The Abdus Salam
International Centre for Theoretical Physics

SMR 1646 - 15

Conference on
Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and
Non-Commutative Geometry in Condensed Matter Physics and Field Theory
1 - 4 March 2005

Topological spin transport of relativistic particles

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TOPOLOGICAL SPIN TRANSPORT OF RELATIVISTIC PARTICLES

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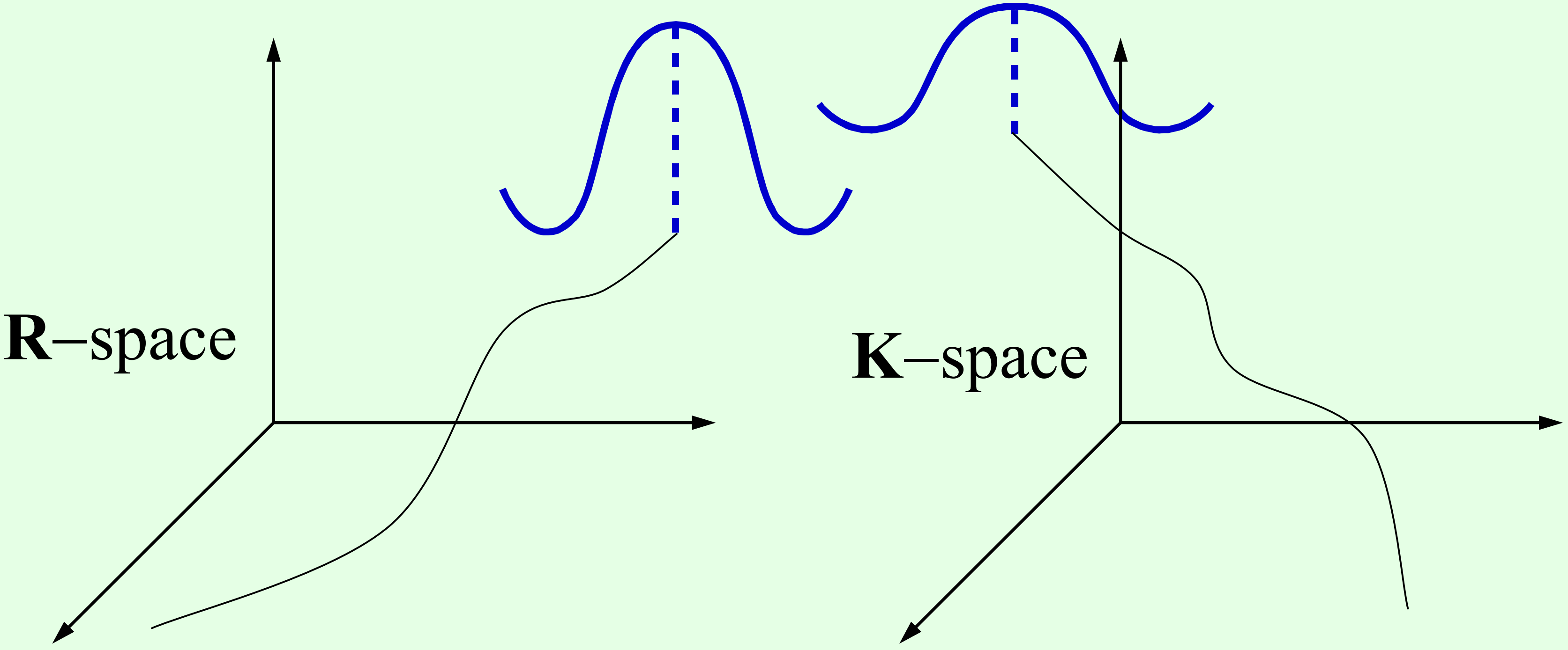
- Introduction
    - topological spin transport
    - general semi-classical motion equations
- Relativistic electron in electromagnetic field
    - diagonalization of Dirac equation
    - Berry’s and total U(2) gauge fields
    - origin of spin-orbit interaction
    - motion equations for relativistic electron
    - simple examples
- Photons in inhomogeneous medium
    - diagonalization of Maxwell equations
    - motion equations for photons
    - spin-orbit interaction of photons
- Conclusions

GENERAL

Semiclassical evolution of a particle with spin

semiclassical particle = wave packet

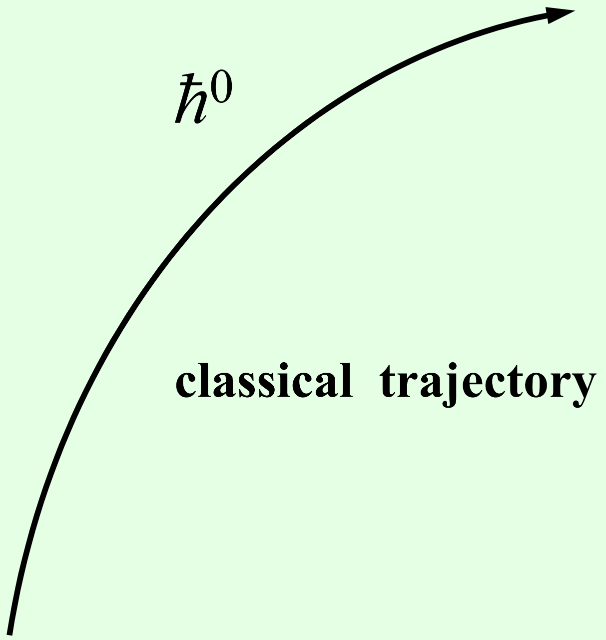


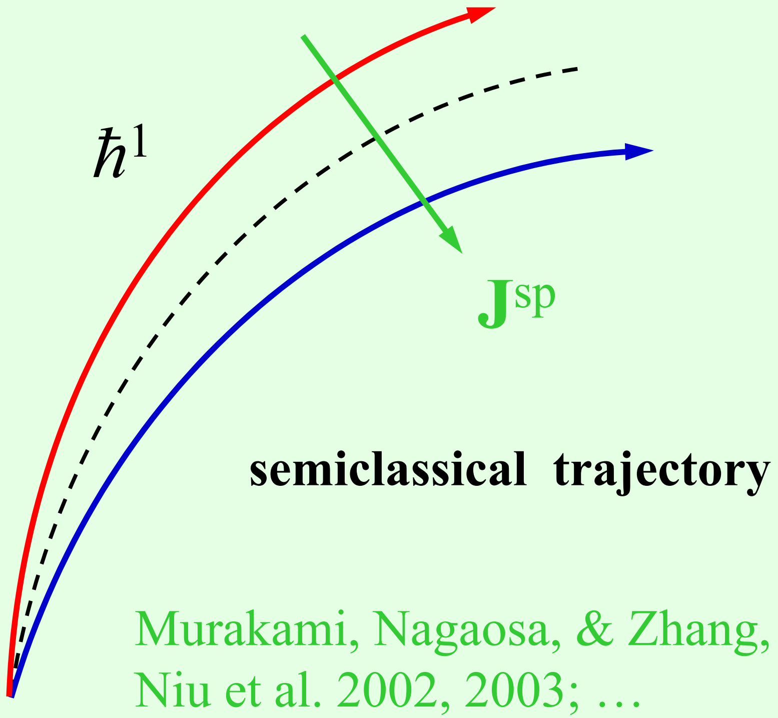
⇒ trajectory of center in classical phase space

