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Conference on
Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and
Non-Commutative Geometry in Condensed Matter Physics and Field Theory
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Four Dimensional Quantum Hall Effect in Phase Space

(Topological Effects in Spin Orbit Coupling Systems)

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Topological Effects in Spin Orbit Coupling Systems

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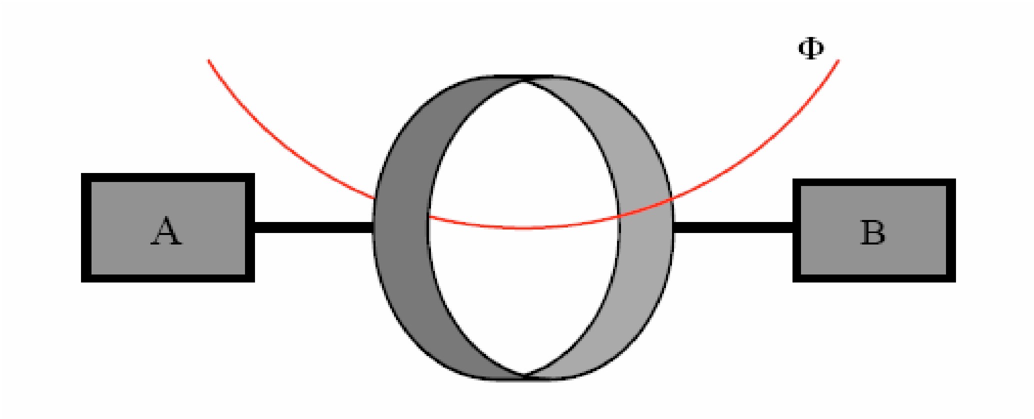
Quantum Hall Effect



Quantum Hall Conductance:

$$n = 2\,\mathrm{Im} < \partial_\phi \psi \mid \partial_\theta \psi >$$

$$\sigma_H = ne^2 / h$$

Collaborators

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Outline

- Motivation
- Spin orbital coupling systems
- Topological orbit angular momentum Hall current in Rasha-Dresselhaus S-O coupling system
- 4D Quantum Hall Effect
- Hidden 4D QHE Structure in Luttinger S-O Coupling systems

Motivation

- New topological effects

- Can we realize 4DQHE in solid state systems?
  --- Emergent physics at low energy

- Connection between 4DQHE and spin orbit coupling systems

Spin Orbit Coupling

- spin orbit coupling

$$H = \frac{\hbar}{4m^2c^2}(\vec{\nabla}V \times \vec{P}) \cdot \sigma$$

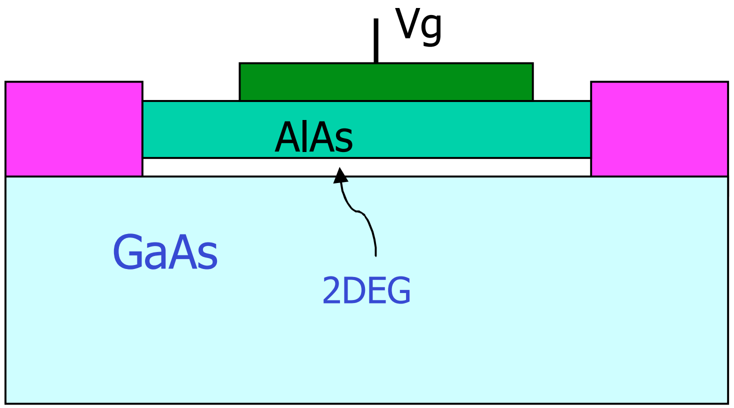
+

Solid state physics: band structure

Rasha-Dresselhaus S-O coupling

- Bychkov-Rashba Spin-Orbital Coupling in two dimensional electron gas
  -- due to confinement in third direction

$$H = \alpha(\vec{P}_y \cdot \sigma_x - \vec{P}_x \cdot \sigma_y)$$



- Dresselhaus Spin Orbital Coupling in two dimensional electron gas
  – due to lacking of inversion symmetry

