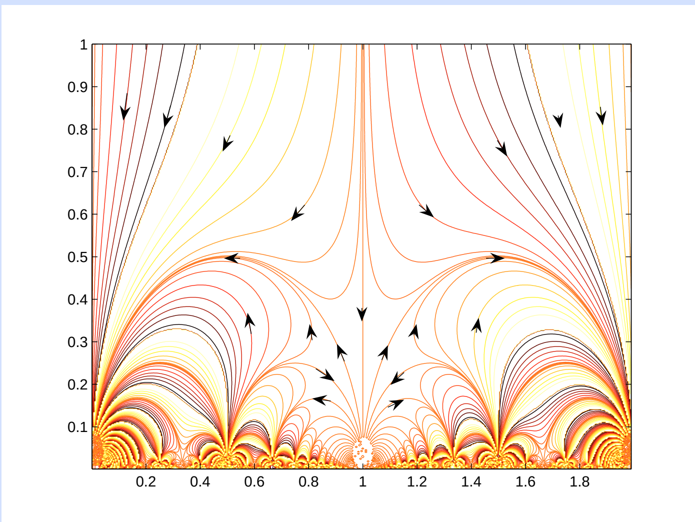
with $\vartheta_3(\tau) := \sum_{n=0}^{\infty} e^{i\pi n^2 \tau}$ (Jacobi ϑ-function).

- $\beta(\tau)$ is a Modular Form (of weight -2):

$$\beta(\gamma(\tau)) = \left(\frac{\mathbf{1}}{\mathbf{c}\tau+\mathbf{d}}\right)^{\mathbf{2}} \beta(\tau)$$

$$\beta(\tau) = -(\mathbf{u}-\mathbf{1})\frac{\mathbf{d}\tau}{\mathbf{d(u-1)}}$$

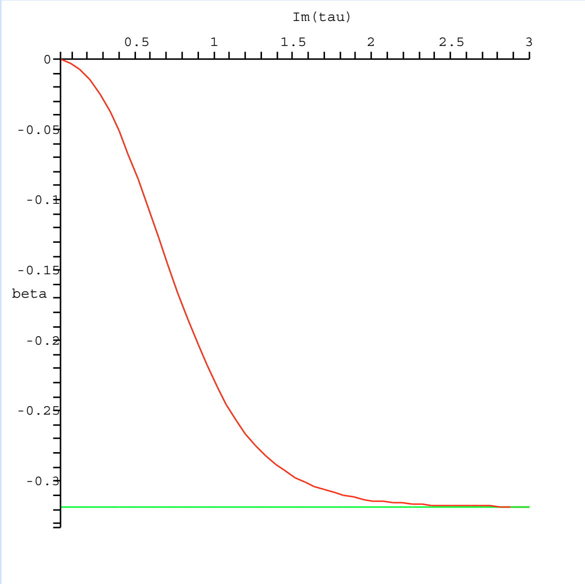




$$\beta(\tau) = -\frac{\mathbf{i}}{\pi}\frac{\mathbf{1}}{\theta_3(\tau)^4}$$

$$\beta(\tau_D) = \frac{\mathbf{i}}{\pi}\frac{\mathbf{1}}{\theta_3(\tau_D)^4}$$

$\theta = 0$



$$\frac{\mathbf{dg}}{\mathbf{ds}} \approx \frac{\mathbf{g^3}}{\mathbf{8\pi^2}}$$

$$\frac{\mathbf{dg_D}}{\mathbf{ds}} \approx -\frac{\mathbf{g_D^3}}{\mathbf{8\pi^2}}$$

Schwinger-Zwanziger Quantisation Condition

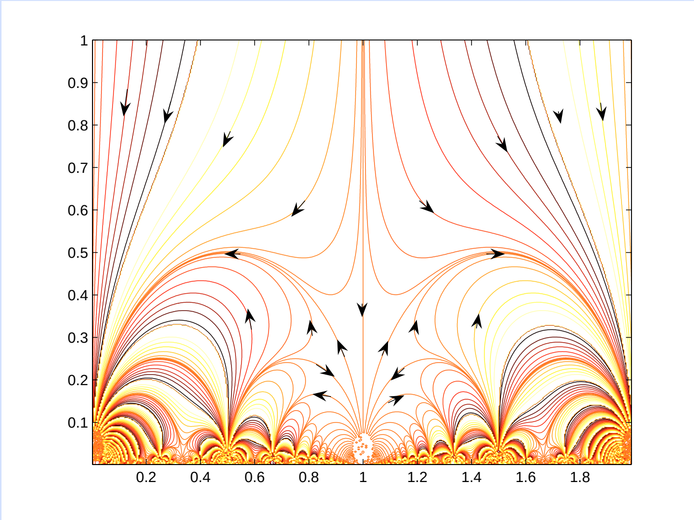
Dyons with electric charges Q_1 and Q_2 and magnetic charges G_1 and G_2 satisfy

$$\mathbf{Q_1G_2 - Q_2G_1 = 4\pi n} \quad n \in \mathbf{Z}$$

Let $Q_1 = q_1 g$, $Q_2 = q_2 g$, $M_1 = p_1\frac{4\pi}{g}$ and $M_2 = p_2\frac{4\pi}{g}$ with $p_1, p_2, q_1, q_2 \in \mathbf{Z}$. Then