Usually: classical trajectories (with accuracy of $\hbar^0$)

phase (polarization) with accuracy of $i\hbar^{-1}(\hbar^0 + \hbar^1)$

Recently: trajectories with accuracy of $\hbar^1$ as well



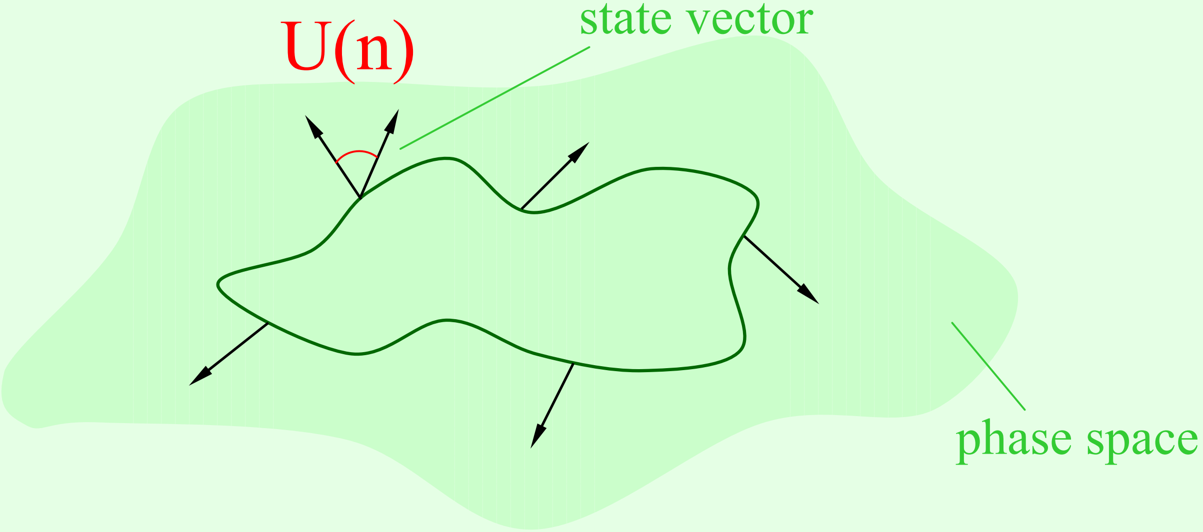


Murakami, Nagaosa, & Zhang, 2003;
Niu et al. 2002, 2003; …

TOPOLOGICAL SPIN TRANSPORT

Parallel transport

$\hbar$-order "spin" terms are connected with geometry of parallel transport of state vector over the phase space.



$\Rightarrow$ (Berry) connection, curvature,

BERRY GAUGE FIELD

Relativistic form of Hamiltonian equations

$$\hat{\mathrm{H}}\left(P^{\alpha}, R^{\alpha}\right) = \hat{H}\left(\mathbf{P}, \mathbf{R}, t\right) - P^{0}c$$ – Hamiltonian

$$\hat{\mathrm{H}}\,\psi = 0$$ – Schrödinger equation

$$\dot{P}^{\alpha} = -\frac{i}{\hbar}\left[P^{\alpha}, \hat{\mathrm{H}}\right]$$

– Hamiltonian equation

$$\dot{R}^{\alpha} = -\frac{i}{\hbar}\left[R^{\alpha}, \hat{\mathrm{H}}\right]$$

$$P^{\alpha} = -i\hbar\partial_{R_{\alpha}}\ , \quad \left[R^{\alpha}, P^{\beta}\right] = i\hbar g^{\alpha\beta}$$ – commutation

metric tensor (– 1,1,1,1)

Introduction of a gauge field

$$P^\alpha \rightarrow \hat{p}^\alpha = P^\alpha - \hat{A}_{r_\alpha}$$

$$R^\alpha \rightarrow \hat{r}^\alpha = R^\alpha + \hat{A}_{p_\alpha}$$

$$\left(\hat{A}_{p_0} \equiv 0\right)$$

canonical variables

covariant variables

gauge potential

$$\left[\hat{p}^\alpha, \hat{p}^\beta\right] = i\hbar \hat{F}_{r_\alpha r_\beta} \qquad \left[\hat{r}^\alpha, \hat{r}^\beta\right] = i\hbar \hat{F}_{p_\alpha p_\beta}$$

$$\left[\hat{r}^\alpha, \hat{p}^\beta\right] = i\hbar g^{\alpha\beta} - i\hbar \hat{F}_{p_\alpha r_\beta}$$

– commutation

gauge field

Semiclassical motion equations

with accuracy of $\hbar$, for classical values:

$$\boxed{\begin{aligned} \dot{p}^{\alpha} &= -g^{\alpha\beta}\partial_{r^{\beta}}\,\mathrm{H} + F_{r_{\alpha}r_{\beta}}\partial_{p^{\beta}}\,\mathrm{H} - F_{r_{\alpha}p_{\beta}}\partial_{r^{\beta}}\,\mathrm{H} \\ \dot{r}^{\alpha} &= g^{\alpha\beta}\partial_{p^{\beta}}\,\mathrm{H} + F_{p_{\alpha}p_{\beta}}\partial_{r^{\beta}}\,\mathrm{H} - F_{p_{\alpha}r_{\beta}}\partial_{p^{\beta}}\,\mathrm{H} \end{aligned}}$$

ordinary part

field terms

where $P^{\alpha}, R^{\alpha} \rightarrow$ classical values,

and $F_{r_{\alpha}r_{\beta}} = (\chi \,|\, \hat{F}_{r_{\alpha}r_{\beta}} \,|\, \chi)$, $\mathrm{H} = (\chi \,|\, \hat{\mathrm{H}} \,|\, \chi)$, etc.