$$H = \beta(\vec{P}_x \cdot \sigma_x - \vec{P}_y \cdot \sigma_y)$$

- 2 dimension
- Spin-1/2

Luttinger Spin-Orbit coupling

p-orbit (x,y,z)× spin ½ = 6 states

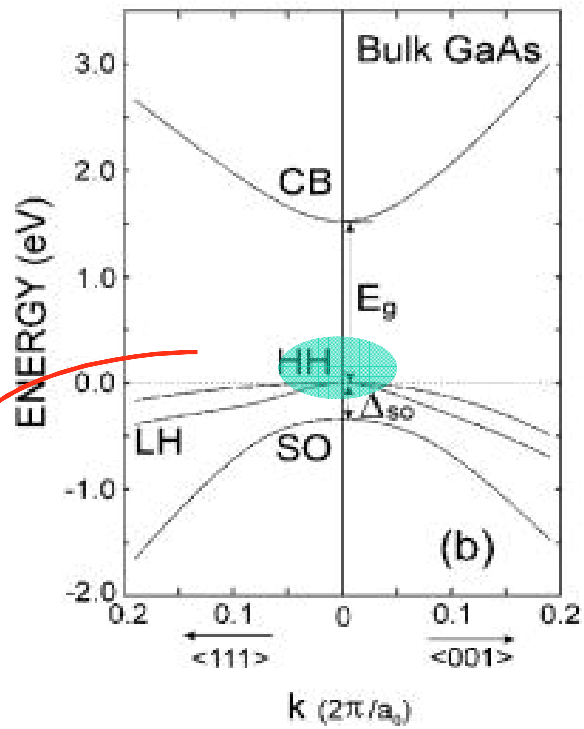
+ spin-orbit coupling

3 bands

Luttinger Hamiltonian

$$H = \frac{1}{2m}\left[\left(\gamma_1 + \frac{5}{2}\gamma_2\right)k^2 - 2\gamma_2\left(\vec{k}\cdot\vec{S}\right)^2\right]$$

($\vec{S}$: spin-3/2 matrix, describing the $P_{3/2}$ band)