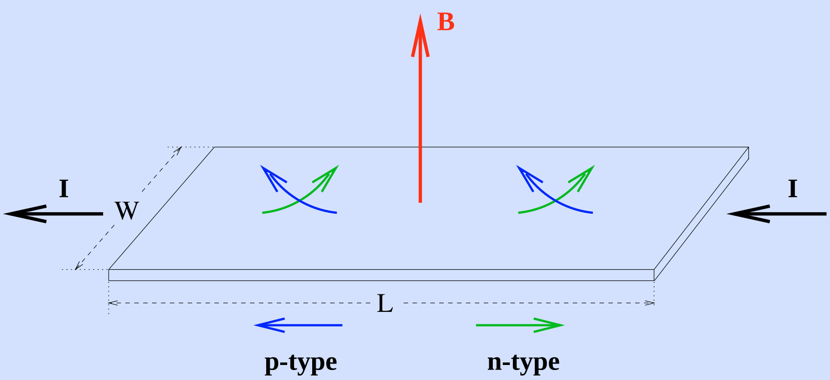
$$\mathbf{p_1q_2 - p_2q_1 = n} \quad n \in \mathbf{Z}$$

- At $u \to \infty$ ($\tau \to i\infty$) squarks have $q = 1, p = 0$
- At $u \to 1$ ($\tau \to 0$) monopoles have $q = 0, p = 1$
- At $u \to -1$ ($\tau \to 1$) dyons have $q = 1, p = 1$.



- On the real axis, there are fixed points at:
  $\tau = 0$ ($q = 0, p = 1$), $\tau = 1$ ($q = 1, p = 1$), $\tau = 2$ ($q = 2, p = 1$), etc.
- All fixed points can be obtained by acting with an element of $\Gamma(2)$ either on $\tau = 0$, $\tau = 1$ or $\tau = i\infty$.
- There are fixed points for strong coupling at all rational values of $\theta/2\pi = q/p$; p odd for infra-red fixed-points and p even for ultra-violet fixed points.

The Quantum Hall Effect

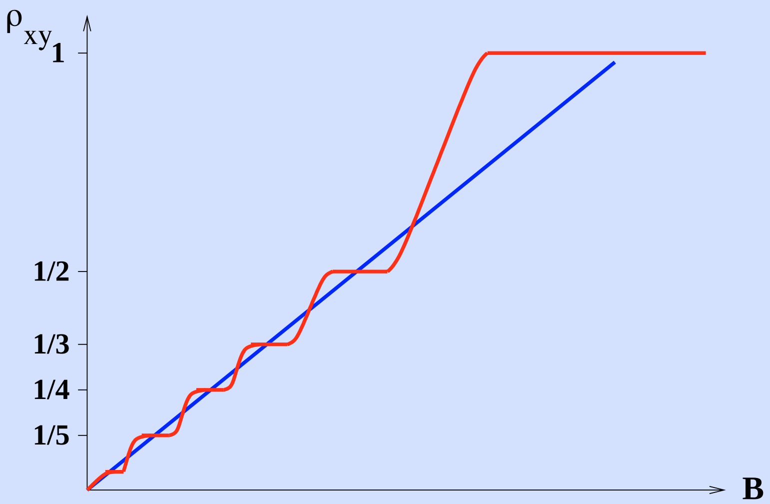


- Longitudinal (Ohmic) voltage V_{xx}, transverse (Hall) voltage V_{xy}.
  Ohmic resistance $R_{xx} = V_{xx}/I$, Hall resistance $R_{xy} = V_{xy}/I$.
  Ohmic resistivity $\rho_{xx} = (W/L)R_{xx}$, Hall resistivity $\rho_{xy} = R_{xy}$.

- Classically $\rho_{xy} = -\frac{B}{en}$.

  ($n =$ charge carrier density)

Duality in Yang-Mills and the Quantum Hall Effect 12



- Classically $\rho_{xy} = -\frac{B}{en}$.

  ($n =$ charge carrier density)

- For low T and high purity, resistance is quantised,

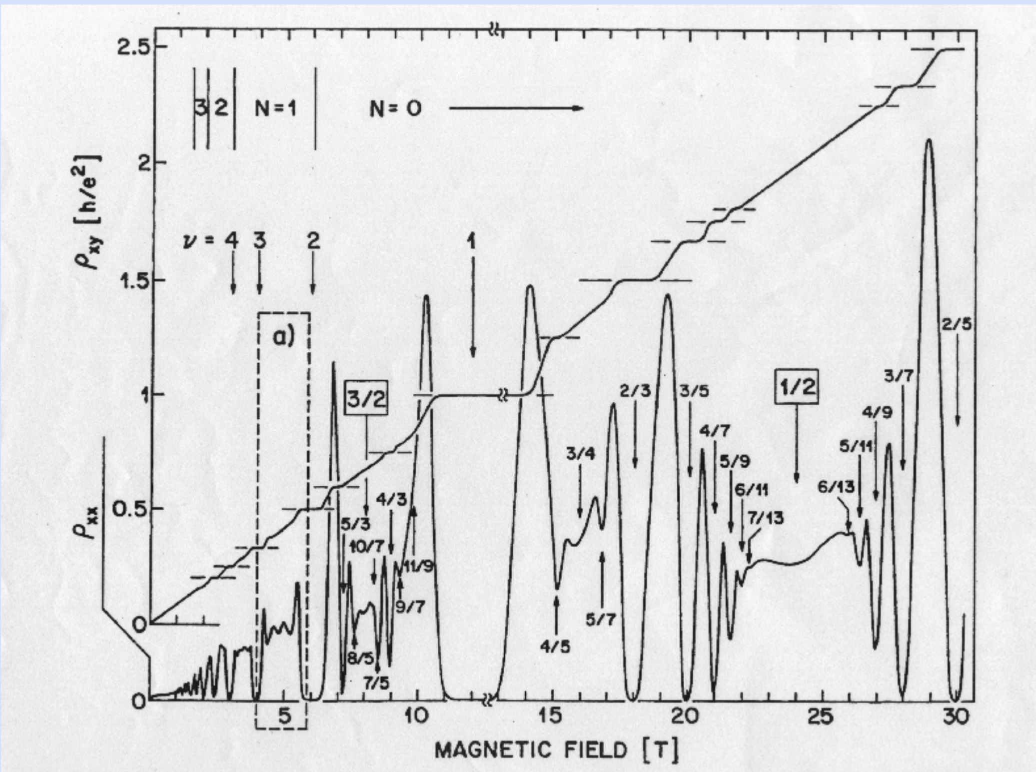
  $R_H = h/e^2 = 25.812807449(86)\ k\Omega$ (von Klitzing (1980)).

