Angular Momentum as a Fedosov Quantization on a Sphere

Philip Tillman

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ICTP Trieste

0. Overview

- Introduction to Moyal $*$.

- Generalization of Moyal: The Fedosov $*$.

- An example of the the Fedosov $*$ on S^2

1. Introduction

The Moyal $*$:

x^a, p_b are classical variables.

$$f(x,p) * g(x,p) \stackrel{def}{=} f \exp\left(\frac{i\hbar}{2}\omega^{AB}\overleftarrow{\partial}_A \overrightarrow{\partial}_B\right) g$$

$$= \quad fg + \frac{i\hbar}{2}\omega^{AB}(\partial_A f)(\partial_B g) + \cdots$$

where $\partial_A = \frac{\partial}{\partial q^A}$, $q^A = (x^a, p_a)$ and given any function f and g.

$\omega = \omega_{AB} dq^A dq^B = dp_a \wedge dx^a \quad , \quad A, B = 1, \ldots, 2n \quad a, b = 1, \ldots, n$

we note,

$$\left[x^a, x^b\right]_* = [p_a, p_b]_* = 0 \ , \ [x^a, p_b]_* = i\hbar\delta^a_b$$

Ordinary QM:

An arbitrary operator given by

$$\widehat{A} = \sum_{m,n} A_{a_1 \cdots a_m b_1 \cdots b_n} \widehat{x}^{a_1} \ldots \widehat{x}^{a_m} \widehat{p}^{b_1} \ldots \widehat{p}^{b_n}$$

can be mapped to:

$$A = \sum_{m,n} A_{a_1 \cdots a_m b_1 \cdots b_n} x^{a_1} * \cdots * x^{a_m} * p^{b_1} * \cdots * p^{b_n}$$

via the Weyl Transform (invertible and 1-1)

Moyal $*$ reproduces QM:

$$\boxed{\widehat{H} \leftrightarrow H \;,\; \widehat{\rho} \leftrightarrow \rho \;\; \text{(density matrices)} \;,\; Tr\left(\widehat{A}\right) \leftrightarrow \int \frac{\omega^n}{n!} A}$$

2. Generalizing Moyal: The Fedosov $*$

Let M is the phase space associated to N.

Fedosov gives an complicated algorithm to construct $*$ by a perturbative expansion. (I won't go over details –not enough time).

$$D_A, \omega_{AB} \rightarrow \begin{array}{c}\text{at each point introduce} \\ \text{operators } \widehat{y}^A \text{ st. } \left[\widehat{y}^A, \widehat{y}^B\right] = i\hbar\omega^{AB}\end{array}$$

$\rightarrow$ some complicated stuff here $\rightarrow \left[x^a, x^b\right]_*$, $[p_a, p_b]_*$, $[x^a, p_b]_*$

The properties of the Fedosov $*$ are,

- $* : C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$ an associative but noncommutative product.

- Invariant under all smooth coordinate transformations.

- In the limit $\hbar \to 0^{+}$ $*$ becomes the pointwise multiplication of functions M.

- To first order in $\hbar$ the commutator is the Poisson bracket: $[f, g]_{*} = i\hbar \{f, g\} + \mathcal{O}\left(\hbar^{2}\right)$.

- We could construct the Fedosov $*$ perturbatively given any smooth manifold N.

- When the configuration space $N = E^n$ ($\mathbb{R}^n$ with Euclidean metric δ_{ab}) we get the Moyal $*$.

*Note: There are some ambiguities in the construction.

3. An Example: $\mathbf{S}^2$

We calculated the $*$ exactly and got the following commutators,

$$\boxed{\left[x^a, x^b\right]_* = 0 \ , \ [x_a, L_b]_* = i\hbar\varepsilon_{abc}x^c \ , \ [L_a, L_b]_* = i\hbar\varepsilon_{abc}L^c}$$

$$\underline{x} \cdot \underline{x} = 1 \quad , \quad \underline{x} \cdot \underline{L} = 0$$

Most of the ambiguities are related by basis transformations.

$$x^a \rightarrow U * x^a * U^{-1}$$

$$L_b \rightarrow U * L_b * U^{-1}$$

*Note that in general we will have commutators that possibly involve the phase space curvature R and ω.

3. Conclusion

- The Fedosov $*$ seems to be a natural generalization of Moyal.

- We construct an exact solution and write the commutators in the case of S^2.

- Hopefully we can find more exact solutions for other manifolds.

4. References

[1] Boris Fedosov, *Deformation Quantization and Index Theory,* Akademie, Berlin 1996.

[2] M. Gadella, M. A. del Olmo and J. Tosiek, *Geometrical Origin of the $*-$product in the Fedosov Formalism,* hep-th/0405157v1.

[3] J. Hancock, M. Walton, B. Wynder, *Quantum Mechanics Another Way,* physics/0405029v1.

[4] H. Omori, Y. Maeda and A. Yoshioka, *Lett. Math. Phys.* **26**, 285 (1992).

[5] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego 1994

[6] A. Connes, M. Flato and D. Sternheimer, *Lett. Math. Phys.* **24**, 1 (1992).

[7] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.

[8] N. Woodhouse, *Geometric Quantization*, Oxford Univ. Press, New York 1980.

[9] J. E. Moyal, Proc. Camb. Phil. Soc. 45, 99 (1949).