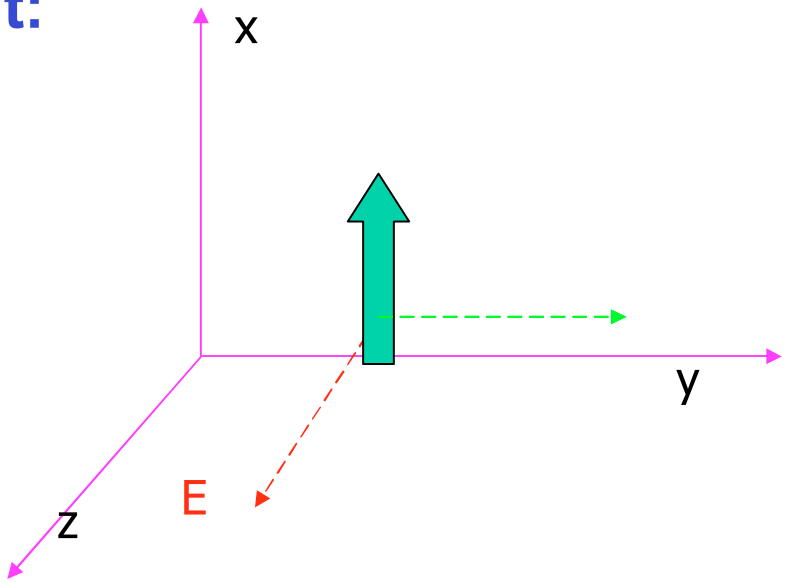


- 3 dimension
- Spin 3/2

Spin Hall Current

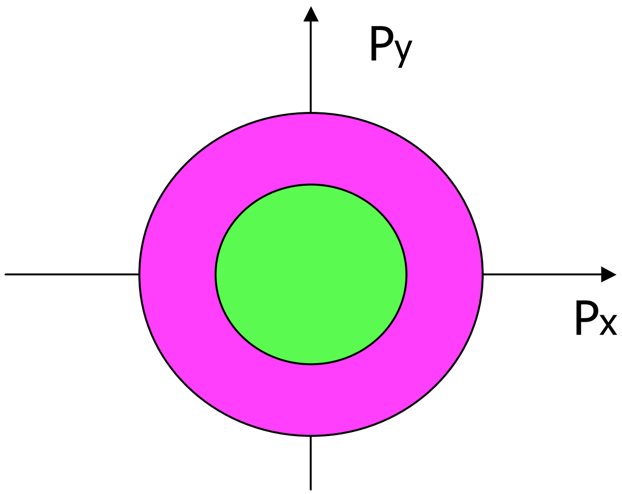
Definition of spin current:

$$J_j^i = \frac{1}{2}\{S_i, V_j\}$$



$$J_j^{\ i} = \sigma_{spin}\, \varepsilon_{ijk} E_k$$

Spin Hall Current in R-D Systems



$$H = \frac{p^2}{2m} + \alpha(p_x\sigma_y - p_y\sigma_x) + \beta(p_x\sigma_x - p_y\sigma_y)$$

$$\sigma_x^z = \begin{cases} -\frac{\mathrm{e}}{8\pi} & \alpha > \beta \\ 0 & \alpha = \beta \\ \frac{\mathrm{e}}{8\pi} & \alpha < \beta \end{cases}$$

Kubo Formula

$$< J_i^j >= \sum \mathrm{Im} \frac{< P(t), \lambda \,|\, J_i^j \,|\, P(t), \lambda' >< P(t), \lambda' \,|\, \frac{\partial}{\partial t} \,|\, P(t), \lambda >}{E_{\lambda'}(p(t)) - E_{\lambda}(p(t))}$$

Orbit Angular Momentum Hall Current

Definition:

$$O_j^i = \frac{1}{2}\{L_i, V_j\} \longrightarrow \theta_x^z = \frac{\mathrm{e}}{8\pi}\frac{\alpha^2+\beta^2}{|\alpha^2-\beta^2|}$$

$$J_j^i = \frac{1}{2}\{S_i, V_j\} \longrightarrow \sigma_x^z = \begin{array}{ll} -\frac{\mathrm{e}}{8\pi} & \alpha > \beta \\ 0 & \alpha = \beta \\ \frac{\mathrm{e}}{8\pi} & \alpha < \beta \end{array}$$

β=0: **[Lz +Sz, H]=0**

α=0 : **[Lz-Sz , H] =0**

Gauge in Momentum Space

- Unitary transformation:

$$H' = U(\vec{k})HU^+(\vec{k}) = \frac{p^2}{2m} + \lambda(p_x, p_y)\sigma_z + U(\vec{k})V(\vec{x})U^+(\vec{k})$$