  ($\nu := nh/eB$, filling factor)

Brian Dolan – NUIM ICTP, 3rd March 2005

- Conductivity tensor $\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$ is the inverse of

  the resistivity tensor, $\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$ (assuming isotropy $\rho_{xx} = \rho_{yy}$).
- Using complex coordinates, $\mathbf{z} = \mathbf{x} + \mathbf{iy}$: $\rho = \rho_{\mathbf{xy}} + \mathbf{i}\rho_{\mathbf{xx}}$ and $\sigma = \sigma_{\mathbf{xy}} + \mathbf{i}\sigma_{\mathbf{xx}}$, $\sigma = -\rho^{\mathbf{-1}}$.
- Quantum Hall states have: $|\sigma_{xy}| = p/q$ with q **odd**, $\sigma_{xx} = 0$.
  (units with $e^2/h = 1$)
- Im$\sigma > 0$ (stability).



Tsui (1990)

Law of Corresponding States

- Physical properties of Quantum Hall Sates

  are related when: $\sigma_{\mathbf{xy}} \rightarrow \sigma_{\mathbf{xy}} + \mathbf{1}$, **Landau Level Addition**

  $\frac{\mathbf{1}}{\sigma_{\mathbf{xy}}} \rightarrow \frac{\mathbf{1}}{\sigma_{\mathbf{xy}}} + \mathbf{2}$, **Flux Attachment**

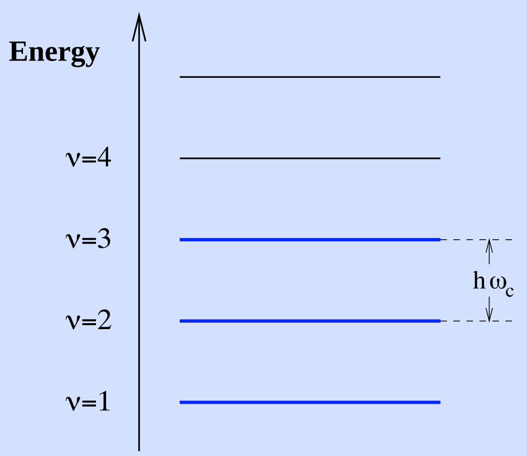
  $\sigma_{\mathbf{xy}} \rightarrow \mathbf{1} - \sigma_{\mathbf{xy}}$, **Particle-Hole Interchange**

- More generally ($\sigma := \sigma_{xy} + i\sigma_{xx}$)

$$\left.\begin{array}{c} \sigma \rightarrow \sigma + \mathbf{1} \\ -\frac{\mathbf{1}}{\sigma} \rightarrow -\frac{\mathbf{1}}{\sigma} + \mathbf{2} \end{array}\right\} \boldsymbol{\Gamma}_{\mathbf{0}}(\mathbf{2}) \subset \boldsymbol{\Gamma}(\mathbf{1})$$

$$\sigma \rightarrow \mathbf{1} - \bar{\sigma}$$ (Outer Automorphism)

(Kivelson, Lee + Zhang (1992), Lütken + Ross (1992), BD + Burgess (2001), Witten (2003)).



Free particles in transverse $\mathbf{B}$,

Schrödinger Equation $\Leftrightarrow$ Harmonic Oscillator.

Energy levels (Landau Levels) equally spaced, degeneracy/unit area $g = |B/e|$.

Filling factor, $\nu := n/g = |1/\rho_{xy}| = |\sigma_{xy}|$ (when $\sigma_{xx} = 0$, $e^2/h = 1$).

$\nu =$ integer $\Rightarrow$ **Energy Gap**, $\Delta E = \hbar\omega_c$.

Expect $\nu \rightarrow \nu + 1$ is a **symmetry**: $\sigma_{\mathbf{xy}} \rightarrow \sigma_{\mathbf{xy}} + \mathbf{1}$.

Maxwell – Chern-Simons Theory

The classical relation

$$B = -en\rho_{xy} \Rightarrow \sigma_{xy}B = J^0$$

($J^0 = en$ and $\sigma_{xx} = 0$)

can be derived from

$$\mathcal{L}_{eff}[A_0] = -\sigma_{xy}A_0B + A_0J^0 \to \mathcal{L}_{eff}[A] = -\frac{\sigma_{xy}}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + A_\mu J^\mu.$$

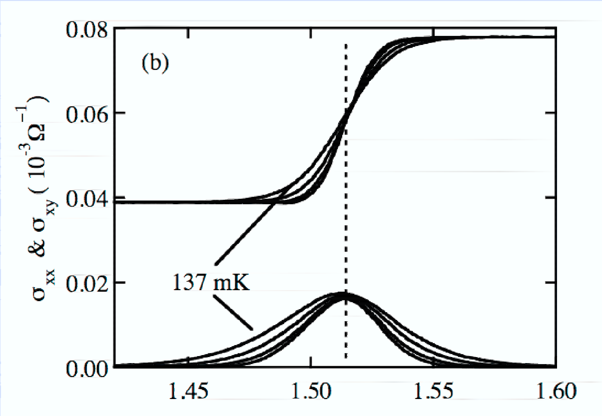
Including Ohmic conductivity, $\sigma_{xx} = i\lim_{\omega\to 0}(\omega\epsilon(\omega))$