vector of polarization

ELECTRON

Semi-classical diagonalization of Dirac equation

$$\hat{\mathrm{H}}\left(P^{\alpha},R^{\alpha}\right)=\hat{\alpha}_{\beta}\left(P^{\beta}-\frac{e}{c}\mathscr{A}^{\beta}\left(R^{\alpha}\right)\right)c+\hat{\beta}mc^{2}$$

electromagnetic potential

modified Foldy-Wouthuysen transformation: $\psi \to \hat{U}^{\dagger}\psi$

$$\hat{\mathrm{H}} \to \hat{U}\,\hat{\mathrm{H}}\,\hat{U}^{\dagger}$$

$$\hat{U}\left(\tilde{\mathbf{p}}\right)=\frac{\bar{E}_{\tilde{p}}+mc^{2}+\hat{\beta}\hat{\alpha}\tilde{\mathbf{p}}c}{\sqrt{2\bar{E}_{\tilde{p}}\left(\bar{E}_{\tilde{p}}+mc^{2}\right)}}$$

where

$$\tilde{p}^{\alpha}=P^{\alpha}-\frac{e}{c}\mathscr{A}^{\alpha},\quad \bar{E}_{\tilde{p}}=E_{\tilde{p}}-\frac{e\hbar c}{2E_{\tilde{p}}}\hat{\Sigma}\mathscr{H},$$

$$E_{\tilde{p}}=\sqrt{m^{2}c^{4}+\tilde{p}^{2}c^{2}}$$

Adiabatic reduction

$$\begin{pmatrix} + & + & \cdot & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & - & - \\ \cdot & \cdot & - & - \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{matrix} + & + \\ + & + \end{matrix} & 0 \\ 0 & \begin{matrix} - & - \\ - & - \end{matrix} \end{pmatrix}$$

– adiabatic operator

positive (negative) energy 2x2 sectors

$$\Rightarrow \quad \begin{pmatrix} + & + \\ + & + \end{pmatrix}$$

– 2x2 operator for electron

with accuracy of $\hbar$

Covariant coordinates an gauge potentials

$$\hat{p}^{\alpha} = P^{\alpha} - \frac{e}{c}\mathscr{A}^{\alpha} - \hbar\hat{A}_{R_{\alpha}}$$

$$\hat{r}^{\alpha} = R^{\alpha} + \hbar\hat{A}_{P_{\alpha}}$$

– covariant variables
$U(2)$ – invariance

Electromagnetic potential

Berry gauge potential

$$\hat{A}_{P_{\alpha}} = \left(0, \hat{\mathbf{A}}_{\mathbf{P}}\right) \qquad \hat{A}_{R_{\alpha}} = -\frac{e}{c}\left(\partial_{R_{\alpha}}\mathscr{A}^{\beta}\right)\hat{A}_{P_{\beta}}$$

$$\hat{\mathbf{A}}_{\mathbf{P}} = \frac{\left(\tilde{\mathbf{p}}\times\hat{\boldsymbol{\sigma}}\right)c^{2}}{2E_{\tilde{p}}\left(E_{\tilde{p}}+mc^{2}\right)}$$

$\Rightarrow$ 2 gauge potentials: U(1) on r^{α} + SU(2) on (p^{α},r^{α})

$\Rightarrow$ total U(2)=U(1)xSU(2) gauge field on (p^{α},r^{α})

Electron’s Hamiltonian (in covariant variables)

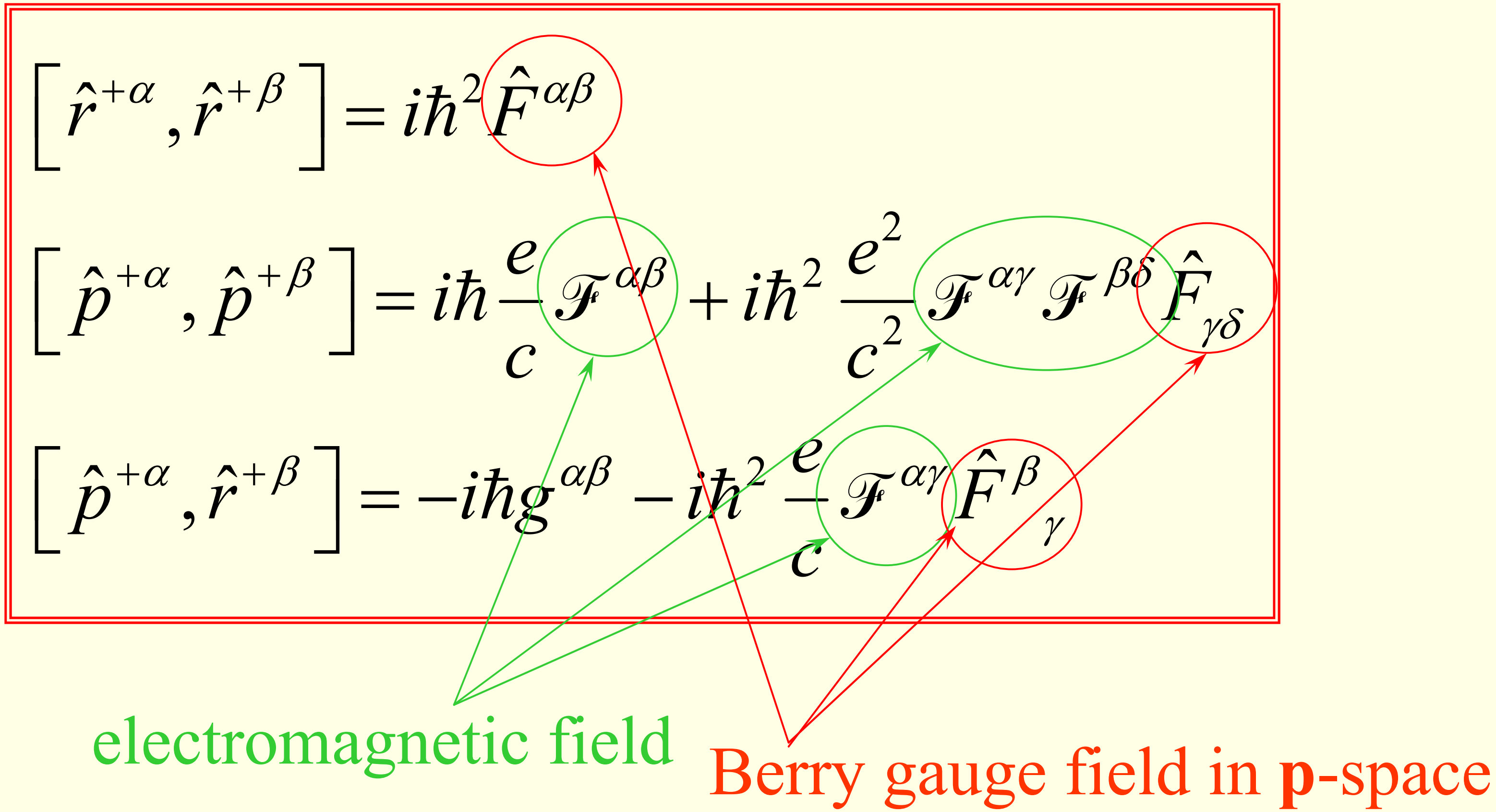
$$\boxed{\hat{\mathrm{H}}\left(\hat{p}^{\alpha},\hat{r}^{\alpha}\right)=\underbrace{E_{\hat{p}}-\hat{p}^{0}c}_{\text{‘free electron’}}+\underbrace{\Delta\hat{E}}_{\text{interaction with magnetic field}}}$$

$$\Delta\hat{E}=-\frac{ec}{E_{\hat{p}}}\left(\frac{\hbar\hat{\boldsymbol{\sigma}}}{2}+\hat{\mathbf{L}}\right)\mathscr{H}$$