$$U(\vec{k})V(i\partial_k)U^+(\vec{k}) = V(\vec{D})$$

$$D_i = i\frac{\partial}{\partial k_i} - A_i$$

$$A_i = -iU(\vec{k})\frac{\partial}{\partial k_i}U^+(\vec{k})$$

- Projection to one band in adiabatic limit

$$H_{\pm}^{eff} = \frac{p^2}{2m_\lambda} \pm \lambda(p) + V_{\pm}(\vec{x})$$

$$X_i \equiv D_i = i\frac{\partial}{\partial p_i} - \tilde{A}_i(\vec{p})$$

Nontrivial gauge after Projection

Gauge field: an angular momentum flux

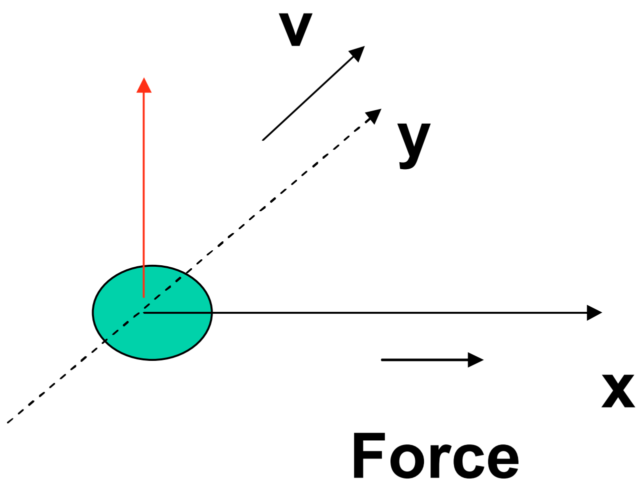
- Gauge field

$$\vec{A} = \langle L_z \rangle \left(\frac{p_y}{p^2}, -\frac{p_x}{p^2}\right)$$

$$\oint \vec{A} \cdot d\vec{p} \neq 0$$

$$F \equiv [x, y] = 0$$

$$\int F dp_x dp_y \neq 0$$



Topological Orbit Hall Current

Derivation of orbit hall current:

$$\int < Lz > \dot{y} dp_x dp_y = -ieE\int dp_x dp_y < Lz > [y, H]$$
$$= eE\int dp_x dp_y < Lz > [y, x]$$

Orbit Hall conductance:

$$\theta_x^z = e\int dp_x dp_y < Lz > [y, x] = \frac{\mathrm{e}}{8\pi} \frac{\alpha^2 + \beta^2}{|\alpha^2 - \beta^2|}$$

Spin Hall conductance from this derivation is zero!
No topological Spin Hall conductance!

Conclusion

- Spin Hall effect in R-D case is not topological effect, However, orbit Hall current is purely topological effect
- Orbit Hall current is larger than spin Hall current in general in R-D case
- Orbit Hall current does not change sign in all coupling parameter region.
- When Rashba coupling is larger than Dresselhaus coupling, spin Hall current and orbit Hall current run in opposite direction. The magnetization direction near edge due to the total angular momentum Hall current is opposite to the prediction from the spin Hall current
- Due to the topological nature, the orbit Hall current is expected to survive in the weak disorder limit.