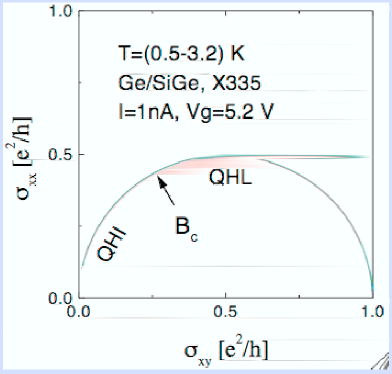
$$\mathcal{L}_{eff}[A] = -\frac{\epsilon}{4}F^2 - \frac{\sigma_{xy}}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + A_\mu J^\mu.$$

$$\boxed{\mathcal{L}_{\mathbf{eff}}[\mathbf{A}] \approx \frac{\mathbf{i}\sigma_{\mathbf{xx}}}{\mathbf{4}\omega}\mathbf{F^2} - \frac{\sigma_{\mathbf{xy}}}{\mathbf{4}}\epsilon^{\mu\nu\rho}\mathbf{A}_\mu\mathbf{F}_{\nu\rho} + \mathbf{A}_\mu\mathbf{J}^\mu.}$$

Scaling and Crossover

- Hall Plateaux $\Leftrightarrow$ Different Phases of 2-D Electron Gas
- ν: $\mathbf{q_1}/\mathbf{p_1} \to \mathbf{q_2}/\mathbf{p_2}$ is a Quantum Phase Transition, Fisher (1990) $\xi \approx |\Delta B|^{-\nu_\xi}$, $\Delta B = B - B_c$
- Simple scaling $\Rightarrow \sigma(\mathbf{T}, \mathbf{\Delta B}, \mathbf{n_i}) = \sigma(\mathbf{\Delta B}/\mathbf{T}^\mu, \mathbf{n_i}/\mathbf{T}^{\mu'})$ (n_i is impurity density).
- At low temperatures $\sigma(\mathbf{\Delta B}/\mathbf{T}^\mu, \mathbf{n_i}/\mathbf{T}^{\mu'}) \to$ $\boxed{\sigma(\mathbf{\Delta B}/\mathbf{T}^\mu)}$ ($\mu =$ scaling dimension). Pruisken (1988)
- Experimentally μ is the same for every transition: super-universality. Wei et al. (1988)



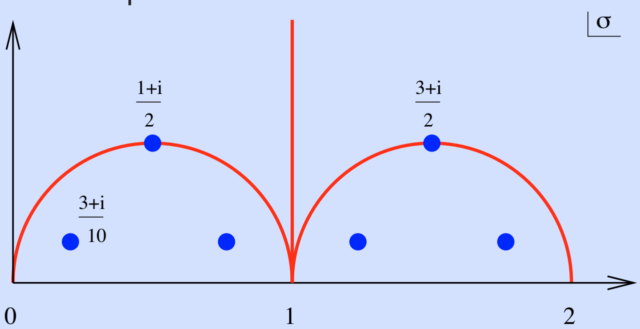


Shahar et al. (1997)
cond-mat/9611011

Hilke et al. (1999)
cond-mat/9810217

Quantum Hall Selection Rule

- Action of $\Gamma_0(2)$ commutes with the crossover flow $\Rightarrow$ fixed points of $\Gamma_0(2)$ are fixed points of the crossover.



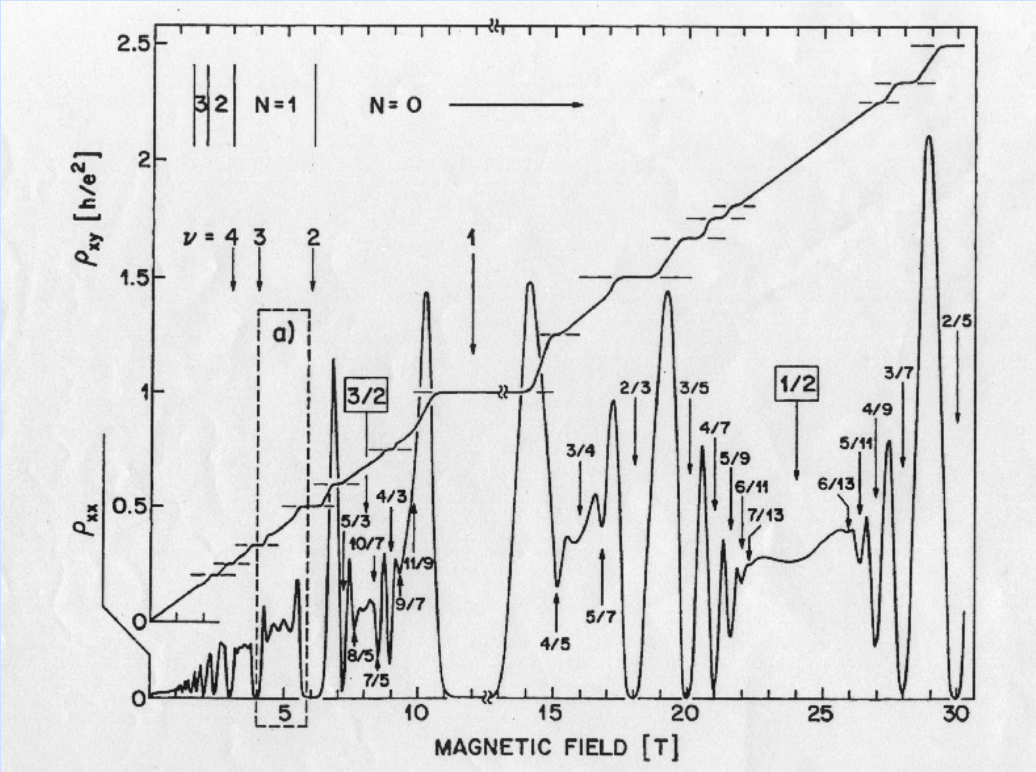
$\nu: 1 \to 2$ has $\sigma_c = \frac{3+i}{2}$

- Any $\nu: q_1/p_1 \to q_2/p_2$ can be obtained from $\nu: 0 \to 1$ by some $\gamma \in \Gamma_0(2)$.