– energy of spin and wave packet interaction
with the magnetic field (corrected Zeeman energy)

Chang & Niu, 1996

Commutators of covariant variables and U(2) gauge field



$$\boxed{\begin{aligned}
\left[\hat{r}^{+\alpha},\hat{r}^{+\beta}\right] &= i\hbar^{2}\hat{F}^{\alpha\beta}\\
\left[\hat{p}^{+\alpha},\hat{p}^{+\beta}\right] &= i\hbar\frac{e}{c}\mathscr{F}^{\alpha\beta}+i\hbar^{2}\frac{e^{2}}{c^{2}}\mathscr{F}^{\alpha\gamma}\mathscr{F}^{\beta\delta}\hat{F}_{\gamma\delta}\\
\left[\hat{p}^{+\alpha},\hat{r}^{+\beta}\right] &= -i\hbar g^{\alpha\beta}-i\hbar^{2}\frac{e}{c}\mathscr{F}^{\alpha\gamma}\hat{F}^{\beta}{}_{\gamma}
\end{aligned}}$$

electromagnetic field

Berry gauge field in **p**-space

$$\hat{\mathbf{F}}=-\frac{c^{4}}{2E_{\tilde{p}}^{3}}\left[m\hat{\boldsymbol{\sigma}}+\frac{\left(\hat{\boldsymbol{\sigma}}\tilde{\mathbf{p}}\right)\tilde{\mathbf{p}}}{E_{\tilde{p}}+mc^{2}}\right]$$

– ‘magnetic field’
in **p**-space

Important note

Total U(2) gauge field on the phase space is not the sum of the electromagnetic U(1) and Berry gauge SU(2) fields.

Only potentials are additive values, while the field is a non-linear function of the potential in non-Abelian case.

Hence, we have to calculate the total U(2) field that essentially takes commutators of two potentials into account.

Trajectory motion equations (4D)

electromagnetic force

$$\dot{p}^{\alpha} = -\partial_{r_\alpha} \mathrm{H} + \frac{e}{c}\mathscr{F}^{\alpha\beta}\dot{r}_{\beta}$$

$$\dot{r}^{\alpha} = \partial_{p_\alpha} \mathrm{H} - \hbar F^{\alpha\beta}\dot{p}_{\beta}$$

‘topological Berry force’

Trajectory motion equations (3D)

$$\boxed{\begin{aligned}\dot{\mathbf{p}} &= -\partial_{\mathbf{r}}\Delta E + e\mathcal{E} + \frac{e}{c}\dot{\mathbf{r}}\times\mathcal{H} \\ \dot{\mathbf{r}} &= \frac{\mathbf{p}c^2}{E_p} + \partial_{\mathbf{p}}\Delta E - \hbar\dot{\mathbf{p}}\times\mathbf{F}\end{aligned}}$$

‘Lorentz force’ in **p**-space

$$\mathbf{F}(\mathbf{p},\mathbf{S}) = -\frac{c^4}{2E_p^3}\left[m\mathbf{S} + \frac{(\mathbf{S}\mathbf{p})\mathbf{p}}{E_p + mc^2}\right]$$

unit spin vector

Trajectory motion equations (‘entangled’)

resolving the motion equations with respect to $\dot{\mathbf{p}}$, $\dot{\mathbf{r}}$:

$$\dot{\mathbf{p}} = -\partial_{\mathbf{r}}\Delta E + e\mathcal{E} + \frac{ec}{E_p}\mathbf{p}\times\mathcal{H}$$

$$+\frac{e}{c}\partial_{\mathbf{p}}\Delta E\times\mathcal{H} + \hbar\frac{e^2}{c}\left(\mathbf{F}\times\mathcal{E}\right)\times\mathcal{H} + \hbar\frac{e^2}{E_p}\left(\mathbf{F}\mathcal{H}\right)\mathbf{p}\times\mathcal{H}$$

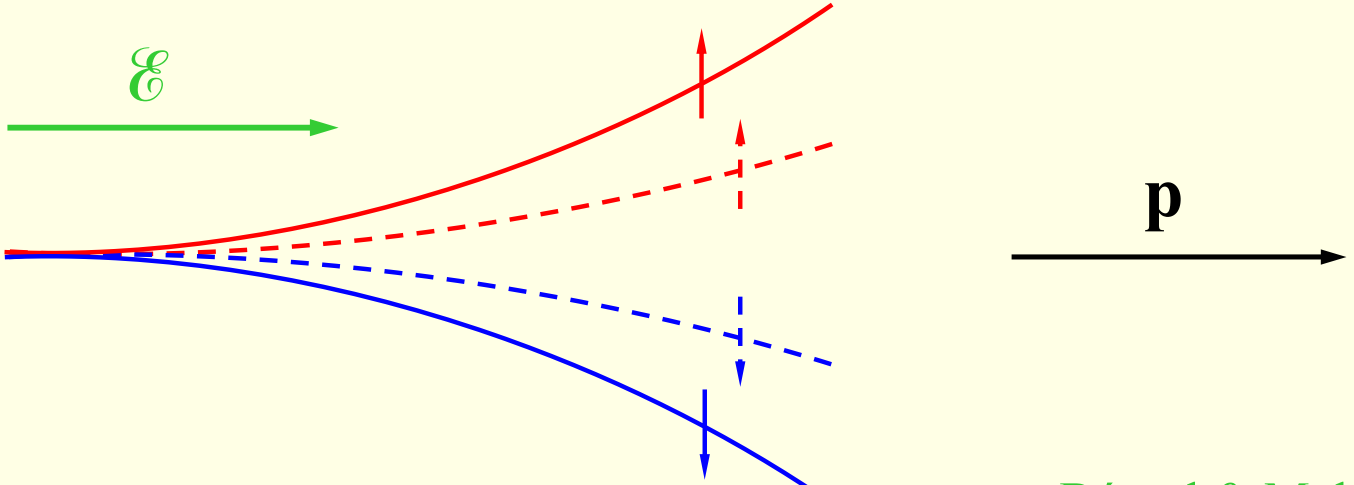
$$\dot{\mathbf{r}} = \frac{\mathbf{p}c^2}{E_p} + \partial_{\mathbf{p}}\Delta E + \hbar e\mathbf{F}\times\mathcal{E} + \hbar\frac{ec}{E_p}\mathbf{F}\times\left(\mathbf{p}\times\mathcal{H}\right)$$