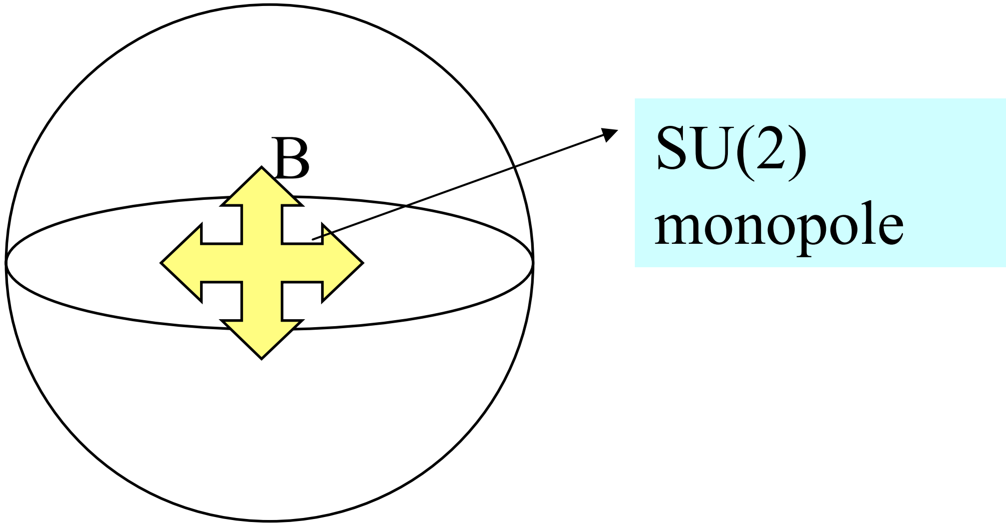
4DQHE

$$H = \frac{\hbar^2}{2MR^2} \sum_{a<b} \Lambda^2_{ab}$$

$$L_{ab} = \Lambda_{ab} - i f_{ab}$$

$$\Lambda_{ab} = -i(x_a D_b - x_b D_a)$$

$$[L_{ab}, L_{cd}] = -i\delta_{bc} L_{ad} + \cdots$$



"Lowest Landau levels" on S^4

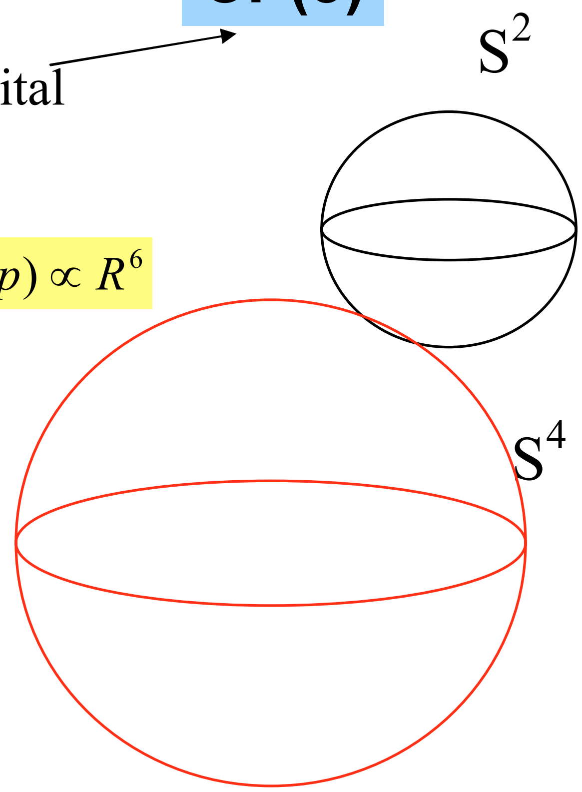
- **Ground degeneracy:** $$D(p) = \frac{1}{6}(p+1)(p+2)(p+3)$$

- **Thermodynamic limit** $$l_0^2 = \lim_{R\to\infty} \frac{R^2}{P}$$

CP(3)

  - The problem is effectively six dimensional, given by orbital space S^4 times the isospin space S^2.
  - Filling fraction is given by $$D(p) \propto p^3 \propto R^6 \Rightarrow N \propto D(p) \propto R^6$$
  - Magnetic length: $$\nu = N / D(p)$$

QH: $$l_0^2 = \frac{\hbar c}{eB} = \frac{R^2}{S}$$



Noncommutative Geometry

- *physics at the lowest laudau level*

- **Noncommutative Geometry in 4D:**

$$[X_\mu, X_\nu] = i l_0^2 \eta_{\mu\nu}^i n^i$$

Magnetic translations commute up to a SU(2) factor.
A noncommutative geometric structure which treats all space coordinates symmetrically

4D Quantum Hall Conductance

- **Effective Hamiltonian at LLL: (No kinetic energy)**

$$H = V(X)$$

- **Quantum Hall Conductance:**

$$\dot{X}_\mu = l_0^{\,2} \eta_{\mu\nu}{}^{i} n^{i} \frac{\partial V(X)}{\partial X_\upsilon}$$

Requirements

- A six dimensional space

- Low energy emergent physics
- **Luttinger spin orbit coupling**

- **A natural choice: Phase space which has six dimensions**

- **Spin Hall effect essentially comes from the physics of 4DQHE**

Gauge structure in momentum space

- Unitary transformation:

$$H' = U(\vec{k})HU^{+}(\vec{k}) = \frac{1}{2m}\left[\left(\gamma_1 + \frac{5}{2}\gamma_2\right)k^2 - 2\gamma_2 k^2 S_z^{\ 2}\right] + U(\vec{k})V(\vec{x})U^{+}(\vec{k})$$