$\gamma(0) = q_1/p_1, \gamma(1) = q_2/p_2 \Rightarrow \gamma = \begin{pmatrix} q_2 - q_1 & q_1 \\ p_2 - p_1 & p_1 \end{pmatrix}$ and

$\det\gamma = 1 \Rightarrow$

Selection Rule: $\left|\mathbf{q_2 p_1 - q_1 p_2}\right| = \mathbf{1}$ (BD (1998)).



Tsui (1990)

Holomorphic β-functions

- Let l_T =scattering length, an let $s(l_T)$ be monotonic in l_T (and T) and assume, in analogy with $N = 2$ SUSY Yang-Mills, that

$$\frac{\mathbf{d}\sigma}{\mathbf{ds}} = \beta(\sigma)$$

is holomorphic. ($\beta(\sigma, \overline{\sigma})$: C. Burgess and A. Lütken (1998), BD (1998), Taniguchi (1998))

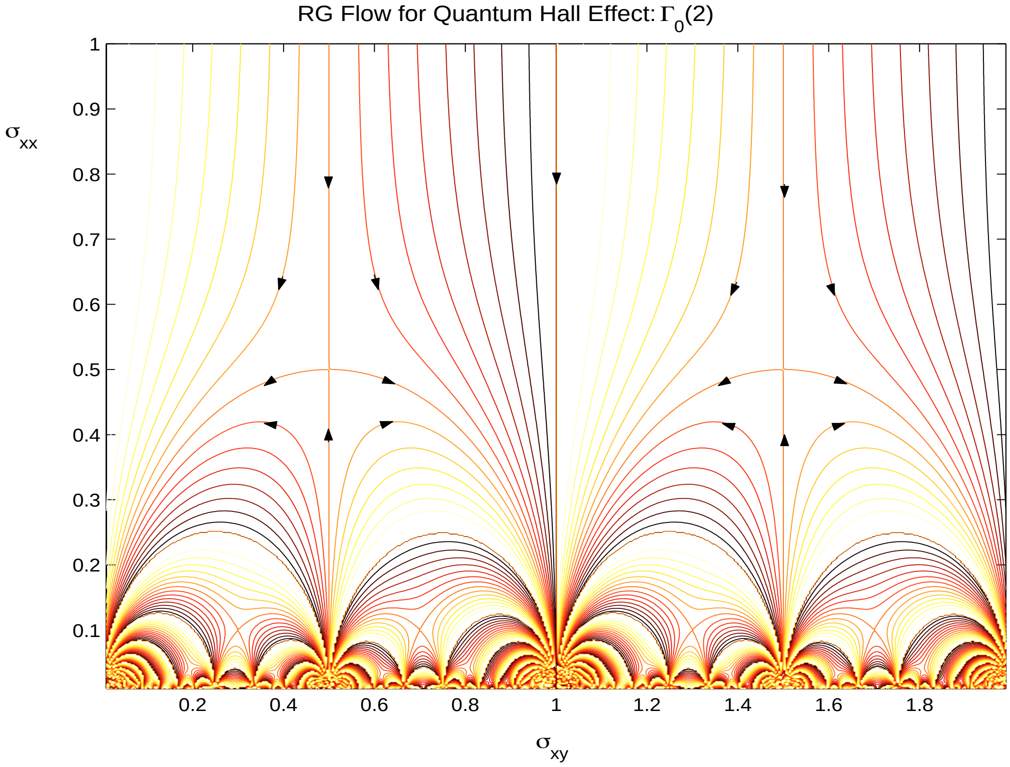
$\Rightarrow$ $\beta(\sigma)$ is a modular form, (of weight -2),

$$\beta(\gamma(\sigma)) = \frac{\mathbf{1}}{(\mathbf{c}\sigma+\mathbf{d})^{\mathbf{2}}}\beta(\sigma).$$

- Further assume: i) as $\sigma_{xx} \to \infty$ β is finite; $\beta_{xy} \to 0$ and $\beta_{xx} < 0$ ii) $\beta \to 0$ as fast as possible at the plateaux (attractive fixed points) and iii) there are no fixed points other than those of $\Gamma_0(2)$. Then

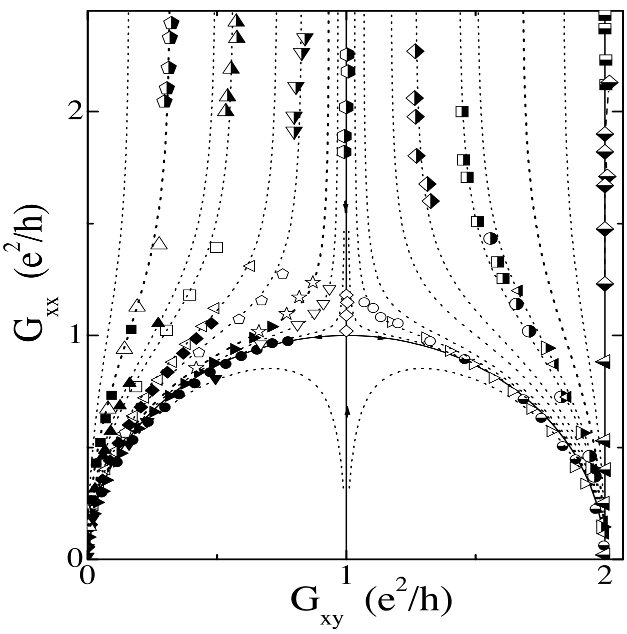
$$\beta(\sigma) = -\frac{\mathbf{i}}{\pi}\frac{\mathbf{1}}{\vartheta_{\mathbf{3}}^{\mathbf{4}}+\vartheta_{\mathbf{4}}^{\mathbf{4}}}.$$