topological terms

Corollary 1: non-relativistic electron in electric field

$$\boxed{\dot{\mathbf{p}} = e\mathcal{E} \qquad \dot{\mathbf{r}} = \frac{\mathbf{p}}{m}\left(1 - \frac{p^2}{2m^2c^2}\right) - \frac{e\hbar}{2m^2c^2}\mathbf{S}\times\mathcal{E}}$$

from Pauli Hamiltonian: $\dot{\mathbf{R}} = \frac{\mathbf{p}}{m}\left(1 - \frac{p^2}{2m^2c^2}\right) - \frac{e\hbar}{4m^2c^2}\mathbf{S}\times\mathcal{E}$



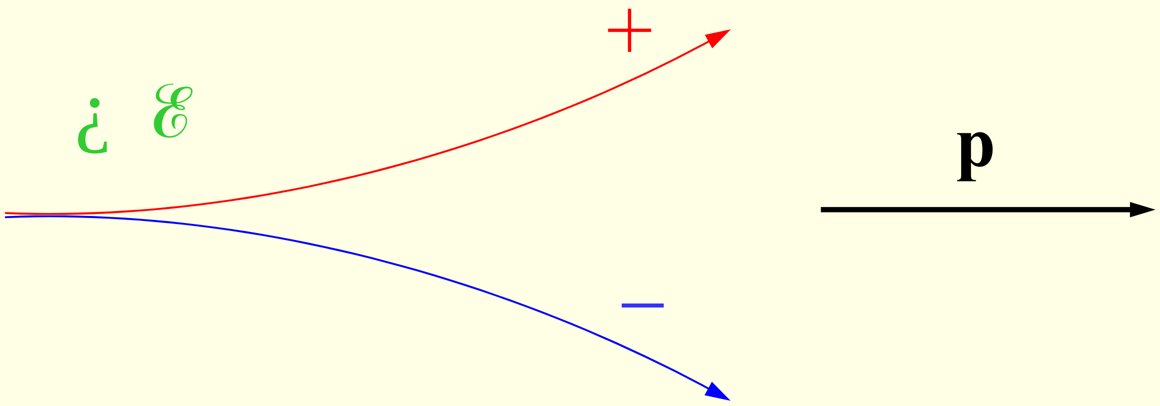
Bérard & Mohrbach, 2004

Corollary 2: ultra-relativistic electron in electric field

$$\boxed{\dot{\mathbf{r}} = c\frac{\mathbf{p}}{p}\left(1-\frac{m^2c^2}{2p^2}\right) - \mu e\hbar\frac{\mathbf{p}\times\mathcal{E}}{p^3}}$$

SP/*2P* – helicity

$$\mathbf{F} = -\mu\frac{\mathbf{p}}{p^3}$$ – magnetic monopole Berry gauge field

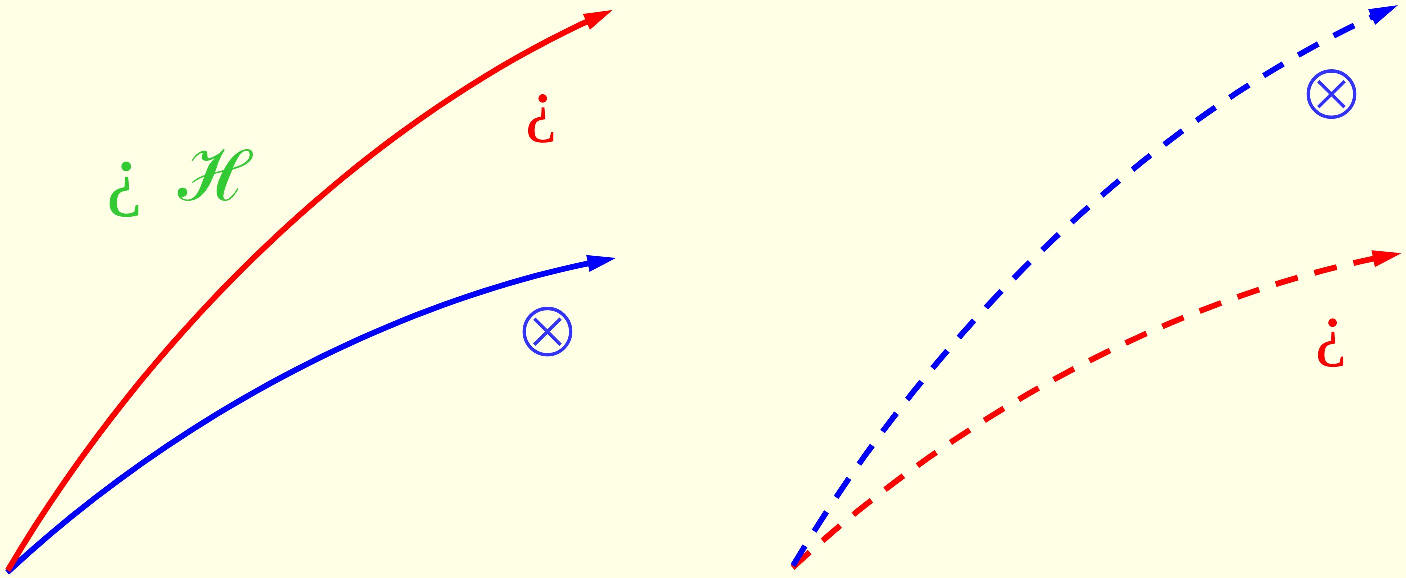


Corollary 3: non-relativistic electron in magnetic field

$$\boxed{\dot{\mathbf{p}} = \frac{e}{mc}\left(1-\frac{p^2}{2m^2c^2} - \nu\frac{e\hbar\mathcal{H}}{2m^2c^3}\right)\mathbf{p}\times\mathcal{H} + \frac{e}{c}\partial_{\mathbf{p}}\Delta E\times\mathcal{H}}$$