$$U(\vec{k})V(i\partial_k)U^{+}(\vec{k}) = V(\vec{D})$$

$$D_i = i\frac{\partial}{\partial k_i} - A_i$$

$$A_i = -iU(\vec{k})\frac{\partial}{\partial k_i}U^{+}(\vec{k})$$

gauge field in k!
But it is a trivial gauge

- Projection to one band in adiabatic limit

$$H^{\text{eff}} = \frac{k^2}{2m_\lambda} + V(\vec{x})$$

$$X_i \equiv D_i = i\frac{\partial}{\partial k_i} - \widetilde{A}_i(\vec{k})$$

Nontrivial gauge after Projection

- Dynamic equation

$$\dot{X}_i = \frac{k_i}{m_\lambda} - [X_i, X_j]\frac{\partial V}{\partial X_j}$$

$$J_i^j(\lambda) = iTr\{S_{i\lambda}[X_{j\lambda}, X_{k\lambda}]\}eE_k$$

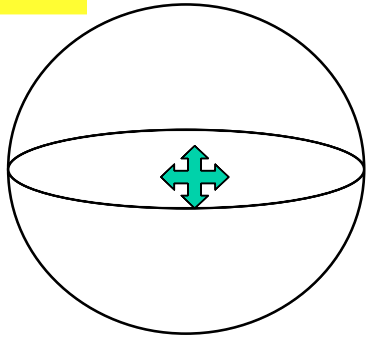
Topological term

Effective Hamiltonian

$$H^{\text{eff}} = \frac{k^2}{2m_\lambda} + V(\vec{x})$$

$$X_i \equiv D_i = i\frac{\partial}{\partial k_i} - \widetilde{A}_i(\vec{k})$$

- The gauge potential rises from the conventional berry phase in adiabatic processes.



(Dirac monopole)

The key mathematical structure is the noncommutative geometry:

$$[x_i, x_j] = -iF_{ij}$$

$$F_{ij} = \varepsilon_{ijk}\lambda\frac{k_k}{k^3}$$

Noncommutative Structure on Phase Space

- Define:

$$X_\mu = (x, y, z, \frac{\lambda}{k}), X_4 = \frac{\lambda}{k}$$

$$\vec{n} = \frac{\vec{k}}{k}$$

- **Noncommutative geometry :**

$$[X_\mu, X_\nu] = i l_0{}^2 \eta_{\mu\nu}{}^i n^i$$

$$l_0 = \lambda \frac{1}{k^2}$$

Effective Hamiltonian For Luttinger Coupling

- Effective Hamiltonian is simply a pure potential in 4d space (no ``momentum”)