$\vartheta_3(\sigma) := \sum_{n=0}^{\infty} e^{i\pi n^2\sigma}$, $\vartheta_4(\sigma) := \sum_{n=0}^{\infty}(-1)^n e^{i\pi n^2\sigma}$



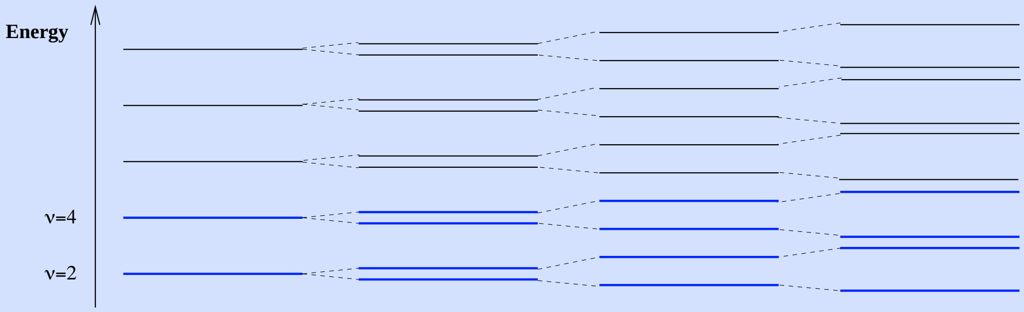
Attractive fixed points at $\sigma_{xy} = q/p$, p odd; repulsive points for p even.

In the composite boson picture p is the number of vortices.



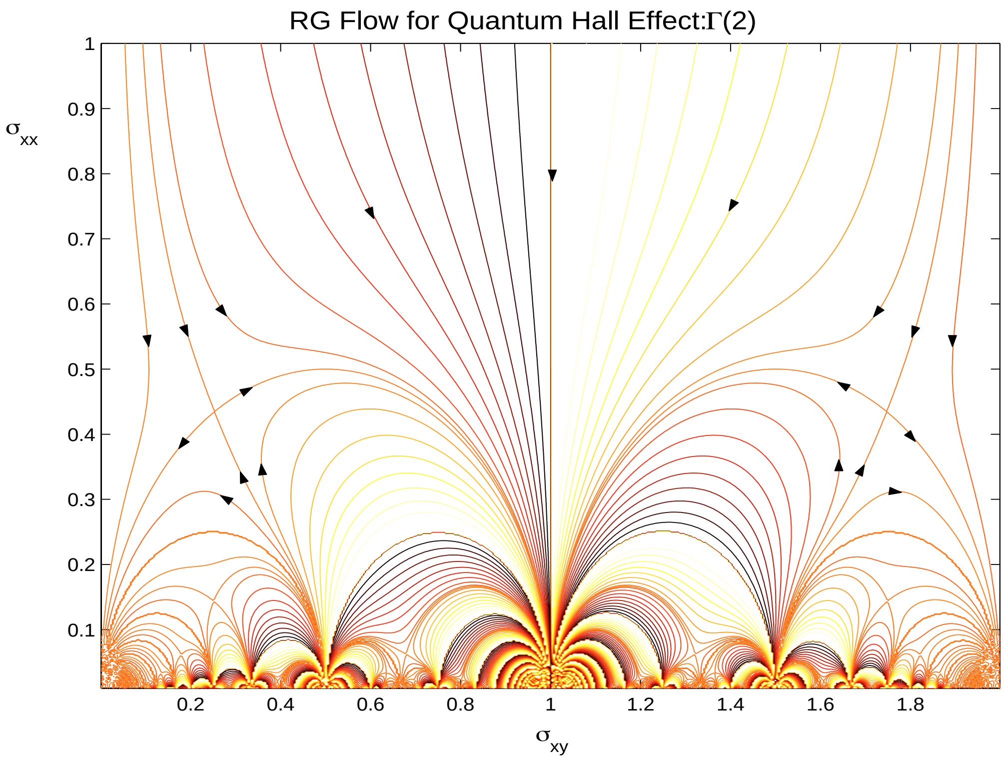
S.S. Murzin et al., cond-mat/0204206

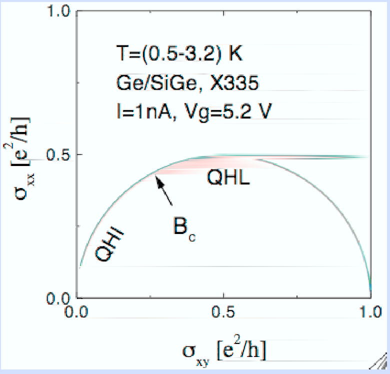
Including spin:

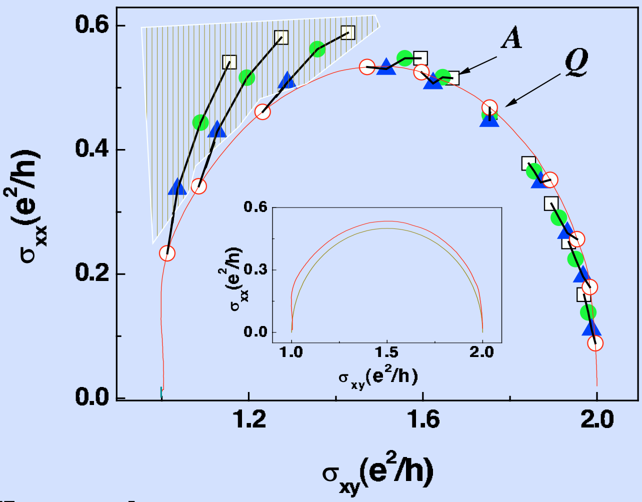


Spins poorly split	Spins well split	Spins poorly split
$\nu \to \nu + 2$	$\nu \to \nu + 1$	$\nu \to \nu + 2$
$\sigma_{xy} \to \sigma_{xy} + 2$	$\sigma_{xy} \to \sigma_{xy} + 1$	$\sigma_{xy} \to \sigma_{xy} + 2$
$\mathbf{\Gamma(2)}$	$\mathbf{\Gamma_0(2)}$	$\mathbf{\Gamma(2)}$