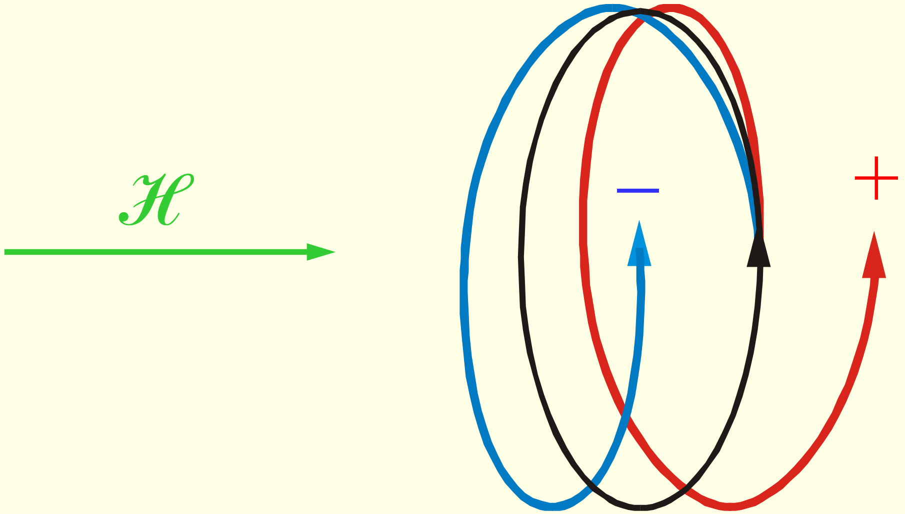
"+"

$\mathbf{S}\mathcal{H}/\mathcal{H}$ – spin projection



Corollary 4: ultra-relativistic electron in magnetic field

$$\boxed{\dot{\mathbf{r}} = c\frac{\mathbf{p}}{p}\left(1-\frac{m^2c^2}{2p^2}\right)+\partial_{\mathbf{p}}\Delta E - \bigcirc\!\!\!\!\!\!\mu\frac{e\hbar}{p^2}\mathcal{H}}$$



What about preceding theories,
Pauli Hamiltonian, spin-interaction et al.?

$$\boxed{\hat{H}^{(\text{Pauli})} = \frac{P^2}{2m} - \frac{e\hbar}{2mc}\hat{\boldsymbol{\sigma}}\mathscr{H} - \frac{e\hbar}{4m^2c^2}\hat{\boldsymbol{\sigma}}\left(\mathscr{E} \times \tilde{\mathbf{p}}\right)} \quad ?$$

Electron’s Hamiltonian (in canonical variables)

$$\boxed{\hat{\mathrm{H}}\left(P^{\alpha},R^{\alpha}\right)=E_{\tilde{p}}-\tilde{p}^{0}c+\Delta\hat{E}\underbrace{-\hbar\hat{\mathbf{A}}_{\mathbf{p}}\dot{\tilde{\mathbf{p}}}}_{\text{topological term}}}$$

or

$$\boxed{\hat{\mathrm{H}}=E_{\tilde{p}}\underbrace{-\frac{e\hbar c}{2E_{\tilde{p}}}\hat{\boldsymbol{\sigma}}\mathscr{H}}_{\text{Zeeman term}}\underbrace{-e\hbar\frac{c^{2}}{2E_{\tilde{p}}\left(E_{\tilde{p}}+mc^{2}\right)}\hat{\boldsymbol{\sigma}}\left(\mathscr{E}\times\tilde{\mathbf{p}}\right)}_{\text{SOI term}}-\tilde{p}^{0}c}$$

$\Rightarrow$ Generalization of Pauli Hamiltonian

Motion equation for polarization

$$\boxed{|\,\dot{\chi}) = i\left[-\Delta\hat{E} + \hat{\mathbf{A}}_{\mathbf{P}}\dot{\tilde{\mathbf{p}}}\right]|\,\chi)}$$ – motion equation

SU(2) evolution of polarization vector $|\chi)$

$$\boxed{|\,\chi) = \mathscr{P}\exp\left[-i\int_{0}^{t}\Delta\hat{E}\,dt' + i\int_{C_{\tilde{\mathbf{p}}}}\hat{\mathbf{A}}_{\mathbf{P}}\,d\tilde{\mathbf{p}}\right]|\,\chi_0)}$$ – solution

chronolodgical ordering

Bolte & Keppeler, 1999

non-Abelian Berry phase

What Spin-Orbit Interaction is?

$$\boxed{\mathrm{SOI} \equiv -e\hbar\hat{\mathbf{A}}_{\mathbf{p}}\mathcal{E}}$$

(without magnetic field: Mathur, 1991
Bérard & Mohrbach, 2004)

~~SOI~~
~~of 'electrodynamical' nature~~

≠

SOI
of topological nature

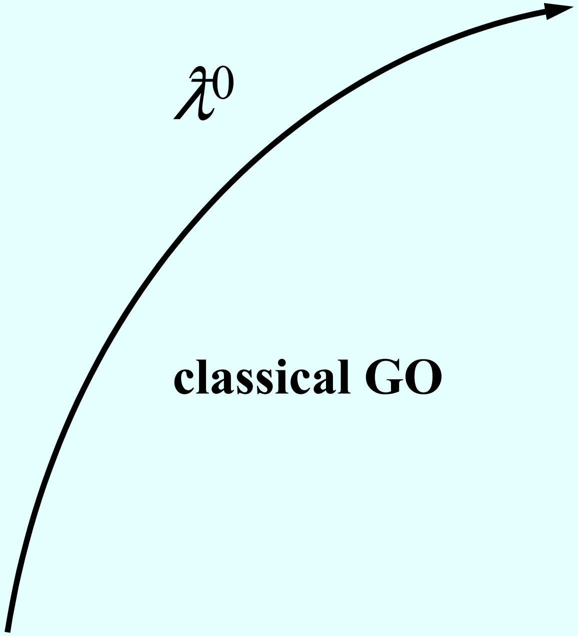
PHOTONS

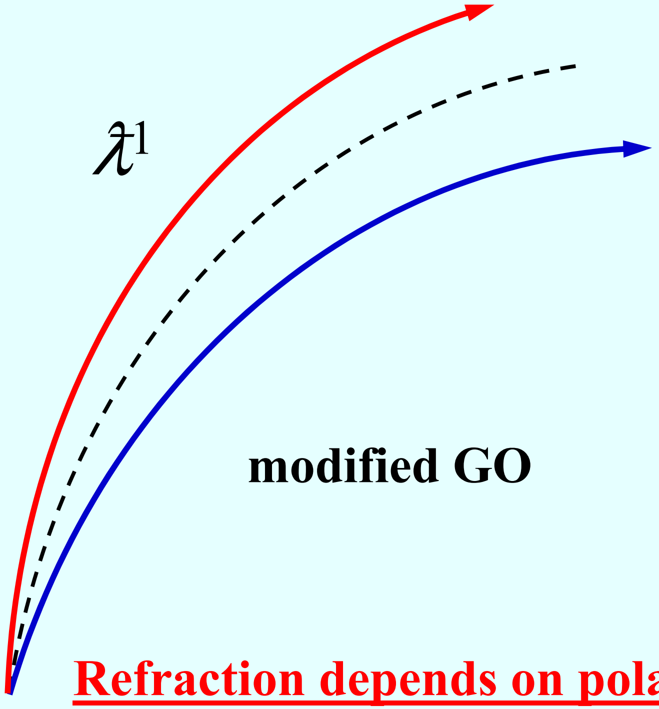
photons = electromagnetic wave

spin state = polarization

$$\hbar \rightarrow \lambdabar = c / \omega$$

semiclassics ⟶ geometrical optics





Refraction depends on polarization!