$$H^{\mathrm{eff}} = \frac{1}{8m_\lambda X_4^2} + V(X_1, X_2, X_3)$$

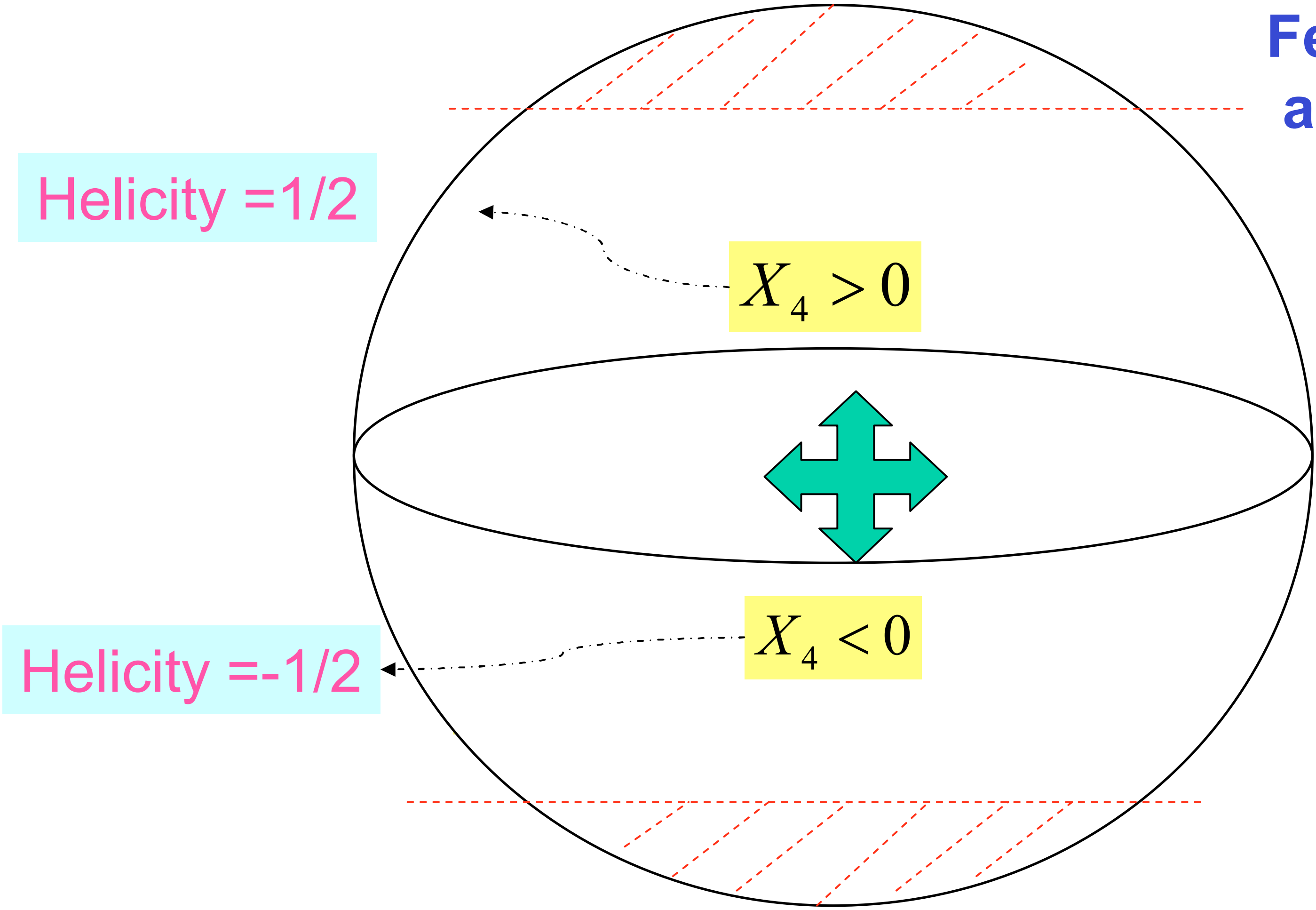
With

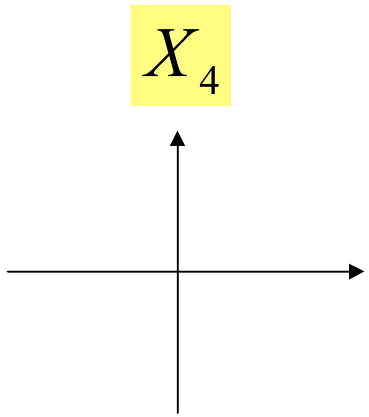
$$[X_\mu, X_\nu] = i l_0{}^2 \eta_{\mu\nu}{}^i n^i$$

An exact formulation of 4DQHE at the LLL!!!

Structure of the 4D liquid

Fermi surface as an edge in 4D





$$H^{\mathrm{eff}} = \frac{1}{8mX_4^2}$$

m is finite

What happens at m = infinite

- m= infinite: band is completely flat. There is no potential, i.e. H=0. $\gamma_1 = -2\gamma_2$

- Based on 4DQHE, a new liquid state (non Fermi liquid) could be formed.

- Since electrons are like membranes in the liquid, excitations with factional statistics could emerge.