Georgelin et al. (1997); BD (2000)







C. F. Huang et al.

Infinite Hierarchies in Bosonic Systems

- When particles are bosonic we get a **different** group:

$$\left.\begin{array}{c} \sigma \to -\frac{1}{\sigma} \\ -\frac{1}{\sigma} \to -\frac{1}{\sigma} + 2 \end{array}\right\} \Gamma_\theta$$

$$\sigma \to 1 - \bar{\sigma}$$

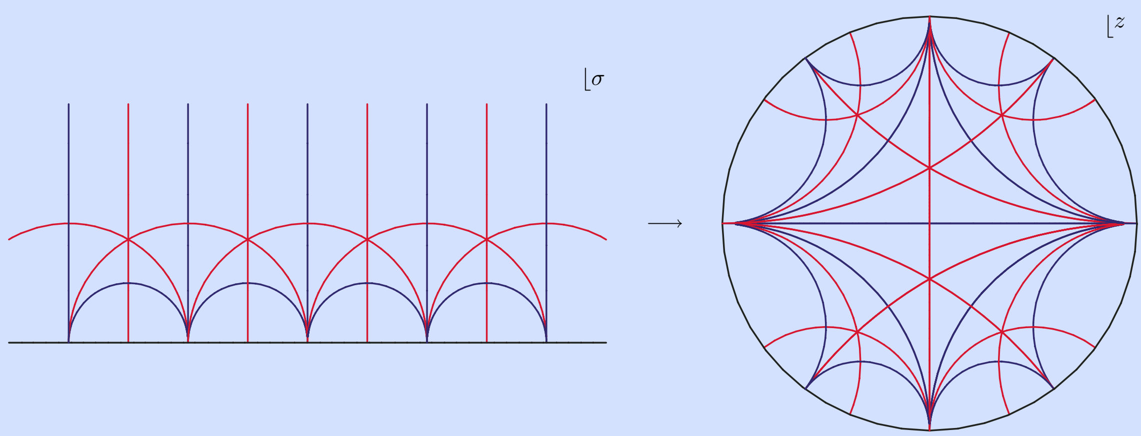
  Characterised by either: a and d both odd, c and d both even; or a and d both even, c and d both odd.

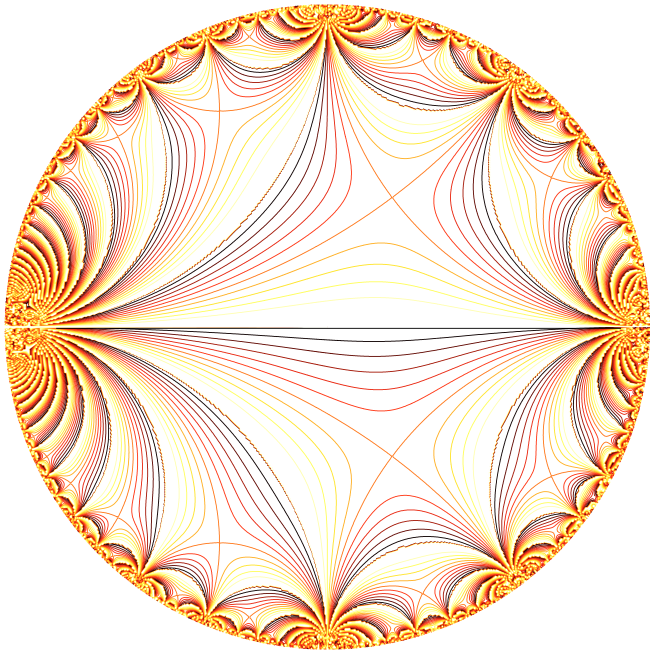
- – Fixed points at $\sigma = i$ $(\sigma_{xx} = 1, \sigma_{xy} = 0)$ and its images under Γ_θ.
  – Attractive fixed points at $\sigma = q/p$ $(\sigma_{xx} = 0)$ with pq even and repulsive fixed points when pq is odd. In particular even integers are stable and odd integers are unstable.
  – Realisable in 2-d bosonic systems: e.g. high mobility thin film superconductors; Josephson junction arrays

Summary

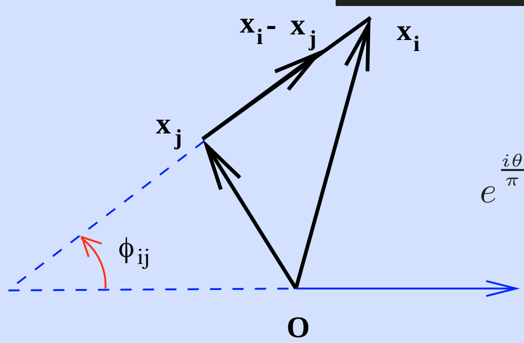
- $N = 2$ SUSY Yang-Mills is a 4-dimensional version on the QHE.
- Duality in $N = 2$ SUSY Yang-Mills manifests itself as $\mathbf{\Gamma(2)}$.
  Infinite hierarchy of vacua with monopoles and dyons, $\theta = \mathbf{q}/\mathbf{p}$.
  Callan-Symanzik β-functions are **modular forms** of weight -2.
- Duality in the quantum Hall effect also manifests itself as $\mathbf{\Gamma(2)}$ or $\mathbf{\Gamma_0(2)}$.
  Infinite hierarchy of vacua for different quantum Hall plateaux, $\sigma_{\mathbf{xy}} = \mathbf{q}/\mathbf{p}$.
  Selection rule $|\mathbf{q_1 p_2} - \mathbf{q_2 p_1}| = \mathbf{1}$.
  Correct topology for crossover. Particle-hole symmetry predicts semi-circle rule.
- When pseudo-particles are bosonic we get a **different** group, $\mathbf{\Gamma_0(2)} \rightarrow \mathbf{\Gamma}_\theta$.
  Expect a similar infinite hierarchy in high mobility, 2-D superconductors.