Maxwell equations

$$\boxed{\left[\mathrm{curl}\,\mathrm{curl} - \bar{\lambda}^2 n^2\right]\mathcal{E} = 0}$$ – Maxwell equations

$$n(\mathbf{R}) = \sqrt{\varepsilon(\mathbf{R})}$$ – refractive index

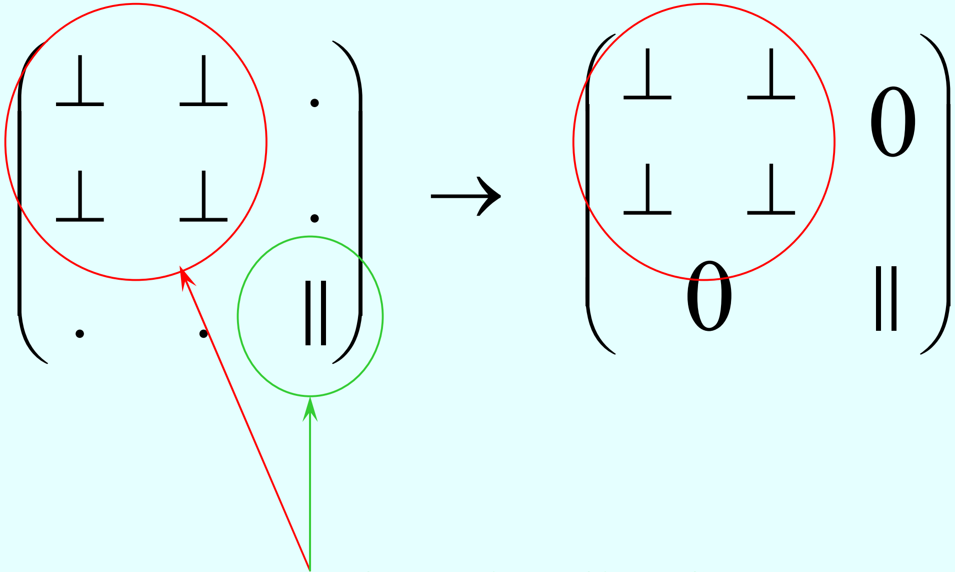
$$\mathbf{P} \equiv \mathbf{p} = -i\bar{\lambda}\partial_{\mathbf{R}}$$ – momentum operator

$$\boxed{\hat{\mathrm{H}}\mathcal{E} = 0\,, \quad \hat{\mathrm{H}}(\mathbf{p},\mathbf{R}) = \left[p^2 - n^2(\mathbf{R})\right] - \hat{Q}(\mathbf{p})}$$

$$\hat{Q}_{ij} = p_i p_j$$ non-diagonal part

Diagonalization and adiabatic reduction

$\mathcal{E} = \hat{U}^{\dagger}\left(\mathbf{p}\right)\mathcal{E}'$ – transformation



$$\begin{pmatrix} \perp & \perp & \cdot \\ \perp & \perp & \cdot \\ \cdot & \cdot & \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \begin{matrix} \perp & \perp \\ \perp & \perp \end{matrix} & 0 \\ 0 & \parallel \end{pmatrix}$$

– adiabatic approximation

transverse (longitudinal) waves

Covariant coordinates and Berry gauge field

$$\begin{pmatrix} \mathcal{E}'_x \\ \mathcal{E}'_y \end{pmatrix} \to \begin{pmatrix} \mathcal{E}'_x - i\mathcal{E}'_y \\ \mathcal{E}'_x + i\mathcal{E}'_y \end{pmatrix}$$ – basis of circular waves $\mu = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\mathbf{r} = \mathbf{R} + \lambda\!\!\!^{-}\mathbf{A}^{\mu}$ – covariant coordinates $\mathrm{U}(1)\times\mathrm{U}(1)$

$$\boxed{\mathbf{F}^{\mu} = \partial_{\mathbf{p}} \times \mathbf{A}^{\mu} = -\mu\frac{\mathbf{p}}{p^{3}}}$$ – Berry gauge field
(‘magnetic monopole’)

$$\boxed{\vartheta_B = \mu \int_{C_{\mathbf{p}}} \mathbf{A} d\mathbf{p}}$$ – Berry phase

Hamiltonian

$$\boxed{\mathrm{H}(\mathbf{p},\mathbf{r}) = c\left[p - n(\mathbf{r})\right]}$$

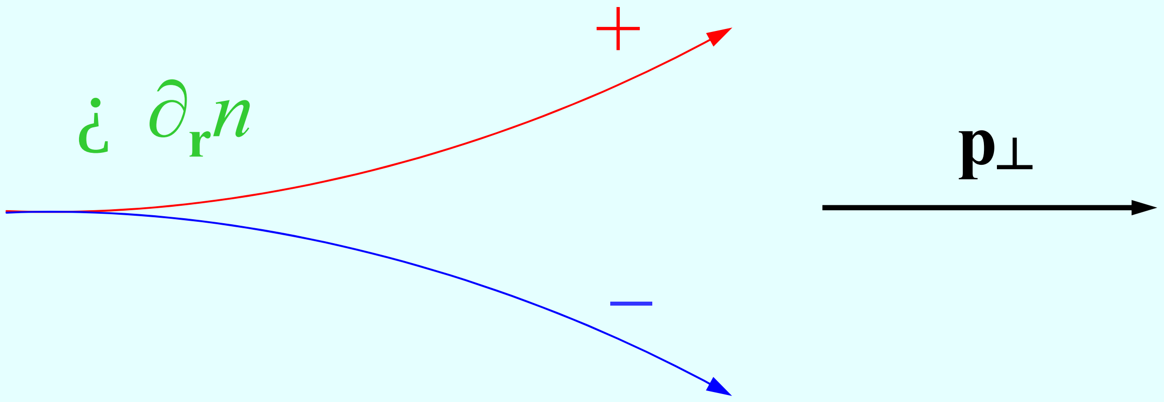
$$\boxed{\left[r_i, r_j\right] = i\lambdabar^2 \varepsilon_{ijk} F_k^{\mu} = -i\lambdabar^2 \mu\, \varepsilon_{ijk} \frac{p_k}{p^3}}$$