The symmetries of the modular group are beautifully exhibited by transforming to $z = \frac{1+i\sigma}{1-i\sigma}$, (Poincaré map):





Statistical Gauge Field



$$\widetilde{\Psi}(\mathbf{x}_1,\dots,\mathbf{x}_N) = e^{\frac{i\theta}{\pi}(\sum_{i<j}(\Sigma_{i<j}\phi_{ij})}\Psi(\mathbf{x}_1,\dots,\mathbf{x}_N)$$

Interchange $i \leftrightarrow j$, $\phi_{ij} \longrightarrow \phi_{ij} + \pi \Rightarrow$ phase changes by

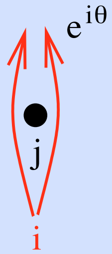
θ, $\begin{cases} \theta = 2\pi k & \text{Identity} \\ \theta = \pi(2k+1) & \text{Fermions} \leftrightarrow \text{Bosons} \end{cases}$

Well defined provided particles never overlap (repulsive core).

In Hamiltonian $\boxed{-i\hbar\nabla - e\mathbf{A} \rightarrow -i\hbar\nabla - e\mathbf{A} - e\mathbf{a}}$ where $(\alpha = 1, 2)$

$$a_\alpha(\mathbf{x}_i) = \frac{\hbar\theta}{e\pi}\sum_{j;j\neq i}\nabla^{(i)}_\alpha\phi_{ij} = -\frac{\hbar\theta}{e\pi}\sum_{j;j\neq i}\epsilon_{\alpha\beta}\frac{(x_i - x_j)^\beta}{|\mathbf{x}_i - \mathbf{x}_j|^2}$$

$$\Rightarrow \frac{1}{2}\epsilon^{\beta\alpha}\nabla^{(i)}_\beta a_\alpha(\mathbf{x}_i) = \frac{\hbar\theta}{e}\sum_j \delta^{(2)}(\mathbf{x}_i - \mathbf{x}_j) \rightarrow \frac{\hbar\theta}{e}n(\mathbf{x}_i)$$



Particle i moving in background of particle j picks up Aharanov-Bohm phase $e^{i\theta}$ — interactions are identical to $\theta = 0$ for all $\theta = 2k\pi$.

- $\theta \to \theta \pm 2\pi$ is a symmetry of the interactions.
- $\theta = \pm\pi$ is a map: Fermions $\leftrightarrow$ Bosons.

In continuum $b := (1/2)\epsilon^{\beta\alpha}\nabla_\beta a_\alpha = \frac{\hbar\theta}{e^2} J^0$

Covariantly: $\mathbf{S}[\mathbf{a}] = -\frac{\mathbf{e^2}}{\mathbf{4\hbar}\theta}\epsilon^{\mu\nu\lambda}\int \mathbf{d^3x}\, \mathbf{a}_\mu \partial_\nu \mathbf{a}_\lambda + \int \mathbf{d^3x}\, \mathbf{a}_\mu \mathbf{J}^\mu$

Two descriptions of the same physics: $\mathbf{\Gamma}[\mathbf{A}]$ and

$$\widetilde{\mathbf{\Gamma}}_\theta[\mathbf{A}, \mathbf{a}] := \mathbf{\Gamma}[\mathbf{A} + \mathbf{a}] - \frac{\mathbf{e^2}}{\mathbf{4\hbar}\theta}\int \mathbf{d^3x}\, \epsilon^{\mu\nu\lambda}\mathbf{a}_\mu \partial_\nu \mathbf{a}_\lambda.$$

For $\theta = 2k\pi$ this is a **symmetry**; $\theta = (2k+1)\pi$ interchanges **Fermions** $\leftrightarrow$ **Bosons**.

Gaussian functional integral over $\mathbf{a}$ gives

$$\mathcal{L}'_{\mathbf{eff}}[\mathbf{A}] = -\frac{\pi'_1}{\mathbf{4}}\mathbf{F^2} - \frac{\pi'_3}{\mathbf{2}}\epsilon^{\mu\nu\lambda}\mathbf{A}_\mu \partial_\nu \mathbf{A}_\lambda$$

Conductivities $\Rightarrow$ $-\frac{\mathbf{1}}{\sigma'} = -\frac{\mathbf{1}}{\sigma} + \mathbf{2}$ $(k = -1)$.

Semi-circle Law

- In general $\beta(\sigma, \bar\sigma)$ and $\beta(\gamma(\sigma), \gamma(\bar\sigma)) = \frac{\mathbf{1}}{(\mathbf{c}\sigma + \mathbf{d})^\mathbf{2}}\beta(\sigma, \bar\sigma)$.
- Change variables from σ to $f(\sigma) := -\frac{\vartheta_3^4\vartheta_4^4}{\vartheta_3^4 - \vartheta_4^4}$. $f(\gamma(\sigma)) = f(\sigma)$ is invariant under $\Gamma_0(2)$

$$\beta_\mathbf{f}(\mathbf{f}, \bar{\mathbf{f}}) := \frac{\mathbf{df}}{\mathbf{ds}}.$$

- Let $q := e^{i\pi\sigma}$, then $f(\bar q) = \overline{f(q)}$ and $\sigma_{xy} \to -\sigma_{xy}$ is $q \to \bar q$.
- Law of Corresponding States $\Rightarrow$ $\overline{\beta_f(f, \bar f)} = \beta_f(\bar f, f)$ $\Rightarrow$ β_f is real if f is real.

Any curve on which f is real is an integral curve of the flow