– non-commutativity

Motion (ray) equations

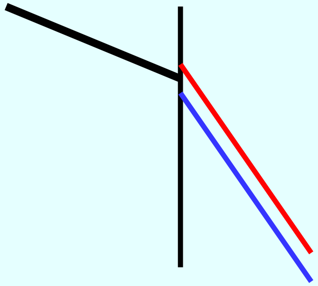
$$\dot{\mathbf{p}} = -\partial_{\mathbf{r}}\,\mathrm{H} \qquad \dot{\mathbf{r}} = \partial_{\mathbf{p}}\,\mathrm{H} + \lambdabar\left(\mathbf{F}^{\mu} \times \dot{\mathbf{p}}\right)$$

$$\dot{\mathbf{p}} = c\,\partial_{\mathbf{r}} n \qquad \dot{\mathbf{r}} = c\frac{\mathbf{p}}{p} - \lambdabar\mu\left(\frac{\mathbf{p}}{p^{3}} \times \dot{\mathbf{p}}\right)$$

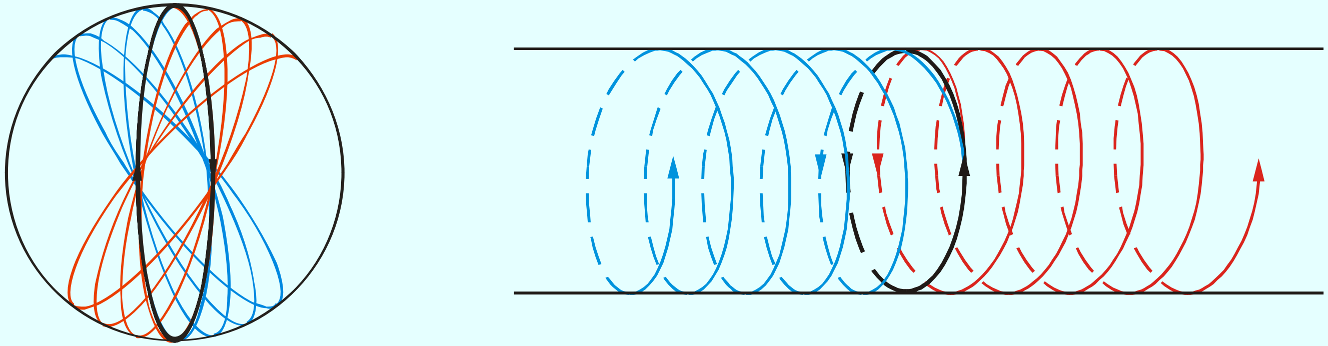


Examples of topological spin transport of photons

1. Ray deflection during refraction at the boundary (correction to the Snellius law)
   Fedorov, 1955; …



2. Spin transport in circular waveguides Zeldovich, 1990; Bliokh & Bliokh, 2004



3. Spin transport in periodic medium Onoda et al., 2004; Bliokh & Freilikher, 2004

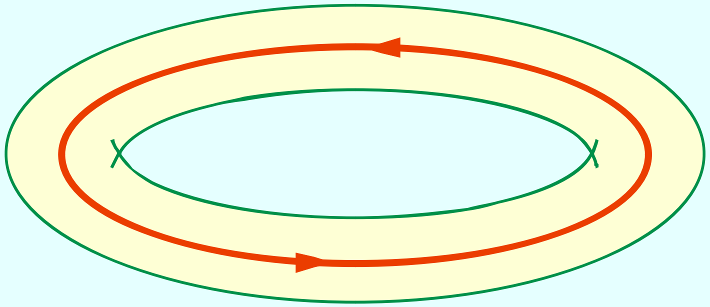
Hamiltonian (in canonical coordinates)

$$\boxed{\mathrm{H}\left(\mathbf{p},\mathbf{R}\right)=c\left[\,p-n\left(\mathbf{R}\right)\right]-\bar{\lambda}\mathbf{A}^{\mu}\dot{\mathbf{p}}}$$

spin-orbit interaction
of photons !

Liberman & Zeldovich, 1992;
Bliokh et al., 2004

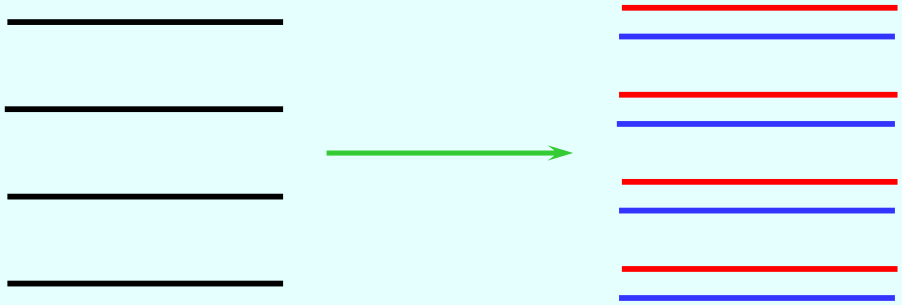
Spin-orbit interaction of photons: fine splitting of levels



– ring resonator

⇒ orbital motion of photons

Taking SOI, $-\lambdabar\mathbf{A}^{\mu}\dot{\mathbf{p}}$, into account leads to the fine splitting of levels of ring resonator:



New quantum number: μm, $\boxed{\text{(helicity)x(orbital moment)}}$

Summary

In summary, the central results of the talk are the consistent semiclassical description and the motion equations for (relativistic) particles with a spin. We have derived **fundamentally new motion equations for a relativistic electron in an external electromagnetic field, and those for photons in a smoothly inhomogeneous dielectric medium.** The equations demonstrate that the additional external field, Berry topological field, affects the particles' motion. This leads to the various manifestations of the single phenomenon: **Topological Spin Transport of Particles**.

The talk is based on the results of the following original papers:

PHOTONS

1. K.Yu. Bliokh and Yu.P. Bliokh, *JETP Lett.* **79**, 519 (2004); K.Yu. Bliokh and Yu.P. Bliokh, *Phys. Rev. E* **70**, 026605 (2004).
2. K.Yu. Bliokh and Yu.P. Bliokh, *Phys. Lett. A* **333**, 181 (2004).
3. K.Yu. Bliokh and V.D. Freilikher, cond-mat/0405384 (to appear in *Phys. Rev. B).*
4. K.Yu. Bliokh and D.Yu. Frolov, physics/0412084.

ELECTRONS AND GENERAL

1. K.Yu. Bliokh and Yu.P. Bliokh, quant-ph/0404144 (to appear in *Ann. Phys. (N.Y.)*)
2. K.Yu. Bliokh, quant-ph/0501183.

THANK YOU !